Primordial black holes from inflation. Julián Rey, DESY, 11/06/24.

Overview

PBHs form in the early Universe from mechanisms other than stellar collapse. They can make up the entirety of DM if their masses are

$$10^{-16}M_{\odot} \lesssim M_{\rm PBH} \lesssim 10^{-11}M_{\odot}. \tag{1}$$

Forming them requires large density fluctuations, which could be produced during inflation. Inflation naturally produces light PBHs (natural here is not used in the technical sense, these models are fine-tuned).

> **Optional content**
> Additional details and content that is not essential to the discussion are presented in gray boxes throughout the notes. These can be skipped.

The CMB puzzles

Suppose the Universe is dominated by a perfect fluid with $p = w\rho$ and the spacetime is FLRW,

$$ds^2 = -dt^2 + a^2\left(\frac{dr^2}{1-k_c r^2} + r^2 d\Omega^2\right). \tag{2}$$

The constant k_c is the *spatial curvature*, a free parameter. The evolution of a is determined by

$$H^2 = \frac{\rho}{3M_p^2} - \frac{k_c}{a^2}, \tag{3}$$

$$\dot{H} + H^2 = -\frac{1}{6M_p^2}(\rho + 3p). \tag{4}$$

The first equation is $\Omega - 1 = \Omega_k$, with $\Omega_k = k_c/(aH)^2$. We know from CMB that $\Omega_k \ll 1$, but for conventional sources $(aH)^{-1}$ grows, so we must have $k_c \ll 1$. This is the *flatness problem*. Let us set $k_c = 0$.

The maximum distance a photon can travel from the Big Bang until time t is ($ad\tau = dt$)

$$\Delta r = \Delta\tau = \int_0^t \frac{dt'}{a(t')} \propto \frac{1}{aH} \propto a^q, \qquad q = \frac{1}{2}(1+3w). \tag{5}$$

We see that indeed $(aH)^{-1}$ grows for $w \geq 0$. A patch with size $R_p(t_0) = R_0$ today will have size $R_p(t) = a(t)R_0$ at time t. But the horizon grows as a^q, so any patch that is connected today must have been disconnected at some point in the past. CMB measurements show temperature is almost uniform across the visible Universe. How this equilibrium was achieved if the patch was causally disconnected at some point is known as the *horizon problem*.

Inflation consists of a phase with decreasing $(aH)^{-1}$. The observable Universe would be causally connected at some point in the past, and Ω_k is driven to a small value, solving both problems. Such a period requires $q < 0$, or $w \leq -1/3$.

Single-field inflation

A phase with $w \leq -1/3$ is easy to achieve. Consider a single field minimally coupled to gravity,

$$\mathcal{L} = \frac{1}{2}M_p^2 R + \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi). \tag{6}$$

The pressure and energy density are (assuming the field is homogeneous)

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{7}$$
$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{8}$$

Thus, if the kinetic energy is negligible we have $p/\rho = -1$. The picture is that the field is slowly rolling towards the minimum of the potential. The slow-roll parameters measure how negligible the kinetic energy is,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{\dot{\phi}^2}{M_p^2 H^2}, \qquad \eta = -\frac{1}{2}\frac{\dot{\epsilon}}{H\epsilon}. \tag{9}$$

Inflation happens for $\epsilon < 1$. The regime $\epsilon \ll 1$ and $|\eta| \ll 1$ is known as slow-roll inflation (from CMB observations we know the field must have been in slow-roll on these scales). Once the field is close to the minimum of the potential inflation ends.

The power spectrum

To describe inhomogeneities and CMB anisotropies, we perturb the metric and stress-energy tensor.

Cosmological perturbations

The perturbed metric is

$$ds^2 = -a^2(1+2\varphi)d\tau^2 + 2a^2\partial_i B dx^i d\tau + a^2\Big[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + h_{ij}\Big]dx^i dx^j. \tag{10}$$

Vector perturbations are not produced in single-field inflation to leading order, and quickly decay with the expansion, so we neglect them. Tensor modes are transverse and traceless, ${h^i}_i = 0$ and $\partial_i {h^i}_j$=0.

The only degree of freedom in the matter sector is the scalar field ϕ, so all perturbations of the stress-energy tensor can be expressed in terms of $\delta\phi$ and its derivatives.

Perturbations mix with each other when we change coordinates. If we switch $x^\mu \to x^\mu + \xi^\mu$, the metric perturbations change as

$$\varphi \to \varphi + aH\alpha + \alpha', \tag{11}$$
$$\psi \to \psi - aH\alpha, \tag{12}$$
$$E \to E + \beta, \tag{13}$$
$$B \to B - \alpha + \beta'. \tag{14}$$

The τi component of the stress-energy tensor is

$$\delta T^{\tau}_{\ i} = (\rho + p)\partial_i(\delta v + B) = \frac{1}{a}\partial_i \delta q. \tag{15}$$

Under a coordinate transformation,

$$\delta q \to \delta q - a(\rho + p)\alpha. \tag{16}$$

From the stress-energy tensor for a scalar field, we find that

$$\delta q = -\dot{\phi}\delta\phi. \tag{17}$$

By choosing α and β appropriately, we can make $E = B = 0$. In the absence of anisotropic stress (the traceless ij component of the stress-energy tensor), one component of Einstein's equations reads

$$(E' - B)' + 2aH(E' - B) + \psi - \varphi = 0. \tag{18}$$

Thus, in the absence of anisotropic stress (which is indeed the case for single-field inflation) and in Newtonian gauge we have $\psi = \varphi$.

The field perturbation $\delta\phi$ and Newtonian potential ψ can be combined into a single gauge-invariant variable (independent of coordinate changes)

$$-\mathcal{R}_k = \frac{H}{\rho + p}\dot{\phi}\delta\phi_k + \psi_k \tag{19}$$

known as the *comoving curvature perturbation*. This is not surprising since during inflation there is only one scalar degree of freedom. This variable can be connected to observables such as energy density fluctuations $\delta\rho/\rho$ or CMB temperature anisotropies δT, and is conserved for $k \ll aH$. The amplitude is given by the dimensionless *power spectrum*

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2. \tag{20}$$

This variable evolves according to the Mukhanov-Sasaki equation

$$\frac{d^2}{dN^2}\mathcal{R}_k + (3 - \epsilon - 2\eta)\frac{d}{dN}\mathcal{R}_k + \frac{k^2}{a^2H^2}\mathcal{R}_k = 0, \tag{21}$$

with $dN = Hdt$ the number of e-folds. The fact that $\mathcal{R}$ is conserved outside the horizon is clear from here.

Canonical quantization

It is convenient to write the Mukhanov-Sasaki equation in conformal time and in terms of the variable $v = -z\mathcal{R}$, with $z = \phi'/H$. The result is

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0, \tag{22}$$

with

$$\frac{z''}{z} = a^2H^2\Big[2 + \mathcal{O}(\epsilon)\Big], \tag{23}$$

where $\mathcal{O}(\epsilon)$ denotes slow-roll suppressed terms.

The Lagrangian that gives rise to this equation is

$$\mathcal{L} = \frac{1}{2}\left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right] \tag{24}$$

The conjugate momentum to v is $\pi = v'$. We can then quantize v in the usual way by imposing the equal-time commutation relations

$$[\hat{v}(\tau, \boldsymbol{x}), \hat{\pi}(\eta, \boldsymbol{y})] = i\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{25}$$

In Fourier space,

$$\hat{v}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\Big[\hat{a}_{\boldsymbol{k}} v_k(\tau) e^{i\boldsymbol{k}\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{k}} v^\star_k(\tau) e^{-i\boldsymbol{k}\boldsymbol{x}}\Big]. \tag{26}$$

Defining the vacuum state is tricky because the Hamiltonian is time-dependent. The key point to notice now is that, in the far past, the z''/z term in the equation of motion is negligible, and we have

$$v_k'' + k^2 v_k = 0. \tag{27}$$

Thus, deep within the horizon the mode functions look just like those of Minkowski,

$$v_{k\gg aH} = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{28}$$

Asking that the vacuum in the far past coincides with the Minkowski vacuum, we find the Bunch-Davies initial condition

$$\mathcal{R}_k = -\frac{e^{-ik\tau}}{2M_p a\sqrt{k\epsilon}}, \tag{29}$$

valid for $k \gg aH$. There are two solutions on superhorizon scales $k \ll aH$,

$$\mathcal{R}_k(N) \simeq \mathcal{R}_k(N_c) + \frac{d}{dN}\mathcal{R}_k(N_c)\int_{N_c}^{N} \exp\left\{\int_{N_c}^{\hat{N}}\Big[2\eta(\tilde{N}) - 3\Big]d\tilde{N}\right\}d\hat{N}, \tag{30}$$

where N_c is the time of horizon crossing. In standard slow-roll inflation, the second one is a decaying mode. Fluctuations freeze at horizon crossing $k = aH$ and

$$\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2}\left|\frac{e^{-ik\tau}}{2M_p a\sqrt{k\epsilon}}\right|^2 = \frac{H^2}{8\pi^2 M_p^2\epsilon}. \tag{31}$$

Black hole mass and abundance

The correct calculation of these quantities is an open question. What follows is just an educated estimate.

The PBH mass is of the order of the mass contained in a Hubble patch at the time of collapse,

$$M_{\rm PBH} = \gamma \frac{4}{3}\pi r_{\rm H}^3 \rho = 4\pi\gamma \frac{M_p^2}{H}, \tag{32}$$

where $\gamma \simeq \mathcal{O}(1)$. We assume that the PBHs form after inflation, during a stage with generic equation of state $p = w\rho$ which ends at time t_r (e.g. reheating). For generality, we do not assume that entropy is conserved in this stage, but it is in the subsequent radiation era. We assume collapse occurs instantaneously at t_k, when the fluctuation with wavenumber $k = a(t_k)H(t_k)$ re-enters the horizon.

We can determine how the Hubble scales in the first stage from the Friedmann equation and use entropy conservation $g_{\star s}(T)a^3T^3 = \text{const.}$ afterwards. We find, after some algebra,

$$M_{\rm PBH} = 4\pi\gamma M_p^2 \left(\frac{\pi^2}{90} g_\star(T_r) \frac{T_r^4}{M_p^2}\right)^{\frac{1}{1+3w}} \left(\frac{1}{k^3}\frac{g_{\star s}(T_0)T_0^3}{g_{\star s}(T_r)T_r^3}\right)^{\frac{1+w}{1+3w}}, \tag{33}$$

Let $\beta = \rho_{\rm PBH}/(\gamma\rho)$ denote the ratio of the energy density in a Hubble patch that ends up in the form of PBHs to the total energy density at the time of collapse. Following the same procedure, the fraction of dark matter in the form of PBHs today $f_{\rm PBH} = \Omega^0_{\rm PBH}/\Omega^0_{\rm DM}$ is

$$f_{\rm PBH} = \gamma\beta \frac{\Omega^0_\gamma}{\Omega^0_{\rm DM}} \left[\left(\frac{g_{\star s}(T_r)}{g_{\star s}(T_0)}\right)^{1/3} \frac{T_r}{T_0}\right]^{\frac{1+9w}{1+3w}} \left(\frac{M_p^2 k^2}{T_r^4}\frac{90}{\pi^2 g_\star(T_r)}\right)^{\frac{3w}{1+3w}}. \tag{34}$$

For $w = 1/3$, both equations become independent of T_r and we have the scaling $M_{\rm PBH} \propto k^{-2}$. Light (unconstrained) PBHs are produced when small-scale fluctuations re-enter the horizon.

The collapse fraction

Assuming that fluctuations $\delta = \delta\rho/\rho$ are Gaussian (which is not correct) and according to Press-Schechter (also not correct),

$$\beta = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\delta_c}^{\infty} e^{-\delta^2/(2\sigma^2)} d\delta. \tag{35}$$

By means of the *gradient expansion* (not perturbation theory), we can relate the overdensity δ to $\mathcal{R}$ at leading order in $k^2/(aH)^2$,

$$\delta = -\frac{4(1+w)}{5+3w}\left(\frac{1}{aH}\right)^2 e^{-5\mathcal{R}/2}\nabla^2 e^{\mathcal{R}/2} \simeq \frac{2(1+w)}{5+3w}\left(\frac{k}{aH}\right)^2 \mathcal{R}. \tag{36}$$

The variance smoothed over a scale k is then

$$\sigma^2 = \frac{4(1+w)^2}{(5+3w)^2} \int \frac{dq}{q}\left(\frac{q}{k}\right)^4 \mathcal{P}_\mathcal{R}(q) W^2(q/k), \tag{37}$$

where we take $W(x) = e^{-x^2/2}\sqrt{2/\pi}$. We are effectively killing off small scales. Superhorizon modes are frozen, so we are only interested in the *initial amplitude* required to form a black hole later on, once the collapse dynamics take place. By taking a scale-invariant spectrum with the CMB amplitude $\mathcal{A}_\flat = 2 \cdot 10^{-9}$, one can check that the amount of PBHs produced is completely negligible. We need to enhance the spectrum. Suppose the spectrum has a sharp peak at $k_\sharp$ (remember, $k_\sharp \simeq 10^{14}\mathrm{Mpc}^{-1}$ must be large so we obtain unconstrained masses)

$$\mathcal{P}_\mathcal{R} = k_\sharp \mathcal{A}_\sharp \delta(k - k_\sharp). \tag{38}$$

The integrals can then be done explicitly. We find that, to get $f_{\mathrm{PBH}} = 1$, we need $\mathcal{A}_\sharp \simeq 10^{-2}$.

Enhancing the spectrum
We saw that in single field inflation the spectrum was $\mathcal{P}_\mathcal{R} \propto 1/\epsilon$. Thus, slowing down the inflaton will produce an enhancement. It turns out this formula cannot be used to estimate the spectrum, because slow-roll breaks down.

Consider a potential with an inflection point. The inflaton is initially in slow-roll, providing a flat spectrum on large scales. As it reaches the inflection point, it decelerates, decreasing ϵ, but also making $\eta \gtrsim 3$ and changing the decaying mode to a growing mode (so the modes now briefly evolve outside the horizon!). In the subsequent phase it accelerates again with $\eta \lesssim 1$ until $\epsilon = 1$ and inflation ends. The Mukhanov-Sasaki equation *must be solved numerically.*

The peak is on small scales, so these models naturally produce unconstrained PBHs and are consistent with the CMB. However, they are extremely fine-tuned.

The phase with $\eta \gtrsim 3$ is known as ultra-slow-roll. We refer to the subsequent phase, which turns out to be the most important one for non-Gaussianities, as constant-roll.

Beyond the naive estimate
A few comments are in order

- This treatment is outdated. We do not care if there is a large overdensity, but rather if the mass distribution fits inside its Schwarzschild radius. We should adopt threshold statistics on the *compaction function* instead of δ, defined as the mass excess over the areal radius in a region of size $R(r,t) \equiv a(t)r$,
$$\mathcal{C}(r,t) = \frac{2G_\mathrm{N}\delta M(r,t)}{R(r,t)}. \tag{39}$$

The compaction function
For spherically-symmetric collapse, the metric is

$$ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1-K(r)r^2} + r^2 d\Omega^2\right] \tag{40}$$

This spacetime is isotropic but not homogeneous (otherwise it would be maximally symmetric and K would be constant, as in FLRW). This metric is not valid throughout the collapse, but only initially, while the perturbation that will later induce collapse is still superhorizon.

We expect a black hole to form whenever $\mathcal{C} \sim 1$, with some wiggle room in the exact number depending on the shape of the overdensity. We can calculate the mass excess by integrating the energy density,

$$\begin{aligned} \delta M(r,t) &= \int \delta\rho(\hat{r},t)\, dV \\ &= 3M_p^2 H^2 \int_0^R \delta(\hat{r},t) 4\pi \hat{R}^2 d\hat{R} \\ &= 3M_p^2 a^3 H^2 \int_0^R \delta(\hat{r},t) 4\pi \hat{r}^2 d\hat{r}, \end{aligned} \tag{41}$$

where we have switched to spherical coordinates, used $\delta = \delta\rho/\rho$ and the Friedmann equation $\rho = 3M_p^2 H^2$. The compaction function is then

$$\begin{aligned} \mathcal{C}(r,t) &= \frac{2G_{\mathrm{N}}}{ar} 3M_p^2 a^3 H^2 \int_0^r \delta(\hat{r},t) 4\pi \hat{r}^2 d\hat{r} \\ &= \frac{3(aH)^2}{r} \int_0^r \delta(\hat{r},t) \hat{r}^2 d\hat{r}, \end{aligned} \tag{42}$$

using $M_p^2 = (8\pi G_{\mathrm{N}})^{-1}$. Inflationary observables are computed using the following metric instead, which includes the nonlinear generalization of $\mathcal{R}$ in comoving gauge,

$$ds^2 = -dt^2 + a^2 e^{2\mathcal{R}(\hat{r})} \Big(d\hat{r}^2 + \hat{r}^2 d\Omega^2 \Big). \tag{43}$$

Demanding that this metric be equal to (40) and comparing the three-dimensional Ricci (on hypersurfaces of constant time) obtained for each, we find the relation

$$\frac{2}{a^2} \frac{1}{r^2} \frac{d}{dr} \Big[r^3 K(r) \Big] = R_3 = -\frac{8}{a^2} e^{-5\mathcal{R}(\hat{r})/2} \nabla^2 e^{\mathcal{R}(\hat{r})/2}. \tag{44}$$

In cosmology, we typically expand Einstein's equations in powers of the perturbations (for instance $\mathcal{R}$) and solve them order by order. An alternative is to use the so-called gradient expansion, where we expand in powers of $k^2/(aH)^2$ and keep everything fully non-linear in the perturbations instead. This is useful if we want to describe superhorizon physics, because the results are then exact in the limit $k \to 0$.

Assuming radiation and using eqs. (42), (44), and (36), we obtain

$$\begin{aligned} \mathcal{C}(r,t) &= \frac{3(aH)^2}{r} \int_0^r \delta(\hat{r},t) \hat{r}^2 d\hat{r} \\ &= \frac{2}{3} r^2 K(r). \end{aligned} \tag{45}$$

Demanding both of our metrics to be equal we find the conditions

$$r = e^{\mathcal{R}(\hat{r})}\hat{r} \qquad \text{and} \qquad \frac{dr^2}{1-K(r)r^2} = e^{2\mathcal{R}(\hat{r})}d\hat{r}^2. \tag{46}$$

Thus, differentiating the first expression and using the second one,

$$r^2K(r) = -\partial_{\hat{r}}\mathcal{R}(\hat{r})\hat{r}\Big[2+\partial_{\hat{r}}\mathcal{R}(\hat{r})\hat{r}\Big]. \tag{47}$$

Finally, we obtain the relation between the compaction function and the curvature perturbation,

$$\mathcal{C}(r,t) = -\frac{2}{3}r\partial_r\mathcal{R}\Big(2+r\partial_r\mathcal{R}\Big), \tag{48}$$

where we have dropped the hats for simplicity. In principle we can connect the statistics of $\mathcal{R}$ to those of $\mathcal{C}$.

- The calculation of the threshold for collapse is highly nontrivial. Numerical simulations yield $\delta_c \sim \mathcal{C}_c \sim 0.4$. This quantity depends on the curvature profile, but defining a universal threshold for $\mathcal{C}$ seems possible.
- The fluctuations are heavily not Gaussian. The inflaton has some intrinsic non-Gaussianity. It is nonlinearly related to $\mathcal{R}$, which generates more. And $\mathcal{R}$ is nonlinearly related to δ (or $\mathcal{C}$), which generates even more.

Non-Gaussianities

To calculate non-Gaussianities we need to expand the inflaton action beyond quadratic order in perturbations and use the *in-in formalism*. The nonlinear generalization of $\mathcal{R}$ is shown in eq. (43) in comoving gauge ($\delta\phi = 0$). The calculation can also be performed in the flat gauge, where there is no Newtonian potential in the metric and the only degree of freedom is $\delta\phi$. The catch is we lose the connection to the conserved, gauge-invariant $\mathcal{R}$.

The calculation is most easily done in flat gauge, with a later transformation to comoving gauge. There are then two sources of non-Gaussianity. The first is intrinsic non-Gaussianity of $\delta\phi$, arising from the self interaction terms

$$S_3 = \int d^4xa^3\Big[-\frac{V_3}{6}\delta\phi^3 + \mathcal{O}(\epsilon)\Big]. \tag{49}$$

These are proportional to time derivatives of η. Gravitational interactions are slow-roll suppressed. The gauge transformation can be obtained to all orders by using the δN formalism. At leading order in the gradient expansion, the result is

$$\eta\mathcal{R} = \log\Big(1-\eta\delta\phi/\sqrt{2\epsilon}\Big), \tag{50}$$

with η evaluated shortly after the USR phase ends. For a smooth USR phase, the self-interactions of the inflaton are negligible and the non-Gaussianity comes mainly from the gauge transformation, which leads to an exponential tail in the PDF of $\mathcal{R}$,

$$P(\mathcal{R}) = \left|\frac{d\delta\phi}{d\mathcal{R}}\right| P(\delta\phi) \qquad \rightarrow \qquad P(\mathcal{R}) \sim e^{\eta\mathcal{R}}. \tag{51}$$

Since this PDF is valid only at leading order in the gradient expansion, and collapse actually depends on the derivatives of $\mathcal{R}$ through the compaction function, it is not clear at present how to reconcile both approaches.

Forming a PBH is quite difficult, so they are rare events. Non-Gaussianities are important because when we form a PBH we integrate only over the tail of the distribution, see eq. (35). As we mentioned earlier, using δ is not good enough, however, and we have to deal with $\mathcal{C}$ instead. Since $\mathcal{R}$ is non-linearly related to $\mathcal{C}$, the latter becomes non-Gaussian.

- Press-Schechter assumes we only care if the overdensity is above a certain threshold. What you actually want is to have a local peak, so we should not just impose $\delta > \delta_c$ (or the equivalent for the compaction function), but also that the first derivative vanishes and the Hessian is negative definite. The PDF becomes much more complex.

Peak theory

Schematically, in the context of peak theory the number density of peaks is given by an integral

$$n_p(\mathcal{C}) = \int P(\mathcal{C}, \nabla\mathcal{C}, \nabla^2\mathcal{C}) d(\mathcal{C}) d(\nabla\mathcal{C}) d(\nabla^2\mathcal{C}). \tag{52}$$

The compaction function $\mathcal{C}$ and its derivatives are heavily correlated, so the PDF does not separate and this is a difficult integral even assuming Gaussianity. The problem of finding the statistics for Gaussian δ was solved by BBKS many decades ago. The case of $\mathcal{C}$ has been worked out recently for Gaussian $\mathcal{R}$.

- For stiff equations of state these estimates are fine, but for matter domination other effects come into play. Collapse takes longer, and nonsphericity and angular momentum begin to play a role.

- Except for MD, these considerations should not alter the fact that $\mathcal{A}_\sharp \simeq 10^{-2}$, so as an order of magnitude estimate the naive treatment is correct. It is important to keep in mind that *the picture is not complete, however.*

The induced GW signal

Let us expand tensor modes in Fourier space

$$h_{ij}(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} (h_k^+ e_{ij}^+ + h_k^\times e_{ij}^\times), \tag{53}$$

where e^s_{ij} are transverse, traceless polarization tensors. At the linear level, in the absence of anisotropic stress, tensor modes decouple from the other degrees of freedom and propagate freely. At second order, their equation of motion is sourced by terms quadratic in scalar modes,

$$h_k^{s\prime\prime} + 2aHh_k^{s\prime} + k^2 h_k^s = S_k^s, \tag{54}$$

where $\mathcal{H} = aH$ and we have written derivatives in conformal time ($' = d/d\tau$ with $d\tau = dt/a$). The source is, in Newtonian gauge,

$$S_k^s = \int \frac{d^3p}{(2\pi)^3} e^s_{ij}(\boldsymbol{k}) p_i p_j \left[8\psi_p \psi_{|\boldsymbol{k}-\boldsymbol{p}|} + \frac{16}{3(1+p/\rho)} \left(\psi_p + \frac{\psi_p'}{\mathcal{H}} \right) \left(\psi_{|\boldsymbol{k}-\boldsymbol{p}|} + \frac{\psi'_{|\boldsymbol{k}-\boldsymbol{p}|}}{\mathcal{H}} \right) \right]. \tag{55}$$

The equation can be solved by means of Green's function techniques.

Solution for tensor modes

The solution to the equation of motion is

$$h_k^s(\tau) = \int_0^\tau d\hat{\tau} G_k(\tau, \hat{\tau}) S_k^s(\hat{\tau}), \tag{56}$$

where the Green's function is

$$G_k(\tau, \hat{\tau}) = \frac{h_1(\tau)h_2(\hat{\tau}) - h_1(\hat{\tau})h_2(\tau)}{h_1'(\hat{\tau})h_2(\hat{\tau}) - h_2'(\hat{\tau})h_1(\hat{\tau})}. \tag{57}$$

and h_1 and h_2 are two linearly independent solutions to the free equation of motion. Assuming radiation, $aH = 1/\tau$ and the solutions turn out to be very simple trigonometric functions. In Newtonian gauge, in the absence of anisotropic stress and in the presence of a perfect fluid, the metric perturbation obeys the equation

$$\varphi_k'' + 3\left(1 + \frac{p}{\rho}\right) aH \varphi_k' + \frac{p}{\rho} k^2 \varphi_k = 0. \tag{58}$$

The solutions are Bessel functions. The initial conditions on superhorizon scales can be related to $\mathcal{R}$ computed at the end of inflation,

$$\varphi_k(0) = \frac{3+3w}{5+3w} \mathcal{R}_k. \tag{59}$$

The solution can then be written as

$$h_k^s = \frac{1}{k^2} \int \frac{d^3p}{(2\pi)^3} e^s_{ij}(\boldsymbol{k}) p_i p_j \mathcal{R}_p \mathcal{R}_{|\boldsymbol{k}-\boldsymbol{p}|} I_k(\tau, p, |\boldsymbol{k}-\boldsymbol{p}|), \tag{60}$$

where I is a time integral of the Green's function and the scalar modes, and $\mathcal{R}$ is the frozen, superhorizon, curvature perturbation computed at the end of inflation. The GW energy density is given by

$$\Omega_{\rm GW} = \frac{1}{24} \frac{k^2}{a^2 H^2} \langle \mathcal{P}_h \rangle, \tag{61}$$

where the brackets denote a time average which must be taken due to the stochasticity of the signal. The power spectrum is

$$\langle h_k^s h_p^t \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_h(k)(2\pi)^3 \delta^{st} \delta^3(\boldsymbol{p}+\boldsymbol{k}). \tag{62}$$

Squaring h, assuming that $\mathcal{R}$ is Gaussian and using Wick's theorem, we find

$$\mathcal{P}_h(\tau, k) = \int_0^\infty dy \int_{|1-y|}^{1+y} dz \left[\frac{4y^2 - (1+y^2-z^2)^2}{8yz}\right]^2 \mathcal{P}_\mathcal{R}(ky)\mathcal{P}_\mathcal{R}(kz) I^2(\tau, ky, kz), \tag{63}$$

where we have performed one of the angular integrals, and switched variables to $y = p/k$ and $z = |\boldsymbol{k} - \boldsymbol{p}|/k$ for the two remaining integrals.

The sin and cos in I_k drop out after averaging. We define

$$\frac{1}{2} J^2(y, z) = \lim_{k\tau\to\infty} \frac{k^2}{a^2H^2} \langle I_k(\tau, ky, kz)^2 \rangle \left[\frac{4y^2 - (1+y^2-z^2)^2}{8yz}\right]^2. \tag{64}$$

The final result for the GW energy density after accounting for the redshift is

$$\Omega_{\rm GW}(T_0, k) = \frac{\Omega_\gamma(T_0)}{48} \frac{g_\star(T)}{g_\star(T_0)} \left(\frac{g_{\star s}(T_0)}{g_{\star s}(T)}\right)^{4/3} \int_0^\infty dy \int_{|1-y|}^{1+y} dz\, \mathcal{P}_\mathcal{R}(ky)\mathcal{P}_\mathcal{R}(kz) J^2(y, z), \tag{65}$$

with J^2 a complicated function resulting from the geometry of the momenta, the time evolution of the scalar modes, and the Green's function. This integral can be done explicitly for a sharp spectrum. A peak in the scalar spectrum induces a peak in the GW spectrum. For unconstrained PBHs, this peak falls in the LISA band.