Infrared subtraction at NNLO and beyond

Matteo Marcoli   Loop Summit 2 - Cadenabbia   21/07/2025

Newton
International
Fellowship

Work in collaboration with:
Elliot Fox, Nigel Glover
Xuan Chen, Petr Jakubcik, Giovanni Stagnitto

Overview

- Introduction: s.o.t.a. of fixed-order calculations in QCD
- Generalized final-state antenna functions at NNLO
- First steps and results at N^3LO with antenna subtraction
- Summary and outlook

INTRODUCTION

State of the art of NNLO calculations: 2→3 processes

Sector improved residue subtraction, STRIPPER

- Isolated photon production in association with a jet pair through next-to-next-to-leading order in QCD — pp→ɣjj
- NNLO QCD corrections to event shapes at the LHC — pp→jjj, LC
- Next-to-next-to-leading order QCD corrections to $Wb\bar{b}$ production at the LHC (PHYSICAL REVIEW D 106, 074016 (2022)) — pp→Wbb, LC, mb=0
- Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC (PHYSICAL REVIEW LETTERS 127, 152001 (2021)) — pp→jjj, LC
- NNLO QCD corrections to diphoton production with an additional jet at the LHC — pp→ɣɣj, LC
- NNLO QCD corrections to three-photon production at the LHC — pp→ɣɣɣ, LC

qT slicing, MATRIX

- Precise predictions for $t\bar{t}H$ production at the LHC: inclusive cross section and differential distributions — pp→Htt, soft approx., massification (LC)
- Precise Predictions for the Associated Production of a W Boson with a Top-Antitop Quark Pair at the LHC (PHYSICAL REVIEW LETTERS 131, 231901 (2023)) — pp→Wtt, soft approx., massification (LC)
- Associated production of a W boson and massive bottom quarks at next-to-next-to-leading order in QCD (PHYSICAL REVIEW D 107, 074032 (2023)) — pp→Wbb, massification (LC)
- Higgs Boson Production in Association with a Top-Antitop Quark Pair in Next-to-Next-to-Leading Order QCD (PHYSICAL REVIEW LETTERS 130, 111902 (2023)) — pp→Htt, soft approx., massification (LC)
- Triphoton production at hadron colliders in NNLO QCD — pp→ɣɣɣ, LC

Antenna subtraction, NNLOJET

- Precise Predictions for Event Shapes in Diphoton Production at the LHC (PHYSICAL REVIEW LETTERS 134, 171901 (2025)) — pp→ɣɣj
- Automation of antenna subtraction in colour space: gluonic processes — pp→jjj, LC, gluons

qT slicing + PS, MiNNLOPS

- Higgs boson production in association with massive bottom quarks at NNLO+PS — pp→Hbb, massification
- Next-to-next-to-leading order event generation for Z-boson production in association with a bottom-quark pair — pp→Zbb, massification (LC)

Matteo Marcoli

State of the art of NNLO calculations: formalisms and tools

Antenna subtraction

Thomas' talk

NNLOJET: a parton-level event generator for jet cross sections at NNLO QCD accuracy

NNLOJET Collaboration

Generalised antenna functions for higher-order calculations

Elliot Fox, Nigel Glover and Matteo Marcoli

Antenna subtraction for processes with identified particles at hadron colliders

Leonardo Bonino, Thomas Gehrmann, Matteo Marcoli, Robin Schürmann

The colourful antenna subtraction method

T. Gehrmann, E.W.N. Glover and M. Marcoli

A general algorithm to build real-radiation antenna functions for higher-order calculations

Oscar Braun-White, Nigel Glover and Christian T Preuss

Nested soft-collinear subtraction

Towards a general subtraction formula for NNLO QCD corrections to processes at hadron colliders: final states with quarks and gluons

Federica Devoto, Kirill Melnikov, Raoul Röntsch, Chiara Signorile-Signorile, Davide Maria Tagliabue, Matteo Tresoldi

A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation

Federica Devoto, Kirill Melnikov, Raoul Röntsch, Chiara Signorile-Signorile and Davide Maria Tagliabue

CoLoRFul subtraction

NNLOCAL: completely local subtractions for color-singlet production in hadron collisions

V. Del Duca, C. Duhr, L. Fekésházy, F. Guadagni, P. Mukherjee, G. Somogyi, F. Tramontano and S. Van Thurenhout

CoLoRFul for hadron collisions: Integrating the counterterms

S. Van Thurenhout, V. Del Duca, C. Duhr, L. Fekésházy, F. Guadagni, P. Mukherjee, G. Somogyi and F. Tramontano

Local analytic sector subtraction

Sandro's talk

Advances in Local Analytic Sector Subtraction: massive NLO and elements of NNLO automation

Gloria Bertolotti, Giovanni Limatola, Paolo Torrielli and Sandro Uccirati

NNLO subtraction for any massless final state: a complete analytic expression

Gloria Bertolotti, Lorenzo Magnea, Giovanni Pelliccioli, Alessandro Ratti, Chiara Signorile-Signorile, Paolo Torrielli and Sandro Uccirati

NNLO grids with STRIPPER

HighTEA: high energy theory event analyser

Michał Czakon, Zahari Kassabov, Alexander Mitov, Rene Poncelet and Andrei Popescu

State of the art of NNLO calculations

So, is NNLO **solved**? In principle yes ...

STRIPPER: given the relevant amplitudes and enough computational resources, the NNLO calculation is streamlined

But:

refine IR schemes, efficiency

- prohibitive computational cost

improve resources (ML, GPUs, grids, ...)

different techniques

- missing cross-validation

NNLO tools

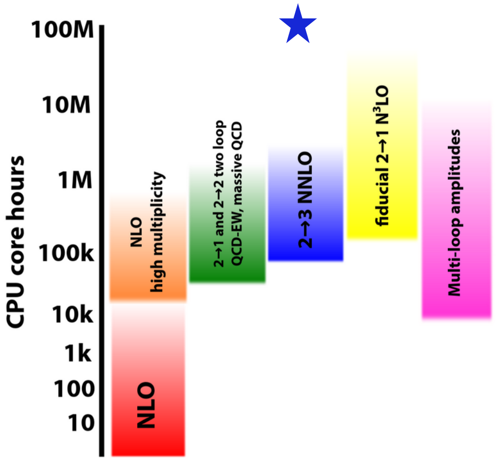
cooperation

2-loop automation

- still a long way to NNLO event generation

PS matching

$pp \to jjj$ event shapes with STRIPPER



[Febres Cordero,von Manteuffel,Neumann '22]

State of the art of N³LO calculations

Inclusive:

- gg→H [Anastasiou,Durh,Dulat,Herzog,Mistleberger '15] [Mistleberger '18]
- gg→HH [Chen,Li,Shao,Wang '19,'20]
- VBF H [Dreyer,Karlberg '16]
- VBF HH [Dreyer,Karlberg '18]
- pp→γ/Z/W [Durh,Dulat,Mistleberger '20] [Durh,Mistleberger '21]
- bb→H
- pp→Z/W H [Baglio,Durh,Mistleberger,Szafron '22]

Fully differential:

- gg→H [Cieri,Chen,Ghermann,Glover,Huss '18] * [Billis,Dehnadi,Ebert,Michel,Tackmann '21] * [Chen,Gehrmann,Glover,Huss,Mistleberger,Pelloni '21] †
- pp→γ/Z/W [Chen,Gehrmann,Glover,Huss,Yang,Zhu '21,'22] * [Chen,Gehrmann,Glover,Huss,Monni,Re,Rottoli,Torrielli '22] * [Campbell,Neumann '22,'23] *
- DIS [Currie,Gehrmann,Glover,Huss,Niehues,Vogt '18] †
- H→bb [Mondini,Schiavi,Williams '19] †
- e^+e^-→jj [Chen,Jakubcik,MM,Stagnitto '25] ‡

Several phenomenologically relevant results despite the extreme complexity.

Available techniques are applicable to limited cases.
New approaches must be developed for more complicated processes.

* qT slicing

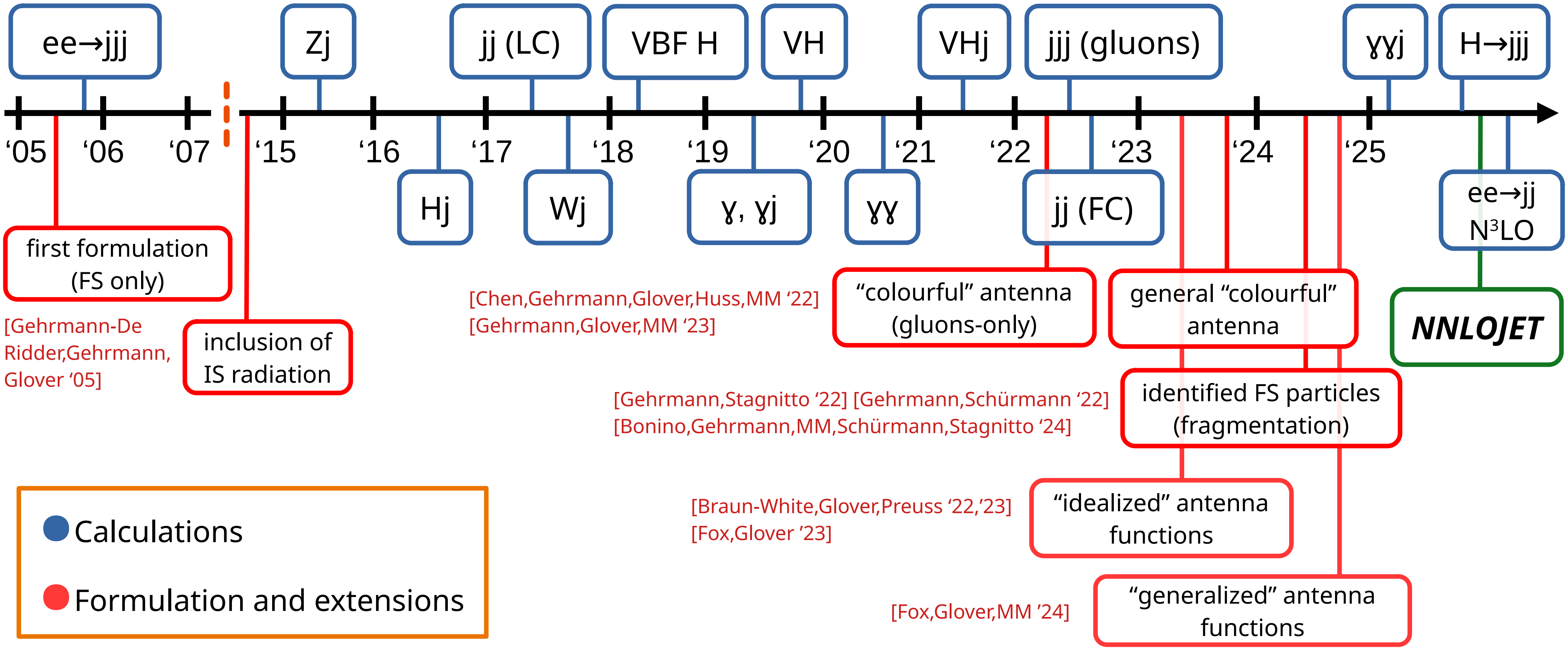
† Projection-to-Born

‡ Antenna Subtraction

Antenna subtraction

[NNLOJET collaboration: Huss *et al.* ‘25]

Successfully applied at NNLO to a variety of processes within the ***NNLOJET*** Monte Carlo framework



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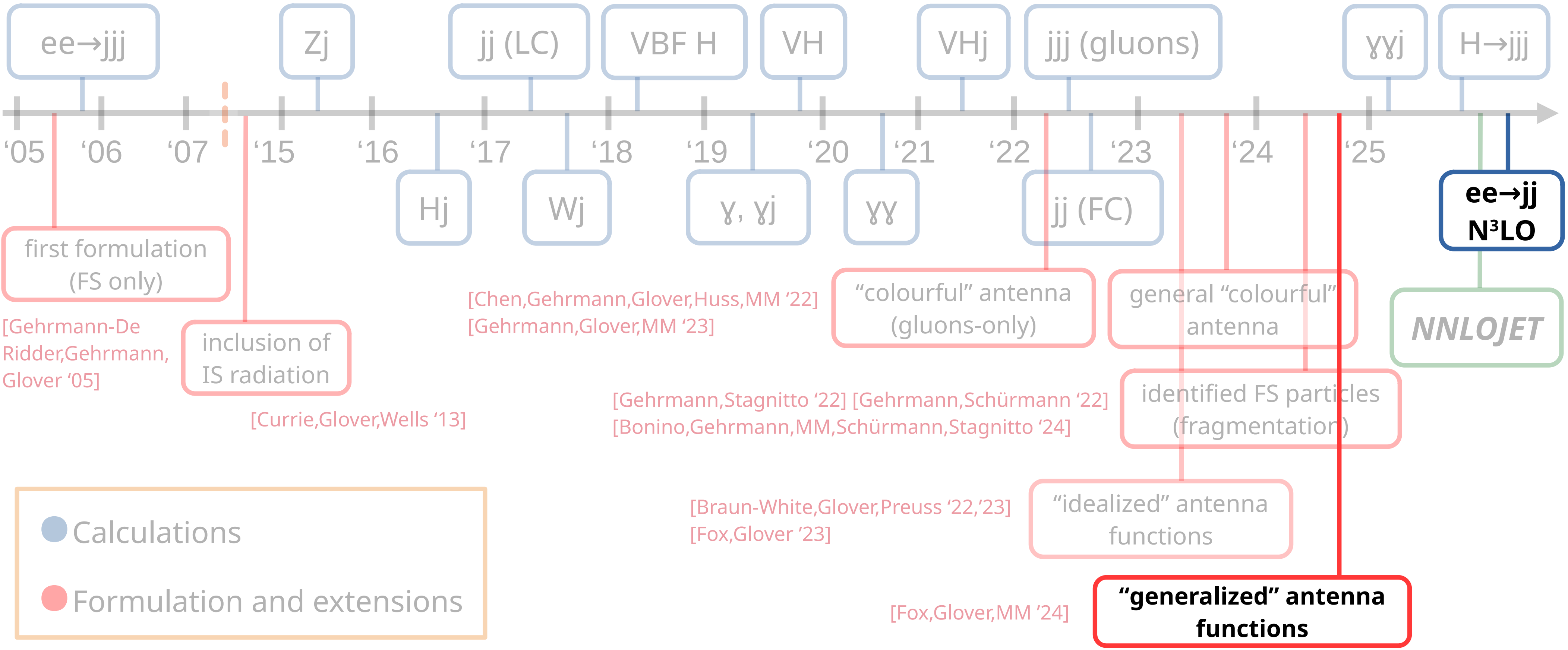
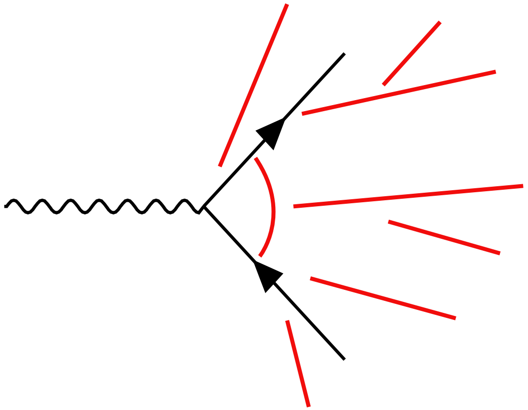


GENERALIZED FINAL-STATE ANTENNA FUNCTIONS AT NNLO

work with Elliot Fox and Nigel Glover, JHEP 12 (2024) 225

Standard antenna functions (FSR)

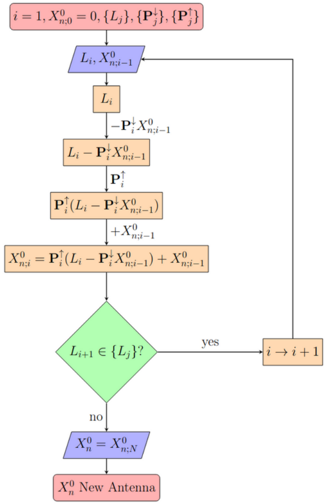


Describe unresolved emissions between **two hard radiators**

Traditionally extracted from colour-ordered matrix elements for colour-singlet decay

[Gehrmann-De Ridder, Gehrmann,Glover ‘04,’05]

[Braun-White,Nigel,Preuss ‘22,’23]



Recently: **designer antenna algorithm** to build antenna functions directly from unresolved factors

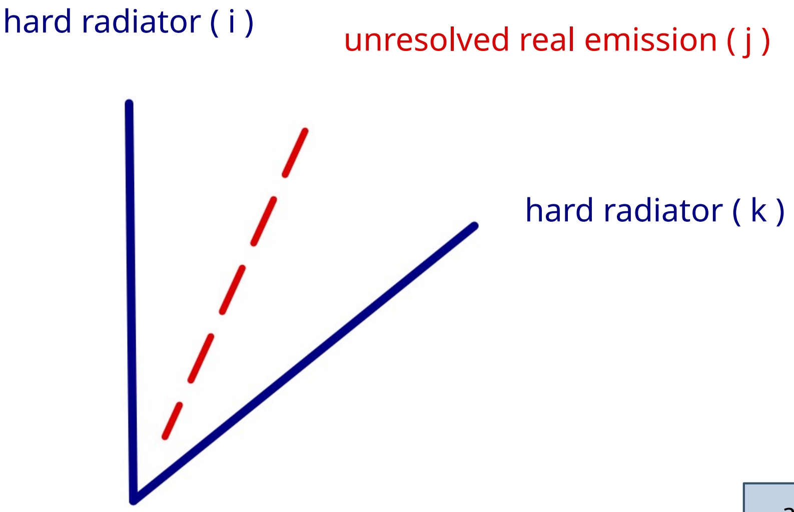
- Simpler expressions
- Better isolation of IR limits
- Removal of some unphysical singularities

Generalized antenna functions [Fox,Glover,MM ‘24]

Two-hard-radiator antenna functions work very well for some configurations, **less well** for others

What **emission topologies** can they easily describe?

NLO: one unresolved emission → only one possible topology



colour-ordered squared amplitude for real emision

$$M^0_{n+1}(\ldots, i, j, k, \ldots)$$

IR limits

$$X^0_3(i^h, j, k^h) M^0_n(\ldots, (\widetilde{ij}), (\widetilde{jk}), \ldots)$$

3→2 momentum mapping:
- momentum conservation
- on-shellness
- IR limits

[Kosower ‘97,‘02]

antenna function

Born-level squared amplitude

Generalized antenna functions [Fox,Glover,MM ‘24]

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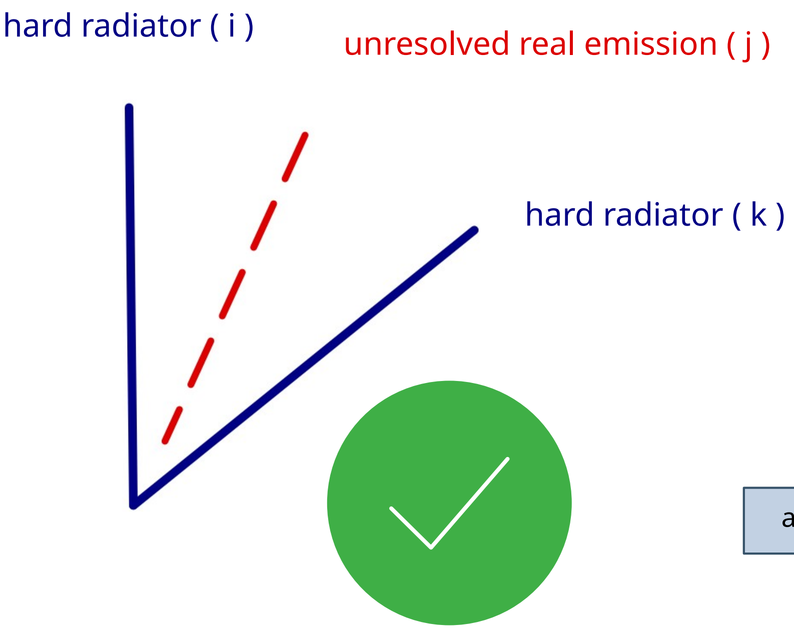

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antenna function

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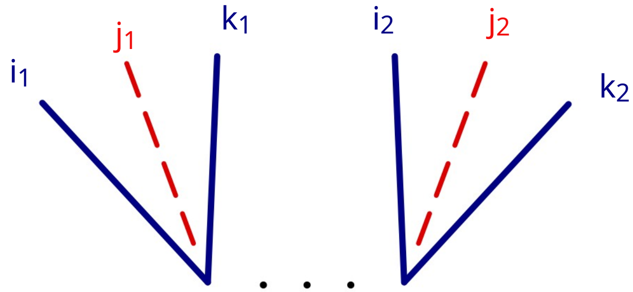
NNLO: two unresolved emissions → multiple topologies

colour-unconnected emissions: no shared hard radiatior



$$M^0_{n+2}(\ldots, i_1, j_1, k_1, \ldots, i_2, j_2, k_2, \ldots)$$

fully **iterated** structure

$$X^0_3(i_1^h, j_1, k_1^h) X^0_3(i_2^h, j_2, k_2^h) M^0_n(\ldots, (\widetilde{i_1 j_1}), (\widetilde{j_1 k_1}), \ldots, (\widetilde{i_2 j_2}), (\widetilde{j_2 k_2}), \ldots)$$

Generalized antenna functions [Fox,Glover,MM '24]

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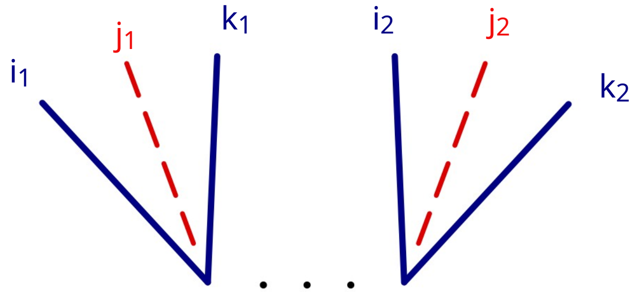

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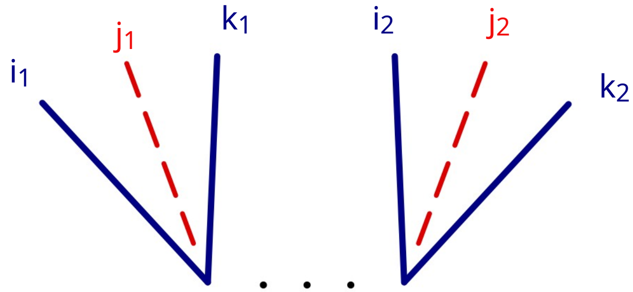
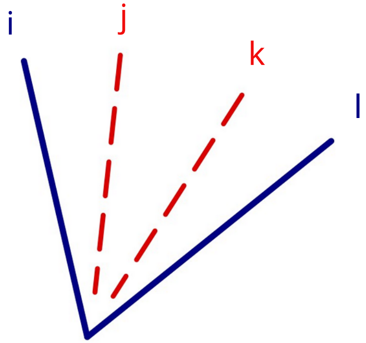


$$M_{n+2}^0(\ldots, i_1, j_1, k_1, \ldots, i_2, j_2, k_2, \ldots)$$

$$X_3^0(i_1^h, j_1, k_1^h) X_3^0(i_2^h, j_2, k_2^h) M_n^0(\ldots, (\widetilde{i_1 j_1}), (\widetilde{j_1 k_1}), \ldots, (\widetilde{i_2 j_2}), (\widetilde{j_2 k_2}), \ldots)$$

fully **iterated** structure

colour-connected emissions: both hard radiators shared



$$M_{n+2}^0(\ldots, i, j, k, l, \ldots)$$

$$X_4^0(i^h, j, k, l^h) M_n^0(\ldots, (\widetilde{ijk}), (\widetilde{jkl}), \ldots)$$

4→2 momentum mapping

Generalized antenna functions [Fox,Glover,MM ‘24]

NNLO: two unresolved emissions → multiple topologies

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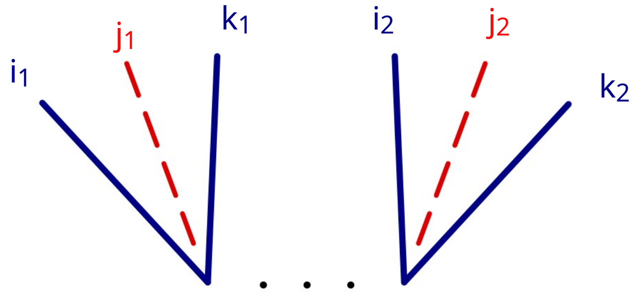

$$M^0_{n+2}(\dots,i_1,j_1,k_1,\dots,i_2,j_2,k_2,\dots)$$

$$X^0_3(i_1^h,j_1,k_1^h)X^0_3(i_2^h,j_2,k_2^h)M^0_n(\dots,(\widetilde{i_1j_1}),(\widetilde{j_1k_1}),\dots,(\widetilde{i_2j_2}),(\widetilde{j_2k_2}),\dots)$$

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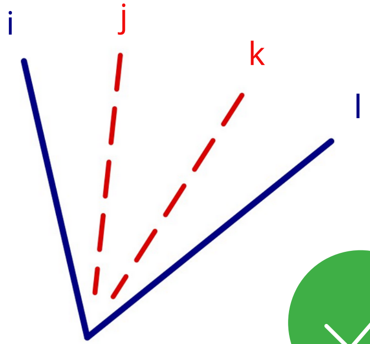


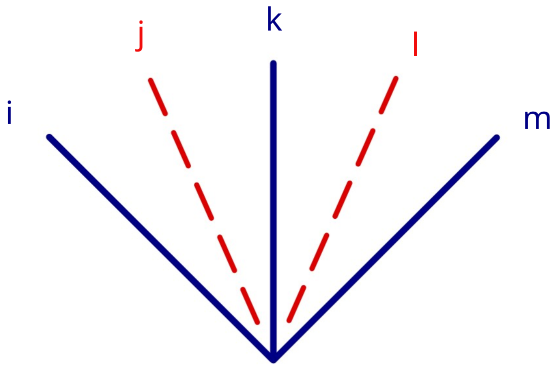
$$M^0_{n+2}(\dots,i,j,k,l,\dots)$$

$$X^0_4(i^h,j,k,l^h)M^0_n(\dots,(\widetilde{ijk}),(\widetilde{jkl}),\dots)$$

4→2 momentum mapping

Generalized antenna functions [Fox,Glover,MM ‘24]

There is more … **almost colour-connected** emissions: only one shared hard radiator

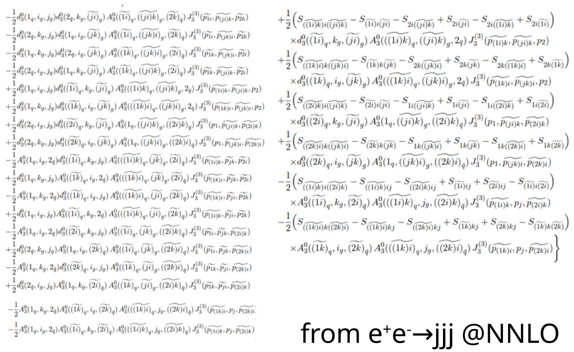


NOT fully iterated: the two emissions “feel” each other through the recoil on the shared radiator

traditional antenna functions can be used, but a **very complicated** sequence of iterated structures is needed, plus **Large-Angle-Soft-Terms**

[Gehrmann De-Ridder,Gehrmann,Glover,Heinrich ‘07] [Weinzierl ‘08] [Currie,Glover,Wells ‘13]

most complicated and inefficient sector of antenna subtraction



from $e^+e^- \to jjj$ @NNLO

Generalized antenna functions [Fox,Glover,MM '24]

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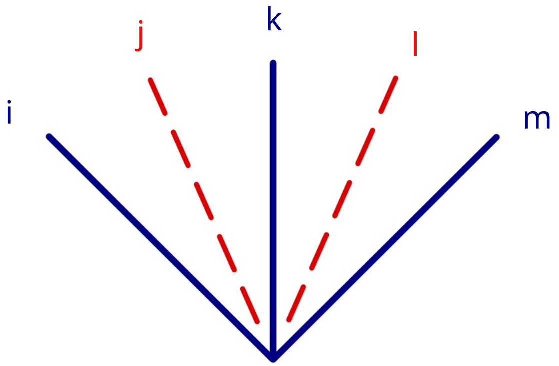

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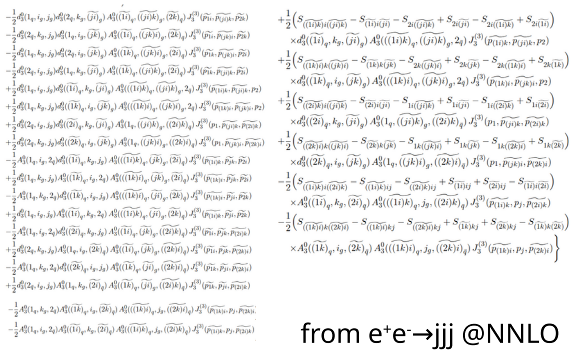


from $e^+e^-\to jjj$ @NNLO

Ideally we want:

$$M^0_{n+2}(\ldots,i,j,k,l,m\ldots)$$

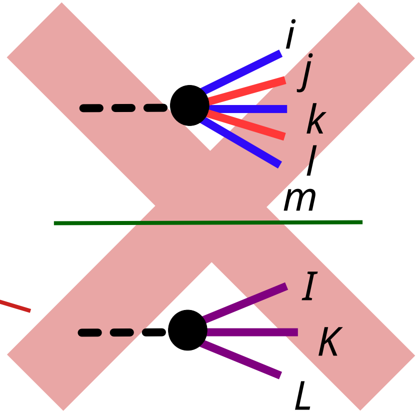
$$X^0_{5,3}(i^h,j,k^h,l,m^h)M^0_n(\ldots,(\widetilde{ijk}),(\widetilde{ijklm}),(\widetilde{klm}),\ldots)$$

5→3 momentum mapping

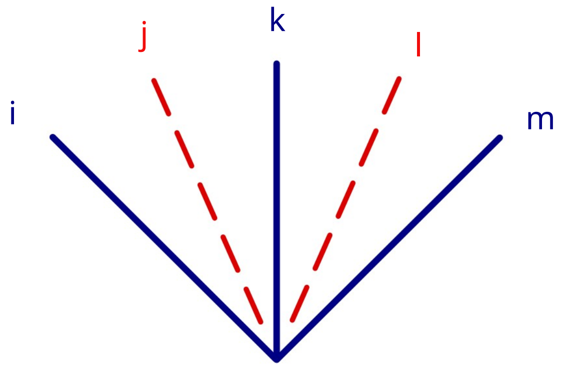
generalized three-hard-radiator antenna function

Generalized antenna functions [Fox,Glover,MM ‘24]

Not possible with matrix element-based antenna functions



non-trivial function of the three-particle phase space



[Braun-White,Nigel,Preuss ‘22,’23]

With the **designer antenna algorithm**, it is possible to construct antenna functions with **more than two hard radiators: generalized antenna functions.**

Analytical integration made particularly simple thanks to a convenient choice of **5→3 momentum mapping**.

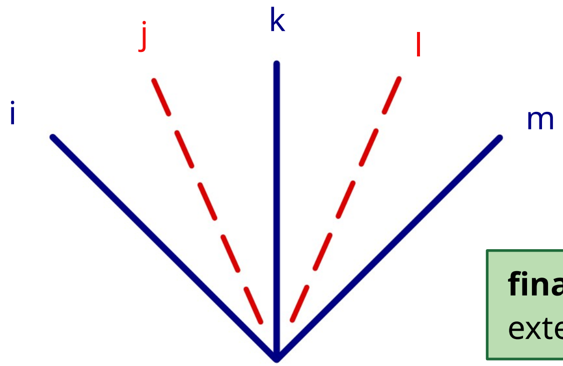
$$\text{map}_{5\to3}: \quad \begin{aligned} p_I &= p_i + p_j - \frac{s_{ij}}{s_{ik}+s_{jk}} p_k \\ p_K &= \left(1 + \frac{s_{ij}}{s_{ik}+s_{jk}} + \frac{s_{lm}}{s_{lk}+s_{mk}}\right) p_k \\ p_M &= p_l + p_m - \frac{s_{lm}}{s_{lk}+s_{mk}} p_k \end{aligned}$$

iterated dipole mapping

Type	Name and parton content	Reconstructed hard partons	$X_3^0 \otimes X_3^0$
quark-antiquark-gluon	$A_{5,3}^0(i_q^h, j_g, k_g^h, l_g, m_{\bar{q}}^h)$	$qg\bar{q}$	$D \otimes \overline{D}$
	$B_{5,3}^0(i_{\bar{q}}^h, j_g, k_Q^h, l_{\bar{Q}}, m_q^h)$	$qg\bar{q}$	$A \otimes \overline{E}$
	$\widetilde{A}_{5,3}^0(i_{\bar{q}}^h, j_\gamma, k_q^h, l_g, m_g^h)$	$\bar{q}qg$	$A \otimes D$
	$\widetilde{B}_{5,3}^0(i_{\bar{q}}^h, j_\gamma, k_q^h, l_{\bar{Q}}, m_Q^h)$	$\bar{q}qg$	$A \otimes E$
quark-antiquark-quark	$\widetilde{\widetilde{A}}_{5,3}^0(i_q^h, j_\gamma, k_{\bar{q}}^h, l_g, m_Q^h)$	$q\bar{q}Q$	$A \otimes A$
quark-gluon-gluon	$D_{5,3}^0(i_q^h, j_g, k_g^h, l_g, m_g^h)$	qgg	$D \otimes F$
	$E_{5,3}^{0(a)}(i_q^h, j_{\bar{Q}}, k_Q^h, l_g, m_g^h)$	qgg	$E \otimes D$
	$E_{5,3}^{0(b)}(i_q^h, j_g, k_{\bar{Q}}^h, l_Q, m_g^h)$	qgg	$A \otimes \overline{G}$
	$E_{5,3}^{0(c)}(i_q^h, j_g, k_g^h, l_{\bar{Q}}, m_Q^h)$	qgg	$D \otimes G$
	$E_{5,3}^{0,(d)}(i_Q^h, j_{\bar{Q}}, k_q^h, l_g, m_g^h)$	gqg	$\overline{E} \otimes D$
	$K_{5,3}^0(i_q^h, j_{\bar{Q}}, k_Q^h, l_{\bar{R}}, m_R^h)$	qgg	$E \otimes E$
gluon-gluon-gluon	$F_{5,3}^0(i_g^h, j_g, k_g^h, l_g, m_g^h)$	ggg	$F \otimes F$
	$G_{5,3}^{0(a)}(i_{\bar{q}}^h, j_q, k_g^h, l_g, m_g^h)$	ggg	$\overline{G} \otimes F$
	$G_{5,3}^{0(b)}(i_g^h, j_{\bar{q}}, k_q^h, l_g, m_g^h)$	ggg	$G \otimes D$
	$H_{5,3}^{0(a)}(i_{\bar{Q}}^h, j_Q, k_g^h, l_{\bar{q}}, m_q^h)$	ggg	$\overline{G} \otimes G$
	$H_{5,3}^{0(b)}(i_g^h, j_{\bar{Q}}, k_Q^h, l_{\bar{q}}, m_q^h)$	ggg	$G \otimes E$

[Fox,Glover,MM ‘24]

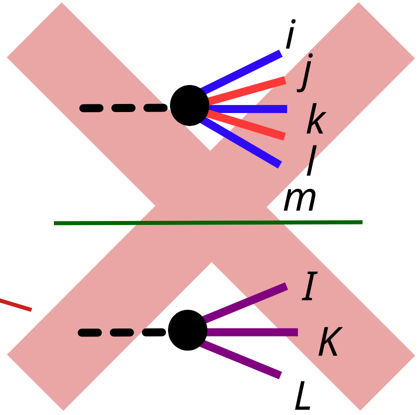
Generalized antenna functions [Fox,Glover,MM '24]



final-state radiation only, extension to ISR in progress

Not possible with matrix element-based antenna functions

non-trivial function of the three-particle phase space



[Braun-White,Nigel,Preuss '22,'23]

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quark-antiquark-quark	$\widetilde{\widetilde{A}}_{5,3}^0(i_q^h, j_\gamma, k_{\bar{q}}^h, l_g, m_Q^h)$	$q\bar{q}Q$	$A \otimes A$
quark-gluon-gluon	$D_{5,3}^0(i_q^h, j_g, k_g^h, l_g, m_g^h)$	qgg	$D \otimes F$
	$E_{5,3}^{0(a)}(i_q^h, j_{\bar{Q}}, k_Q^h, l_g, m_g^h)$	qgg	$E \otimes D$
	$E_{5,3}^{0(b)}(i_q^h, j_g, k_{\bar{Q}}^h, l_Q, m_g^h)$	qgg	$A \otimes \overline{G}$
	$E_{5,3}^{0(c)}(i_q^h, j_g, k_g^h, l_{\bar{Q}}, m_Q^h)$	qgg	$D \otimes G$
	$E_{5,3}^{0,(d)}(i_Q^h, j_{\bar{Q}}, k_q^h, l_g, m_g^h)$	gqg	$\overline{E} \otimes D$
	$K_{5,3}^0(i_q^h, j_{\bar{Q}}, k_Q^h, l_{\bar{R}}, m_R^h)$	qgg	$E \otimes E$
gluon-gluon-gluon	$F_{5,3}^0(i_g^h, j_g, k_g^h, l_g, m_g^h)$	ggg	$F \otimes F$
	$G_{5,3}^{0(a)}(i_{\bar{q}}^h, j_q, k_g^h, l_g, m_g^h)$	ggg	$\overline{G} \otimes F$
	$G_{5,3}^{0(b)}(i_g^h, j_{\bar{q}}, k_q^h, l_g, m_g^h)$	ggg	$G \otimes D$
	$H_{5,3}^{0(a)}(i_{\bar{Q}}^h, j_Q, k_g^h, l_{\bar{q}}, m_q^h)$	ggg	$\overline{G} \otimes G$
	$H_{5,3}^{0(b)}(i_g^h, j_{\bar{Q}}, k_Q^h, l_{\bar{q}}, m_q^h)$	ggg	$G \otimes E$

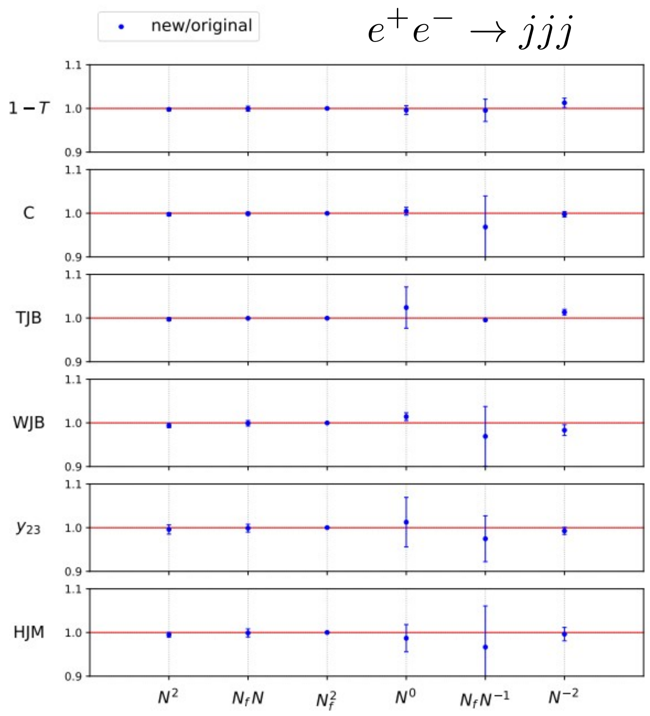
[Fox,Glover,MM '24]

Generalized antenna functions: validation and applications

NNLO correction to event shapes in e^+e^- annihilation:

- perfect agreement with original method
- simpler subtraction terms
- up to 5-10x faster

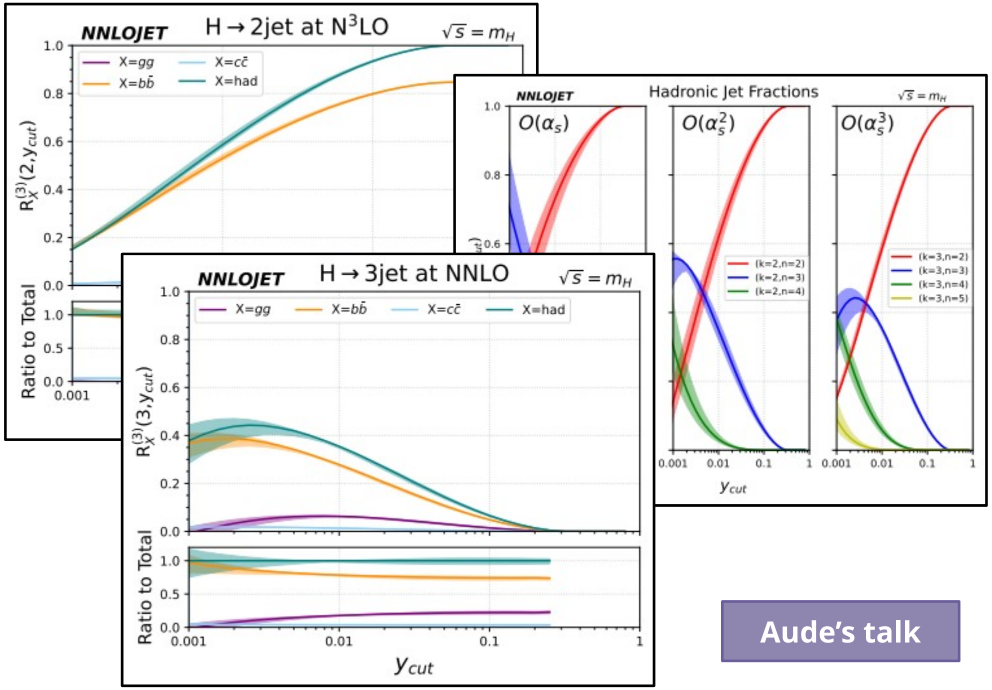
event shapes



[Fox,Glover,MM '24]

Hadronic Higgs decays:

- differences between H→bb and H→gg
- jet rates at order α_s^3 (3jet @NNLO, 2jet @N^{3}LO)



Aude's talk

[Fox,Gehrmann-De Ridder,Gehrmann,Glover,MM,Preuss '25]

FIRST STEPS AND RESULTS AT N^3LO WITH ANTENNA SUBTRACTION

work with Xuan Chen, Petr Jakubcik and Giovanni Stagnitto
2505.10618, 2507.12537

Jet production at lepton colliders: $e^+e^- \to jj$ @N^{3}LO

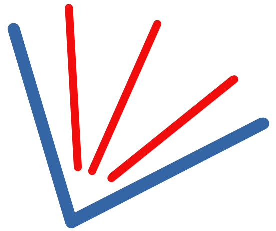
[Chen,Jakubcik,MM,Stagnitto ‘25]

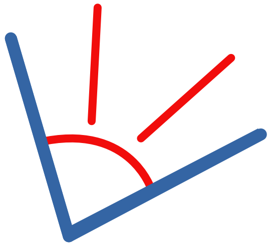
Simple process:

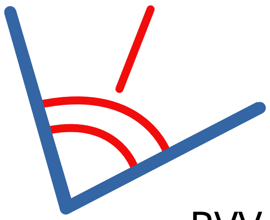
- only **q-$\bar{q}$** N^{3}LO antenna functions;
- only **dipole-like correlations** at N^{3}LO;

Goals:

- definition of **N^{3}LO antenna functions**;
- exploration of **numerical challenges** (IR stability of loop amplitudes);
- preparation of **computational framework** for more complicated processes;







The calculation builds on top of $e^+e^- \to jjj$ @NNLO in ***NNLOJET***

[Gehrmann,Glover,Huss, Nieuhes,Zhang ‘17]

Analytic integration of FSR N^{3}LO antenna functions in:

[Jakubcik,MM,Stagnitto ‘22]
[Chen,Jakubcik,MM,Stagnitto ‘23]

standard matrix-element based antenna functions

Local subtraction at N³LO

Subtraction at N³LO:

$$\mathrm{d}\sigma_{N^3LO} = \int_n [\mathrm{d}\sigma^{VVV} - \mathrm{d}\sigma^{W}] + \int_n [\mathrm{d}\sigma^{RVV} - \mathrm{d}\sigma^{U}] + \int_{n+1} [\mathrm{d}\sigma^{RRV} - \mathrm{d}\sigma^{T}] + \int_{n+2} [\mathrm{d}\sigma^{RRR} - \mathrm{d}\sigma^{S}]$$

triple-virtual subtraction term

double-virtual real subtraction term

double-real-virtual subtraction term

triple-real subtraction term

with:

$$\mathrm{d}\sigma^{S} = \mathrm{d}\sigma^{S_1} + \mathrm{d}\sigma^{S_2} + \mathrm{d}\sigma^{S_3}$$

$$\mathrm{d}\sigma^{T} = \mathrm{d}\sigma^{V_1 S_1} + \mathrm{d}\sigma^{V_1 S_2} - \int_1 \mathrm{d}\sigma^{S_1}$$

$$\mathrm{d}\sigma^{U} = \mathrm{d}\sigma^{V_2 S_1} - \int_1 \mathrm{d}\sigma^{V_1 S_1} - \int_2 \mathrm{d}\sigma^{S_2}$$

$$\mathrm{d}\sigma^{W} = -\int_1 \mathrm{d}\sigma^{V_2 S_1} - \int_2 \mathrm{d}\sigma^{V_1 S_2} - \int_3 \mathrm{d}\sigma^{S_3}$$

[Chen,Jakubcik,MM,Stagnitto '25]

See also [Chen,MM '25]

Challenge: numerical stability of

one-loop double-unresolved quantities — Rescue-system to trigger **quadruple precision**

two-loop single-unresolved quantities — Rescue-system to trigger **Taylor expansions** of special functions

Results

Basic checks: inclusive cross section

N³LO coefficient

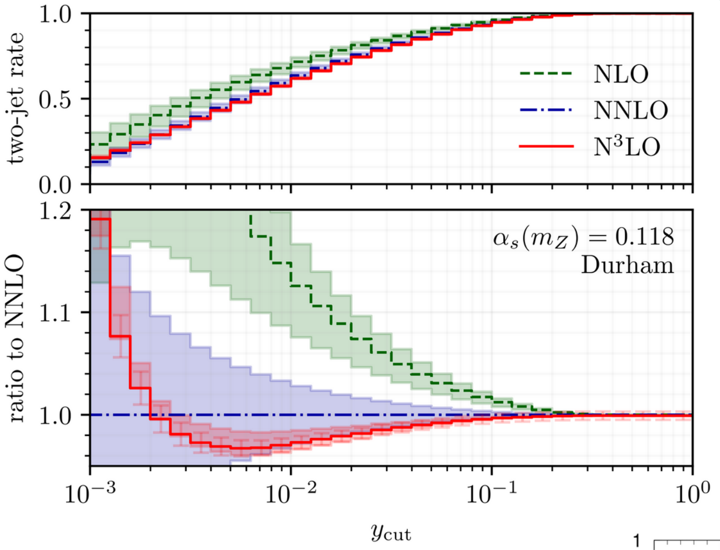
$$\sigma^{(3)} = \sigma_0 \left(\frac{\alpha_s}{2\pi}\right)^3 (-105 \pm 11)$$

Monte Carlo error.
Not so small for inclusive quantities due to large cancellations: not the *most clever* way to compute inclusive *XS*.

$$\sigma^{(3)}_{\text{exact}} = \sigma_0 \left(\frac{\alpha_s}{2\pi}\right)^3 (-102.14\dots)$$

[Chetyrkin,Künn,Kwiatkowski '95]

Two-jet rate at order O(α_s^3) (direct calculation)



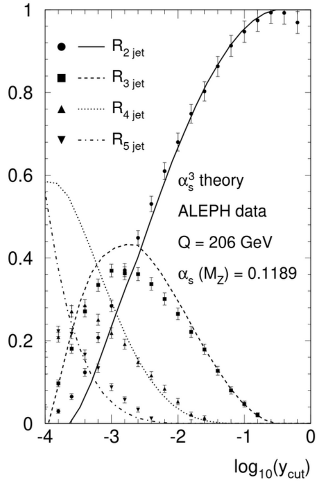
No need of previous knowledge of N³LO quantities. Fiducial cuts can be implemented too

Full agreement with indirect calculation of

[Gehrmann-De Ridder, Gehrmann,Glover,Heinrich '08]

$$R_n^{(3)}(y_{\text{cut}}) = \frac{\Gamma^{(3)}_{\gamma^* \to \text{n jets}}(y_{\text{cut}})}{\Gamma^{(3)}_{\gamma^* \to \text{hadrons}}}$$

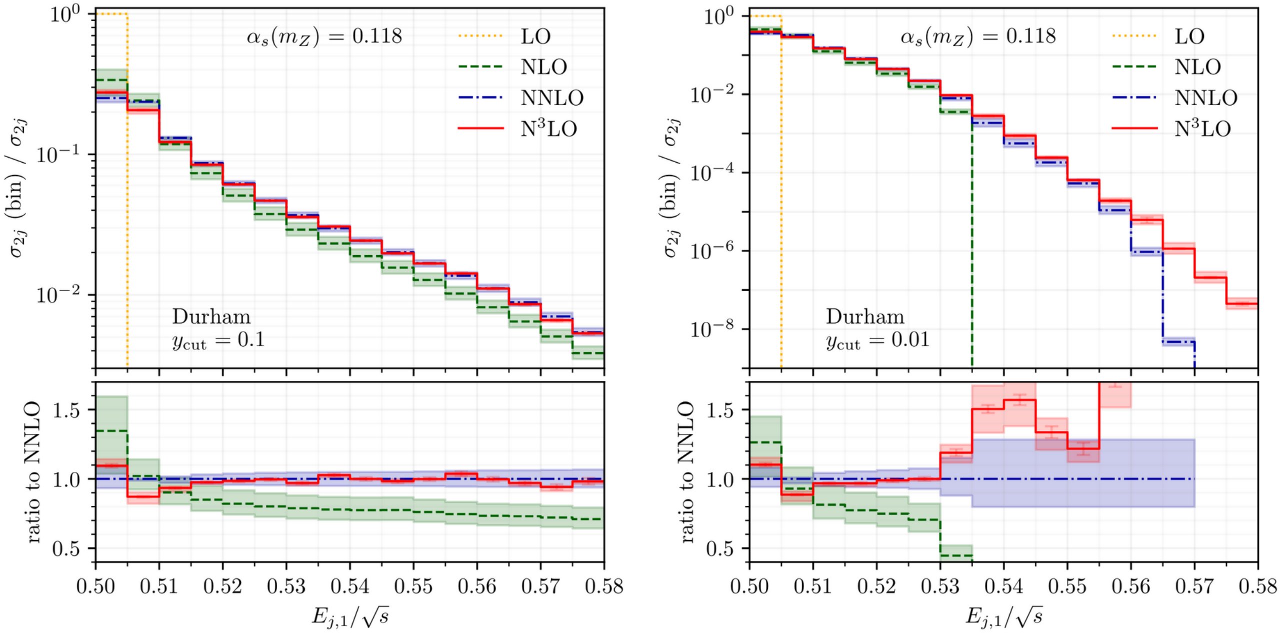
$$R_2^{(3)}(y_{\text{cut}}) = 1 - \sum_{n=3}^{5} R_n^{(3)}(y_{\text{cut}})$$



Results

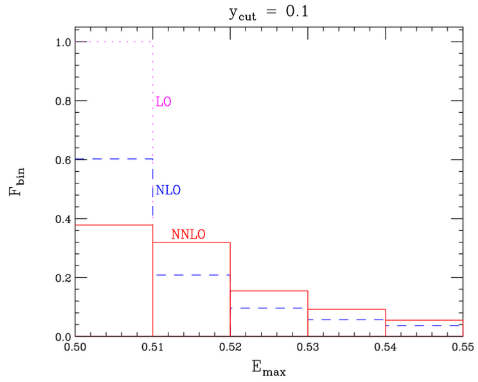
Leading-jet energy:

- defined on 2-jet events, bin-integrated;
- lower orders vanish quickly at large E_{j1} for smaller y_{cut} because energetic jets recoil against multiple emissions which are more likely to be clustered as three or more jets;
- again, the whole distribution can be obtained by combining $e^+e^- \to jjj$ @NNLO with the N^{3}LO inclusive XS.
  It's a proof-of-principle application.



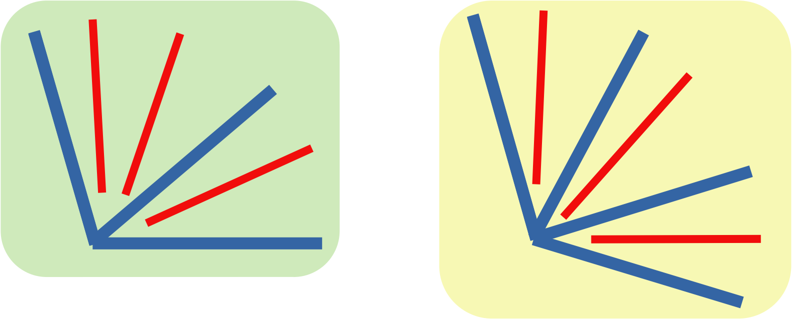
Full agreement up to NNLO with

[Anastasiou,Melnikov,Petriello '04]

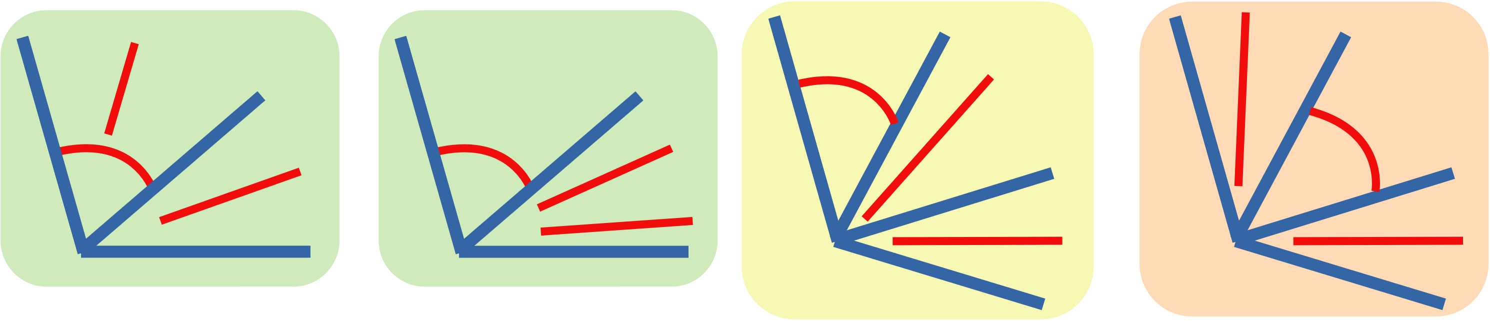


How to proceed? Generalized antenna functions at N^{3}LO

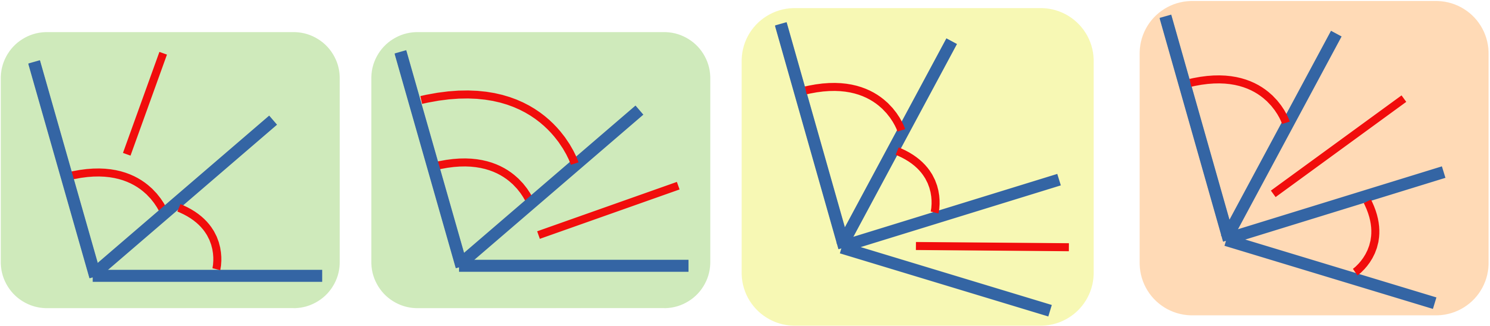
RRR:



RRV:



RVV:



Starting from:
$e^+e^- \to 3j$
$pp \to V+j$

Starting from:
$e^+e^- \to 4j$
$pp \to 2j$

Special case for:
$e^+e^- \to 3j$
$pp \to V+j$
General case for:
$e^+e^- \to 4j$
$pp \to 2j$

SUMMARY AND OUTLOOK

NNLO may be **solved**, but still not so accessible. Several ongoing efforts towards automation and generalization. N^3LO is very challenging, but (more than) first steps made in this direction already.

Generalized antenna functions yield a simpler and more efficient formulation of final-state IR subtraction. Outlook: extension to hadronic processes.

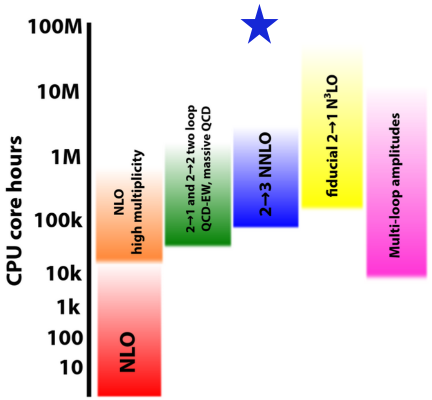
First application of antenna subtraction to a fully-differential **N^3LO** calculation. Outlook: gradual extension to more complicated processes.

Thank you very much for your attention!

BACKUP SLIDES

Fixed-order calculations in QCD

pp → jjj event shapes with STRIPPER



[Febres Cordero,
von Manteuffel,Neumann '22]

$$\sigma = \sigma_0 + \left(\frac{\alpha_s}{2\pi}\right)\sigma_1 + \left(\frac{\alpha_s}{2\pi}\right)^2\sigma_2 + \left(\frac{\alpha_s}{2\pi}\right)^3\sigma_3 + \ldots$$

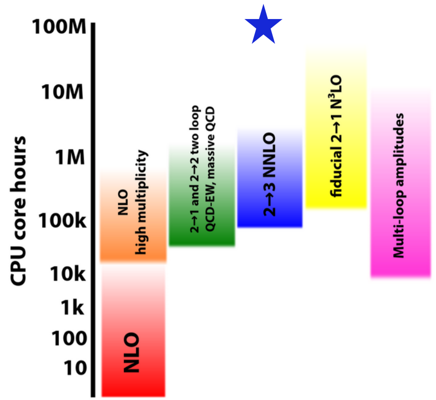
Leading Order (LO)	Next-to-Leading Order (NLO)	Next-to-Next-to Leading Order (NNLO)	Next-to-Next-to-Next-to Leading Order (N^3LO)
	O(10%) - O(100%)	O(1%) - O(10%)	≤ O(1%)

>> accuracy, reduction of theory uncertainties

>>> complexity, manpower, computational cost

Fixed-order calculations in QCD

pp → jjj event shapes with STRIPPER



[Febres Cordero,
von Manteuffel,Neumann '22]

$$\sigma = \sigma_0 + \left(\frac{\alpha_s}{2\pi}\right)\sigma_1 + \left(\frac{\alpha_s}{2\pi}\right)^2\sigma_2 + \left(\frac{\alpha_s}{2\pi}\right)^3\sigma_3 + \dots$$

Leading Order (LO)	Next-to-Leading Order (NLO)	Next-to-Next-to Leading Order (NNLO)	Next-to-Next-to-Next-to Leading Order (N^3LO)
	O(10%) - O(100%)	O(1%) - O(10%)	≤ O(1%)

>> accuracy, reduction of theory uncertainties

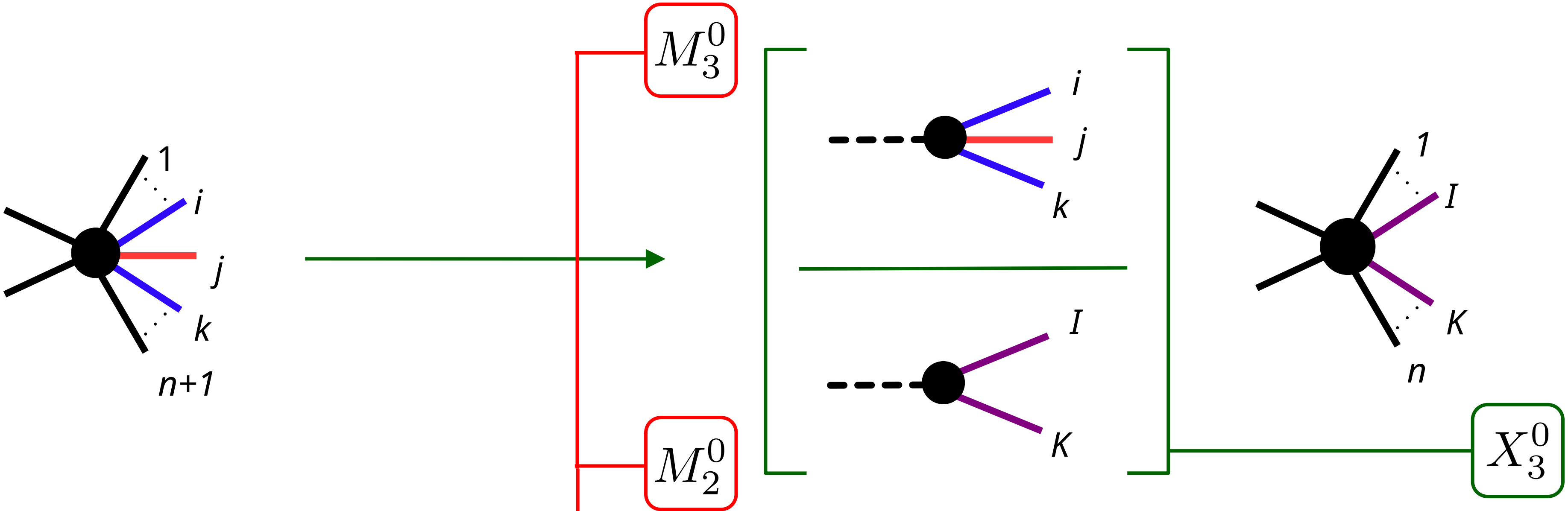
>>> complexity, manpower, computational cost

Conditions to claim that infrared subtraction is **solved** at a N^kLO?

STRIPPER

	NLO	NNLO	N^3LO
Given matrix elements and resources, the N^kLO correction can be computed	✓	✓	✗
New N^kLO calculations are streamlined	✓	✓	✗
Different groups/approach available (validations, efficiency, ...)	✓	●	✗
N^kLO calculations can be easily reproduced and performed by anyone	✓	✗	✗
N^kLO event generation is possible	✓	✗	✗

NLO: three-parton tree-level antenna functions



$$X_3^0 = \frac{M_3^0}{M_2^0}$$

$$\mathcal{X}_3^0 \propto \int \mathrm{d}\Phi_3 \, X_3^0$$

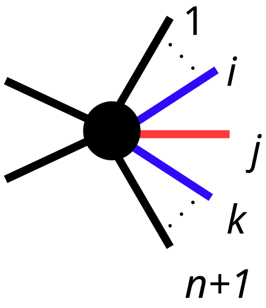
Matrix elements for **colour singlet** decay.

Integrated antenna function

Encapsulates the divergent behaviour when parton **j** becomes **soft** or **collinear** to **i**, **k**.

Antenna subtraction at NLO

Real



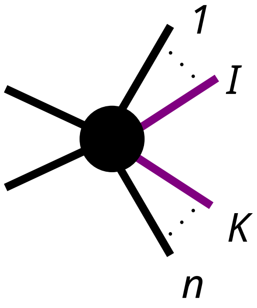
parton j soft or collinear to i or k

factorization properties of QCD

3-parton tree antenna

$$X_3^0(i,j,k)\;\cdot$$

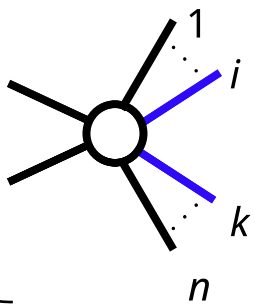
Antenna function



Analytical integration over PS of the unresolved radiation

Virtual

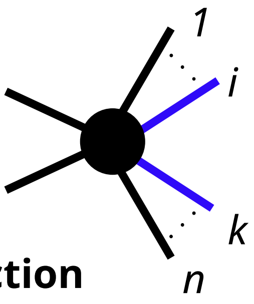
Poles [



] = Poles [

$$\mathcal{X}_3^0(s_{ik})\;\cdot$$

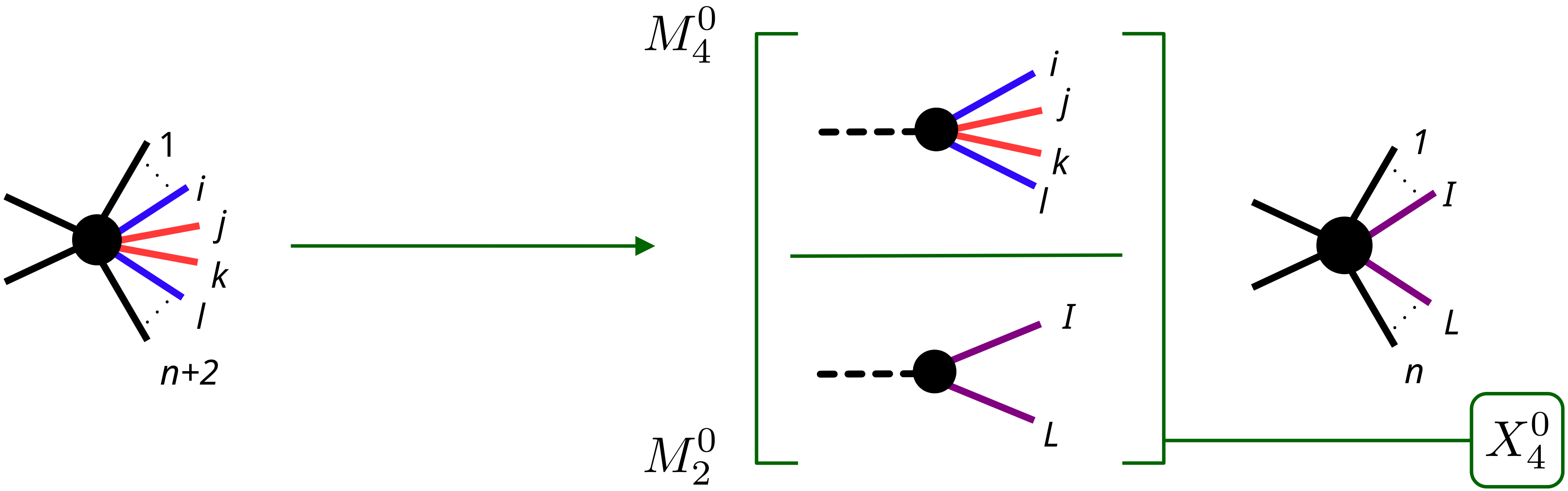
Integrated antenna function



]

[Gehrmann-De Ridder,Gehrmann,Glover ‘05] [Currie,Glover,Wells ‘13]

NNLO: four-parton tree-level antenna functions

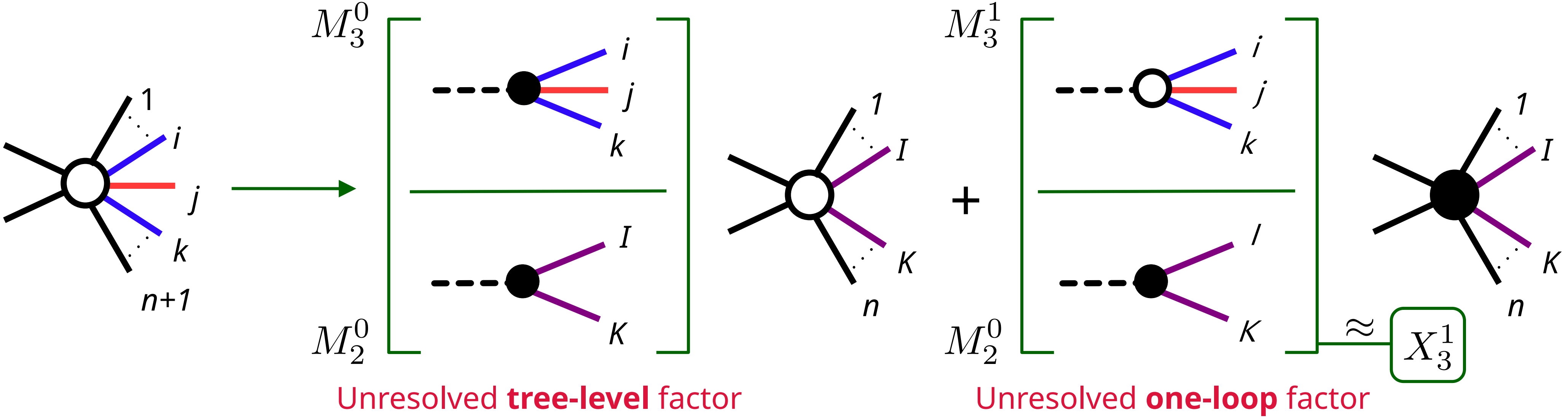


$$X_4^0 = \frac{M_4^0}{M_2^0}$$

$$\mathcal{X}_4^0 \propto \int \mathrm{d}\Phi_4 \, X_4^0$$

Four-parton tree-level antenna functions are extracted analogously to the three-parton ones

NNLO: three-parton one-loop antenna functions



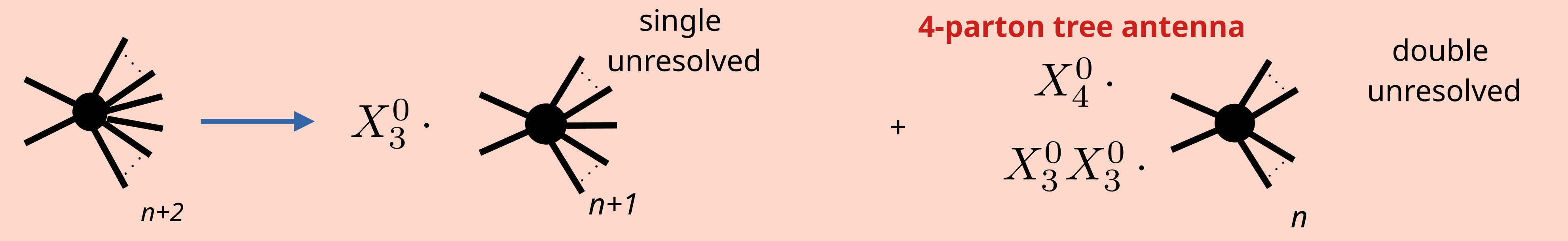
Three-parton one-loop antenna defined removing from the one-loop decay matrix element the unresolved tree-level configuration:

$$X_3^1 = \frac{M_3^1}{M_2^0} - X_3^0 \frac{M_1^2}{M_2^0}, \qquad \mathcal{X}_3^1 \propto \int \mathrm{d}\Phi_3 \, X_3^1$$

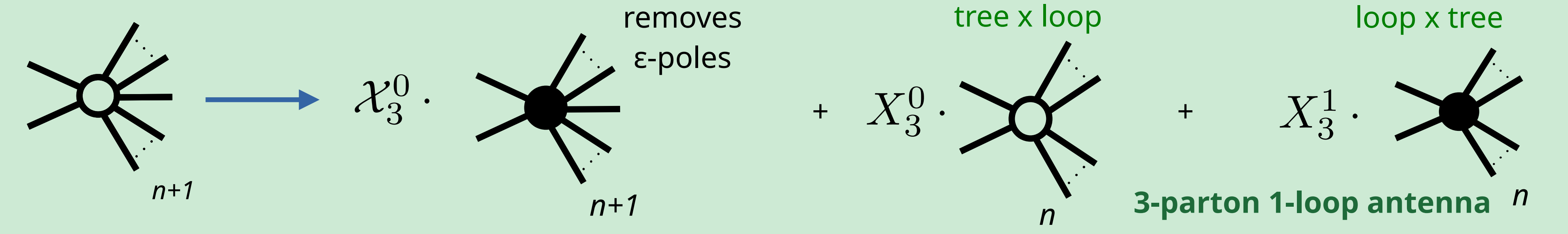
Antenna subtraction at NNLO

[Gehrmann-De Ridder,Gehrmann,Glover ‘05] [Currie,Glover,Wells ‘13]

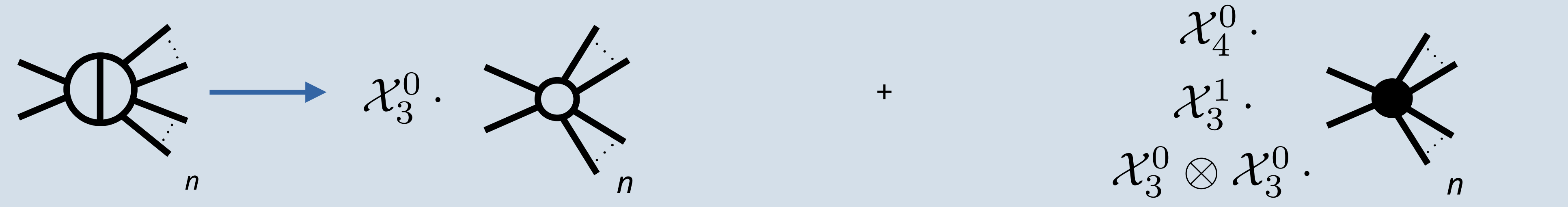
RR:



RV:



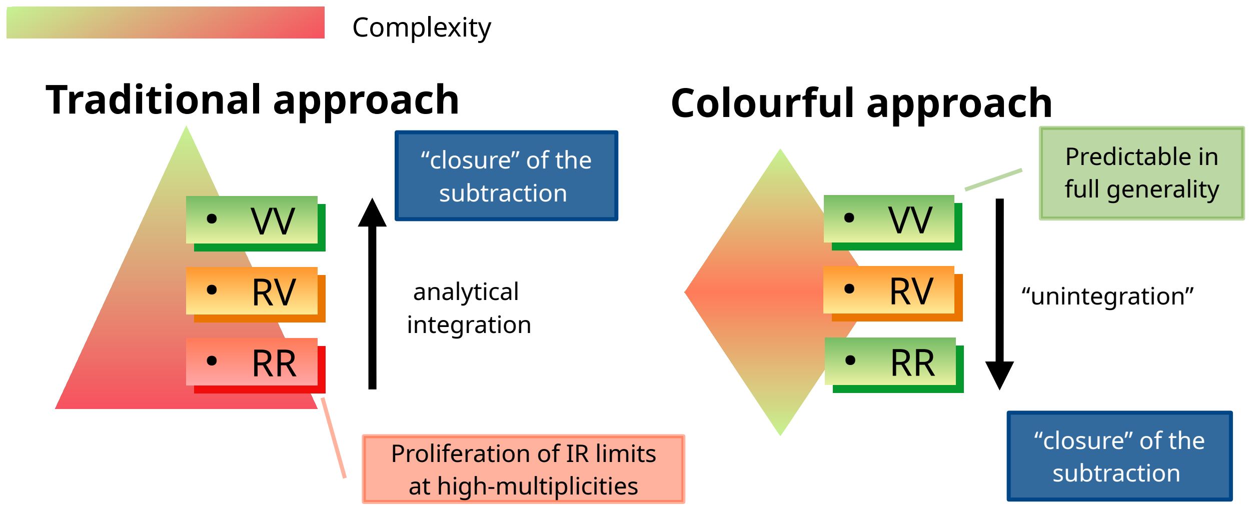
VV:



Colourful antenna subtraction

Ultimate goal: combine **generalized antenna functions** with the **colourful antenna subtraction** method

[Chen,Gehrmann,Glover,Huss,MM ‘22]
[Gehrmann,Glover,MM ‘23]



$$\mathrm{d}\sigma^T = \mathcal{N}_V \int \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_n \, J_n^n(\Phi_n) \, \left[2\,\langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle\right]$$

one loop

$$\begin{aligned}\mathrm{d}\sigma^U = \mathcal{N}_{VV} \int \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_n J_n^n(\Phi_n) \\ \times 2\Big\{\langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^1_{n+2}\rangle + \langle A^1_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle - \frac{\beta_0}{\epsilon}\langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle \\ -\langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}\otimes\boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle + \langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(2)}|A^0_{n+2}\rangle\Big\}\end{aligned}$$

two loops

Exploit **universal** IR singularity structure in virtual corrections to **systematically** construct real-radiation counterterms

Mapping (in)dependence [Fox,Glover,MM '24]

mapping to absorb the recoil of unresolved radiation:

$$\{p\} \to \{\widetilde{p}\}$$

Let's consider:

- **n_p** momenta **{p}** involved in an unresolved configuration
- **n_q** spectator momenta **{q}**

generic subtraction term

$$d\sigma^S \propto \int dPS_{n_p+n_q}(\{p\},\{q\})\, X(\{p\})\, M(\{\widetilde{p}\},\{q\})\, J^{(n_{\widetilde{p}}+n_q)}_{n_{jets}}(\{\widetilde{p}\},\{q\})$$

full phase-space — unresolved factor (antenna function) — resolved matrix element — measurement function

selects n_{jets} jets applies fiducial cuts

The mapping is chosen to induce a **factorization of the phase space**

$$dPS_{n_p+n_q}(\{p\},\{q\}) = dPS_X(\{p\}/\{\widetilde{p}\})\, dPS_{n_{\widetilde{p}}+n_q}(\{\widetilde{p}\},\{q\})$$

unresolved phase-space

resolved phase-space: the measurement function acts on it

$$d\sigma^S \propto \int dPS_{n_{\widetilde{p}}+n_q}(\{\widetilde{p}\},\{q\}) \left[\int X(\{p\}) dPS_X(\{p\}/\{\widetilde{p}\})\right] M(\{\widetilde{p}\},\{q\})\, J^{(n_{\widetilde{p}}+n_q)}_{n_{jets}}(\{\widetilde{p}\},\{q\})$$

$$= \int dPS_{n_{\widetilde{p}}+n_q}(\{\widetilde{p}\},\{q\})\, \mathcal{X}(\{\widetilde{p}\})\, M(\{\widetilde{p}\},\{q\})\, J^{(n_{\widetilde{p}}+n_q)}_{n_{jets}}(\{\widetilde{p}\},\{q\})$$

integrated unresolved factor

two mappings are **equivalent** if the yield the same $\mathcal{X}(\{\tilde{p}\})$

Mapping (in)dependence [Fox,Glover,MM '24]

two hard radiators: $n_{\widetilde{p}} = 2\,, \qquad \{p_1, \ldots, p_{n_p}\} \to \{\widetilde{p}_A, \widetilde{p}_B\}$

$$s_{1\ldots n_p} \equiv (p_1 + \cdots + p_{n_p})^2 = (\widetilde{p}_A + \widetilde{p}_B)^2 \equiv s_{AB}$$

$$\mathcal{X}\left(\{\widetilde{p}_A, \widetilde{p}_B\}\right) = C(\epsilon)(s_{AB})^{\alpha}$$

any momentum-conserving mapping gives same result

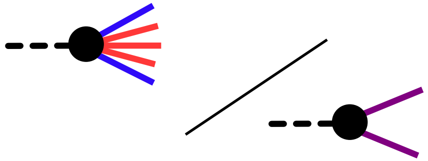
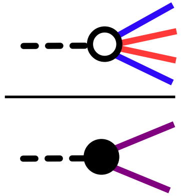
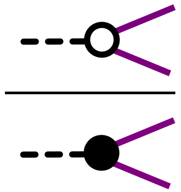
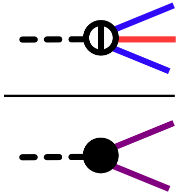
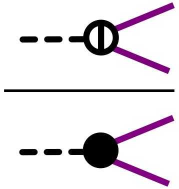
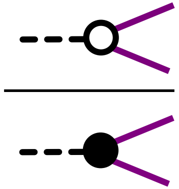
three hard radiators: $n_{\widetilde{p}} = 3\,, \qquad \{p_1, \ldots, p_{n_p}\} \to \{\widetilde{p}_A, \widetilde{p}_B, \widetilde{p}_C\}$

$$s_{1\ldots n_p} \equiv (p_1 + \cdots + p_{n_p})^2 = (\widetilde{p}_A + \widetilde{p}_B + \widetilde{p}_C)^2 \equiv s_{ABC}$$

$$\mathcal{X}\left(\{\widetilde{p}_A, \widetilde{p}_B, \widetilde{p}_C\}\right) = \sum_i C_i(\epsilon)(s_{AB})^{\alpha_i}(s_{AC})^{\beta_i}(s_{BC})^{\gamma_i} + \ldots$$

many "unfixed" scales, different result for different mappings

N³LO antenna functions

		Antenna	Integrated antenna
RRR:		$X_5^0 = \frac{M_5^0}{M_2^0}$	$\mathcal{X}_5^0 \propto \int \mathrm{d}\Phi_5 \, X_5^0$
RRV:	 $-X_4^0 \cdot$ 	$X_4^1 = \frac{M_4^1}{M_2^0} - X_4^0 \frac{M_2^1}{M_2^0}$	$\mathcal{X}_4^1 \propto \int \mathrm{d}\Phi_4 \, X_4^1$
RVV:	 $-X_3^0 \cdot$  $-X_3^1 \cdot$ 	$X_3^2 = \frac{M_3^2}{M_2^0} - X_3^0 \frac{M_2^2}{M_2^0} - X_3^1 \frac{M_2^1}{M_2^0}$	$\mathcal{X}_3^2 = \int \mathrm{d}\Phi_3 X_3^2$

Analytic integration of final-state N³LO antenna functions

Integration of **renormalized matrix elements** for colour-singlet decay over the **fully inclusive phase space**:

$$\int \mathrm{d}\Phi_5 M_5^0 , \quad \int \mathrm{d}\Phi_4 M_4^1 , \quad \int \mathrm{d}\Phi_3 M_3^2 , \quad \int \mathrm{d}\Phi_2 M_2^3$$

[Jakubcik,MM,Stagnitto '22]
[Chen,Jakubcik,MM,Stagnitto '23]

Two-parton three-loop, for validation:

$$\int \mathrm{d}\Phi_5 M_5^0 + \int \mathrm{d}\Phi_4 M_4^1 + \int \mathrm{d}\Phi_3 M_3^2 + \int \mathrm{d}\Phi_2 M_2^3 =$$ finite N³LO inclusive XS

Master integrals from

[Gituliar,Magerya,Pikelner '18]
[Magerya,Pikelner '19]

Reverse unitarity: $$2\pi i \delta^+(p^2) \rightarrow \frac{1}{p^2 - i0} - \frac{1}{p^2 + i0}$$

[Cutkosky '60]
[Anastasiou, Melnikov '02,'03]

- Phase space and (genuine) loop integrals addressed simultaneously;
- **Systematic treatment of all four layers** within a common framework;