Thermal Ads/CFT lecture notes

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16 April 2024

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1 Thermal AdS/CFT

1.1 Introductory Words

In this lecture we will go beyond the conformal and maximally supersymmetric version of the AdS/CFT correspondence. We will introduce an energy scale to the CFT and a black hole to the bulk and observe the consequences.

Lets start with remembering the initial statement of the correspondence.

Holographic Principle: In the context of semi-classical quantum gravity, the information restored in a volume V_{d+1} is encoded in the boundary area A_d. This is motivated by the Bekenstein bound. Bekenstein bound states that the maximum entropy of the entropy stored in a volume V_{d+1} is given as $S = A_d/(4G_N)$ which is measured with respect to planck scale and the Newton constant.

Then we can understand the AdS/CFT correspondence as a realisation of the holographic principle such that the TypeIIB string theory which is reduced to a five dimensional quantum gravity theory by Kaluza-Klein reduction on a five-sphere is related to the four-dimensional conformal field theory which lives on the boundary of the five-dimensional space. The fundamental parameters of the string theory and the field theory is related as

$$g_{YM}^2 = 2\pi g_s \;, \qquad \frac{1}{2g_{YM}^2 N} = \frac{\alpha'^2}{L^4} \tag{1}$$

There is a way to see this correspondence as a strong/weak coupling duality which will interest us more in this lecture. In the strongly coupled large N limit of the gauge theory, string length should be very small compared to the radius of curvature (close string perspective).
To go beyond the conformal realm, first I will explain what I mean by thermal CFTs. Then we will look at the black hole thermodynamics in the AdS which admits the thermal CFT as its boundary. Finally we will bring these two parts together in the ten-dimensional context by using black branes.
In this lectures my main sources are

- Alberto Zaffaroni's lecture notes, *Introduction to the AdS/CFT correspondence*
- Christopher P. Herzog's lecture notes, AdS/CFT
- Gauge/Gravity Duality book by Johanna Erdmenger and Martin Ammon
- Pioneering papers by Juan Maldecena and Edward Witten.
  - J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int.J.Theor.Phys. 38 (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th]
  - E. Witten, "Anti-de Sitter space and holography," Adv.Theor.Math.Phys. 2 (1998) 253–291, arXiv:hep-th/9802150 [hep-th]
  - E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv.Theor.Math.Phys. 2 (1998) 505–532, arXiv:hep-th/9803131 [hep-th]

1.2 Thermal CFT

To break the conformal invariance we introduce a temperature and thermalize the states. This introduces a scale for the energy and breaks the conformal invariance. Practically, we can achieve this by performing Wick rotation and then compactifying the Euclidean time direction on a circle, S^1. The claim is the partition function of the CFT on the manifold $\mathcal{M} \times S^1$ will be equivalent to a thermal partition function.

$$\mathrm{Tr}_{\mathcal{H}}(e^{-\beta \mathcal{H}}) = \int_{\mathcal{M}\times S^1} d[\Phi] e^{-S_E(\Phi)} \ . \tag{2}$$

Here, the relation between the inverse temperature and the radius is given as $2\pi R_{S^1} = \beta = \frac{1}{T}$. By looking at the quantum mechanical example we can demonstrate how the radius of the circle is related to the inverse temperature. Recall that the thermal partition function is given as

$$\mathcal{Z} = Tr(e^{-\mathcal{H}/T}) = \sum_n e^{iE_n/T} \tag{3}$$

is the sum over all eigenstates $|n\rangle$ with the energies E_n. On the other hand, we start with the standard path integral formula with Lorentzian signature, the time evolution expressing the propagator as a sum over paths connecting two points,

$$\langle x|e^{-i\mathcal{H}t}|x'\rangle = \int_{x(0)=x}^{x(t)=x'} [dx(t)] e^{iS[x(t)]} \ . \tag{4}$$

If we wick rotate to Euclidean signature and take the trace we can identify these two expressions.

$$\begin{aligned}
\mathrm{Tr}(e^{-\beta\mathcal{H}}) &= \int dx\, \langle x|e^{-\beta H}|x\rangle \\
&= \int dx \int_{x(0)=x}^{x(-i\beta)=x} [dx(t)] e^{iS[x(t)]} \\
&= \int dx \int_{x(0)=x}^{x(\beta)=x} [dx(t_E)] e^{-S_E[x(t_E)]} \\
&= \int_{x(t_E)=x(t_E+\beta)} [dx(t_E)] e^{-S_E[x(t_E)]}
\end{aligned} \tag{5}$$

where $t = -it_E$ and the final path integral in taken over all paths that are periodic in time. This formula is easily generalized to a quantum field theory, expressing the thermal partition function as a Euclidean path integral with fields that are periodic of period $\beta = \frac{1}{T}$ in the Euclidean time. If we would be more careful, a more complete comparison reveals that we need to compactify the Euclidean time and take periodic boundary conditions for bosons and antiperiodic boundary conditions for fermions. This also indicates that the supersymmetry is broken.
Thermal correlation functions of the thermal CFTs is defined as

$$\langle\mathcal{O}\rangle_\beta = \mathrm{Tr}\left(\frac{\exp(-\beta\mathcal{H})}{\mathrm{Tr}\exp(-\beta\mathcal{H})}\mathcal{O}\right) . \tag{6}$$

Furthermore, the thermal Greens function is defined by

$$G^{\mathcal{C}}(x_1,\cdots,x_n) = \langle T_{\mathcal{C}}\phi(x_1)\cdots\phi(x_n)\rangle_\beta , \tag{7}$$

here, the green's function is defiend for the complex time, t_i. Therefore, how to defien the time ordering is tricky. We are required to choose a curve on the complex plane which is parametrized by the real number and $T_{\mathcal{C}}$ is there to indicate this important detail.
Not much is known about the thermal CFTs yet!

1.3 Thermal AdS

We want to understand the thermal CFTs from the bulk perspective. To have a holographic description of the thermal CFT we consider a black-hole in the AdS_5 because it has the $S^3 \times S^1$ space-time as its boundary.
We start with the following form of the bulk metric.

$$\frac{ds^2}{L^2} = \frac{1}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + b^2 d\Omega_3^2\right) \tag{8}$$

such that

$$d\Omega_3^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1\sin^2\theta_2 d\phi^2 \tag{9}$$

Here, we use the form of the metric which as the inverse of the radius compared to the Poincare patch of the metric, $z = L^2/r$. Therefore, the conformal boundary is at $z = 0$ and the Poincare horizon is at

$z \to \infty$. And L is still the same radius of the curvature. The Einstein equations are

$$R_{AB} = -\frac{4}{L^2} G_{AB} \tag{10}$$

and the $\theta_1\theta_1$ component of this equation gives a first order differential equation for $f(z)$

$$f' - \frac{4}{z} f + \frac{2z}{b^2} + \frac{4}{z} = 0 \tag{11}$$

and the most general solution for $f(z)$ is given as

$$f(z) = 1 + \frac{z^2}{b^2} + cz^4 \tag{12}$$

Note that b is the radius of S^3 and β is proportional to the radius of S^1 as explained in the previous section. Furthermore, we can recover the Poincare patch by taking $b \to \infty$ and we would zoom in to a small region on the S^3. And finally we may introduce a black hole by choosing c. If we choose,

$$c = -\frac{1 + z_h^2/b^2}{z_h^4} \tag{13}$$

then we introduce a black hole into bulk space time such that its horizon is located at

$$g_{tt}(z = z_h) = 0 \tag{14}$$

To obtain the Hawking temperature we demand the Euclidean metric to be regular at $z = z_h$. We also performed Wick rotation.

$$\frac{ds_E^2}{L^2} = \frac{1}{z_h^2}\left(f'(z_h)(z - z_h)d\tau^2 + \frac{dz^2}{f'(z_h)(z - z_h)} + \cdots \right) \tag{15}$$

We introduce a new radial parameter,

$$r = \frac{2L}{z_h\sqrt{|f'(z_h)|}}\sqrt{z_h - z} \tag{16}$$

such that the metric takes the following form

$$ds_E^2 = \frac{f'(z_h)^2}{4} r^2 d\tau^2 + dr^2 + \cdots \tag{17}$$

At this point we can actually see how the period of the radial coordinate enters to the the Euclidean path integral as an overall factor and replaces the inverse temperature in the thermal partition function definition. When we introduce an angular variable with period 2π,

$$d\theta = \frac{|f'(z_h)|}{2} d\tau \tag{18}$$

we can read the Hawking temperature as the inverse of the period of τ variable,

$$T_H = \frac{|f'(z_h)|}{4\pi} \tag{19}$$

$$= \frac{1}{\pi z_h} + \frac{z_h}{2\pi b^2} \tag{20}$$

Observe that every-value of T_H there are two z_h values. The relatively larger value of z_h corresponds to relatively small black hole because the horizon is further from the boundary an has a smaller area.The smaller black holes are similar to the black holes in flat space-time but in this context the larger ones have positive specific heat (and this means they are stable).
We can deepen our understanding by also computing the free-energy from the Euclidean action. In a gravitational theory we have the following description of the partition function.

$$Z_{grav} = e^{-S} \tag{21}$$

We wnat to interpret it as a thermal partition function therefore, we identify

$$S(Euclidean) = \frac{F}{T} \tag{22}$$

We start with the 5d Euclidean gravity action which is given in terms of three-pieces.

$$S_{grav} = S_{EH} + S_{GH} + S_{ctr} \tag{23}$$

We compute each of these contributions one-by-one.

$$S_{EH} = -\frac{1}{2\kappa^2}\int d^5x\sqrt{-G}\left(R + \frac{12}{L^2}\right) \tag{24}$$

$$= -\frac{L^3}{2\kappa^2}\text{Vol}\left(S^3\right)\frac{1}{T}\int_{\epsilon}^{z_h} dz\left(-\frac{8}{z^5}\right)b^3 \tag{25}$$

$$= -\frac{L^3}{\kappa^2}\text{Vol}\left(S^3\right)\frac{1}{T}\left(\frac{1}{z_h^4} - \frac{1}{\epsilon^4}\right)b^3 \tag{26}$$

Here, we took $R = -20/L^2$. Then we move on to Gibbons-Hawking term:

$$S_{GH} = -\frac{1}{\kappa^2}\int d^4x\sqrt{-\gamma}K \tag{27}$$

such that

$$K = \frac{1}{\sqrt{-G}}\partial_z\sqrt{-G}\left(-\frac{z}{L}\sqrt{f}\right) \tag{28}$$

$$= -z^5\partial_z\left(\frac{\sqrt{f}}{z^4}\frac{1}{L}\right) \tag{29}$$

Therefore, the Gibbons-Hawking term of the action is given as

$$S_{GH} = \frac{L^3}{\kappa^2}b^3\text{Vol}\left(S^3\right)\frac{1}{T}\left(z\sqrt{f}\partial_z\left(\frac{\sqrt{f}}{z^4}\right)\right)_{z=\epsilon} \tag{30}$$

$$= -\frac{L^3}{\kappa^2}b^3\text{Vol}\left(S^3\right)\frac{1}{T}\left(\frac{4}{\epsilon^4} + \frac{3}{b^2\epsilon^2} - \frac{2(1+z_h^2/b^2)}{z_h^4} + \cdots\right) \tag{31}$$

Finally the action of the counter-term brings the following expression to the sum.

$$S_{ctr} = \frac{1}{\kappa^2}\int d^4x\sqrt{-\gamma}\left(\frac{3}{L} + \frac{L}{4}R\right) \tag{32}$$

$$= \frac{L^3}{\kappa^2}b^3\text{Vol}\left(S^3\right)\frac{1}{T}\frac{1}{z^4}\sqrt{f}\left(3 + \frac{3}{2}\frac{z^2}{b^2}\right)_{z=\epsilon} \tag{33}$$

$$= \frac{L^3}{\kappa^2}b^3\text{Vol}\left(S^3\right)\frac{1}{T}\left(\frac{3}{\epsilon^4} + \frac{3}{b^2\epsilon^2} + \frac{3}{8}\left(\frac{1}{b^4} - \frac{4}{z_h^4} - \frac{4}{b^2 z_h}\right)\right) \tag{34}$$

We observe that the powers $\frac{1}{\epsilon}$ is divergent but they cancel. Therefore we are left with the total expression,

$$S_{grav} = \frac{L^3}{\kappa^2}\text{Vol}\left(S^3\right)\frac{1}{T}b^3\left(\frac{3}{8b^4} - \frac{1}{2z_h^4} + \frac{1}{2b^2 z_h^2}\right) \tag{35}$$

From the on-shell action we deduce the free-energy for the black hole and the AdS space.

$$F_{bh} = \frac{L^3}{\kappa^2}\text{Vol}\left(S^3\right)b^3\left(\frac{3}{8b^4} - \frac{1}{2z_h^4} + \frac{1}{2b^2 z_h^2}\right) \tag{36}$$

$$F_{Ads} = \frac{L^3}{\kappa^2}\text{Vol}\left(S^3\right)b^3\left(\frac{3}{8b^4}\right) \tag{37}$$

If we consider the difference of these free-energies we obserev the following.

$$\Delta F = \frac{L^3}{\kappa^2}\text{Vol}\left(S^3\right)\frac{b^3}{z_h^4}\left(\frac{z_h^2}{b^2} - 1\right) \tag{38}$$

Then we define the entropy as

$$S = \frac{dF}{dT} = \frac{dF}{dz_h}\left(\frac{dT}{dz_h}\right)^{-1} \tag{39}$$

$$= \frac{L^3}{\kappa^2}\text{Vol}(S^3)b^3\frac{4\pi}{z_h^3} \tag{40}$$

Then heat capacity

$$C = T\frac{dS}{dT} = T\frac{dS}{dz_h}\left(\frac{dT}{dz_h}\right)^{-1} \tag{41}$$

$$= \left(\frac{1}{\pi z_h} + \frac{z_h}{2\pi b^2}\right)\frac{L^3}{\kappa^2}\text{Vol}(S^3)b^3\left(-\frac{12\pi}{z_h^4}\right)\left(-\frac{1}{\pi z_h^2} + \frac{1}{2\pi b^2}\right)^{-1} \tag{42}$$

We realize that when $z_h < b$ blackhole state is favored and when $b < z_h$ thermal AdS is favored. Mention the $b \to \infty$ limit to connect to the last part.

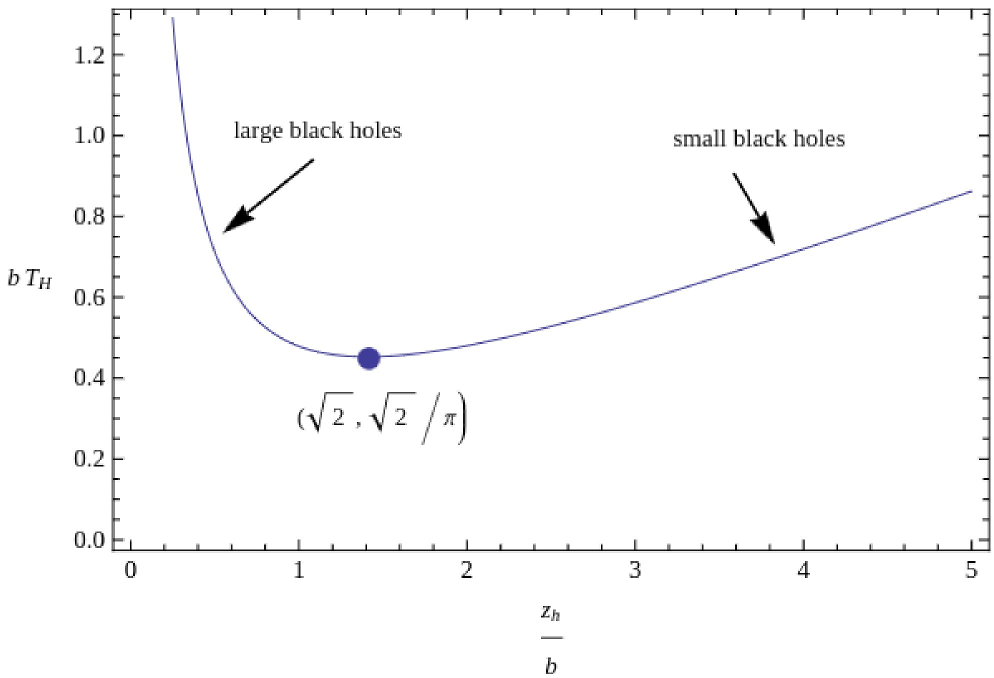
1.4 Concluding remarks - The black three-brane and the thermal SYM

Now we take a brief look at the AdS/CFT correspondence which is describing the $\mathcal{N} = 4$ SYM with the gauge group $SU(N)$ on $\mathbb{R}^3 \times S^1$. Therefore, we start with TypeII-B string theory on $AdS_5 \times S^5$ with N D3-branes.

$$ds^2 = \frac{R^2}{z^2}\left(\sum_{i=1}^{3} dx_i^2 + \left(1 - \frac{z^4}{z_h^4}\right)d\tau^2 + \frac{1}{\left(1 - \frac{z^4}{z_h^4}\right)}dz^2\right) + R^2 d\Omega_5 \tag{43}$$

$R = \sqrt{4\pi g_s N \alpha'}$, The metric is a product of the metric we discussed in the previous section under the limit $b \to \infty$ and the metric of 5-sphere. The geometry has a horizon at $z = z_h$ and the radius is $R_0 = \frac{z_h}{2}$. We already study the thermodynamics of such a system. Here we can combine that analysis with the relations between the gauge theory and quantum gravity parameters. The connection between Einstein terms is given as the following.

$$\frac{1}{(2\pi)^7(\alpha')^4 g_s^2}\int d^{10}x\sqrt{g}R \longrightarrow \frac{R^3\text{Vol}(S^5)}{(2\pi)^7(\alpha')^4 g_s^2}\int d^5x\sqrt{G}R \tag{44}$$



such that

$$\frac{1}{16\pi G_N} = \frac{1}{(2\pi)^7(\alpha')^4 g_s^2} R^5 \pi^3 \tag{45}$$

Then we can use the holographic dictionary to obtain the following relations.

$$\sqrt{x} = \frac{L^2}{\alpha'} \,, \qquad \frac{x}{N} = 4\pi g_s \quad \rightarrow \quad g_s^2 = \frac{R^8}{16\pi^2 N^2 (\alpha')^4} \tag{46}$$

therefore,

$$\frac{R^3}{4G_N} = \frac{N^2}{2\pi} \tag{47}$$

Then we conclude that the free-energy of $\mathcal{N} = 4$ SYM in large N should be

$$F = -\frac{\pi^2}{8} N^2 T^4 V \tag{48}$$

which correctly scales like N^2, which is proportional to the number of degrees of freedom. In a free theory, we could compute the free energy of a gas of free gluons, fermions and scalars in thermal equilibrium at temperature T .

Finally recall that we also can consider the $\mathcal{N} = 4$ SYM on $S^3 \times S^1$. In this case we have two possible bulk geometries as we discussed before. One is the Schwarzschild black hole in AdS with the evert horizon located at z_h. And the other is the thermal AdS for $z_h = 0$. Since both of these solutions also require the same spin structure we conclude that there are two saddle point contributions to the gravitational part integral and we need to figure out which one dominates. Hawking and Page discovered that there is a phase transition with the critical temperature

$$T_c = \frac{3}{2\pi R} \tag{49}$$

such that for $T > T_c$ the black hole solution dominates and the free energy scales as N^2. When $T_c > T$ the thermal AdS solution dominates and free-energy is scaled as $\mathcal{O}(1)$. This is interpreted as a confinement/deconfinement phase transition in the gauge theory side.