Holographic entanglement entropy

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Abstract. These notes review the concept of entanglement entropy within the framework of the AdS/CFT correspondence. The discussion includes an introduction to the Ryu-Takayanagi formula and its subsequent generalizations, which offer a geometric interpretation of entanglement entropy.

Contents

1 Introduction

We begin by presenting the motivations behind the study of entanglement entropy within the holographic framework. These motivations stem from the thermodynamic interpretation of black holes and the evolution of the AdS/CFT correspondence as a powerful tool for probing the underlying degrees of freedom of black holes.

Note that, throughout these notes, all the formulas are given in *natural units*:

$$c = \hbar = k_{\mathrm{B}} = 1\,. \tag{1.1}$$

1.1 Black holes as thermodynamic objects

Although black holes seem very different to thermodynamic systems, certain insights into their behavior suggest intriguing parallels. The key observation is the famous *Bekenstein-Hawking* entropy formula [1, 2], which asserts that the entropy $S_{\rm BH}$ of a black hole is directly proportional to the *area* A_Σ of its event horizon Σ:

$$S_{BH} = \frac{A_\Sigma}{4G_N} \,, \tag{1.2}$$

where G_N represents the Newton constant of gravity. In the classical scenario (i.e., without Hawking radiation), this entropy satisfies the 1st and 2nd laws of thermodynamics:

- **1st law**: (1.2) satisfies

  $$TdS = dM - \Omega dJ \,, \tag{1.3}$$

  where M denotes the mass of the black hole, J its angular momentum, and Ω is the rotational velocity of the horizon. Additionally, one needs to identify the temperature with the *surface gravity* of the black hole:

  $$T \propto \kappa \,, \tag{1.4}$$

- **2nd law**: due to the fact that the event horizon always increases, (1.2) satisfies

  $$\Delta S = S_{\rm final} - S_{\rm initial} \geq 0 \,. \tag{1.5}$$

This is notoriously the first example of a *holographic* formula, as the entropy of the black hole correlates with the *area* of its horizon rather than its *volume*, contrary to naive expectations. One challenge for pushing this line of thoughts further is that it is unclear how to quantify the number of microstates of a black hole within a generic geometry, thus keeping the identification of the surface gravity as temperature an intriguing coincidence.

If the gravitational theory is coupled to a quantum field theory, the entropy (1.2) should be reformulated as [3]

$$S = \frac{A_\Sigma}{4G_N} + S_{\rm out} \,, \tag{1.6}$$

where $S_{\rm out}$ encodes the contribution from the environment, namely the *quantum fields*. This formulation sets the stage for the famous *Hawking information paradox*.

1.2 The AdS/CFT perspective

The problem of counting the microstates of a black hole becomes tractable through the lens of the *AdS/CFT correspondence*, which can be summarized as follows [4]:

> The degrees of freedom of a conformal field theory (*without* gravity) in a flat d-dimensional spacetime are related to the degrees of freedom of a *gravitational* theory in $(d + 1)$-dimensional Anti-de-Sitter spacetime.

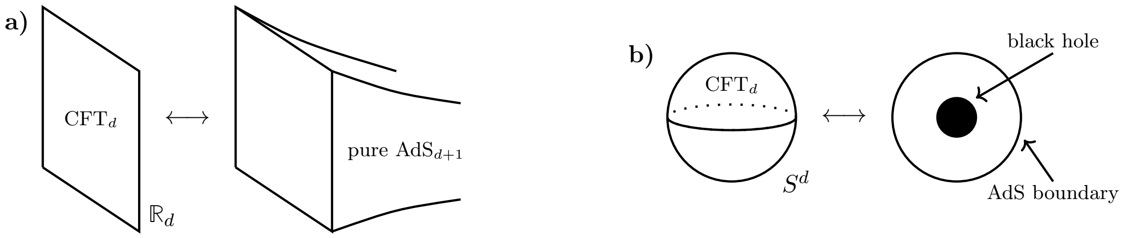

Figure 1: *Two illustrative examples of the AdS/CFT correspondence.* ***a)*** *represents a CFT in d dimensions and in flat Euclidean space, dual to pure AdS spacetime in* $(d+1)$ *dimensions.* ***b)*** *depicts a thermal CFT (with finite density), corresponding to a compactification on a sphere* S^d*, and dual to a black hole in AdS spacetime.*

This duality provides a new way of understanding the entropy of a black hole (and, more broadly, of arbitrary regions in a theory of gravity). For instance, the central charge c of a conformal field theory (roughly representing its number of degrees of freedom) can be correlated with the *geometry* of the AdS spacetime. In the case of $(1+1)$ dimensions, the relation (which will be needed later) is precisely [5]

$$c = \frac{3R}{2G_N^{(3)}}\,, \tag{1.7}$$

where R denotes the radius of the AdS$_3$ spacetime, and $G_N^{(d+1)}$ is the Newton constant in the $(d+1)$-dimensional spacetime. Remarkably, we can view the Bekenstein-Hawking formula (1.2) as a special instance of the AdS/CFT correspondence (in the near-horizon limit).

The aim of these notes is to use the holographic principle to establish a connection between the *entropy of black holes* in gravitational systems and the *entanglement entropy* in quantum field theory. As we will see, this discussion extends far beyond the domain of black holes and even entanglement. The notes here are based on several reviews (see in particular [6–8]).

2 Entanglement entropy in quantum field theory (20 mins)

We start the discussion with an introduction to the concept of entanglement and its associated entropy in quantum systems. This section follows mostly the flow and notation of [7].

2.1 Quantum subsystems

2.1.1 Quantum states

We consider in the following a quantum system U, artificially divided into two *subsystems* A and B (depicted in Figure 2), such that

$$U = A \cup B\,, \tag{2.1}$$

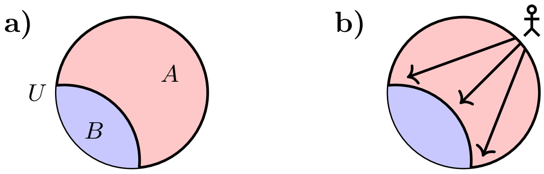

Figure 2: *Figure **a)** illustrates the decomposition of the quantum system U in subsystems A and B, as described in (2.1). **b)** shows the analogy to a black hole, as perceived by an observer located in region A.*

with $B := A^c$ the complement of A. Conceptually, the subsystems should be viewed in the following way: we imagine an observer only having access to rhe subregion A, and who remains oblivious to any signal coming from B. While this setup is more general, an analogy can be drawn between this situation and the scenario of a black hole, with B representing the interior of the black hole horizon (see also Figure b in 2).

A state $|\Psi\rangle$ in the Hilbert space of the full system U can be expressed as a superposition of states residing in regions A and B:

$$|\Psi\rangle = \sum_{m,n} c_{mn} |\psi_m^A\rangle \otimes |\psi_n^B\rangle \ , \tag{2.2}$$

where the coefficients c_{mn} satisfy the normalization condition

$$\sum_{m,n} c_{mn} c_{mn}^* = 1 \, . \tag{2.3}$$

In a pure state, the *quantum state* of the full system is described by the *density matrix* ρ

$$\rho = |\Psi\rangle \langle\Psi| \ . \tag{2.4}$$

Our objective is to describe the quantum state of subsystem A. This state can (always) be expressed as a *mixed state*:

$$|\Psi\rangle = \sum_i p_i |\psi_i^A\rangle |\psi_i^B\rangle \ , \tag{2.5}$$

for which the density matrix of subsystem A reads

$$\rho_A = \sum_i p_i^2 |\psi_i^A\rangle \langle\psi_i^A| \ . \tag{2.6}$$

This is known as *Schmidt decomposition*. Here, the coefficients p_i represent probabilities, reflecting the classical uncertainty about the state of the system. They satisfy the condition

$$\sum_i p_i^2 = 1 \, . \tag{2.7}$$

For an operator $\mathcal{O}_A$ living in subsystem A, the expectation value is given by

$$\langle\, \mathcal{O}_A \,\rangle_A = \mathrm{tr}_A\, \mathcal{O}_A \rho_A = \sum_i p_i^2 \langle\psi_i^A|\mathcal{O}_A|\psi_i^A\rangle \ . \tag{2.8}$$

This can equivalently be computed from the perspective of the full system:

$$\langle\, \mathcal{O}_A \otimes \mathbb{1}\, \rangle_U = \mathrm{tr}_U((\mathcal{O}_A \otimes \mathbb{1})\rho)\,. \tag{2.9}$$

Consequently, the density matrix ρ_A can be expressed as

$$\rho_A = \mathrm{tr}_B\, \rho\,. \tag{2.10}$$

This implies that *any* quantum subsystem can be represented as a mixed state. Conversely, any mixed state of system A can be *purified* by introducing an auxiliary system.

2.1.2 Purification and the thermofield double state

It is apparent from the preceding discussion that a thermal state within a system A can be *purified* by considering a system with *two* copies of the Hilbert space, which we denote by $\mathcal{H}_A$ and $\mathcal{H}_B$. This construct is called a *thermofield double state*, which can be expressed as

$$|\Psi_{\mathrm{TFD}}\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\beta E_n/2}\, |\psi_n^A\rangle \otimes |\psi_n^B\rangle\ , \tag{2.11}$$

where the factor Z_β is the partition function of the thermal state of A, necessary in (2.11) for normalization purposes. This state is a special case of (2.2), and it is important in the context of the AdS/CFT correspondence, where it can be interpreted as two *non-interacting* copies of the same CFT, but entangled at the times $t_A = t_B = 0$ [9].[1] As we have learned in the previous section, acting on $|\Psi_{\mathrm{TFD}}\rangle$ with, say, the Hamiltonian H_A of the subsystem A effectively probes the mixed state of subsystem A.

2.2 Entropy from entanglement

We now consider the question of *entanglement* and its corresponding *entropy* in the subsystems described above.

2.2.1 Entanglement entropy

The (sub)systems A and B are deemed to be in an *entangled state* if

$$|\Psi\rangle \neq |\psi^A\rangle \otimes |\psi^B\rangle\ . \tag{2.12}$$

In words, the systems are not entangled if a (pure) state $|\Psi\rangle$ of the full system U can be expressed as a factorization of states living in A and B.

The *entanglement entropy* associated with this state is defined as

$$S_A = -\,\mathrm{tr}_A\, \rho_A \log \rho_A\,. \tag{2.13}$$

This entropy (also known as the *von Neumann* or *fine-grained* entropy) provides a convenient measure of the degree of entanglement in the wave function $|\Psi\rangle$. More precisely, the quantity e^{S_A} quantifies the *number of entangled states*. Note that (2.13) vanishes for the total system.

[1] We have two times since the spacetime itself is doubled.

We have assumed so far that the Hilbert space is *finite-dimensional*. This is however not true in quantum field theory, and we expect the entanglement entropy to be UV divergent. To address this, we introduce a *UV cutoff* ε_{UV}. Assuming the QFT to be *scale-invariant*, the only scales that we expect to appear in the entropy of a region A are the length of the region L_A and the cutoff ε_{UV}. For the simple case of $d = 2$, it can be shown that the entanglement entropy takes the *universal* form

$$S_A = K \log \frac{L_A}{\varepsilon_{\text{UV}}} + \dots\,, \tag{2.14}$$

where K is a theory-dependent coefficient independent of both L_A and ε_{UV}.

2.2.2 Properties of the entanglement entropy

We now list several properties satisfied by the entanglement entropy (2.13).

- **Triangle inequalities**: for a system $U = A \cup B$, the following inequalities hold:

$$|S_A - S_B| \leq S_U\,, \qquad S_A + S_B \geq S_U\,. \tag{2.15}$$

- **Strong subadditivity**: for given subsystems A, B and C, the entanglement entropy (2.13) satisfies the inequality

$$S_{A \cup B \cup C} + S_B \leq S_{A \cup B} + S_{B \cup C}\,. \tag{2.16}$$

  While not immediately obvious in this form, it can be interpreted as *an observer in A knows more about the system $B \cup C$ than it knows about B alone.*

- **Intensive quantity**: if the full system $U = A \cup B$ is in a pure state, then

$$S_A = S_B\,. \tag{2.17}$$

  Note that this equality is violated at finite temperature $T = \beta^{-1}$.

2.2.3 Examples

We now discuss a couple of examples, that we also consider later in their holographic setups.

$\mathbf{CFT_2}$ in vacuum. We start with the case of a $(1+1)$-dimensional CFT. We define a segment (the region A) at $t = 0$, defined by the points

$$x_1 = -\frac{L_A}{2}\,, \qquad x_2 = +\frac{L_A}{2}\,, \tag{2.18}$$

as depicted in Figure 3. The associated entanglement entropy can be determined using the *replica method*, as computed in [10] (see also [11]). This method consists of calculating the entropy for n copies of the CFT to derive an analytical expression, for which it is possible to take the limit $n \to 1$.

The result follows the universal form mentioned earlier, that only depends on the *central charge* c, the length of the segment L_A, and the UV cutoff ε_{UV}:

$$S_A = \frac{c}{3} \log \frac{L_A}{\varepsilon_{\text{UV}}}\,. \tag{2.19}$$

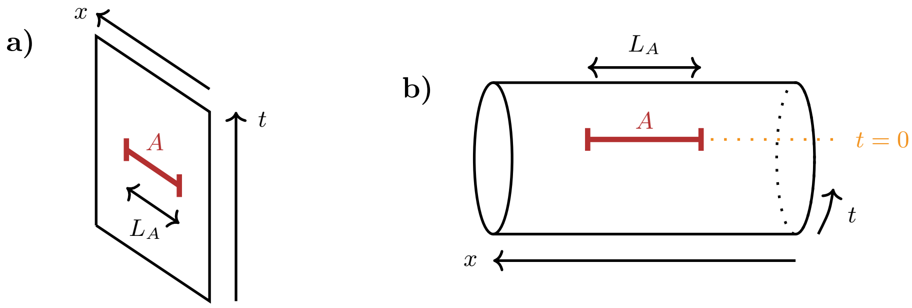

Figure 3: ***a)*** *Illustration of a region A in a two-dimensional CFT, chosen to have length L_A for a fixed time $t=0$.* ***b)*** *In a two-dimensional CFT at finite temperature, the time dimension is compactified and has length $\beta = T^{-1}$. We consider a segment at $t=0$ of length L_A in the x-direction.*

$\mathbf{CFT_2}$ at finite temperature. We now consider a thermal state ρ_A. In two dimensions, this corresponds to a cylindrical spacetime, with $\beta = T^{-1}$ the length of the thermal circle. As mentioned above, this can be described using the thermofield double state formalism. For a segment at $t=0$ of length L_A, the entanglement entropy is given by [10]

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi\varepsilon_{\mathrm{UV}}}\sinh\left(\frac{\pi L_A}{\beta}\right)\right). \tag{2.20}$$

Note that here, we have considered the spatial length of the segment to be much greater than the temperature:

$$\frac{\beta}{L_A} \ll 1\,. \tag{2.21}$$

3 Holographic entanglement entropy (30 mins)

We now shift our focus to entanglement entropy from a holographic point of view. We aim to present a *geometric* interpretation of (2.13) in AdS spacetime, and apply it to the examples previously considered. This section is mostly based on [12] and [6].

3.1 Entropy from spacetime geometry

3.1.1 The Ryu-Takayanagi formula

Our starting point is the *Ryu-Takayanagi formula* (or RT formula), conjectured in [12], to calculate the entanglement entropy of a region A of the boundary CFT in the corresponding AdS dual. For simplicity, we consider the *static* case (i.e., $dt=0$), which allows us to focus on a time slice (say $t=0$). Moreover, we ignore the quantum corrections, and thus set $S_{\mathrm{out}}=0$. The entropy is then given by

$$S_A = \min_{\tilde{A}} \frac{A_{\gamma_A}}{4G_N^{(d+1)}}\,, \tag{3.1}$$

where γ_A is an *extremal surface* of codimension 2 (meaning it has dimension $d-1$). Here, $\min_{\tilde{A}}$ means that we should choose the extremal surface that minimizes the surface of $\tilde{A}$

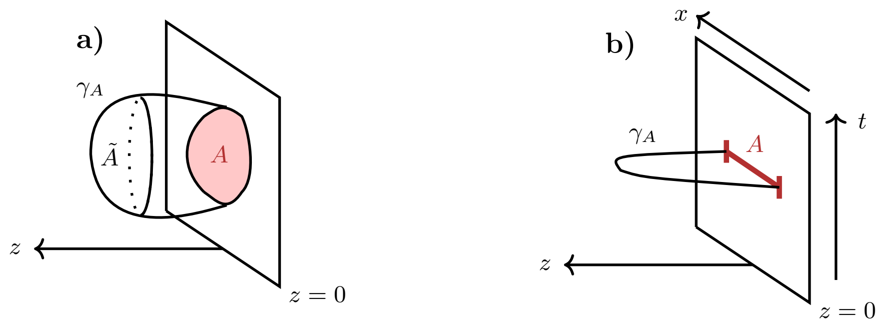

Figure 4: ***a)*** *Illustration of the Ryu-Takayanagi formula given in* (3.1). *For a given region* A *of the* CFT_d*, the entropy of the quantum subsystem is proportional in the holographic description to the area of an extremal surface* γ_A *of dimension* $d-1$*, which coincides on the boundary with the boundary of* A*. As an example, in a CFT living in* $(1+1)$ *dimensions (Figure* ***b)****), the entropy of the region* A *represented here by a segment* $x \in [-L_A/2, +L_A/2]$ *at a fixed time* $t = 0$ *is proportional to the length of the geodesic* γ_A *extending in the* z*-direction.*

(see Figure 4). At the boundary, the surface γ_A coincides with the boundary of region A:

$$\partial\gamma_A = \partial A\,. \tag{3.2}$$

Finally, we impose a *homology* condition: γ_A must be continuously deformable to A.

An intuitive understanding of γ_A can be formulated in the case of AdS_3, where it has dimension 1. In this case, the minimization procedure corresponds to minimizing a *geodesic* action, making A_{γ_A} the *length* of the geodesic (see Figure 4 b).

The RT formula (3.1) is important, as it provides a geometric interpretation of the entanglement entropy (2.13). It suggests that the minimal surface γ_A acts as a *holographic screen* for an observer with access restricted to region A. This proposal exhibits the same properties as the entanglement entropy (see Section 2.2.2).

3.1.2 Examples

We now consider two applications of the Ryu-Takayanagi formula, which can be compared to the results of Section 2.2. These are based on the original paper [12] and on [9].

Pure AdS$_3$. We would like to rederive (2.19) from the holographic dual of CFT_2 in vacuum, i.e., pure AdS_3. In the static case, we can set $t = 0$ and the metric is given by

$$ds^2 = \frac{R^2}{z^2}(dx^2 + dz^2)\,, \tag{3.3}$$

with R the radius of AdS.

We need to calculate the geodesic length between the points located at $x_1 = -L_A/2$ and $x_2 = +L_A/2$. Minimizing the geodesic action

$$S = \int ds\,, \tag{3.4}$$

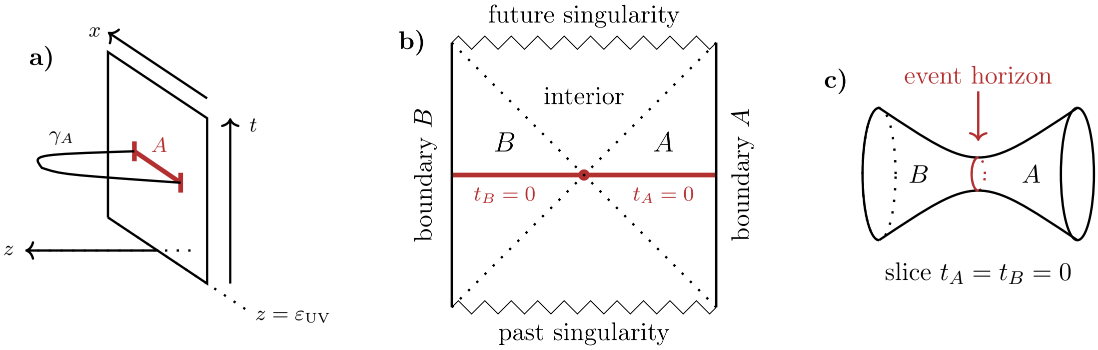

Figure 5: ***a)*** *illustrates the cutoff* ε_{UV} *used to regularize the length* L_{γ_A} *of the geodesic. The dual to an eternal black hole in* AdS_d *is the product of two non-interacting* CFT_d*. In Figure* ***b)****, the Penrose diagram shows how the geometry of* ***c)*** *leads to two boundaries, suggesting the duality to* $CFT_d \times CFT_d$*, i.e., a thermofield double state.*

we find that the solution is a *half-circle* (see Figure 5), and that the *length* L_{γ_A} of the geodesic is given by

$$L_{\gamma_A} = 2R \int_0^\infty \frac{dz}{z} \sqrt{1 + x'(z)^2} \,. \tag{3.5}$$

It turns out that this result is *divergent*, aligning with our expectation from the entanglement entropy in QFT. We therefore regularize the integral by introducing a cutoff $\varepsilon_{\mathrm{UV}}$, such that

$$z \geq \varepsilon_{\mathrm{UV}} \,, \tag{3.6}$$

as illustrated in Figure 5. We obtain

$$L_{\gamma_A} = 2R \log \frac{L_A}{\varepsilon_{\mathrm{UV}}} \,. \tag{3.7}$$

We now only have to input this result in the RT formula (3.1) to find

$$S_A = \frac{R}{2G_N^{(3)}} \log \frac{L_A}{\varepsilon_{\mathrm{UV}}} \,, \tag{3.8}$$

which perfectly matches the CFT_2 result (2.19), after using the $\mathrm{AdS}_3/\mathrm{CFT}_2$ dictionary (1.7).

The eternal AdS black hole. As we have stated before, a black hole in AdS spacetime is dual to a *thermal* CFT. Interestingly, the maximally-extended Schwarzschild solution is dual to a thermofield double state, as it can be seen from Figure 5 b and c. This can be understood as follows: a maximally extended Schwarzschild black hole has *two boundaries* in AdS, and the entropy of the black hole corresponds to the entanglement between the two exteriors.

The extremal surface γ_A in this case is the event horizon of the black hole. The RT formula gives

$$S_A = \frac{A_\Sigma}{4G_N^{(d+1)}} \,, \tag{3.9}$$

which obviously coincides with the Bekenstein-Hawking formula (1.2).

In three dimensions, this situation corresponds to the *BTZ black hole*. The length of the geodesic line can be found in a similar way as for the CFT_2 in vacuum, and reads

$$L_{\gamma_A} = 2R \log\left(\frac{\beta}{\pi\varepsilon_{\mathrm{UV}}} \sinh \frac{\pi L_A}{\beta}\right), \tag{3.10}$$

which, after using the RT formula and (1.7), matches the thermal CFT result (2.20).

3.1.3 Generalized entropy

As mentioned earlier, the RT formula is limited to the static case and does not account for cases with the gravitational theory being coupled to quantum fields, such as Hawking radiation. We discuss here generalizations of the holographic entanglement entropy.

Before proceeding, we give a short historical summary of the developments surrounding the holographic entanglement entropy:

- 2006 (Ryu-Takayanagi, RT): the conjecture of the RT formula for the static case without quantum fields [12];
- 2007 (Hubeny-Rangamani-Takayanagi, HRT): extension of the RT formula to time-dependent situations, still without quantum fields [13];
- 2013 (Lewkowycz-Maldacena, LM): derivation of the RT formula and of certain cases of the HRT extension, extension to quantum fields [14].

We now introduce the concept of *generalized entropy* for black holes [14]:

$$S_{\mathrm{gen}}(\mathrm{BH}) = \frac{A_{\gamma_A}}{4G_N^{(d+1)}} + S_{\mathrm{out}}, \tag{3.11}$$

where S_{out} accounts for the von Neumann entropy contribution from the environment, including quantum fields (also gravitons). It is conjectured that this formula, when applied to the Universe, satisfies the 2nd law of thermodynamics:

$$\frac{d}{dt} S_{\mathrm{gen}}(\mathrm{Universe}) \geq 0\,. \tag{3.12}$$

Here $S_{\mathrm{gen}}(\mathrm{Universe})$ represents the sum of $S_{\mathrm{gen}}(\mathrm{BH})$ for all black holes in the Universe. The most general version of holographic entanglement entropy is then given by

$$S_A = \min_{\tilde{A}} \operatorname{ext}_{\tilde{A}} (S_{\mathrm{gen},\tilde{A}}), \tag{3.13}$$

where $\min_{\tilde{A}} \operatorname{ext}_{\tilde{A}}$ implies that the surface γ_A should be a local minimum under spacelike deformations, a local maximum under timelike deformations, and, if multiple such surfaces exist, the one with the minimum entropy is selected. The boundary condition (3.2), as well as the homology condition, remain applicable. We refer to a surface satisfying these conditions as *quantum extremal surface* (or QES). In the presence of a black hole, the minimal surface tends to envelop the horizon, and in this sense (3.13) can be viewed as a generalization of (1.2).

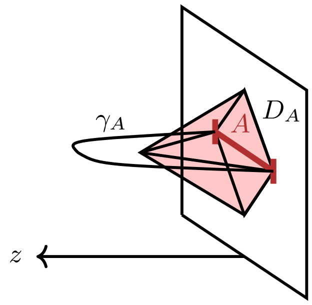

Figure 6: *An illustration of the bulk reconstruction conjecture. Causal wedge reconstruction states that the physics of the causal wedge D_A can be reconstructed from the boundary region A. However, the holographic entanglement entropy formula implies that there exists a bigger region, the entanglement wedge, that can be reconstructed from A.*

3.2 Bulk reconstruction

The AdS/CFT correspondence posits that any physics occurring in the bulk can be reconstructed from the physics of the boundary CFT. However, this dictionary is not yet complete, and one unresolved aspect concerns *causality* (see, e.g., the review [15]).

Consider a region A of the CFT, as discussed in the previous sections. On the boundary, this region defines a *causal diamond* D_A, which includes all states causally related to A. From the AdS perspective, we can define a *causal wedge*, which encompasses all tbe events causally related to the original region A. The *causal wedge reconstruction conjecture* states that any bulk operator within the causal wedge can be reconstructed from operators of the boundary CFT.

However, the RT formula and its generalizations imply the possibility of reconstructing a larger region from A. Indeed, the surface γ_A may extend beyond the causal wedge, forming a new region known as the *entanglement wedge*. The *entanglement wedge reconstruction conjecture* suggests that the reconstructible operators from region A are the operators located in the entanglement wedge, rather than the causal wedge. This conjecture is particularly crucial in the context of the Hawking information paradox, for understanding how the unitary evolution of the CFT translates to its AdS dual, especially in the presence of a black hole.

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