Notes on the Island proposal

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Abstract: In these notes we explore the basics idea of the Island proposal, its relation with AdS/CFT and information paradox. Notes prepared for the Desy theory Workshop (7.05.2024) and Desy theory workshop (14.05.2024).

Contents

1 Introduction and Page curve

In these notes, we will explore the recent *Island* proposal to compute the Entropy of a black hole system. This consist in a formula for the (generalized) von-Neumann entropy [1]

$$S(\rho_{\mathcal{I}}) = \min_{\mathcal{I}} \mathrm{ext}_{\mathcal{I}} \left(\frac{\mathrm{Area}(\partial\mathcal{I})}{4G_N} + S(\tilde{\rho}_{\mathcal{I}\cup R}) \right) . \tag{1.1}$$

The details of this formula will be given later. The above proposal was first motivated by AdS/CFT [1] and then argued from saddles of the gravitational path integral by using the replica trick [2, 3]. The goal of these notes is to give an introduction to the proposal and its use for the computation of the entropy of a black hole system.

Before going on it is worth to remember what the information paradox is and how it is related to the entropy. We remember that the black hole is radiating as first observed by Hawking. This radiation can be understood as a production of particle/anti-particle state near the horizon [4]. We can imagine that the particle escapes from the black hole, while the anti-particle, due to tunneling, is eaten by the black hole (for a more careful analysis based on this idea see [5]). As a conclusion an observer at infinity will effetely experience a thermal state due to the radiation of the black hole. The expected number of radiating particles can be computed, in some approximations, by using a Bose-Einstein statistic in which the temperature is called the Hawking temperature[2]. The Hawking radiation will increase the entropy of the system until the black hole is completely evaporated as pictorially shown in Fig. 1 (blue curve).

Already here, it seems we reached a paradox since the entropy after the complete evaporation of the black hole should be zero because we expect to be in a pure state, nonetheless the Hawking computation shows a constant entropy which strictly greater

[1]We explicitly write here G_N in the formula: in the following we will use $G_N = 1$ where not necessary for the understanding.

[2]A gray body factor is usually necessary to give a realistic quantitative estimation.

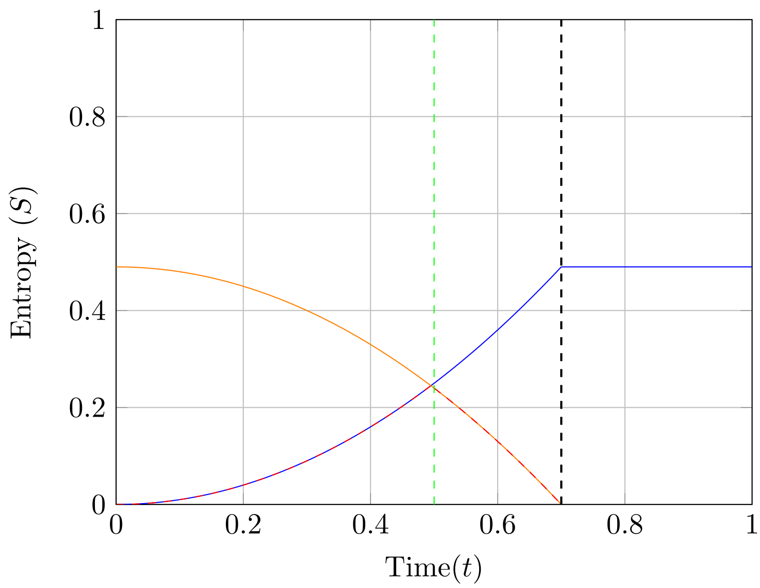


Figure 1: The entropy as a function of time. In blue we pictorially plot the Hawking computation, while in orange the area law. The Page curve is the combination of the two: for $t < t_{\text{Page}}$ (in figure the dashed green line) the entropy follows the Hawking computation, while for $t > t_{\text{Page}}$ the entropy follows the Area law. The black dashed line represent the moment in which the black hole is completely evaporated.

than zero. On the other hand it is possible, by using a thermodynamics interpretation of the black hole to compute its entropy by a very famous formula known as the Area law: this is

$$S_{BH} = \frac{\text{Area}(\partial BH)}{4} \ . \tag{1.2}$$

This is pictorially depicted in Fig.1 (orange curve). It is clear that there is a problem in this picture since after some critical time, known as the Page time (green vertical line in Fig. 1), the entropy of the radiation is bigger then the entropy of the black hole (given by the area law). The contradictions comes from the fact that the Hawking radiation is made out of particles which should be entangled with the particles inside the black hole and therefore we expect the two entropies to be equal. This is one of the incarnations of the information paradox, sometimes called *the entropy paradox*. The expected solution of the problem was given by Page [6]: the correct entropy is expected to follow the Hawking computation up to the Page time and the Area law afterwards (as depicted with a dashed red line in Fig. 1).

The goal of the Islands formula is to give a solution for this problem.

2 The Island proposal

We start describing how the Island proposal works and how it solves the problem. We will present the topic by taking strong inspiration by [7, 8].

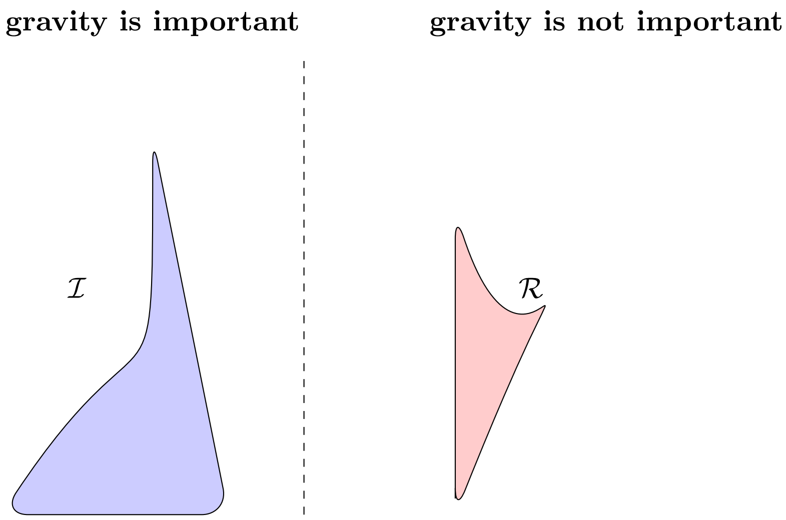


Figure 2: Pictorial representation of the region $\mathcal{R}$ associated with the entropy we want to compute and a region $\mathcal{I}$ which is the Island: see text.

The idea is to distinguish regions of the spacetime where quantum gravity is not contributing dominantly $\mathcal{R}$, for instance very far from the black hole $\mathcal{I}$, and some regions in which gravity is dominant (usually the interior of the black hole itself). We are interested in the von-Neumann entropy associated with the region $\mathcal{R}$, $S(\rho_{\mathcal{R}})$. $\mathcal{R}$ is therefore an input of our problem. $\mathcal{I}$ instead is up to this moment an arbitrary region of the spacetime. The Islands proposal is therefore captured by the following formula for the entropy:

$$S(\rho_{\mathcal{I}}) = \min_{\mathcal{I}} \left(\mathrm{ext}_{\mathcal{I}} \left(\frac{\mathrm{Area}(\partial\mathcal{I})}{4} + S(\tilde{\rho}_{\mathcal{I}\cup R}) \right) \right) \ , \tag{2.1}$$

where we need to distinguish between the exact density matrix of the quantum gravity system ρ, which require the knowledge of the ultraviolet completion of quantum gravity and is therefore a non-accessible information, and the density matrix $\tilde{\rho}$ which is the one computed by standard methods of quantum field theory in curved spacetime. The second term is therefore just the Hawking calculation, but notice that the entropy includes both the region $\mathcal{R}$ and $\mathcal{I}$. The first term instead is reminiscent of the Area law [3] and in fact we will see that it will reproduce the entropy of the black hole for late time ($t > t_{\mathrm{Page}}$). In practice the steps to compute the entropy are the following:

- $\star$ For a generic $\mathcal{I}$ we compute the generalized entropy

$$\frac{\mathrm{Area}(\partial\mathcal{I})}{4} + S(\tilde{\rho}_{\mathcal{I}\cup R}) \ . \tag{2.2}$$

  The first term is just a simple surface area computation, while the second term is a quantum field theory computation since $\tilde{\rho}$ is associated with the density matrix in which gravitational effects are not included but in considering the quantum theory living in curved spacetime.

[3] It is exactly the Area law when $\partial\mathcal{I}$ coincides with the horizon

- ⋆ We need to extremize the generalized entropy computed above with respect to the region $\mathcal{I}$. More precisely the prescription is to maximize in the space direction and maximize in time direction.

- ⋆ If the procedure underlined until here gives more then one region we need to choose the one that minimizes the generalized entropy.

The formula is very similar to the one of the holographic entanglement entropy, but one important difference is that, while in holographic case the region $\mathcal{I}$ is attached to the boundary, here $\mathcal{I}$ is freely floating by itself and this is why it is called the *island*.
The Island proposal is usually related to a correlated but distinct in principle conjecture. In fact observe that the knowledge of ρ could in principle imply that we can reconstruct the physics in region $\mathcal{I}$ ($\mathcal{I}$ is considered now the one that survives the extremization). In fact this second conjecture is something similar to the holographic principle: *all the physics in region $\mathcal{I}$ can be (in principle) reconstruct from $\rho_{\mathcal{R}}$.* It will often be the case that the Island region $\mathcal{I}$ is inside the black hole and therefore this second statement implies that knowing the correct quantum gravity theory in the region $\mathcal{R}$ it is possible to reconstruct the physics in some region inside the black hole.

2.1 Page curve revisited

Let us take into consideration a specific case: a black hole created by the collapse of a star. We consider a region $\mathcal{R}(t)$ in this setup by connecting the space-like infinity with some region very distant from the black hole with respect to the Schwarzschild radius as depicted (in red) in Fig. 3. We are now interested in computing the entropy $S(\rho_{\mathcal{R}})$. Observe that $\mathcal{R} = \mathcal{R}(t)$ is a function of the time. Observe that an extrema for the generalized entropy is always given by the trivial solution in which $\mathcal{I}$ is nothing. This is the simplest case possible in which there is no island. In fact what happend in this case is that $\text{Area}(\partial\mathcal{I}) = 0$ and therefore we have that the first contribution is

$$S^{(\mathcal{I}=0)}(\rho_{\mathcal{R}}) = S(\tilde{\rho}_{\mathcal{R}}) = \text{Hawking result} \ , \tag{2.3}$$

and since $\tilde{\rho}$ is the quantum field theory definition of the density matrix we have that if the island is trivial then the Islands formula simplify to the Hawking computation. In this sense the Islands formula can be thought as a generalization of the Hawking result for the entropy of the radiation.
By solving explicitly for the extremization one realizes that for time t grater then some critical time t_c a new quantum extrema surface appear. This is the first non-trivial island and it is localted near the horizon, in particular slightly inside the black hole.

For this region we have a non-trivial term coming from the area of the boundary of the island $\mathcal{I}$. Since the Island is slightly inside the black hole the area of the boundary of the Island is well approximated by the are of the boundary of the black hole, namely the horizon. Since the black hole is evaporating it is important to fix the time at which we are going to measure the area of the black hole! We will see how to compute this time later, let us just say here that this time depends on the time t defining the region $\mathcal{R}(t)$. Let us focus

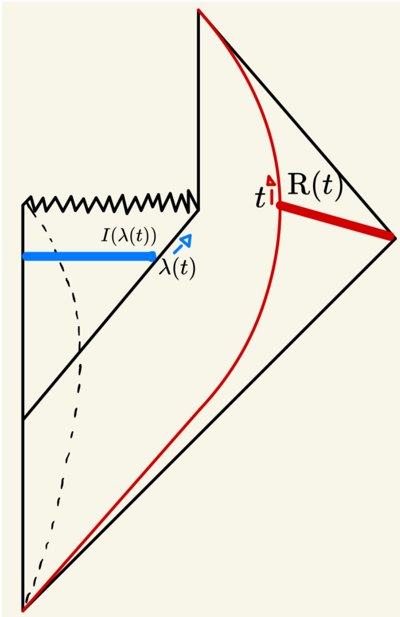


Figure 3: Pictorial representation of a collapsing star generating a black hole. The region $\mathcal{R}$ is depicted in red, while the possible Island $\mathcal{I}$ is in blue.

now on $S(\tilde{\rho}_{\mathcal{I}\cup\mathcal{R}})$. Notice that the each particle radiated by the black hole which is part of the Hawking radiation, there is an entangled anti-particle which is falling in the black hole and exactly lives in region $\mathcal{I}$. This implies that the contribution to the entropy given by the Hawking radiation is perfectly compensated by the contribution given by particle in region $\mathcal{I}$. Therefore we have

$$S^{(\mathcal{I}\neq 0)}(\rho_{\mathcal{R}}) = \frac{1}{4}\text{Area}(\partial I) \sim \frac{1}{4}\text{Area}(\partial BH) \ . \tag{2.4}$$

The conclusion is therefore that

$$S(\rho_{\mathcal{R}}) = \min\{S^{(\mathcal{I}=0)}(\rho_{\mathcal{R}}), S^{(\mathcal{I}\neq 0)}(\rho_{\mathcal{R}})\} = \min\left\{\frac{\text{Area}(\partial BH)}{4}, S_{\text{Hawking}}\right\} \ . \tag{2.5}$$

This prediction perfectly match the expectation by the Page curve. Observe that the non-trivial island appear at $t_c < t_{\text{Page}}$ but it will be subdominant with respect to the Hawking term and only from the page time t_{Page} it will be the contribution dominating the entropy. Let us make some observation:

- The Island formula is consistent with unitarity at late time since the entropy is going to zero while the black hole is evaporating.
- The Page transition is sharp: this is believed to be at this level of approximation (in the sense of the Euclidean path integral). More refined corrections are assumed to smooth out the phase diagram at the Page time.
- From the Island formula one can conclude that the physics in the Island is completely contained inside the region $\mathcal{R}$ (by assuming to know $\rho_{\mathcal{R}}$).

2.2 Where (When) are the Islands? Scrambling of information

Now let us observe that we discussed before that the field's contribution of the Island exactly cancel the Hawking radiation. This means that the amount of information the Island contains is equivalent to the Hawking radiation. The question can be therefore be posed in a different way. Imagine we send a qbit (unit of information) inside the black hole. How much time the qbit is completely *scrambled* in the black hole and is therefore emitted? The answer was partially given by Sekino and Susskind [9] in a famous conjecture. The conjecture is divided in two statement:

- For any quantum system with N degrees of freedom we have that the scrambling time is bounding as
$$t_s \geq \log N \ . \tag{2.6}$$
- Black holes saturate the above bound.

The second statement was also discussed in a paper called *Black holes as mirrors* [10] in which the result happen to be the same. The procedure to derive it is quite different: they imagine to collect radiation until some time $t > t_{\text{Page}}$ and they then imagine to send an extra qbit. The question the authors wanted to answer is: how much time we need to wait until we can read the extra qbit from the radiation? The solution requires the assumption known as *no cloning* which is quite techinical and we are not going to discuss it. However, under the technical assumptions of the paper the solution is that the minimal time after which we expect to read the qbit is

$$t_s = \frac{\beta}{2\pi} \log S \ , \tag{2.7}$$

confirming the conjecture of Sekino and Susskind. Now we will show that this time is connected with the position of the Island. We will argue this in the simplest rotational invariant system in which only rotationally symmetric Cauchy slices are important. Since the area of an infolling lightcone in an evaporating balck hole is monotonically decreasing we can increase the are term

$$\min_{\chi} \frac{\text{Area}(\chi)}{4} \ , \tag{2.8}$$

by pushing the Cauchy slice backwards and outwards along infalling lightray and the maximising Cauchy slice is therefore simply the past lightcone of the boundary. We write the Eddington-Finkelstein coordinates for a static black hole

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2 \ , \qquad f(r) = 1 + \frac{r^2}{l^2} - \frac{16\pi M}{(d-1)\Omega_{d-1} r^{d-2}} \ , \tag{2.9}$$

where $f(r)$ is the solution for the uncharged AdS-Schwarzschild black hole, but the full expression of $f(r)$ is not important in the argument. In the semicalssical limit the evaporation is very slow and we can approximate the metric of the evaporating black hole with the metric for a static balck hole of mass M and Schwarzschild radius r_s slowly varying with infalling time v [4]. In this approximation the radius r_{lc} of an outgoing light-cone satisfies

$$\frac{\mathrm{d}r_{lc}}{\mathrm{d}v} = \frac{f(r)}{2} \sim \frac{2\pi}{\beta}(r - r_s) \ , \tag{2.10}$$

where in the near-horizon region we have

$$\beta = \frac{4\pi}{f'(r_s)} \ . \tag{2.11}$$

We redefine $r' = r_{lc} - r_s$ and we have

$$\frac{\mathrm{d}r'_{lc}}{\mathrm{d}v} \sim \frac{2\pi}{\beta} r'_{lc} - \frac{\mathrm{d}r_s}{\mathrm{d}v} \ . \tag{2.12}$$

At leading order $\mathrm{d}r_s/\mathrm{d}v$ is a constant: we can therefore integrate the above equation to obtain

$$r_{lc} = r_s + Ce^{\frac{2\pi}{\beta}v} + \frac{\beta}{2\pi}\frac{\mathrm{d}r_s}{\mathrm{d}v} \ . \tag{2.13}$$

The choice of C is arbitrary and different choices gives different interesting physics [3], however it is possible to show that the radius r_{lc} of the past lightcone reaches a minimum and begins increasing when it reaches the apparent horizon r_s. This is

$$v = -\frac{\beta}{2\pi}\log\left(\frac{r_s}{\beta\left|\frac{\mathrm{d}r_s}{\mathrm{d}v}\right|}\right) + \mathrm{O}(\beta) \ . \tag{2.14}$$

The only relevant scale is r_s and $\mathrm{d}r_s/\mathrm{d}v = \mathrm{O}(G_N)$ so that

$$v = -\frac{\beta}{2\pi}\log S_{BH} + \mathrm{O}(\beta) \ . \tag{2.15}$$

The island proposal is therefore in agreement with the Sekino-Susskind conjecture. Now we are ready to answer the question where the islands are or better when the island are. In practice we can fix the $\lambda(t)$ function we were keeping general before. We can imagine to send the qbit inside the black hole from infinity. The Island position $\lambda(t)$ coincides to the moment in which the qbit enter the horizon. After $t_s = \beta/(2\pi)\log S$ the black hole will emit the qbit in terms of radiation.

2.3 A simple toy model

There is a simple toy model that describes the physics of the island: this is given by the tensor network in Fig. 5. Here we distinguished four regions:

† R is the region where we collected radiations;

† B is the region far from the black hole but not included in the region R;

† G is a region in which gravity is important but it is not inside the black hole;

† I is the interior of the black hole.

[4] This is sometimes called the Vaidya metric.

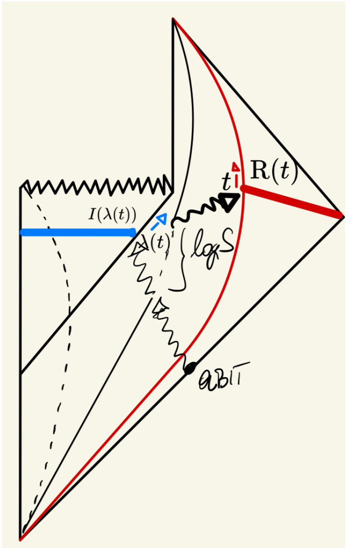


Figure 4: Pictorial representation of the position of the island in the simplest case. The scrambling time $\log(S)$ is perfectly predicted by the island proposal.

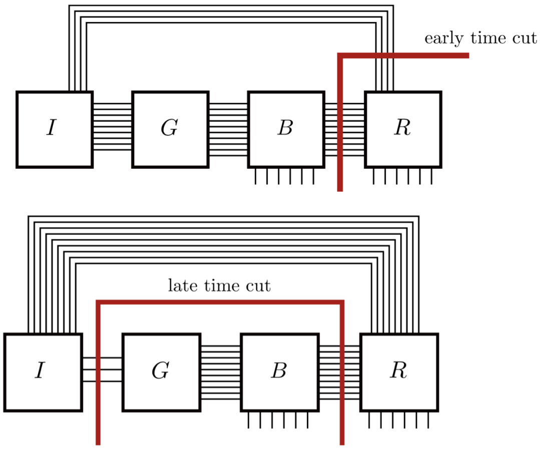


Figure 5: Tensor network describing the island proposal. Figure taken from [11].

The degrees of freedom are here pictorially represented with the lines. In particular the lines going down from R and B represents the field content of the theory, while the lines among the regions represents the gravitational effects which contributes in the entropy as area terms. The Hawking radiation connects the region I to the region R: the evolution of the system will remove some line from R to B since the black hole is evaporating and therefore the area is diminishing, but the amount of Hawking radiation is increasing. There is a simple rule to compute the entropy of a tensor network: this is to cut the region we are interested in and the entropy will simply be the number of connections we have to cut in this procedure. Observe that at early time we can simply cut the lines that contributes for the Hawking radiation. However at late time it is more convenient to cut as in Fig.

5 leaving connected the region I and the region R. In this way we don't have to cut the lines corresponding to the Hawking radiation, but we have to cut the contribution of the area of the black hole, predicting that the late time entropy coincide with the area law. The transition between the two cuts coincide with the Page transition time since it is the moment in which the black hole entropy start be less then the Hawking radiation entropy.

2.4 Example: AdS$_2$/CFT$_1$ example

To give a concrete example we will review the AdS$_2$/CFT$_1$ example given in [12]. This example has three different interpretations if the matter is chosen properly:

- ⋆ two-dimensional gravity: A two dimensional gravity theory coupled to a two dimensional field theory (in our case a two dimensional CFT).
- ⋆ three-dimensional gravity: Three dimensional graivty theory in AdS$_3$ with a dynamical boundary (Planck brane) on part of the space and with rigid boundary on the rest. This interpretation is close to the Randall-Sundrum model.
- ⋆ quantum mechanics:A two dimensional field theory (CFT in our case) with non-conformal boundary degrees of freedom. Arguably this can be though as the fundamental view-point.

Let us be more concrete. Since gravity is topological in two-dimensions, we need to add non-trival dynamics by introducing a dilaton field ϕ. By shifting the dilaton properly, it is possible to absorb the topological Einstein-Hilbert term so that the action is

$$\mathcal{A}[g_{ij}^{(2)},\phi,\chi] = \int d^2y\, \sqrt{-g}\left[\frac{1}{16\pi G_N^{(2)}}\phi R^{(2)} + U(\phi)\right] + \mathcal{A}_{CFT}[g_{ij}^{(2)},\chi]\ . \tag{2.16}$$

If we want this theory to have a holographic dual in AdS$_3$, we need to consider $1 \ll c \ll \phi/(4G_N^{(2)})$ to ensure the correctness of semi-classical limits in two dimensions and have a large radius dual in AdS$_3$. The complete embedding of this two-dimensional theory in the three-dimensional gravity dual is completely discussed in [12].

We go directly to the computation of the fine-grained entropy of the quantum mechanical system by using the extremization prescription. The generalized entropy takes the form

$$S_{\text{gen}}(y) = \frac{\phi(y)}{4G_N^{(2)}} + S_{\text{Bulk}}[\mathcal{I}_y]\ , \tag{2.17}$$

where $\mathcal{I}_y$ is an interval from the point y to the boundary of the two-dimensional space. The term $S_{\text{Bulk}}[\mathcal{I}_y]$ is the von Neuman entropy of that interval. Note that $\phi(y) = \text{Area}^{(2)}$: in fact in two dimensions the area of a point is the coefficient of the curvature term in the action. We need now to extremize this surface. Since the model has also a three-dimensional holographic interpretation, the term $S_{\text{Bulk}}[\mathcal{I}_y]$ can be also be computed by using the Ryu-Takayanagi prescription,i.e.

$$S_{\text{gen}}(y) \sim \frac{\phi(y)}{4G_N^{(2)}} + \frac{\text{Area}^{(3)}(\Sigma_y)}{4G_N^{(3)}}\ , \tag{2.18}$$

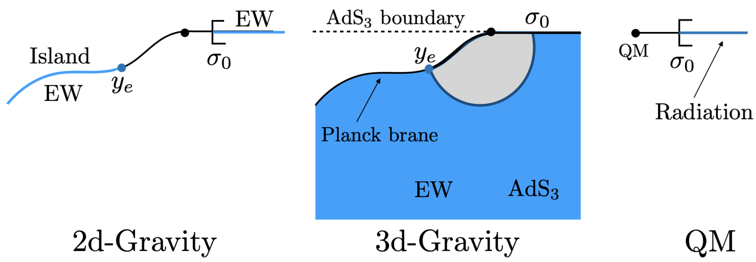


Figure 6: Pictorial representation of the two-dimensional setup, three-dimensional setup and one-dimensional quantum mechanics at late time. Picture taken from [12].

where we are neglecting quantum fluctuations of ϕ and $g_{ij}^{(2)}$. We start with the computation at late time. Observe that since the portion of the area connected with the entanglement of the black hole (entanglement wedges) only covers a portion of the interior then it is tempting to say that the rest is connected with the radiation. This is indeed the island which emerge naturally from this assumption. Basically we have to consider the region from σ_0 on and before y_e in Fig. 6. Observe that in this case the extra dimension that connects the island with the mainland is the third dimension of AdS$_3$: in the fully general derivation this will be the an artefact of the replica wormhole. At late time, we can directly write the generalized entropy as

We then have to consider the case of early time. When we consider the black hole and a bath we can use to approximate the Minkowki space we can think that the brane, called "Cardy" branes which separates the bath from the Plank brane now start falling in the black hole: its distance with the boundary increases and the entropy therefore grows up. This growing is the same as computed from the Hawking radiation and in fact

$$S_{\text{Black hole}} \sim S_{\text{rad}} = \frac{\pi c}{6} \int_0^t \mathrm{d}t' \; T(t') = 2 S_{\text{BH}} (1 - e^{\frac{\kappa}{2} t}) \;, \tag{2.19}$$

where $T(t')$ is the temperature at time t' and S_{BH} is the coarse-grained Bekenstein-Hawking entropy.

2.5 Example: Asymptotically flat 2d gravity

Another interesting example we can discuss to show how the island formula works is the two dimensional asymptotically flat gravity model known as RST (Russo-Susskind-Thorlacius) model [13]. The classical action is given by

$$\mathcal{A} = \frac{1}{2\pi} \int \mathrm{d}^2 x \; \sqrt{-g} \left[e^{-2\phi} \left(R + 4(\nabla\phi)^2 + 4\lambda^2 \right) - \frac{N}{24} \phi R \right] + \mathcal{A}_{\text{CFT}} \;, \tag{2.20}$$

where the CFT is any family of 2d CFT with central charge $c = N$ minimally coulped to gravity. Details on the CFT will influence the equations but not the idea, to be concrete we choose

$$\mathcal{A}_{\text{CFT}} = -\sum_{k=1}^{N} \frac{1}{4\pi} \int \mathrm{d}^2 x \; \sqrt{-g} (\nabla f_k)^2 \;. \tag{2.21}$$

We can use the unit $\lambda = 1$. In the effective action at the quantum level also anomalies will contributes as

$$\mathcal{A}_{\text{anom.}} = -\frac{N}{96\pi}\int \mathrm{d}^2x\ \sqrt{-g}R\Box^{-1}R\ . \tag{2.22}$$

We are intereted in this model in the limit

$$N \to \infty\ , \qquad Ne^{2\phi} = \text{fixed}. \tag{2.23}$$

The theory simplify in conformal gauge $\mathrm{d}s^2 = -e^{2\rho}\mathrm{d}x^+\mathrm{d}x^-$ with $x^\pm = x^0 \pm x^1$ and $R_{+-} = -2\partial_+\partial_-\rho$. In fact after defining

$$\Omega = \frac{12}{N}e^{-2\phi} + \frac{\phi}{2} - \frac{1}{4}\log\frac{48}{N}\ , \qquad \chi = \frac{12}{N}e^{-2\phi} + \rho - \frac{\phi}{2} + \frac{1}{4}\log\frac{3}{N}\ , \tag{2.24}$$

we have that

$$\mathcal{A}_{\text{eff}} = \frac{N}{12\pi}\int \mathrm{d}^2x\ \left(\partial_+\Omega\partial_-\Omega - \partial_+\chi\partial_-\chi + e^{2\chi-2\Omega}\right) + \mathcal{A}_{\text{CFT}}\ . \tag{2.25}$$

After conformal gauging we can choose coordinates in which $\chi = \Omega$ and the equations of motions reads

$$\partial_+\partial_-\Omega = -1\ . \tag{2.26}$$

Different solutions will give different geometries. We are going to review them but we only focus on the one we are interested in which is the evaporating black hole: the solution corresponds to

$$\Omega = -x^+x^- - \frac{1}{4}\log(-4x^+x^-) - M(x^+-1)\Theta(x^+-1)\ . \tag{2.27}$$

In Fig. 7a we depicted the solution. There is a matter shock-wave taken to be at $x^+ = 1$ (or $\sigma^+ = 0$ in Mikowski coordinates) that creates the balck hole: the singularity is associated with $\Omega > 1/4$. It then evaporates until the endpoint of evaporation (EP). The apparent horizon is localted at $\partial_+\Omega = 0$.

Entropy computation: The entropy due to the CFT can be computed by standard methods. In particular Cardy and Calabrese [14] proposed a method to compute it. We will not give detailed explaination but let us comment on the idea. We want to compute

$$S_{\text{ent}}(A) = -\operatorname{Tr}\rho_A\log\rho_A\ . \tag{2.28}$$

The complicated part is the trace, but we imagine to compute more easily the partition function

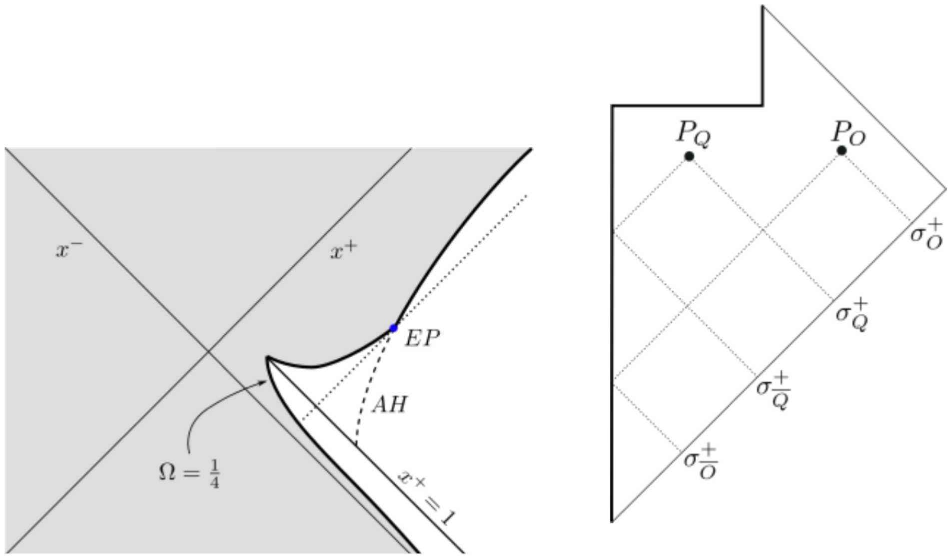
$$Z(A) = \operatorname{Tr}\rho_A\ , \tag{2.29}$$

and in a similar way we can imagine to analytically extend it to

$$Z_n(A) = \operatorname{Tr}\rho_A^n\ . \tag{2.30}$$

Mathematical detail are suppressed here. The trick is now to observe that

$$S_{\text{ent}}(A) = -\lim_{n\to 1}\frac{\partial Z_n(A)}{\partial n}\ , \tag{2.31}$$



(a) Evaporating black hole in Kruskal coordinates. The apparent horizon AH is on the dashed line and the evaporation endpoint is marked EP. The dotted line prior to EP is the event horizon. Figure taken from [11].

(b) Points are labeled by $(\sigma^+_P, \sigma^+_{\bar{P}})$, where $\sigma^+_{\bar{P}}$ is the value of σ^+_P for the image point obtained by reflecting across the timelike boundary. Figure taken from [11].

Figure 7: Evaporating asymptotically flat black hole.

the result is that for an interval of length l the entropy is

$$S_{\rm ent}(A) = \frac{c}{3} \log \frac{l}{\epsilon_{uv}} + \text{const.} \; , \tag{2.32}$$

where ϵ_{uv} is a cutoff that can be thought of as the lattice scale. In a very similar way one can use the replica trick for the curved background case and compute the entropy $S_{\rm ent.}(\mathcal{I} \cup \mathcal{R})$ as required for the island formula. The conclusion is that

$$S(\mathcal{I} \cup \mathcal{R}) = \frac{N}{6} \ln \left[\frac{d^2(P_O, P_Q) d(P_O, P_{\bar{O}}) d(P_Q, P_{\bar{Q}})}{\epsilon^2_{uv} d^2(P_Q, P_{\bar{O}}) e^{-\rho_i(P_Q) - \rho_i(P_O)}} \right] \; , \tag{2.33}$$

where $d^2(P, P') = (\sigma^+_P - \sigma^+_{P'})(\sigma^-_{P'} - \sigma^-_P)$ and ρ_i are the constant contribution of before which in this case will take in account the geometry of the theory. It is conveneint to write $\rho_i = \rho_K + \frac{1}{2}(\sigma^+ - \sigma^-)$. Also the Bekenstein-Hawking entropy can be computed and it is

$$S_{\rm grav} = \frac{N}{6} \left(e^{-2\rho_K} - \frac{\rho_K}{2} \right) + \frac{N}{24} (\log 4 - 1) + \frac{N}{6} \log \epsilon_{UV} \; . \tag{2.34}$$

The derivation is given in [15]. The sum is

$$S(\mathcal{I} \cup \mathcal{R}) = \frac{N}{6} \left[\Omega(P_Q) - \frac{1}{4} + \frac{1}{2}(\sigma^+_{\bar{Q}} - \sigma^-_{\bar{Q}}) + \log(\sigma^+_{\bar{Q}} - \sigma^+_{\bar{O}}) \right] + \frac{N}{6} \log \left(\frac{\sigma^+_{\bar{O}}}{\epsilon_{uv} \sqrt{1 - 4M e^{\sigma^+_{\bar{O}}}}} \right) \; . \tag{2.35}$$

The extremization requires

$$\partial_{\sigma_Q^+} S = \partial_{\sigma_Q^+} S = 0 \ , \tag{2.36}$$

and the solutions are

$$x_Q^- = \frac{M}{4} W_{-1}\left(\frac{4x_O^-}{e^4 M}\right) \ , \qquad x_Q^+ = \frac{1}{4(M + x_Q^-)} \ , \tag{2.37}$$

$$x_Q^- = \frac{M}{4} W_0\left(\frac{4x_O^-}{e^4 M}\right) \ , \qquad x_Q^+ = \frac{1}{4(M + x_Q^-)} \ , \tag{2.38}$$

where $W_n(x)$ is the Lambert W function. The entropy computed in this way needs to be compared with the entropy without the island. In particular we have

$$S = \min\left(S_{\text{ent}}(\sigma_O^-), S_{\text{gen}}(I \cup R)\right) \ . \tag{2.39}$$

For large $M \sim \tilde{\sigma}_O^-$ we have

$$x_Q^+ \sim \frac{3}{4} e^{\tilde{\sigma}_O^-} \ , \qquad x_Q^- + M \sim \frac{1}{3} e^{-\tilde{\sigma}_O^-} \ . \tag{2.40}$$

The Island rule then predicts

$$S = \min \frac{N}{24}\left(2\tilde{\sigma}_O^-, \tilde{\sigma}_{EP}^- - \tilde{\sigma}_O^-\right) + \frac{N}{6}\log\left(\frac{\sigma_O^+}{\epsilon_{uv}}\right) \ , \tag{2.41}$$

where $\tilde{\sigma}_{EP}^- \sim 4M$ is the evaporating endpoint. The page time is $t_P = 4/3M$. $\tilde{\sigma}$ is the retarded time.

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