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Black Hole Superradiance - A Crashcourse

DESY THEORY WORKSHOP SEMINAR

Matthias Koschnitzke
matthias.koschnitzke@desy.de

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*Image: deepai.org result for "Draw a gravitational atom, an atom with a black hole as nucleus, surrounded by a dense cloud resembling the quantum mechanical wavefunctions of the hydrogen atom. Put the whole thing into an astrophysical setting."

1 Introduction

Rotational superradiant scattering, or short **superradiance**, is the process of amplification of waves by scattering of a medium that rotates with superluminal speed compared to the wave and that offers a mechanism for dissipation. Via superradiance, rotational energy is transferred from the medium to the wave. For **black hole (BH) superradiance**, the dissipative medium is the rotating spacetime itself.

It has been theorised first by Penrose [1] that it is possible for particles to enter the ergoregion of a rotating black hole (which is described by the **Kerr metric** [2]), extract energy and angular momentum from the black hole and leave the ergoregion with more energy than the particle initially had.

Zel'dovich first argued [3, 4], that a spherical symmetric wave can be amplified by a rotating medium, i.e. object, that asserts friction to the wave. This can be applied to rotating BHs.

To use superradiance to built up the field it has been theorised to put a BH inside a mirror, confining the scattered wave to the BH and leading to a runaway process of superradiance, which has been called a BH bomb [5].

If a scalar field possesses a mass term, this leads to a natural confinement, leading to a **superradiant instability** which can make the scalar cloud grow until the BH has been spun down [5–7].

Since the bound states of the scalar fields in the Kerr metric resemble hydrogenic states, the superradiant clouds around BHs have been called **gravitational atoms**. Introducing a binary companion gives rise to **gravitational atomic physics** [8, 9].

We are going to derive the most important basic results heuristically, sketch the actual calculations of the superradiance rates, look at the most important ideas for the phenomenology of BH superradiance and discuss level transitions due to a binary companion.

Due to time constraints we are going to ignore the effects that non-zero self interactions [10] or coupling to the SM of the ultralight particles could have [11–13]. We are also not going to discuss superradiance for vector and tensor fields. For a review containing all of the above, see [14].

We work in units where $c = \hbar = 1$, but sometimes I may have forgotten factors of G. In general, due to time constraints, this manuscript will contain **typos**. Sorry in advance.

2 The Kerr metric and hydrogenic states

The Kerr metric in Boyer-Lindquist coordinates is [8]

$$ds^2 \equiv \frac{\Delta}{\rho^2}(dt - a\sin^2(\theta)d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2(\theta)}{\rho^2}(adt - (r^2 + a^2)d\phi)^2, \tag{1}$$

where

$$\Delta \equiv r^2 - 2GMr + a^2 \quad \text{and} \quad \rho^2 \equiv r^2 + a^2\cos^2(\theta). \tag{2}$$

Here M is the mass of the BH, $a \equiv J/M$, where J is the angular momentum. We will also use $\tilde{a} = a/(GM) \leqslant 1$, the dimensionless spin of BH.[I] The event horizon is the positive root of Δ,

[I] In the literature, this is often called χ.

$r_+ = GM + \sqrt{(GM)^2 - a^2}$. The surface where $g_{tt} = 0$ is $r_{\rm E} = GM + \sqrt{(GM)^2 - a^2\cos^2(\theta)}$ and is called the ergosphere. Inside the ergosphere, no observer can be at rest.

The Lagrangian density of a scalar field Ψ with mass μ in a general metric g_{ab} is

$$\mathcal{L} = -\frac{1}{2}g^{ab}\nabla_a\Psi\nabla_b\Psi - \frac{1}{2}\mu^2\Psi^2. \tag{3}$$

Just due to the components of the inverse metric, the actual equations of motions found from this Lagrangian look horrible, so we will not show them. There is no FULL analytic solution for a scalar field in the Kerr metric. The rest of the first half of these notes is to understand some aspects.

We make the non-relativistic ansatz to decouple the oscillations due to the mass

$$\Psi(t,\vec{r}) = \frac{1}{\sqrt{2\mu}}\left(\psi(t,\vec{r})e^{-i\mu t} + \psi^*(t,\vec{r})e^{i\mu t}\right), \tag{4}$$

where ψ is a complex scalar field that varies on timescales much longer than μ^{-1}. The action for ψ then reads

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2\mu}\left(\nabla_a\psi^*\nabla^a\psi + i\mu g^{0a}(\psi^*\nabla_a\psi - \psi\nabla_a\psi^*) + \mu^2(g^{00}+1)\psi^*\psi\right)\right]. \tag{5}$$

When writing the explicit terms, the combination $\alpha \equiv GM\mu$ will pop up, which is dimensionless and for reasons that will become obvious in a moment is called the **gravitational fine-structure constant**. It is the ratio of the event horizon to the Compton wavelength of the scalar field. We define $r_c \equiv (\mu\alpha)^{-1} = (GM/\alpha^2)$ (which we will discuss in the next chapter) and assume $r \sim r_c$. We can now expand the above action into powers of α, and assuming $\alpha \ll 1$, we find an approximate equation of motion for the field far away from the horizon. The leading order contributions give

$$i\frac{\partial}{\partial t}\psi(t,\vec{r}) = \left(-\frac{1}{2\mu}\vec{\nabla}^2 - \frac{\alpha}{r}\right)\psi(t,\vec{r}). \tag{6}$$

This is the Schroedinger equation for a hydrogen atom, with α substituting the electromagnetic fine-structure constant. This dubbed the term **gravitational atom**. The solutions are superpositions of

$$\psi_{nlm}(t,r,\theta,\phi) \simeq e^{-(\omega-\mu)t}R_{nl}(r)Y_{lm}(\theta,\phi), \tag{7}$$

where $R_{nl}(r)$ are the radial wavefunctions of the hydrogen atom, but with r_c being the "gravitational Bohr radius".

Typical values for BH mass and ultralight scalar for small α are:

$$\alpha \approx 0.07\left(\frac{M}{10M_\odot}\right)\left(\frac{\mu}{10^{-12}\,\mathrm{eV}}\right). \tag{8}$$

We see that, as in atomic physics, the states are distinguished by the general quantum number n, the angular momentum l and the "magnetic quantum number" m, i.e. the value of the angular momentum in z-direction. In the non-relativistic regime, the eigenvalues of the levels are given by [8, 15]

$$\omega_{nlm} = \mu\left[1 - \frac{\alpha^2}{2n^2} - \left(\frac{1}{8n} + \frac{6}{2l+1} - \frac{2}{n}\right)\frac{\alpha^4}{n^3} + \frac{16}{2l(2l+1)(2l+2)}\frac{\tilde{a}m\alpha^5}{n^3}\right]. \tag{9}$$

This derivation works only in the limit far from the horizon, and for small α. Therefore it does not capture the effect of the horizon. For the hydrogen atom, the wavefunctions need to be regular at the origin. Here, the wavefunction need to be purely ingoing at the horizon. This will lead to non-zero imaginary parts of the frequency, and hence growing and decaying modes. Before having a small peak into how this arises, we will give some heuristic arguments.

3 Scalar field superradiance basics - the heuristic approach

3.1 A very basic analogy

The basic principle of superradiance can be understood with a non-rotational analogy in one dimension: If Casey jumps with initial horizontal velocity v_i and lands on the floor with (some) friction, the final velocity v_f will be reduced, $v_f < v_i$. If Casey jumps onto a treadmill that runs with velocity v_t that is slower than the horizontal speed $v_t < v_i$ and it has (some) friction, the treadmill will pull Casey back with it, again making $v_f < v_i$. However, when the treadmill runs faster than the initial horizontal speed, $v_t > v_i$, in the rest frame of the running treadmill, Casey's initial speed will be negative, i.e. $v_i' < 0$. This means, when Casey lands, friction will make this negative velocity less negative. (For a realistic treadmill of course it will make it 0 in the treadmill frame.) Back in the "lab"-frame, this leads to $v_f > v_i$. The energy to increase Casey's speed came from the treadmill, of course. This is the basic superradiance argument, and exchanging the jumping person with a wave, and the treadmill with a rotating body is the Zel'dovich argument.

3.2 The Zel'dovich argument

We follow [3]. Consider a scalar field ψ within a dissipative medium, with friction/damping coefficient $\tilde{c}$. In the reference frame where the medium is at rest, the equation of motion for the scalar field is

$$\Box\psi - \tilde{c}\dot{\psi} - m^2\psi = 0. \tag{10}$$

Let us imagine the medium is actually rotating with angular velocity Ω. We then make the ansatz for a cylindrical problem $\psi = f(r)\exp(-i(\omega t + m\phi))$, where ω is the frequency, $m \in \mathbb{Z}$ and r and ϕ are the polar coordinates in the plane orthogonal to the rotation vector. If we look at the field at a specific radius R, and take the $x \equiv R\phi$ direction to be reckoned along this circle, the Lorentz transformation with velocity $v \equiv \Omega R$ into the rotating frame changes the damping term to

$$\begin{aligned}\tilde{c}\partial_t\psi \rightarrow\ & \tilde{c}\gamma(\partial_t\psi - v\partial_x\psi) \\ &= \tilde{c}\gamma(\partial_t - v\frac{1}{R}\partial_\phi)\psi \\ &= -i\tilde{c}\gamma(\omega - \frac{\Omega R}{R}m)\psi \\ &= -i\tilde{c}\gamma(\omega - \Omega m)\psi.\end{aligned} \tag{11}$$

This result shows that when $\Omega m > \omega$, i.e. when the medium rotates faster than the spherical modes of the wave oscillate, the damping term switches sign, i.e. the rotating medium amplifies the wave.

3.3 Growing the cloud: cloud size, growth rates, spin down, and mass

With rather heuristic arguments, we can find the rough size of the cloud, the scaling of the decay rates and the final spin of the black hole after the superradiance condition has been exhausted.

Cloud size: To find the cloud size, we assume that the cloud is non-relativistic and hence the energy is given by $\omega \approx \mu + \frac{\vec{p}^2}{2\mu}$ where $\vec{p}^2 < 0$, because we have a bound state.

We expect the kinetic energy to be of the same order as the potential energy at the radius of the cloud r_c by the virial theorem:

$$\frac{|\vec{p}|^2}{\mu} \sim \frac{GM\mu}{r_c}. \tag{12}$$

For the n-th energy level, we expect the energy level times the wavelength to match the circumference of the cloud, while wavelength and momentum are related as usual

$$n\lambda \sim 2\pi r_c \Rightarrow n\frac{2\pi}{|\vec{p}|} \sim 2\pi r_c. \tag{13}$$

Solving this for $|\vec{p}|$ and plugging it back into Eq. (12), we can solve for the radius of the cloud:

$$r_c \sim \frac{n^2}{GM\mu^2} = \frac{n^2 GM}{(GM\mu)^2} = \frac{n^2 r_g}{\alpha^2}, \tag{14}$$

where $r_g \equiv GM$ is the gravitational radius.

Superadiance rate: While we discuss analytic calculations of the actual superradiance rate in the next section, we are going to give a heuristic argument how to find the scaling with α for a given state:

The superradiance rate is the ratio of (negative) energy change of the BH divided by the mass in the cloud:

$$\Gamma_{\text{SR}} \simeq -\frac{\dot{E}_{\text{BH}}}{M_c}. \tag{15}$$

$\dot{E}_{\text{BH}}$ has to be given by the superradiance condition, the density of bosons at the horizon, which is $\psi^2\big|_{r_+}$, integrated over the horizon, which we approximate by multiplying by r_+^2, times the frequency ω for dimensional consistency:

$$\dot{E}_{\text{BH}} \sim -(\omega - \Omega_H m)\ \psi^2\big|_{r_+} r_+^2 \omega. \tag{16}$$

The scaling of the field, i.e. the radial wavefunction, is roughly $\psi \sim (r/r_c)^l e^{-r/r_c}$. We find the mass of the cloud via $M_c \simeq \mu^2 \int dr r^2 \psi^2$. Integrating this gives

$$M_c \sim \mu^2 r_c^3. \tag{17}$$

Ignoring the exponential function, this gives for the ratio

$$\begin{aligned} \frac{\dot{E}_{\text{BH}}}{M_c} &\sim -(\omega - \Omega_H m)\frac{\left(\frac{r_+}{r_c}\right)^{2l} r_+^2 \omega}{r_c^3 \mu^2} \\ &\sim -(\omega - \Omega_H m)\frac{r_+^{2l+2}}{\mu r_c^{3+2l}}. \end{aligned} \tag{18}$$

where we used that $\omega \approx \mu$. We can now use $r_+ = GM(1 + \sqrt{1 - \tilde{a}^2}) \approx GM$ and the above result $r_c \sim r_g/\alpha^2 = GM/\alpha^2$, and find:

$$\begin{aligned} \frac{\dot{E}_{\text{BH}}}{M_c} &\sim -\frac{(GM)^{2l+2}\alpha^{4l+6}}{\mu (GM)^{2l+3}} \\ &\sim -\frac{\alpha^{4l+6}}{GM\mu}. \end{aligned} \tag{19}$$

Hence, the scaling of the superradiance rate is

$$\Gamma_{\text{SR}} \sim (\omega - \Omega_H m)\alpha^{4l+5}. \tag{20}$$

We see that the rate is extremely sensitive to the value of α, which leads to the fact that for every given BH mass, scalars with mass too small will not grow a cloud with significant density during astrophysical or even cosmological timescales.

We will see that the cloud also decays via GW emission, again on time scales strongly dependent on α. The combination of these two phenomena leaves, for every given BH mass, only a small window in the scalar mass parameter space where BH superradiance is viable.

Spin down and mass loss: The superradiance rate obviously depends on $\Omega_H m - \omega$. Since the BH loses spin during the growth of the cloud, superradiance will stop at one point. It is easy to estimate when this happens.

The energy for scalars in the n-th level is $\omega = \mu(1 - \alpha^2/(2n^2) + \mathcal{O}(\alpha^4)$, but for now it is okay to just set $\omega = \mu$ again. The angular velocity of the horizon is $\Omega_H = \tilde{a}/(2r_+) = \tilde{a}/(2GM(1 + \sqrt{1-\tilde{a}^2})$. Equating this two expressions and dividing by μ gives

$$1 - m\frac{\tilde{a}}{2\alpha(1 - \sqrt{1-\tilde{a}^2})} = 0. \tag{21}$$

The solution for $\tilde{a}$ is

$$\tilde{a}_{\text{sat}} = \frac{4m\alpha}{m^2 + 4\alpha^2}. \tag{22}$$

We see that for small α the $|211\rangle$ state spins the BH down to $\tilde{a} \approx 4\alpha$, while the $|322\rangle$ spins it down further to $\tilde{a} \approx 2\alpha$. Note that when this happens, the superradiance rate of the $|211\rangle$ state turns negative, and this state begins to decay.

Mass of the cloud: The last estimate we want to make is how much mass the cloud extracts, i.e. find M_c/M, the ratio of cloud mass to BH mass. The final angular momentum of the cloud is given by the number of particles M_c/μ times the angular momentum of a single particle m (note that we work in units where $\hbar = 1$), i.e.

$$L = \frac{M_c}{\mu} m. \tag{23}$$

The angular momentum of a black hole is $J = \tilde{a}GM^2$. If the BH is extremal in the beginning, i.e. $\tilde{a} = 1$ we can extract down to $J_{\max} = (1 - 4\alpha)GM^2$, but lets keep it general and say, we extract $j < 1$ of the dimensionless spin of the BH, i.e. $J_{\text{ext}} = jGM^2$. We set this equal to the angular momentum of the cloud:

$$\frac{L}{J_{\text{ext}}} \overset{!}{=} 1 = \frac{M_c}{\mu} m \frac{1}{jGM^2}. \tag{24}$$

Solving this for M_c/M gives

$$\frac{M_c}{M} = \frac{j\alpha}{m}. \tag{25}$$

This bound is derived in vacuum. It can be relaxed in astrophysical environments due to accretion from surrounding matter. We also note that during the whole process, angular momentum and mass is extracted in a way that keeps the BH area theorem in tact. Since a fast rotating BH has a smaller horizon than a non-rotating BH of the same mass, during the spin down the area of the BH grows.

4 Sketch of the Detweiler approximation

The best-known derivation of the superradiance rates for $\alpha \ll l$ (note: not necessarily $\alpha \ll 1$) has been done by Detweiler [7]. After realising that the equation of motion of the scalar field is separable into spherical and radial part, the equation for the radial wavefunction is

$$\Delta \frac{d}{dr}\left(\Delta \frac{dR}{dr}\right) + [\Omega^2(r^2 + a^2)^2 - 4aGMrm\Omega + a^2m^2 - \Delta(\mu^2 r^2 + a^2\Omega^2 + \lambda)]R = 0, \tag{26}$$

where λ is a separation constant coming from the spherical wavefunction. The solution we are looking for is outgoing at infinity and ingoing at the event horizon. Detweiler then approximates the radial equation for small α and with $k^2 \equiv \mu^2 - \omega^2$, $n \equiv M\mu^2/k$ and $x \equiv 2kr$ as

$$\frac{d^2(xR)}{dx^2} + \left[-\frac{1}{4} + \frac{n}{x} - \frac{l(l+1)}{x^2}\right] xR = 0. \tag{27}$$

which resembles the radial wavefunction for the hydrogen atom and has the analytic solution

$$R(x) = x^l e^{-x/2} U(l+1-n, 2l+2, x), \tag{28}$$

where U is a confluent hypergeometric function. For the hydrogen atom, n corresponds to the principal quantum number, which is integer, and satisfies $n = l + 1 + \nu$, where ν is also integer. However, due to boundary condition at the horizon, we expect an instability and therefore a small imaginary part in n, δn:

$$n - l - 1 \equiv \nu + \delta n. \tag{29}$$

For small δn, the solution for $R(x)$ can be expanded into real and imaginary part. Without going to more details, the full radial equation is then simplified in the limit of small r, which again admits analytic solutions, this time in terms of hypergeometric functions. They admit independent solutions that describe ingoing and outgoing solutions at the horizon. Both approximations have overlapping regions, which has been used in [7] by matching the lowest terms of the Taylor expansion for the large r solution, but at small r with the asymptotic behaviour of the near horizon solution, and find a value of δn such that the outgoing solution for the near horizon limit vanishes. Defining the complex frequency $\omega = \omega_{\mathrm{R}} + i\Gamma$ gives the superradiance rate via

$$i\Gamma = \frac{\delta n}{GM}\left(\frac{GM\mu}{l+1+\nu}\right)^3. \tag{30}$$

The result is

$$\Gamma_{nlm} \simeq 2\tilde{r}_+ C_{nl}\, g_{lm} \alpha^{4l+5}(m\Omega_H - \omega_{nlm}), \tag{31}$$

with $g_{lm} \equiv \prod_{k=1}^{l}\left[k^2(1-\tilde{a}^2) + (\tilde{a}m - 2r_+\omega)^2\right]$, $C_{nl} \equiv \frac{2^{4l+1}(n+l)!}{n^{2l+4}(n-l-1)!}\left(\frac{l!}{(2l)!(2l+1)!}\right)^2$, $\tilde{r}_+ = 1 + \sqrt{1-\tilde{a}^2}$, $M\Omega_H = \tilde{a}/(2\tilde{r}_+)$.

For reference, typical inverse superradiance rates are

$$\Gamma_{211}^{-1} \approx 4.6 \times 10^6 \left(\frac{M}{10\,M_\odot}\right)\left(\frac{0.1}{\alpha}\right)^9 \mathrm{s}\,, \tag{32}$$

$$\Gamma_{322}^{-1} \approx 1.8 \times 10^5 \left(\frac{M}{10\,M_\odot}\right)\left(\frac{0.1}{\alpha}\right)^{13} \mathrm{yrs}. \tag{33}$$

The possible regions in the BH Regge plane (showing mass and spin) for superradiance and how BHs move through it is shown in Fig. 1.

There are also analytical results for $\alpha \gg 1$, like [17, 18]. For this, a WKB approximation can be used, which reduces the problem to a Schroedinger equation for $\Psi \equiv (r^2 + a^2)^{1/2} R$ with a potential:

$$\frac{d^2\Psi}{dr_*^2} - V\Psi = 0, \tag{34}$$

where $dr_* = \frac{r^2+a^2}{\Delta} dr$ is the tortoise coordinate, and with

$$V = -\omega^2 + \frac{4r_g ram\omega - a^2m^2}{(r^2+a^2)^2} + \frac{\Delta}{r^2+a^2}\left(\mu^2 + \frac{l(l+1)+k^2a^2}{r^2+a^2} + \frac{3r^2 - 4r_g r + a^2}{(r^2+a^2)^2} - \frac{3\Delta r^2}{(r^2+a^2)^2}\right). \tag{35}$$

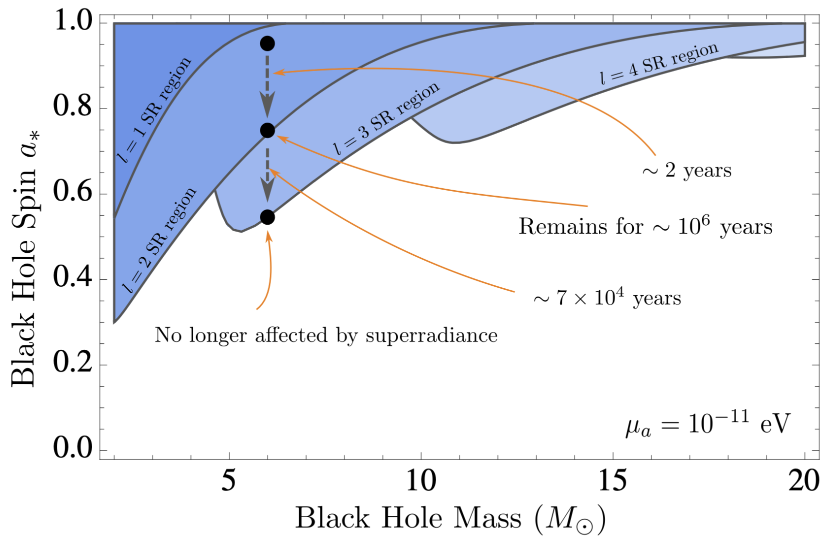

Figure 1: Effect of superradiance for a QCD axion with mass $\mu_a = 10^{-11}$ eV and decay constant $f_a = 6 \times 10^{17}$ GeV. Shaded regions correspond to BH parameters which would result in spin down within a binary lifetime (10^6 years), for $l = 1$ (dark blue) to $l = 5$ (light blue) levels. We also show an example evolution of a $6\,M_\odot$ black hole with initial spin $\tilde{a} = 0.95$. Plot and caption taken from [16].

In written form this may look non-trivial, but plotted for typical parameters as in Fig. 2 it reveals, how the mass term actually leads to a confinement, i.e. provides the natural "mirror" that leads to superradiance, which is why we included it here.

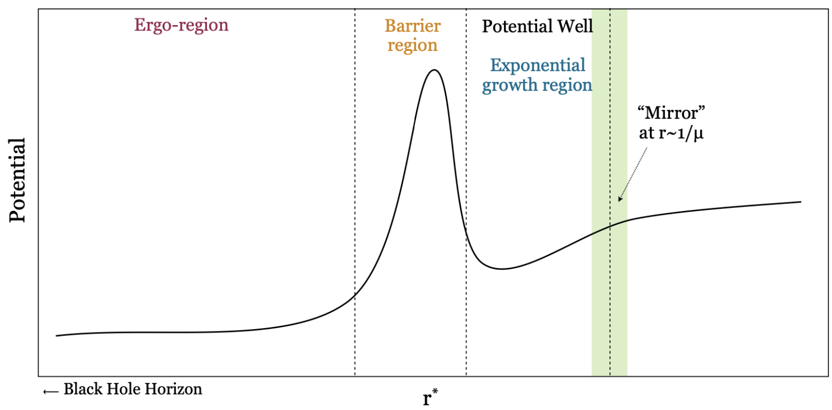

Figure 2: Potential from Eq. (2). Taken from [18].

5 Phenomenology of gravitational atoms (that are not part of binaries)

There are two possibilities to observe (the effects) of a superradiant cloud of a BH that has no binary companion:

- (a) Spin of the BH
- (b) GW emission from the cloud

(a) Superradiance extracts BH spin → observation of highly spinning BHs exclude the corresponding mass of ultralight scalars. We observe spinning stellar black holes as well as supermassive black holes. Therefore, bounds in the corresponding ultralight scalar mass range where clouds should be able to grow on astrophysical time scales have been put e.g. in [16]: Spin measurements of BHs are usually based on the measurement of the innermost stable circular orbit of the accretion disk: the radius at which matter in the disk stops orbiting and rapidly falls into the black hole is a monotonically decreasing function of $\tilde{a}$ that becomes steeper for $\tilde{a} \sim 1$. We show the regions where ultralight scalars are supposedly excluded in Fig. 3. However, these exclusions are not without caveats, since measuring the spins of BHs is not trivial, and also there may be other reasons why the cloud could not grow in the first place, for example caused by perturbations in the surrounding environment.

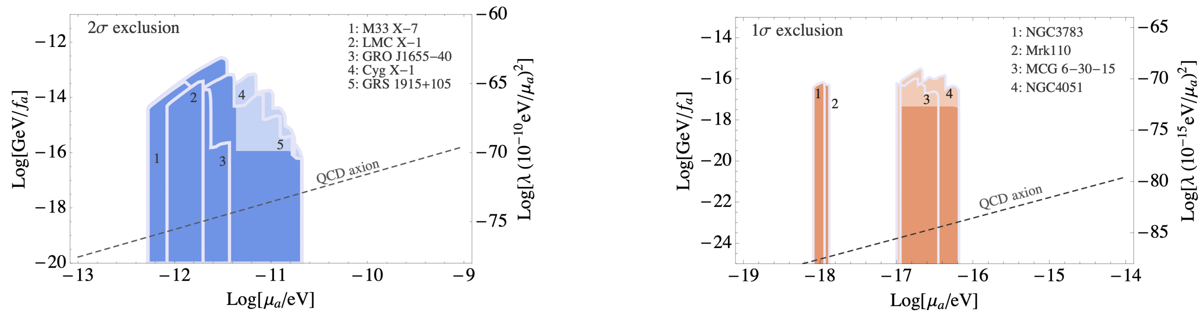

Figure 3: Exclusion regions for ultralight scalars due to observed BHs in the corresponding mass region with large spin that should not be possible if the superradiant clouds exist or once have existed. Taken from [16].

(b) Due to annihilation of two axions into gravitons with rate Γ_a, the cloud decays into GWs over time. The corresponding equation for the particle number is given by

$$\dot{N} = -\Gamma_a N^2. \tag{36}$$

The solution to this is

$$N(t) = \frac{N_0}{1 + N_0 \Gamma_a t} \tag{37}$$

The quantity $\tau \equiv 1/(N_0\Gamma_a)$ is known as the decay time of the cloud. Typical values are: [9, 16, 19]

$$\tau_{211} \approx 1.5 \times 10^8 \,\text{yrs} \left(\frac{M}{10M_\odot}\right)\left(\frac{0.07}{\alpha}\right)^{14}, \tag{38}$$

$$\tau_{322} \approx 1.5 \times 10^8 \,\text{yrs} \left(\frac{M}{10M_\odot}\right)\left(\frac{0.2}{\alpha}\right)^{18}, \tag{39}$$

The monochromatic GW signal from the decaying cloud has a GW-frequency of $f \simeq 2\mu/(2\pi) \simeq 10^3\,\text{Hz}(\mu/10^{-11}\,\text{eV})$. So far there are no exclusions from a non-observation of these monochromatic GW signals, but it has been investigated how this should be possible with aLIGO and LISA. [14, 20] We show estimated GW strains compared to sensitivites of GW observatories in Fig. 4.

6 BH superradiance in binaries: atomic transitions

If we now bring a binary companion, for example a second BH, into the picture, it will act as a gravitational perturbation that mixes the states. The most dramatic effect of this is resonant level mixing, when the orbital frequency matches the energy difference between the two levels

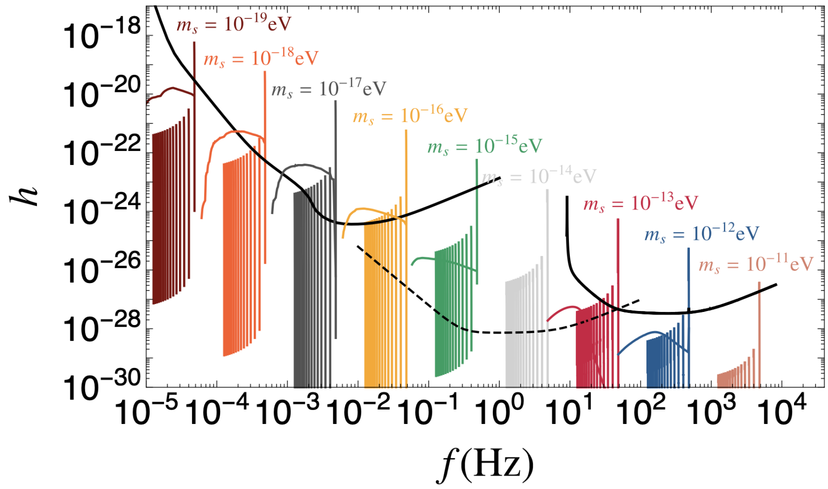

Figure 4: GW strain produced by the clouds compared to the sensitivities of aLIGO and LISA (black thick curves) as well as DECIGO (dashed line), assuming a coherent observation time of four years in all cases. Nearly vertical lines represent BHs with initial spin $\tilde{a} = 0.9$. Each line corresponds to a single source at redshift $z \in (0.001, 3.001)$ (from right to left), and different colors correspond to different boson masses. Thin lines show an optimistic estimate of the stochastic background produced by the whole population of astrophysical BHs. Taken from [20].

divided by the angular momentum difference, at which a so-called **Landau-Zener transition** [21, 22] happens.

We only have time to discuss resonant transitions between bound states on equatorial and ciruclar orbits. There is also non-resonant mixing due to the perturbation [23] and excitation of bound states into unbound states, which is called ionisation and happens inside the cloud and is the ultralight scalar equivalent to dynamical friction [24–26]. For inclined orbits see [24–27], for eccentric orbits, which give rise to additional resonances, see [27, 28]. These references also take the decay of the states into account, which we also ignore.

6.1 Gravitational mixing

We are working in spherical coordinates aligned with the cloud, with coordinates (r, θ, ϕ). The binary companion is at $\vec{\mathsf{R}}_\star(t) \equiv \{\mathsf{R}_\star(t), \Theta_\star(t), \varphi_\star(t)\}$ and has mass $M_\star$. The gravitational potential of this companion can then be decomposed into multipoles [8]:

$$V_\star(t, \vec{r}) = -GM_\star\mu \sum_{l_\star \geqslant 2} \sum_{|m_\star| \leqslant l_\star} \frac{4\pi}{2l_\star + 1} Y^*_{l_\star m_\star}(\Theta_\star, \varphi_\star) Y_{l_\star, m_\star}(\theta, \phi) \left(\frac{r^l_\star}{\mathsf{R}_\star^{l_\star+1}} \Theta(\mathsf{R}_\star - r) + \frac{\mathsf{R}_\star^{l_\star}}{r^{l_\star+1}} \Theta(r - \mathsf{R}_\star) \right). \quad (40)$$

For equatorial orbits ($\Theta_\star = \pi/2$), the spherical harmonic depending on the companion's coordinates becomes a prefactor times $e^{im_\star\varphi_\star}$, which is a crucial ingredient for what follows. To find the mixing with two states $|a\rangle$ and $|b\rangle$, we now have to calculate the matrix element $\langle b| V_\star |a\rangle$.

For resonances that are triggered when the companion is far away from the cloud, we can ignore the part with $\Theta(r - \mathsf{R})$, because in that region, the wavefunctions of the states will be exponentially suppressed.

We rewrite $\mathrm{r} \equiv r/r_c$, $GM_\star\mu = \frac{GM\mu M_\star}{M} \equiv q\alpha$. With this, the tidal interaction is given by [8, 9]

$$\langle b|\, V_\star \,|a\rangle \;\equiv\; \sum_{l_\star=2}^{\infty} \sum_{|m_\star|\leqslant l_\star} \eta_{ab}^{(\star)} e^{-im_\star\varphi_\star}\,, \tag{41}$$

$$\eta_{ab}^{(\star)} \;=\; -\frac{q\alpha}{r_c}\mathrm{R}_\star^{-(l_\star+1)}\frac{4\pi}{2l_\star+1}\left|Y_{(\star)}^*\left(\frac{\pi}{2},\varphi_\star\right)\right| I_r I_\Omega\,,$$

$$I_r \;\approx\; \int^{\infty} d\mathrm{r}\mathrm{r}^2\hat{\mathcal{R}}_b\hat{\mathcal{R}}_a\mathrm{r}^{l_\star} \tag{42}$$

$$I_\Omega \;\equiv\; \int d\Omega Y_b^*(\theta,\phi)Y_{(\star)}(\theta,\phi)Y_a(\theta,\phi)\,, \tag{43}$$

where $(\star) \equiv (l_\star, m_\star)$, $\hat{\mathcal{R}}_c = r_c^{3/2}\mathcal{R}_c$ is the (dimensionless) hydrogenic radial wavefunction. We note that this formula is true for all the terms in the expansion with $l_\star \geqslant 2$, while for $l_\star = 1$, a different formula applies that makes the mixing only non-zero when the companion is inside the cloud.

The integral I_r gives the strength of the mixing, which is determined by the overlap of the wavefunctions. Note that for simplicity, we took the limit of the integral to be ∞, since if the companion is far away, the radial wavefunctions are strongly suppressed for large r anyway.

The second integral, I_Ω does not depend on the position of the companion, has well-known analytic solutions, and for given states $|a\rangle$ and $|b\rangle$ determines, which terms of the infinite sum in Eq. (40) actually can contribute to the mixing. It is only non-zero if the following selection rules are satisfied:

$$\begin{aligned} &(\mathrm{S1}) \quad m_\star = m_b - m_a, \\ &(\mathrm{S2}) \quad l_a + l_\star + l_b = 2p, \quad \text{for } p \in \mathbb{Z}, \\ &(\mathrm{S3}) \quad |l_a - l_b| \leqslant l_\star \leqslant l_a + l_b. \end{aligned} \tag{44}$$

Transitions from states where only m changes have the smallest difference in energy and happen first in the orbital evolution. They are called **hyperfine** transitions. **Fine** transitions are when $n_a = n_n$, but $l_a \neq l_b$. The transitions where n changes are called **Bohr** transitions, as in atomic physics. The selection rules show us, which terms we need to take into account. For example for the fastest growing $|211\rangle$ state, there could be hyperfine transitions to $|21-1\rangle$ and $|210\rangle$. The selection rules make clear that $|211\rangle \to |21-1\rangle$ needs $l_\star = 2$, because it needs to be even, but cannot be larger than 2. So the quadrupole term, $l_\star = 2$, is the only one contributing, with $m_\star = -2$. $|211\rangle \to |210\rangle$ needs to be mediated by the quadrupole perturbation as well, however, the m selection rule tells us that $m_\star = -1$. While $I_\Omega \neq 0$, the spherical harmonic $Y_{2,-1}^\star(\pi/2,\varphi_\star) = 0$, which makes this hyperfine transition only possible for inclined orbits.

6.2 Landau-Zener transitions

Let's assume from the last subsection we found a transition between two states that has non-zero η. For circular orbits[II], the frequency evolution is given by

$$\frac{d\Omega}{dt} = \gamma_0\left(\frac{\Omega}{\Omega_0}\right)^{11/3}, \tag{45}$$

with

$$\gamma_0 = \frac{96}{5}\frac{q}{(1+q)^{1/3}}(GM\Omega_0)^{5/3}\Omega_0^2, \tag{46}$$

[II]We are going to ignore eccentric orbits, as well as orbits that are not aligned with the spin of the central BH and the cloud.

where Ω_0 is any reference frequency. Near a certain resonance frequency, we can linearise the frequency evolution to

$$\Omega = \Omega_0 + \gamma_0 t. \tag{47}$$

This is the crucial ingredient to find Landau-Zener transitions.

We are going to write a general bound state of the cloud as $|\psi(t)\rangle = \sum_i c_i(t)\,|i\rangle$, where c_i is the occupation density of the i-th state. Most of the time we will assume that just one c_i is non-zero. If we ignore the decay or growth of states, we will have $\sum_i |c_i(t)|^2 = 1$. The Schroedinger equation describing the evolution of the cloud can then be written in terms of the occupation densities as

$$i\frac{dc_i}{dt} = \sum_j \mathcal{H}_{ij}(t)c_j, \tag{48}$$

with $\mathcal{H}_{ij} = E_i\delta_{ij} + V_{ij}(t)$. For a system with two states $|a\rangle$ and $|b\rangle$, the Hamiltonian reduces to

$$\mathcal{H} = \begin{pmatrix} -\frac{\Delta E}{2} & \eta_{ab}(t)e^{i\Delta m\varphi_\star(t)} \\ \eta_{ab}(t)e^{-i\Delta m\varphi_\star(t)} & \frac{\Delta E}{2} \end{pmatrix}, \tag{49}$$

with $\Delta E = E_b - E_a$ and $\Delta m = m_b - m_a$, while the off-diagonal terms come from the most dominant mixing term that we found as described in the last subsection. Here, we have ignored the decay rates of the states. From here it is not obvious why there are resonant transitions between the states. The exponential functions are oscillating heavily. Therefore it makes sense to go to the **dressed frame**, the frame that co-rotates with the companion. This can be done via the time-dependent unitary transformation

$$\mathcal{U}(t) = \begin{pmatrix} e^{i\Delta m\varphi_\star/2} & 0 \\ 0 & e^{-i\Delta m\varphi_\star/2} \end{pmatrix}. \tag{50}$$

In the dressed frame, the Hamiltonian becomes

$$\mathcal{H}_{\rm D}(t) = \mathcal{U}^\dagger\mathcal{H}\mathcal{U} - i\mathcal{U}^\dagger\frac{d\mathcal{U}}{dt} = \begin{pmatrix} (\Delta m\Omega(t) - \Delta E)/2 & \eta_{ab}(t) \\ \eta_{ab}(t) & -(\Delta m\Omega(t) - \Delta E)/2 \end{pmatrix}, \tag{51}$$

where $\Omega(t) = \dot{\varphi}_\star(t)$. Since the transformation is unitary, the occupation densities of the states corresponding to the dressed frame Hamiltonian will be same as for the original one.

The resonance condition now becomes obvious, which is when the diagonal terms vanish:

$$\Omega_{\rm res} = \frac{\Delta E}{\Delta m}. \tag{52}$$

We can understand the transition by assuming the linearised behaviour of the frequency, $\Omega = \Omega_0 + \gamma t$ (we have dropped the subscript 0 on γ) and setting $\Omega_0 = \Omega_{\rm res}$. We also assume that $\eta_{ab}(t)$ is constant ($\to \eta$) during the resonance, because the companion inspirals slowly. The dressed Hamiltonian then becomes

$$\mathcal{H}_{\rm D,\,lin}(t) = \frac{\gamma t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{53}$$

whose energy eigenvalues are $E_\pm(t) = \pm\sqrt{(\gamma t/2)^2 + \eta^2}$, while the eigenstates are

$$|E_\pm(t)\rangle = \mathcal{N}_\pm^{-1}(\gamma t/2 \pm \sqrt{(\gamma t/2)^2 + \eta^2}, \eta). \tag{54}$$

Careful investigation of the behaviour in the infinite past and infinite future shows

$$|E_+(-\infty)\rangle = (0,1), \quad |E_-(-\infty)\rangle = -(1,0), \tag{55}$$

while

$$|E_+(+\infty)\rangle = -|E_-(-\infty)\rangle, \quad |E_-(+\infty)\rangle = |E_+(-\infty)\rangle. \tag{56}$$

This means, due to the mixing term, the two eigenstates permutate their identities. This implies that as long as the transition happens slow enough, i.e. adiabatic, so that the system can track its instantaneous eigenstates, a system that starts with a fully populated state $|a\rangle$ completely transfers its population to the other state $|b\rangle$. This is the Landau-Zener transition.

Long after the transition, we have

$$|\langle E_+(\infty)|\psi(\infty)\rangle|^2 = e^{-2\pi z}, \tag{57}$$

where z is the Landau-Zener parameter defined as

$$z \equiv \frac{\eta^2}{\gamma}. \tag{58}$$

It is obvious that if z is larger than unity, we have full transfer of the population. Two examples are shown in Fig. 5.

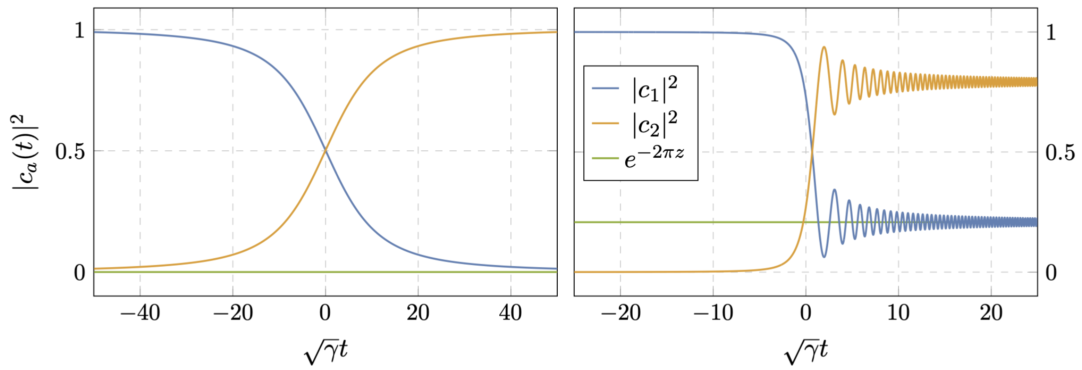

Figure 5: Examples of the population transfer for Landau-Zener transitions. *(left):* Adiabatic, $z > 1$, *(right):* Non-adiabatic, $z < 1$. Taken from [9].

6.3 Backreaction onto the orbit

So far, we have treated the orbital frequency as independent of the transition. This may be true in atomic physics, but not in our system. During the transition, the cloud gains or loses angular momentum and energy, and this will necessarily lead to a **backreaction** onto the orbit.

We can understand the backreaction by writing a full balance equation for energy in the orbit, the cloud and the dissipation term due to GWs[III]:

$$\dot{E}_\mathrm{o} + \dot{E}_\mathrm{c} = \mathcal{F}_\mathrm{GW} \equiv -\frac{32}{5}\frac{G^4M^5q^2(q+1)}{a^5}, \tag{59}$$

where a is the semi-major axis. The orbital energy is given by $E_\mathrm{o} = -\frac{GM^2q}{2a}$, while for the cloud the energy is a sum over the populated states, $E_{\mathrm{c}(i)} \equiv (M_{\mathrm{c},0}/\mu)\epsilon_i|c_i|^2$.

[III] We are ignoring decay rates rates, hence also the backreaction of the BH.

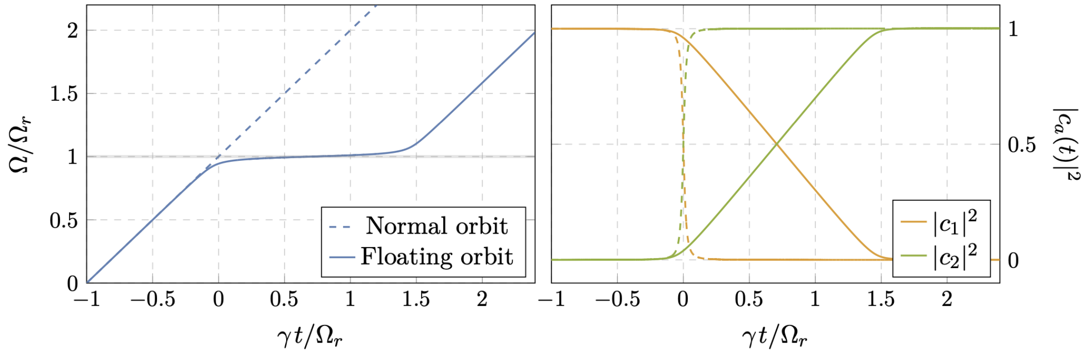

Figure 6: Comparison of a LZ transition without (dashed lines) and with backreaction (solid lines) showing orbital frequency (left) and occupation densities (right). Taken from [9].

With $\Omega = \sqrt{\frac{GM}{a^3}}$, the above equations can then be rewritten as

$$\frac{d\Omega}{dt} = r\gamma_0 \left(\frac{\Omega}{\Omega_0}\right)^{11/3}, \tag{60}$$

$$r \equiv \frac{\dot{E}_o}{\mathcal{F}_{GW}} = 1 - b\frac{\mathrm{sgn}(s\Delta m)\left(\frac{\Omega}{\Omega_0}\right)^{-11/6}}{\sqrt{\gamma_0|\Delta m|}}\frac{d|c_a|^2}{dt}, \tag{61}$$

in terms of the orbital parameters, where

$$b \equiv \frac{3M_{c,0}}{M}|\Delta m|^{3/2}\left(\frac{\Omega}{\Omega_0}\right)^{-3/2}\frac{(1+q)^{1/3}}{\alpha q}\frac{(GM\Omega_0)^{1/3}\,\Omega_0}{\sqrt{\gamma_0}}, \tag{62}$$

and for co-rotating orbits (orbiting the same direction as the BH spins) $s = 1$, otherwise $s = -1$. We see that an "effective" LZ parameters emerges: $\zeta(t) \equiv z/r(t)$, making it a fully nonlinear system.

We see that b is always positive, $\dot{c_a}^2$ is negative during the transition, so if we have a transition on a co-rotating orbit ($s = 1$) in which the cloud loses energy $\Delta E < 0$, Δm must be negative, so the whole second term of r is negative. This means during the transition, the backreaction reduces r. Solving the system numerically, we find that it tends towards zero. This can be analytically understood by seeing the $r \to 0$ corresponds to $\zeta \to \infty$, which means the transition becomes as adiabatic as possible.

When $r = 0$, $\dot{\Omega} = 0$, this means the orbit **floats**. We note the following things:

- A floating orbit of a binary would be a smoking gun signature for the ultralight boson cloud.
- A non-adiabatic transition can become adiabatic due to the backreaction.
- The system is "self-regulatory" in the sense that backreaction makes the transition infinitely adiabatic at best, but cannot lead to outspiral ($r < 0$).

There is of course the opposite effect that for transitions during which the cloud gains energy, and $\Delta m > 0$ for co-rotating orbits $\to r > 1$ and the inspiral happens faster, i.e. the orbit **sinks**.

An example of how backreaction changes the transition for a floating orbit is shown in Fig. 6.

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