Loop Summit 2, Cadenabbia, Italy

21 July 2025

Feynman Integrals in Parameter Space: Positive Integrands in the Minkowski Regime

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THE ROYAL SOCIETY

Outline

Revisiting the Basics

- Feynman Parameter Representation
- Landau Equations
- Contour Deformation

Positive Integrands in the Minkowski Regime

- Concept
- Internal Masses
- Examples

Summary & Outlook

Revisiting the Basics

Parameter Space

Can exchange integrals over loop momenta for integrals over parameters

Feynman Parametrisation

$$[\mathrm{d}\boldsymbol{x}] = \prod_{e \in G} \frac{\mathrm{d}x_e}{x_e} \qquad \boldsymbol{x}^{\nu} = \prod_{e \in G} x_e^{\nu_e}$$

$$J(\boldsymbol{s}) = \frac{\Gamma(\nu - LD/2)}{\prod_{e \in G} \Gamma(\nu_e)} \int_0^{\infty} [d\boldsymbol{x}]\, \boldsymbol{x}^{\nu} \delta\left(1 - \alpha(\boldsymbol{x})\right) \frac{\left[\mathscr{U}(\boldsymbol{x})\right]^{\nu - (L+1)D/2}}{\left[\mathscr{F}(\boldsymbol{x}; \boldsymbol{s}) - i\delta\right]^{\nu - LD/2}}$$

$\mathscr{U}, \mathscr{F}$ homogeneous polynomials of degree L and $L+1$

$\alpha(\mathbf{x})$ arbitrary hyperplane* that bounds the integral in $\mathbb{R}^N_{>0}$ for at least one $x_i > 0$

*generally, integrate over positive projective simplex $\mathbb{P}^{N-1}_{>0}$

Many Other Parametrisations

Schwinger

Lee-Pomeransky Parametrisation Lee, Pomeransky 13; Gardi, Herzog, SJ, Ma, Schlenk 22;...

Baikov & Loop-by-loop Baikov Baikov 96, 96, 05; Britto, Duhr, Hannesdottir, Mizera 24; Frellesvig 24; Correia, Giroux, Mizera 25;...

Feynman Integrals in a Nutshell

$$J(\mathbf{s}) \sim \int_{\mathbb{R}^N_{\geq 0}} \left[\mathrm{d}\boldsymbol{x}\right] \boldsymbol{x}^{\boldsymbol{\nu}} \frac{[\mathscr{U}(\boldsymbol{x})]^{N-(L+1)D/2}}{[\mathscr{F}(\boldsymbol{x};\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1-\alpha(\boldsymbol{x}))$$

Singularities

1. UV/IR singularities when some $x \to 0$ (or $x \to \infty$) simultaneously
2. Thresholds when $\mathscr{F}$ vanishes inside integration region, $\lim_{\delta \to 0^+}$ gives causal (Feynman) prescription

Kinematic Regions

$$\mathcal{U}(\mathbf{x}) = \sum_{T^1} \prod_{e \notin T^1} x_e,$$

$$\mathcal{F}_0(\mathbf{x};\mathbf{s}) = \sum_{T^2} (-s_{T^2}) \prod_{e \notin T^2} x_e, \qquad \mathcal{F}(\mathbf{x};\mathbf{s}) = \mathcal{F}_0(\mathbf{x};\mathbf{s}) + \mathcal{U}(\mathbf{x}) \sum_e m_e^2 x_e,$$

The signs of the monomials of $\mathcal{F}$ depend on **kinematic invariants** and **masses**

If all signs are the same: *manifestly same-sign regime*
If $\forall \mathbf{x} \in \mathbb{R}^N_{>0} : \mathcal{F}(\mathbf{x};\mathbf{s}) > 0$: *same-sign (Euclidean) regime*
If $\forall \mathbf{x} \in \mathbb{R}^N_{>0} : \mathcal{F}(\mathbf{x};\mathbf{s}) < 0$: *same-sign (Pseudo-Euclidean) regime*
Otherwise: *mixed-sign (Minkowski) regime*

For fixed $\mathbf{s} = (s_1, \ldots, s_M, m_1^2, \ldots, m_N^2)$:
The equation $\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$ (variety of $\mathcal{F}$) defines a codim-1 hypersurface

Landau Equations

Landau Equations (parameter space):

Necessary, but not sufficient, conditions to have a singularity

$$1) \quad \mathscr{F}(\boldsymbol{x};\mathbf{s}) = 0$$

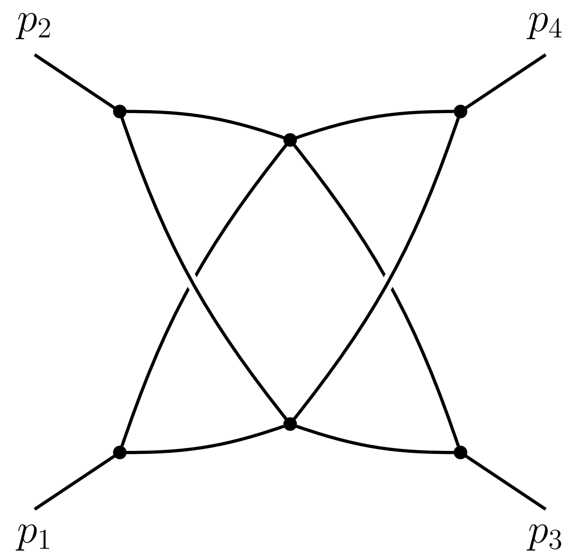
$$2) \quad x_j \frac{\partial \mathscr{F}(\boldsymbol{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$

Leading: $x_j \neq 0 \forall j$

Can think of solutions of leading Landau equations as ``*pinched" surfaces* on which several hypersurfaces $\frac{\partial \mathscr{F}(\boldsymbol{x};\mathbf{s})}{\partial x_j} = 0$ intersect

Method of Regions: Smirnov 91; Beneke, Smirnov 97; Jantzen, A. Smirnov, V. Smirnov 12
Landau Discriminants: Mizera, Telen 22; Fevola, Mizera, Telen 23; Fevola, Mizera, Telen 23
Unitarity Cuts/ Discontinuities: Hannesdottir, Mizera 22; Britto, Duhr, Hannesdottir, Mizera 24
… (your favourite topic here)

3-loop Crown Example



$$= \int_0^\infty \mathrm{d}x_0 \ldots \mathrm{d}x_7 \, \frac{\mathcal{U}(\mathbf{x})^{4\epsilon}}{\mathcal{F}(\mathbf{x};\mathbf{s})^{2+3\epsilon}} \delta(1-x_7)$$

$$\mathcal{U}(\alpha) = x_0x_2x_4 + x_0x_2x_5 + x_0x_2x_6 + (29 \text{ terms})$$

$$\mathcal{F}(\boldsymbol{x};\mathbf{s}) = -s_{12}\ (x_1x_4 - x_0x_5)\,(x_3x_6 - x_2x_7) - s_{13}\ (x_1x_2 - x_0x_3)\,(x_5x_6 - x_4x_7),$$

$$\frac{\partial \mathcal{F}(\boldsymbol{x};\mathbf{s})}{\partial x_0} = s_{12}\ x_5(x_3x_6 - x_2x_7) + s_{13}\ x_3(x_5x_6 - x_4x_7),$$

$$\vdots$$

$$\frac{\partial \mathcal{F}(\boldsymbol{x};\mathbf{s})}{\partial x_7} = s_{12}\ x_2(x_1x_4 - x_0x_5) + s_{13}\ x_4(x_1x_2 - x_0x_3)$$

Can have a leading Landau singularity with *generic kinematics* (arbitrary s_{12}, s_{13}) when each factor of $\mathcal{F}$ vanishes!

Gives rise to new regions when this integral appears in an expansion

Halliday 64; Landshoff 72; Botts, Sterman 89; Gardi, Herzog, Jones, Ma 24

Landau Equations

Landau Equations (parameter space):

Necessary, but not sufficient, conditions to have a singularity

$$1) \quad \mathscr{F}(\boldsymbol{x};\mathbf{s}) = 0$$

$$2) \quad x_j \frac{\partial \mathscr{F}(\boldsymbol{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$

Leading: $x_j \neq 0 \forall j$

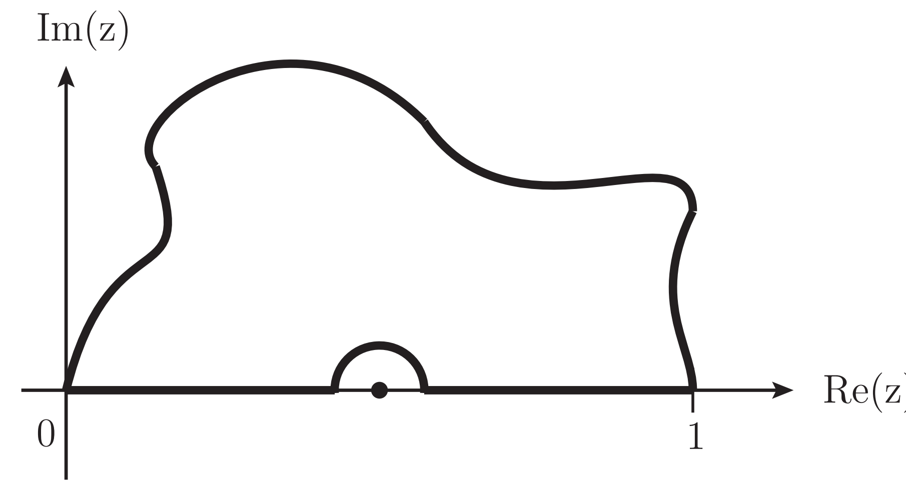
What happens when we satisfy only the first equation?

Let's consider this in the context of direct (numerical) integration in param. space

Contour Deformation

Feynman integral (after integrating δ-func.):

$$J(\mathbf{s}) \sim \int_0^1 [\mathrm{d}\mathbf{x}]\, \mathbf{x}^{\nu} \frac{[\mathcal{U}(\mathbf{x})]^{N-(L+1)D/2}}{[\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta]^{N-LD/2}}$$



$\mathcal{F} \to \mathcal{F} - i\delta$ tells us how to causally deform contour

For numerics we need to **explicitly pick a contour**

Let $\mathbf{z} = \mathbf{x} - i\boldsymbol{\tau}$: $\mathcal{F}(\mathbf{z};\mathbf{s}) = \mathcal{F}(\mathbf{x};\mathbf{s}) - i\sum_j \tau_j \frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j} + \mathcal{O}(\tau^2)$

Choose $\tau_j = \lambda_j\, x_j(1-x_j)\frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j}$ with small $\lambda_j > 0$

$-i\sum_j \left(\frac{\partial \mathcal{F}}{\partial x_j}\right)^2$

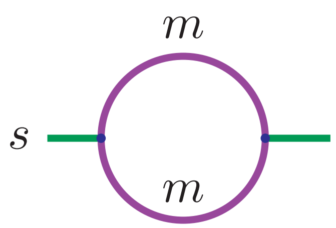
correct deformation sign

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14; Borinsky, Munch, Tellander 23; …

Can also generalise $\lambda_j \to \lambda_j(\mathbf{x})$ and train the deformation with a Neural Network

Winterhalder, Magerya, Villa, SJ, Kerner, Butter, Heinrich, Plehn 22

Contour Deformation: Example

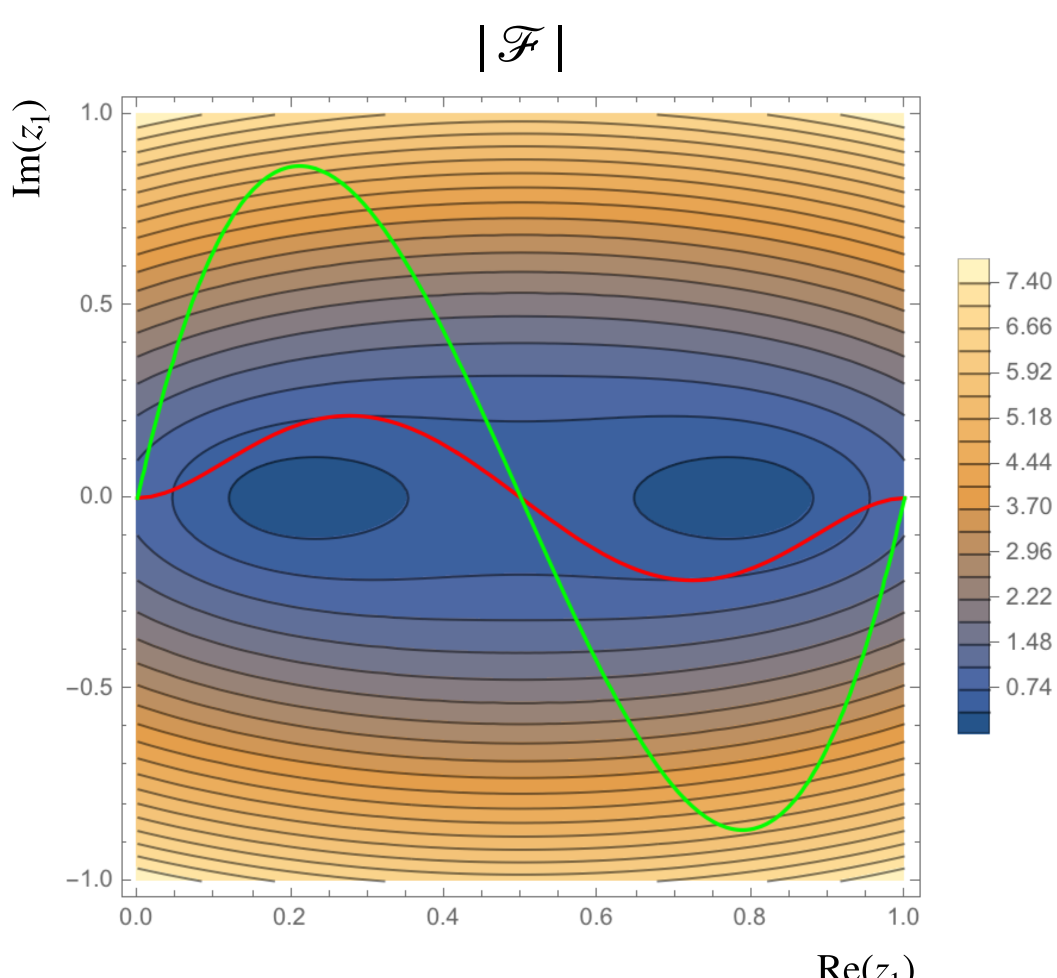


$$= \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2\,\frac{\mathscr{U}(\mathbf{x})^{-2+2\epsilon}}{\mathscr{F}(\mathbf{x};\mathbf{s})^{\epsilon}}\delta(1-x_1-x_2) \to \int_\gamma \mathrm{d}z_1 \frac{\mathscr{U}(z_1)^{-2+2\epsilon}}{\mathscr{F}(z_1;s,m)^{\epsilon}} = \int_0^1 \mathrm{d}x\,|J_z|\,\frac{\mathscr{U}(z_1(x))^{-2+2\epsilon}}{\mathscr{F}(z_1(x);s,m)^{\epsilon}}$$

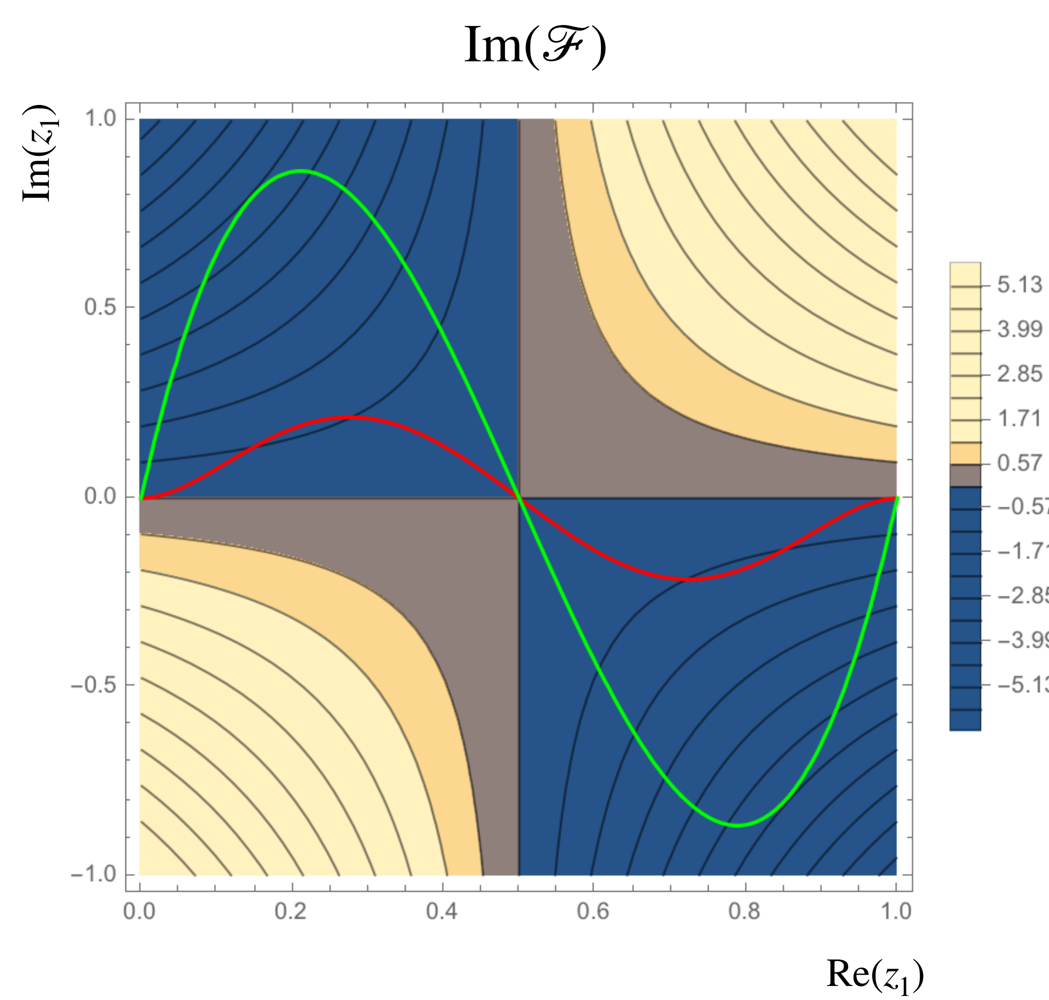
Jacobian det

$$\mathscr{U}(\mathbf{x}) = x_1 + x_2$$

$$\mathscr{F}(\mathbf{x},\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)(x_1 + x_2)$$



$$\mathbf{s}_R = \{s > 4m^2, m^2 > 0\}$$



Contour Deformation

Downsides of contour deformation:

1. Real valued integrand → complex valued integrand (slower numerics)
2. Large and complicated Jacobian from $\mathbf{x} \to \mathbf{z}$ (can be optimised, dual numbers?)
   Borinsky, Munch, Tellander 23; Suggestion by Hirschi
3. Increases variance of function (integrand can be both > 0 and < 0)
   e.g. Janßen, Poncelet, Schumann 25
4. Arbitrary and sensitive to choice of contour
5. Sometimes fails analytically and/or numerically

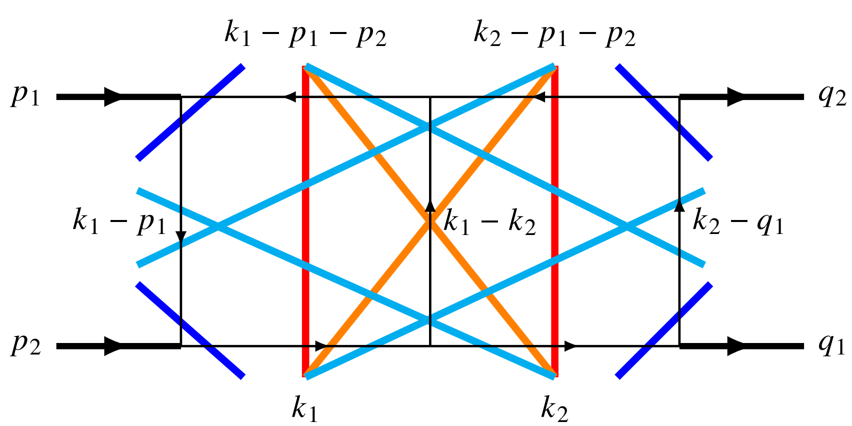
Summary: it is **slow, arbitrary** and can **fail**

Positive Integrands

Avoiding* Contour Deformation

Various efforts to avoid/mitigate contour deformation in numerical contexts

Threshold subtraction (used in context of Loop-Tree Duality)



Locate thresholds of integral/amplitude, subtract using local counterterms

Dispersive/absorptive parts can be computed separately

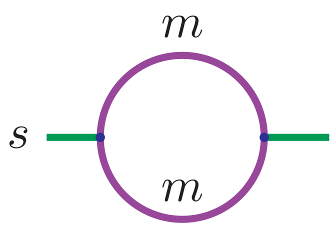
Kermanschah 21; Kermanschah, Vicini 24;
Locally finite amplitudes: Anastasiou, Haindl, Sterman, Yang, Zeng 20; Anastasiou, Sterman 22; Anastasiou, Karlen, Sterman, Venkata 24;

Finite $\delta \neq 0$ but flatten behaviour of integral near $\delta \to 0$ with variable changes

$$J = \lim_{\delta \to 0} \int_{-1}^{1} \mathrm{d}x \, \frac{F(x)}{x + i\delta} \to \frac{-i\pi}{g_\delta} \int_{\delta/\pi}^{1-\delta/\pi} \mathrm{d}\alpha \left(1 + i \frac{x(\alpha;\delta)}{\delta} \right) F(x(\alpha;\delta)), \qquad x = \frac{\delta}{\tan\left[\pi(1-\alpha)\right]}$$

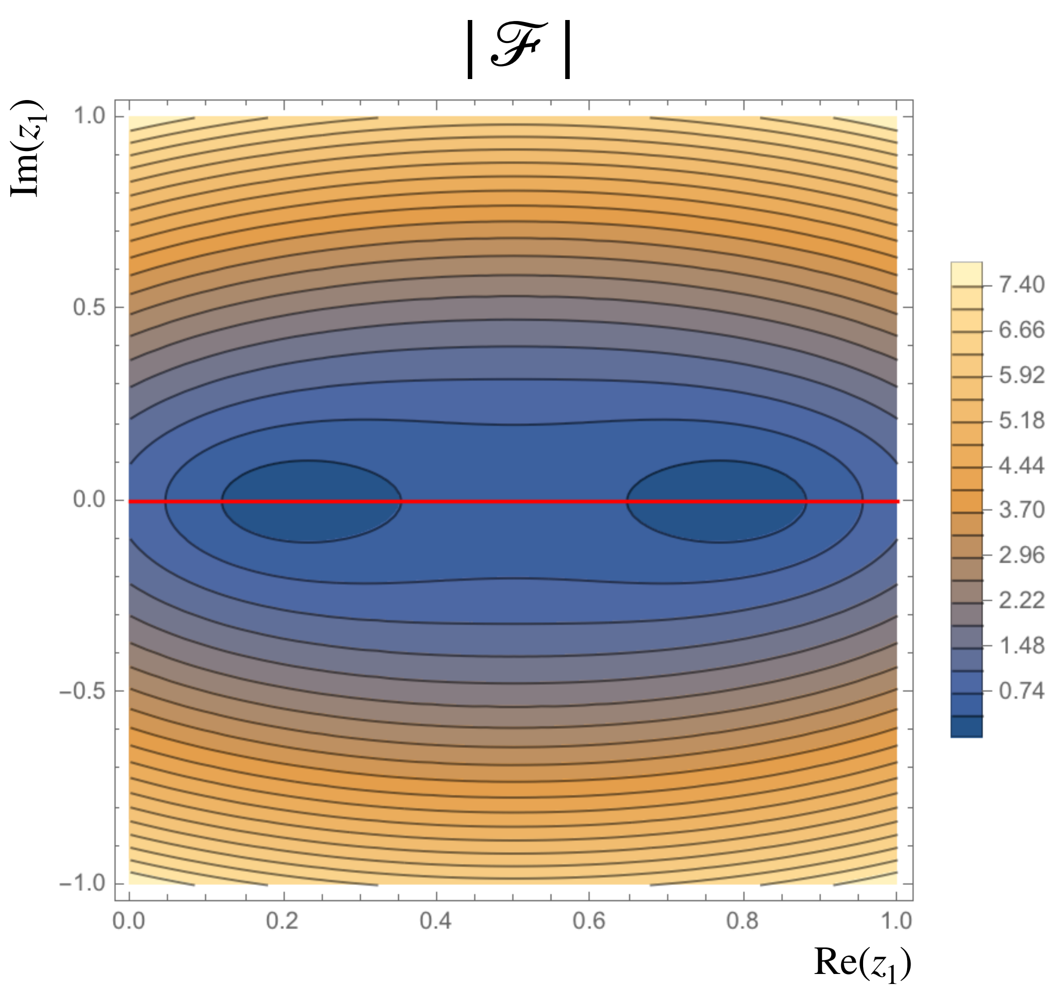
Pittau, Webber 22; Pittau 24; GLoop code

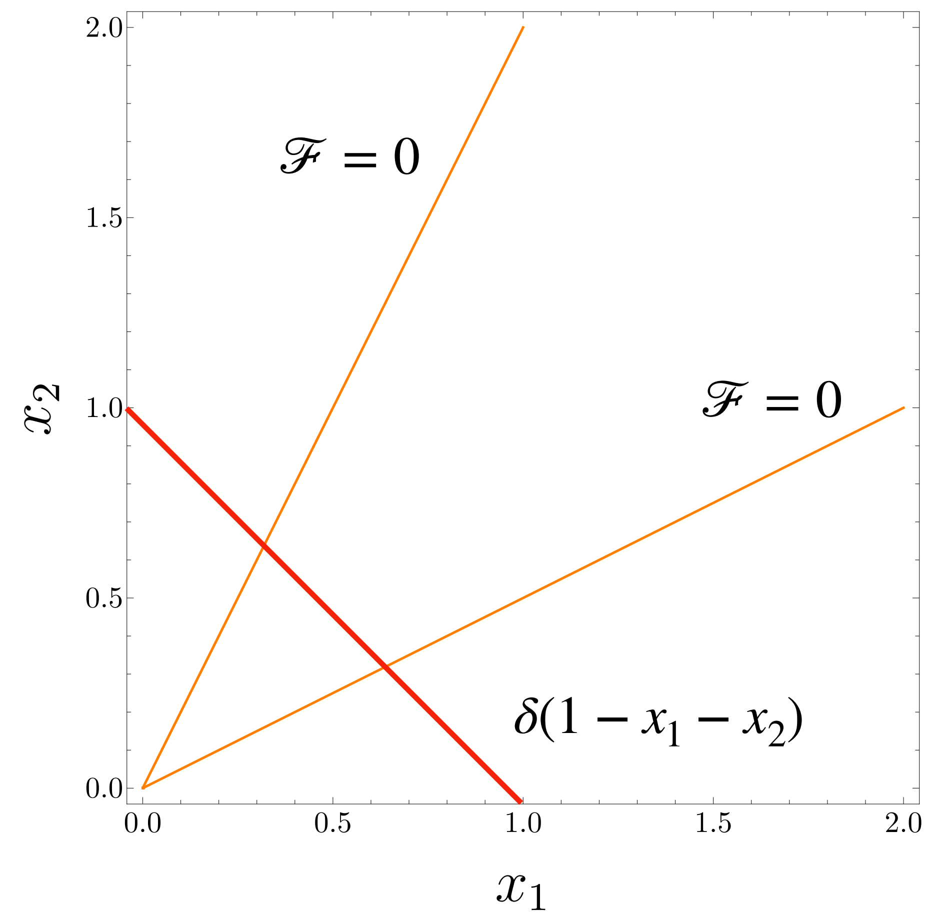
Picture in N-dimensions



$$= \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2\, \frac{\mathscr{U}(\mathbf{x})^{-2+2\epsilon}}{\mathscr{F}(\mathbf{x};\mathbf{s})^{\epsilon}} \delta(1-x_1-x_2)$$

$$\mathscr{U}(\mathbf{x}) = x_1 + x_2$$
$$\mathscr{F}(\mathbf{x},\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)(x_1 + x_2)$$
$$\mathbf{s}_R = \{s > 4m^2, m^2 > 0\}$$





Singularities: live on boundary hypersurface ∩ $\mathscr{F}$-variety

(codim-1) (codim-1)

Avoiding Contour Deformation

Idea:

1. Construct transformations of the Feynman parameters which map $\mathscr{F}(\boldsymbol{x};\mathbf{s}) = 0$ to boundaries of integration for a given kinematic region $\mathbf{s}_R = \{\mathbf{s}_{\min} < \mathbf{s} < \mathbf{s}_{\max}\}$

2. For transformations which make $\mathscr{F}$ non-positive, factor out overall minus sign (using the $i\delta$ prescription to generate the causally correct imaginary part)

3. Stitch together the resulting integrals

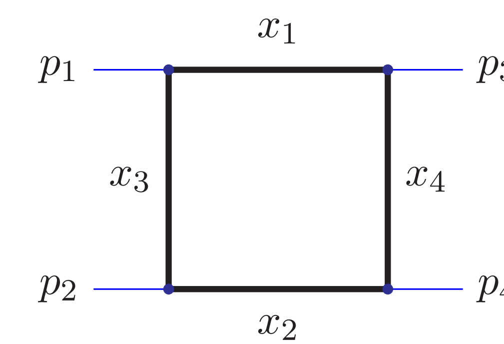
$$\boxed{J(\mathbf{s}) = \sum_{n_+=1}^{N_+} J^{+,n_+}(\mathbf{s}) + \lim_{\delta\to 0^+} (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_-=1}^{N_-} J^{-,n_-}(\mathbf{s})}$$

The individual integrals $\{J^+_{n_+}, J^-_{n_-}\}$ have real non-negative integrands

$\implies$ no contour deformation, trivial analytic continuation, faster to integrate

A First Example

Massless 1-loop On-Shell Box



$$\mathcal{U}(\mathbf{x}) = x_1 + x_2 + x_3 + x_4$$

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -\,s_{12}x_1x_2 - s_{13}x_3x_4$$

$$J_{\text{box}}(\mathbf{s}) = \Gamma(2+\epsilon) \lim_{\delta\to 0^+} I_{\text{box}}(\mathbf{s};\delta),$$

$$I_{\text{box}}(\mathbf{s};\delta) = \int_{\mathbb{R}^4_{\geq 0}} \mathrm{d}\mathbf{x}\, \mathcal{I}(\mathbf{x};\mathbf{s})\, \delta(1-\alpha(\mathbf{x})) = \int_{\mathbb{R}^4_{\geq 0}} \prod_{i=1}^{4} \mathrm{d}x_i \frac{\mathcal{U}(\mathbf{x})^{2\epsilon}}{\left(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta\right)^{2+\epsilon}} \delta\left(1-\alpha(\mathbf{x})\right)$$

Kinematics: $s_{ij} = (p_i + p_j)^2,$

$$\mathbf{s}_{\text{phys}} = \{0 < s_{12} < \infty,\ -s_{12} < s_{13} < 0\}$$

Resolution: $\{\mathcal{F}(\mathbf{x};\mathbf{s}) < 0\} \cup \{0 < \mathbf{x}\} \cup \mathbf{s}_R,$

$$\{-s_{12}x_1x_2 - s_{13}x_3x_4 < 0\} \cup \{0 < x_1,\ 0 < x_2,\ 0 < x_3,\ 0 < x_4\} \cup \mathbf{s}_{\text{phys}},$$

$$\{\underbrace{\frac{-s_{13}x_3x_4}{s_{12}x_2}}_{f(\mathbf{x}_{\neq 1})} < x_1\} \cup \{0 < x_2,\ 0 < x_3,\ 0 < x_4\} \cup \mathbf{s}_{\text{phys}}$$

solve for x_1

$$\mathbf{x}_{\neq i} = \mathbf{x}\backslash\{x_i\} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_N\}$$

Can bisect domain in a single variable x_1 (*Univariate Bisectable*)

$\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$ to $x_1 = 0$: $\quad x_1 \to y_1' = x_1 + f\left(\mathbf{x}_{\neq 1}\right)$ gives $\mathcal{F}^-(\mathbf{x};\mathbf{s}) = s_{12}x_1x_2,$

$\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$ to $x_1 \to \infty$: $\quad x_1 \to y_1 = \dfrac{x_1}{x_1 + x_j} f\left(\mathbf{x}_{\neq 1}\right)$ with $x_j \neq x_1$ gives $\mathcal{F}^+(\mathbf{x};\mathbf{s}) = \dfrac{(-s_{13})x_3x_4^2}{x_1 + x_4},$

A First Example (II)

Resulting integrands product of dim. reg. polynomials that are

Positive in $\mathbb{R}^N_{>0}$ & **Homogeneous** (by construction)

1) $\mathcal{J}^-(\mathbf{x}) = 1, \quad \mathcal{U}^-(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{-s_{13}x_3x_4}{s_{12}x_2}, \quad \mathcal{F}^-(\mathbf{x};\mathbf{s}) = s_{12}x_1x_2$

$$\mathcal{I}^-_{\text{box}}(\mathbf{x};\mathbf{s}) = \mathcal{J}^-(\mathbf{x})\frac{\mathcal{U}^-(\mathbf{x})^{2\epsilon}}{\mathcal{F}^-(\mathbf{x};\mathbf{s})^{2+\epsilon}} = x_1^{-2-\epsilon}(s_{12}x_2)^{-2-3\epsilon}\Big(s_{12}x_2\left(x_1+x_2+x_3+x_4\right) - s_{13}x_3x_4\Big)^{2\epsilon}$$

2) $\mathcal{J}^+(\mathbf{x}) = \frac{(-s_{13})x_3x_4^2}{s_{12}x_2(x_1+x_4)^2}, \quad \mathcal{U}^+(\mathbf{x}) = \frac{x_1}{x_1+x_4}\frac{(-s_{13})x_3x_4}{s_{12}x_2} + x_2 + x_3 + x_4, \quad \mathcal{F}^+(\mathbf{x};\mathbf{s}) = \frac{(-s_{13})x_3x_4^2}{x_1+x_4}$

$$\mathcal{I}^+_{\text{box}}(\mathbf{x};\mathbf{s}) = \mathcal{J}^+(\mathbf{x})\frac{\mathcal{U}^+(\mathbf{x})^{2\epsilon}}{\mathcal{F}^+(\mathbf{x};\mathbf{s})^{2+\epsilon}} = \left(x_1+x_4\right)^{-\epsilon}\left(s_{12}x_2\right)^{-1-2\epsilon}\left(-s_{13}x_3x_4^2\right)^{-1-\epsilon}\Big(s_{12}x_2\left(x_1+x_4\right)\left(x_2+x_3+x_4\right) - s_{13}x_1x_3x_4\Big)^{2\epsilon}$$

Final Result

$$\boxed{\begin{aligned} J_{\text{box}}(\mathbf{s}) &= \Gamma(2+\epsilon)\lim_{\delta\to 0^+} I_{\text{box}}(\mathbf{s};\delta) \\ I_{\text{box}}(\mathbf{s};\delta) &= I^+_{\text{box}}(\mathbf{s}) + (-1-i\delta)^{-2-\epsilon}\, I^-_{\text{box}}(\mathbf{s}) \end{aligned}}$$

Algorithm: Univariate Bisectable Integrals

For univariate bisectable in $\mathbf{s}_R$ integrals, we can formulate simple pseudo-algorithm

Algorithm 1: Univariate Bisection (UB)

Input: $\mathcal{I}(\mathbf{x};\mathbf{s};\delta)$, $\mathbf{s}_R$
Output: $\mathcal{I}^+(\mathbf{x};\mathbf{s})$, $\mathcal{I}^-(\mathbf{x};\mathbf{s})$
foreach $x_i \in \mathbf{x}$ **do**
  Let $r = \texttt{Reduce}[\{\mathcal{F}(\mathbf{x};\mathbf{s}) < 0\} \cup \{0 < \mathbf{x}\} \cup \mathbf{s}_R,\, x_i]$; ← **solve for x_i**
  if $r \sim$ (1) **then**
    Let $\mathcal{I}^-(\mathbf{x};\mathbf{s}) = \mathcal{J}(\mathbf{x}_{\neq i}, y_i)\, \mathcal{I}(\mathbf{x}_{\neq i}, y_i; -\mathbf{s}; 0)$
    Let $\mathcal{I}^+(\mathbf{x};\mathbf{s}) = \mathcal{J}(\mathbf{x}_{\neq i}, y_i')\, \mathcal{I}(\mathbf{x}_{\neq i}, y_i'; \mathbf{s}; 0)$
    return $\mathcal{I}^+(\mathbf{x};\mathbf{s})$, $\mathcal{I}^-(\mathbf{x};\mathbf{s})$
  else if $r \sim$ (2) **then**
    Let $\mathcal{I}^-(\mathbf{x};\mathbf{s}) = \mathcal{J}(\mathbf{x}_{\neq i}, y_i')\, \mathcal{I}(\mathbf{x}_{\neq i}, y_i'; -\mathbf{s}; 0)$
    Let $\mathcal{I}^+(\mathbf{x};\mathbf{s}) = \mathcal{J}(\mathbf{x}_{\neq i}, y_i)\, \mathcal{I}(\mathbf{x}_{\neq i}, y_i; \mathbf{s}; 0)$
    return $\mathcal{I}^+(\mathbf{x};\mathbf{s})$, $\mathcal{I}^-(\mathbf{x};\mathbf{s})$
end
return $\neg$UB *in* $\mathbf{s}_R$

Reduced system conditions:

$$\{0 < x_i < f(\mathbf{x}_{\neq i})\} \quad \cup \quad \{0 < \mathbf{x}_{\neq i}\} \quad \cup \quad \mathbf{s}_R, \qquad (1)$$

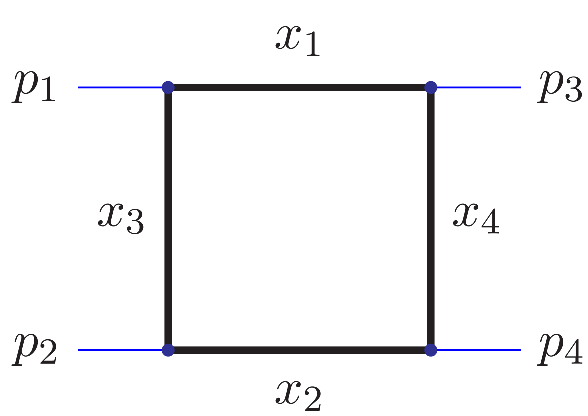
$$\{f(\mathbf{x}_{\neq i}) < x_i\} \quad \cup \quad \{0 < \mathbf{x}_{\neq i}\} \quad \cup \quad \mathbf{s}_R, \qquad (2)$$

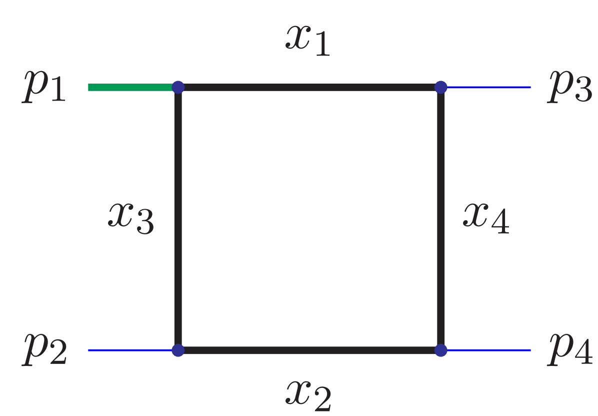
Transformations:

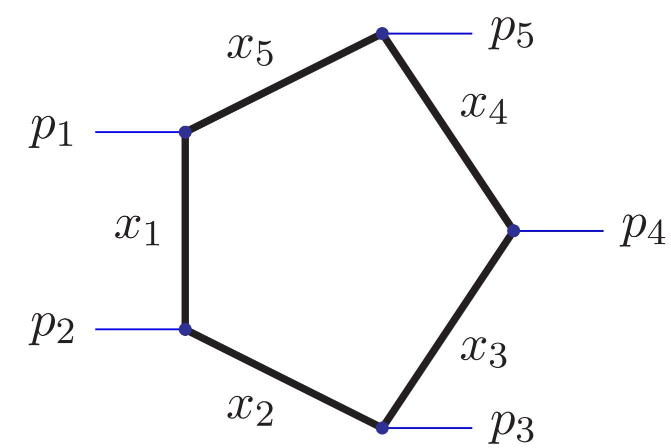
$$y_i = \frac{x_i}{x_i + x_j} f(\mathbf{x}_{\neq i}), \qquad y_i' = x_i + f(\mathbf{x}_{\neq i})$$

Algorithm either finds bisection of $\mathcal{I}(\mathbf{x};\mathbf{s},\delta)$ in $\mathbf{s}_R$ or shows that one does not exist

Some Examples of (Massless) UB Integrals

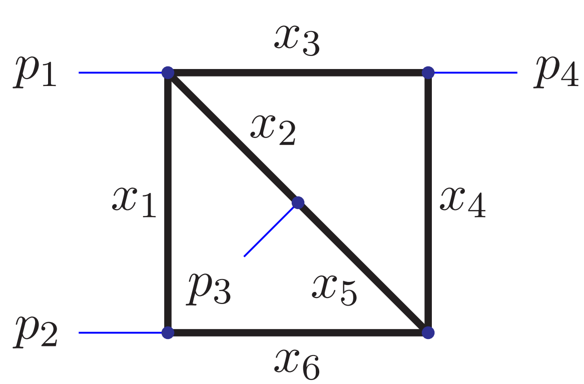


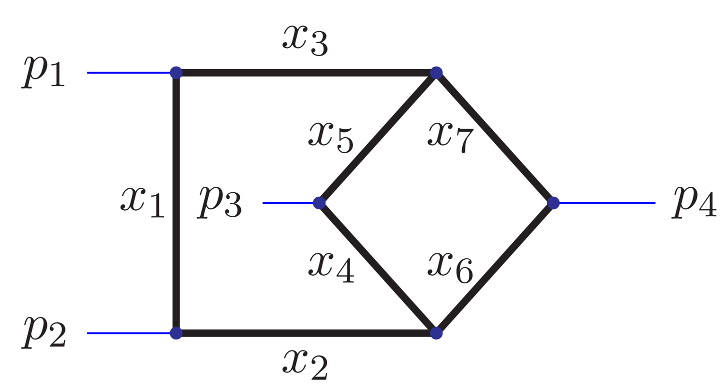




$$\mathbf{s}_{p_1^2>0} = \{0 < p_1^2 < \infty, \quad 0 < s_{12} < \infty, -s_{12} < s_{13} < 0\}$$

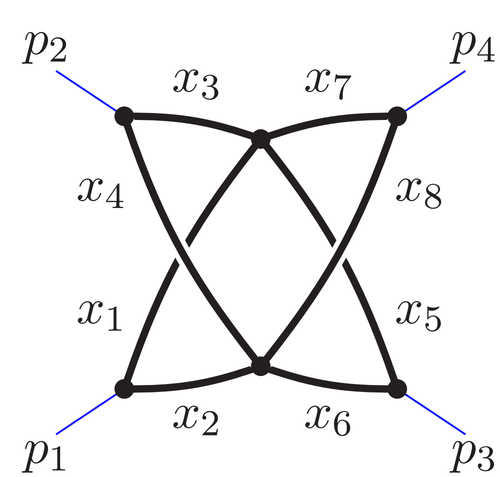
$$\mathbf{s}_R = \{0 < s_{12}, s_{34}, s_{51} < \infty, \quad -\infty < s_{23}, s_{45} < 0\}$$





$$\mathbf{s}_{\text{phys}} = \{0 < s_{12} < \infty, \ -s_{12} < s_{23} < 0\}$$

$$\mathbf{s}_{\text{phys}} = \{0 < s_{12} < \infty, -s_{12} < s_{13} < 0\}$$



* after dissection

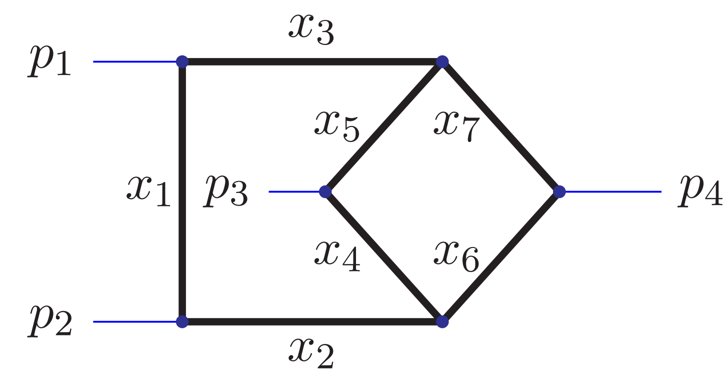
Gardi, Herzog, SPJ, Ma 24

A Second Example

Massless 2-loop Non-planar Box

$$J_{\mathrm{BNP7}} = -\Gamma(3+2\epsilon) \lim_{\delta\to 0^+} I_{\mathrm{BNP7}}$$

$$I_{\mathrm{BNP7}} = \int_{\mathbb{R}^7_{\geq 0}} \prod_{i=1}^{7} \mathrm{d}x_i \frac{\mathscr{U}(\mathbf{x})^{1+3\epsilon}}{\left(\mathscr{F}(\mathbf{x};\mathbf{s}) - i\delta\right)^{3+2\epsilon}} \delta\left(1-\alpha(\mathbf{x})\right)$$



$$\mathscr{U}(\mathbf{x}) = x_1x_4 + x_1x_5 + x_1x_6 + x_1x_7 + x_2x_4 + x_2x_5 + x_2x_6 + x_2x_7 + x_3x_4 + x_3x_5 + x_3x_6 + x_3x_7 + x_4x_6 + x_4x_7 + x_5x_6 + x_5x_7$$

$$\mathscr{F}(\mathbf{x},\mathbf{s}) = -s_{12}(x_3x_4x_6 + x_2x_5x_7 + x_2x_3x_7 + x_2x_3x_6 + x_2x_3x_5 + x_2x_3x_4) - s_{23}x_1x_5x_6 + (s_{12}+s_{23})x_1x_4x_7$$

Kin: $s_{12} + s_{13} + s_{23} = 0,$ ◄- - - - - - - - - **No Euclidean region**

$\mathbf{s}_{\mathrm{phys}} = \{0 < s_{12} < \infty, \; -s_{12} < s_{23} < 0\}$

Res: $\{\mathscr{F}(\mathbf{x};\mathbf{s}) < 0\} \cup \{0 < \mathbf{x}\} \cup \mathbf{s}_R,$

$\{f(\mathbf{x}_{\neq 1}) < x_1 \cup \{0 < \mathbf{x}_{\neq 1}\} \cup \mathbf{s}_{\mathrm{phys}},$

$$f(\mathbf{x}_{\neq 1}) = \frac{s_{12}\left[x_3x_4x_6 + x_2x_5x_7 + x_2x_3\left(x_4+x_5+x_6+x_7\right)\right]}{\left(s_{12}+s_{23}\right)x_4x_7 - s_{23}x_5x_6}$$

Bisect domain in a variable x_1 (*Univariate Bisectable*)

$\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ to $x_1 = 0$: $\quad \mathscr{F}^+(\mathbf{x};\mathbf{s}) = x_1\left[\left(s_{12}+s_{23}\right)x_4x_7 - s_{23}x_5x_6\right]$

$\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ to $x_1 \to \infty$: $\mathscr{F}^-(\mathbf{x};\mathbf{s}) = \dfrac{s_{12}x_7\left[x_3x_4x_6 + x_2x_5x_7 + x_2x_3\left(x_4+x_5+x_6+x_7\right)\right]}{x_1+x_7},$

What About Massive Integrals?

Primary complication

$$\mathscr{F}(\mathbf{x};\mathbf{s}) = \mathscr{F}_0(\mathbf{x};\mathbf{s}) + \mathscr{U}(\mathbf{x};\mathbf{s}) \sum_{j=1}^{N} m_j^2 x_j,$$

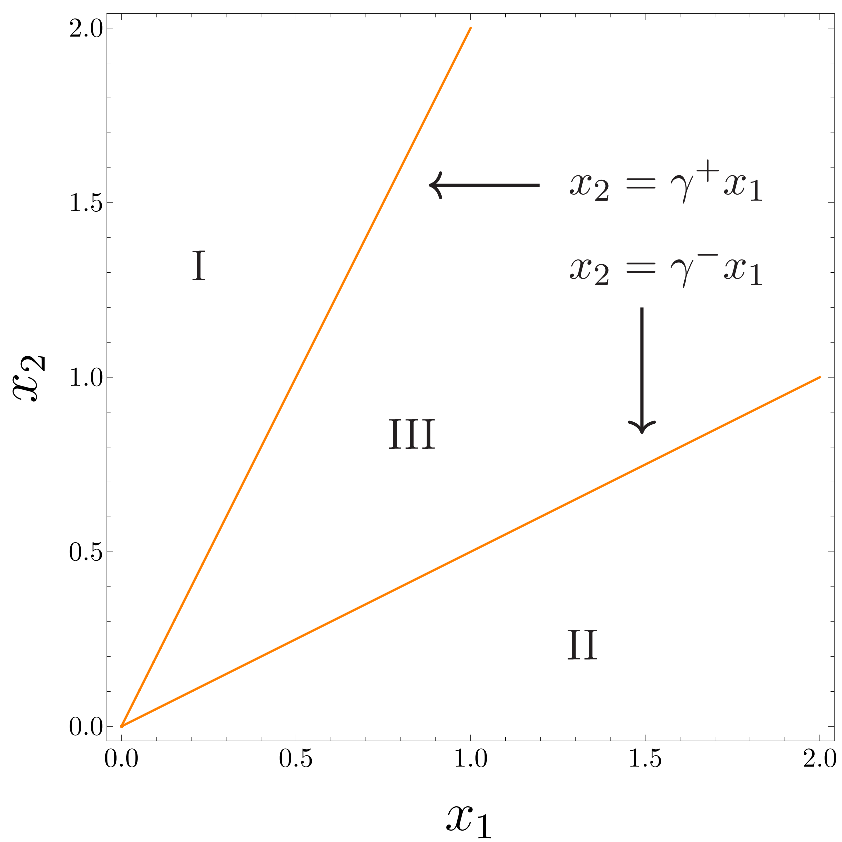
Linear in each x_j **Quadratic in each x_j**

Very easy to encounter non-UB integrals

Solutions for $\mathscr{F}(\mathbf{x};\mathbf{s})$ generically contain square roots

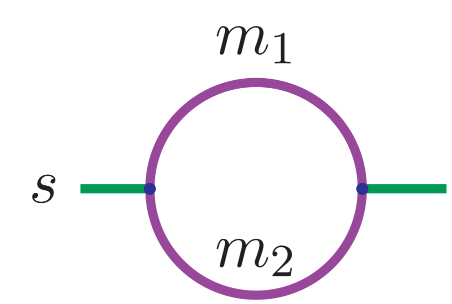
Often more thresholds and therefore more distinct kinematic regions

A First Massive Example



$$\beta^2 = \frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2} \in (0,1)$$

$$\gamma^{\pm} = \frac{1}{\gamma^{\mp}} = \frac{1 \pm \beta}{1 \mp \beta}$$



$$\mathscr{F} = -s x_1 x_2 + (x_1+x_2)\left(m_1^2 x_1 + m_2^2 x_2\right)$$

$$\widetilde{\mathscr{F}} = x_1^2 + x_2^2 - 2\frac{1+\beta^2}{1-\beta^2} x_1 x_2$$

$$\text{I}: \qquad x_2 \to y_2 = x_2 + \gamma^+ x_1, \qquad \widetilde{\mathscr{F}}^{+,1} = x_2\Big(x_2 + \frac{4\beta}{1-\beta^2} x_1\Big),$$

$$\text{II}: \qquad x_1 \to y_1 = x_1 + \gamma^+ x_2, \qquad \widetilde{\mathscr{F}}^{+,2} = x_1\Big(x_1 + \frac{4\beta}{1-\beta^2} x_2\Big),$$

$$\text{III}: \qquad \begin{array}{l} x_2 \to y_2 = x_2 + \gamma^- x_1, \\ x_1 \to y_1 = x_1 + \gamma^- x_2, \end{array} \qquad \widetilde{\mathscr{F}}^{-} = -\frac{16\beta^2}{(1-\beta)(1+\beta)^3} x_1 x_2,$$

$$\boxed{I_{\text{bub},m_1\neq m_2} = I^{+,1}_{\text{bub},m_1\neq m_2} + I^{+,2}_{\text{bub},m_1\neq m_2} + (-1-i\delta)^{-\epsilon}\, I^{-}_{\text{bub},m_1\neq m_2}}$$

A First Massive Example (II)

$$I^{+,1}_{\mathrm{bub},m_1\neq m_2} = (m_1 m_2)^{1-2\epsilon}(1-\beta)^{2-2\epsilon}\int_{\mathbb{R}^2_{\geq 0}} \mathrm{d}x_1\mathrm{d}x_2\, x_2^{-\epsilon}\left(x_2 + \frac{4\beta}{1-\beta^2}x_1\right)^{-\epsilon}\times$$

$$\big(m_1(1-\beta)x_2 + [m_1(1+\beta)+m_2(1-\beta)]x_1\big)^{-2+2\epsilon}\,\delta\left(1-\sum\nolimits_{i=1}^{2}\alpha_i x_i\right)$$

$$I^{+,2}_{\mathrm{bub},m_1\neq m_2} = (m_1 m_2)^{1-2\epsilon}(1-\beta)^{2-2\epsilon}\int_{\mathbb{R}^2_{\geq 0}} \mathrm{d}x_1\mathrm{d}x_2\, x_1^{-\epsilon}\left(x_1 + \frac{4\beta}{1-\beta^2}x_2\right)^{-\epsilon}\times$$

$$\big(m_2(1-\beta)x_1 + [m_2(1+\beta)+m_1(1-\beta)]x_2\big)^{-2+2\epsilon}\,\delta\left(1-\sum\nolimits_{i=1}^{2}\alpha_i x_i\right)$$

$$I^{-}_{\mathrm{bub},m_1\neq m_2} = (4m_1m_2\beta)^{1-2\epsilon}(1-\beta^2)^{\epsilon}\int_{\mathbb{R}^2_{\geq 0}} \mathrm{d}x_1\mathrm{d}x_2\,(x_1x_2)^{-\epsilon}\times$$

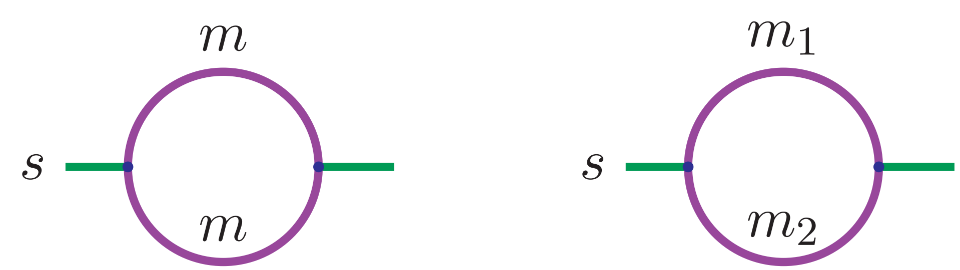
$$\big[(m_1+m_2)(x_1+x_2)-(m_1-m_2)(x_1-x_2)\beta\big]^{-2+2\epsilon}\,\delta\left(1-\sum\nolimits_{i=1}^{2}\alpha_i x_i\right).$$

Resulting integrands are again **positive & homogeneous** (any $\delta(1-\alpha(\mathbf{x}))$ ok)

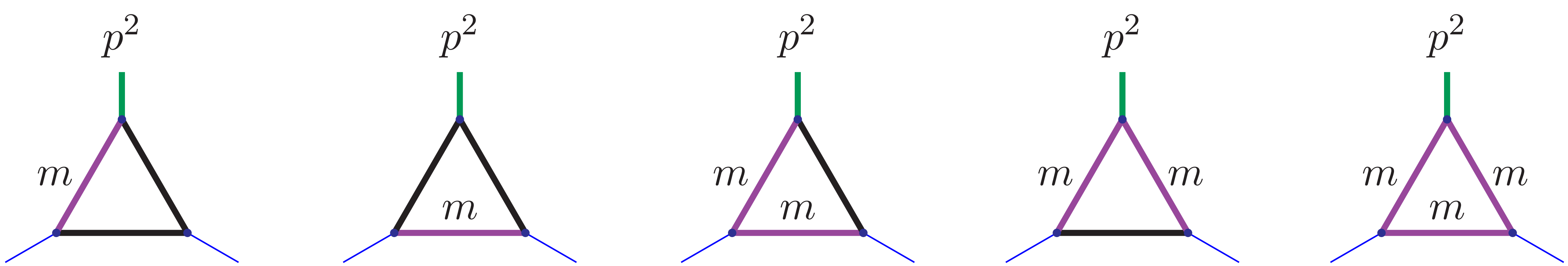
Symmetry of $I^{+,1}_{\mathrm{bub},m_1\neq m_2} \leftrightarrow I^{+,2}_{\mathrm{bub},m_1\neq m_2}$ under simultaneous $(x_1 \leftrightarrow x_2,\, m_1 \leftrightarrow m_2)$ manifest

Some Examples of Massive Integrals

Bubbles



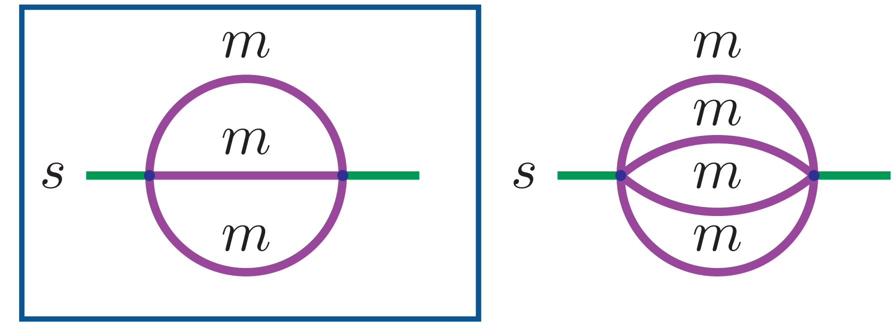
Triangles



Possible to resolve all in a similar* manner to the massive bubble

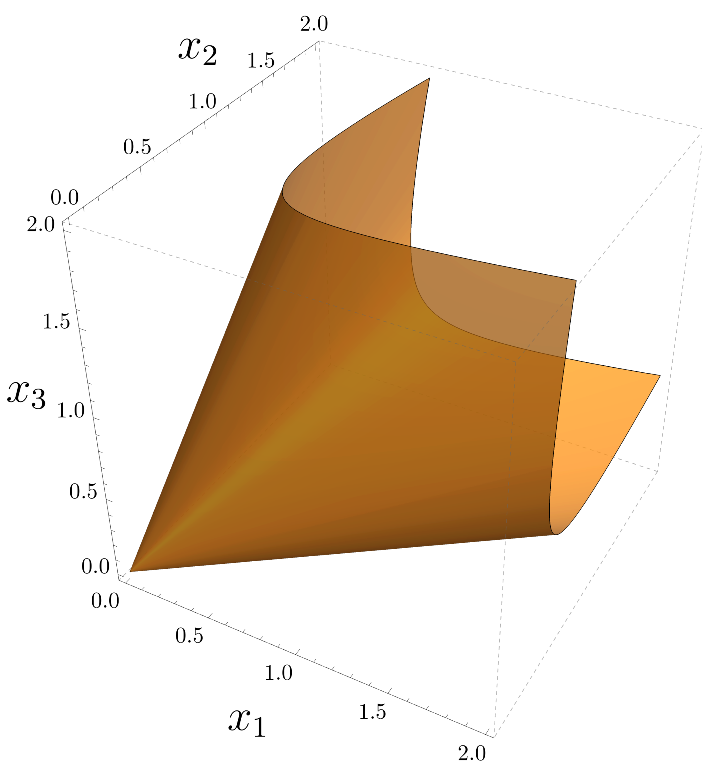
How about something a bit more interesting?

Consider something that involves functions beyond polylogarithms



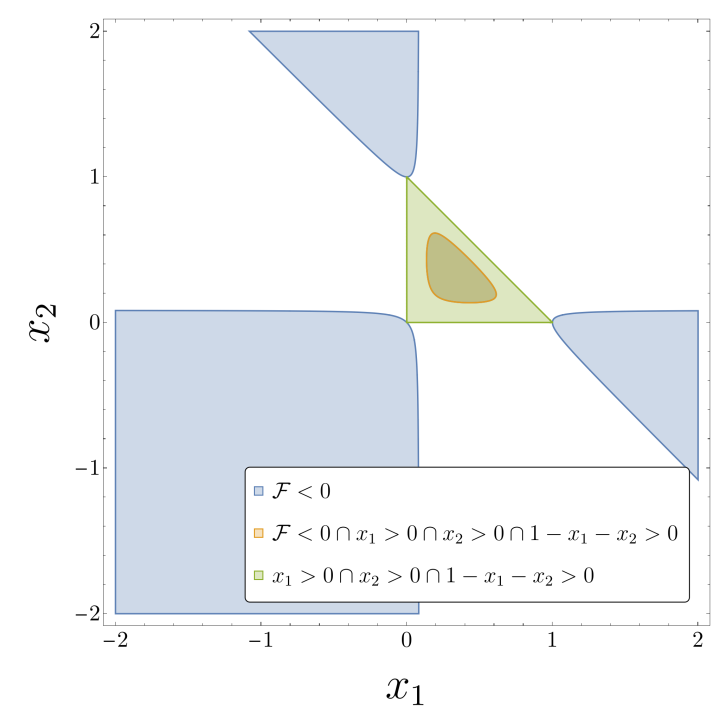
$$J_{\text{sun}} = \lim_{\delta \to 0^+} -\Gamma(-1+2\epsilon)\, I_{\text{sun}}$$

$$I_{\text{sun}} = \int_{\mathbb{R}^3_{\geq 0}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \frac{\left(x_1x_2 + x_2x_3 + x_1x_3\right)^{-3+3\epsilon} \delta\left(1 - \sum_{i=1}^{3} \alpha_i x_i\right)}{\left(-s x_1x_2x_3 + \left(x_1x_2 + x_2x_3 + x_1x_3\right) m^2 \left(x_1 + x_2 + x_3\right) - i\delta\right)^{-1+2\epsilon}}$$

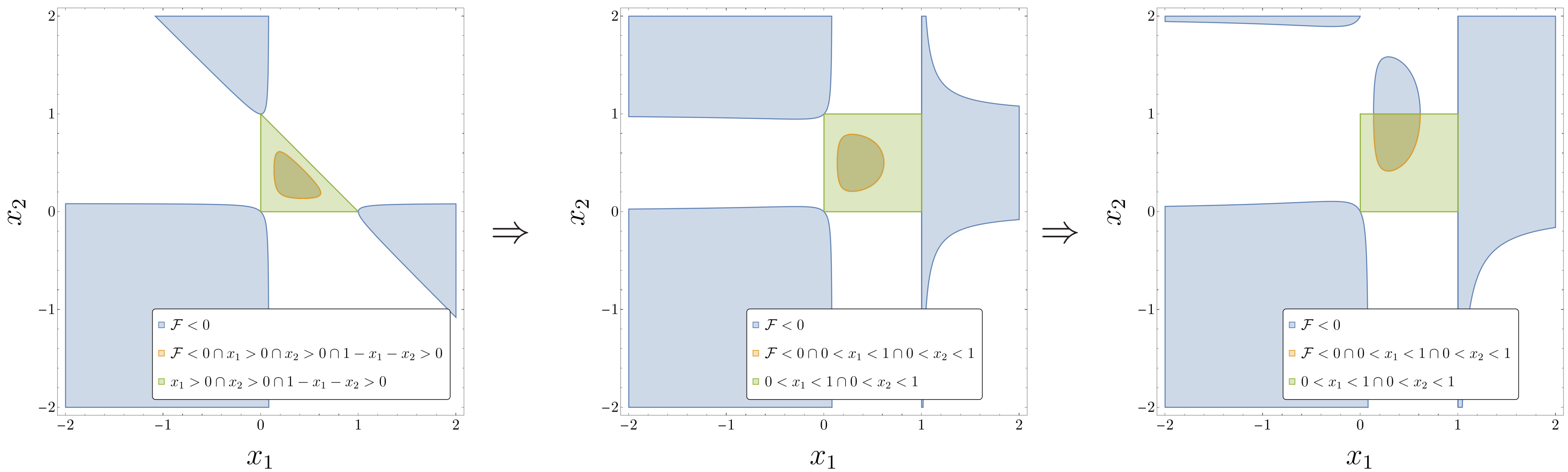


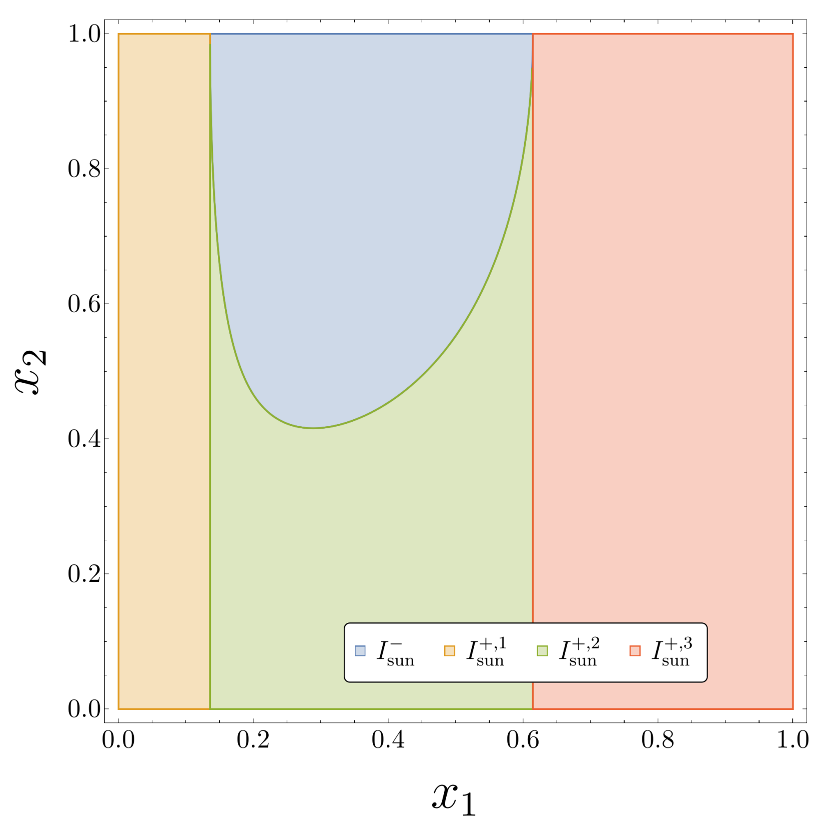
$\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ hypersurface for $s > 9m^2$

$$\delta(1 - \sum_{i=1}^{3} x_i)$$



2-Loop Elliptic Sunrise





Encounter ***algebraic*** (square-root) transformations

Singular structure can *still be factorised* $\Longrightarrow$ can use standard techniques (for examples considered)

$$\boxed{I_{\text{sun}} = \sum_{n_+=1}^{3} I_{\text{sun}}^{+,n_+} + (-1 - i\delta)^{1-2\epsilon}\, I^-_{\text{sun}}}$$

2-Loop Elliptic Sunrise (II)

$$I^{-}_{\text{sun}} = 2^{7-6\epsilon}3^{\frac{1}{2}-\epsilon}\left(\beta^2\right)^{2-2\epsilon}\left(8+\beta^2\right)^{2-2\epsilon}\left(\frac{1-\beta^2}{m^2}\right)^{-1+2\epsilon}\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\left(1-x_1\right)^{\frac{3}{2}-2\epsilon}x_1^{\frac{3}{2}-2\epsilon}x_2^{1-2\epsilon}R^{-}_{\text{sun}}\left(x_1,x_2;\beta\right)$$

singularities
$x_1, x_2 \to 0$
$x_1 \to 1$

finite/regular

Carelessly integrating out δ-functional yields…
inelegant results

$$R^{-}_{\text{sun}}\left(x_1,x_2;\beta\right) = R_1(x_2;\beta)R_2(x_1;\beta)R_3(x_1;\beta)R_4(x_1,\beta)R_5(x_1,x_2;\beta),$$

$$R_1(x_2;\beta) = \bar{x}_2^{1-2\epsilon},$$

$$R_2(x_1;\beta) = \left[-\beta^2+\beta\bar{\beta}\tilde{x}_1+4\right]^{3\epsilon-2},$$

$$R_3(x_1;\beta) = \left[4-\beta\left(2\beta\left(\beta^2+1\right)-3\beta\bar{\beta}^2x_1\bar{x}_1+2\tilde{\beta}\bar{\beta}\tilde{x}_1\right)\right]^{\frac{3}{2}-\epsilon},$$

$$R_4(x_1;\beta) = \left[\beta^2\bar{\beta}^2x_1\bar{x}_1\left(-11\beta^2+3\beta\bar{\beta}\tilde{x}_1+20\right)+4\tilde{\beta}^2\left(\beta^2-\beta\bar{\beta}\tilde{x}_1+4\right)\right]^{1-2\epsilon},$$

$$R_5(x_1,x_2;\beta) = \left[\beta^2\bar{\beta}^2x_1\bar{x}_1\left(x_2\bar{x}_2\left(-\beta^2+\beta\bar{\beta}\tilde{x}_1+4\right)+4\beta\left(3\beta-\bar{\beta}\tilde{x}_1\right)\right)+4\tilde{\beta}\left(\beta^4+7\beta^2-\left(\beta^2+3\right)\beta\bar{\beta}\tilde{x}_1+4\right)\right]^{3\epsilon-3},$$

$$\bar{x}_1 = 1-x_1,\ \tilde{x}_1 = 1-2x_1,\ \bar{x}_2 = 2-x_2, \bar{\beta} = \sqrt{8+\beta^2},\ \tilde{\beta} = 1-\beta^2$$

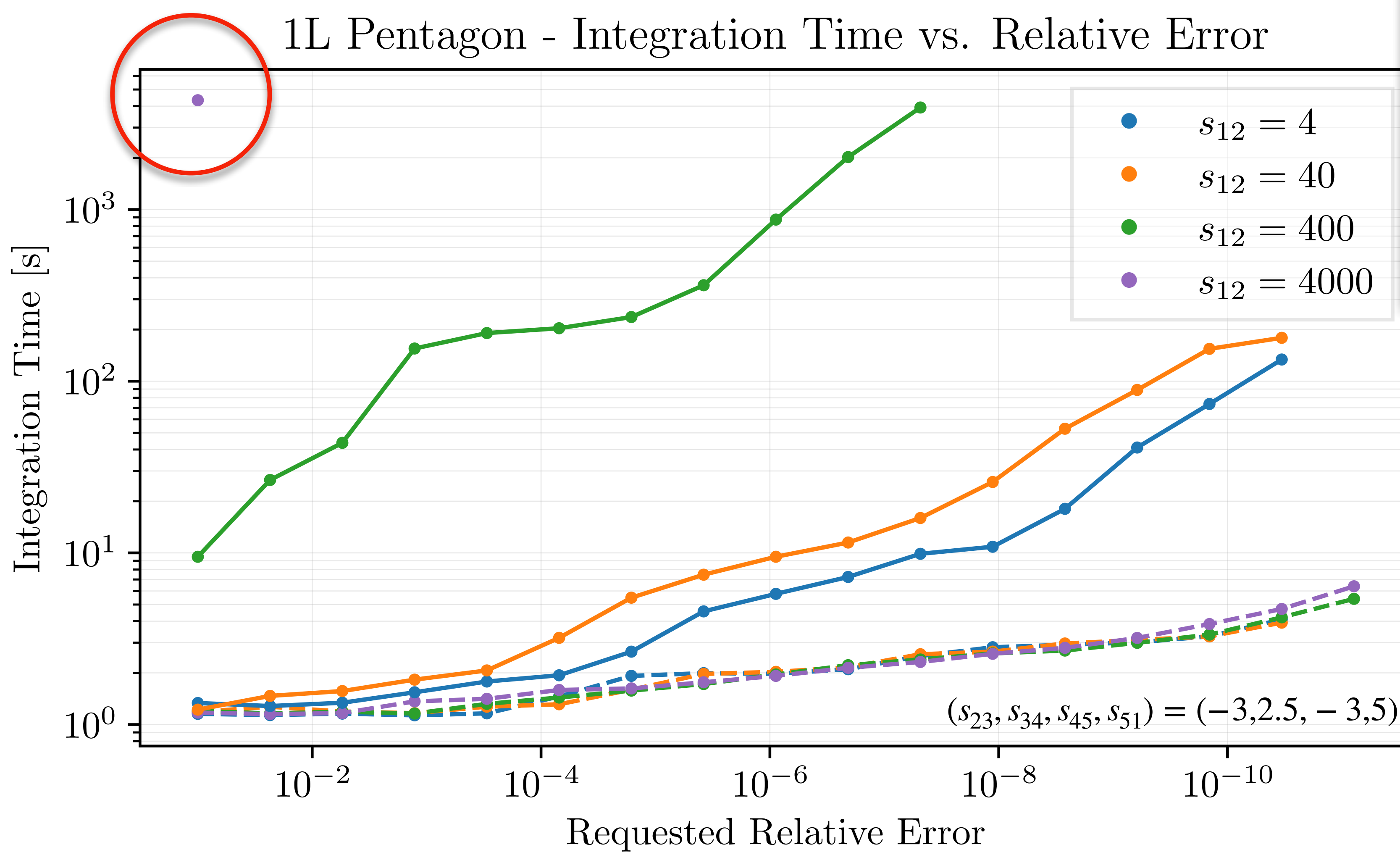
Proof that resolution of elliptic integrals is possible, not an optimal/elegant implementation of this resolution

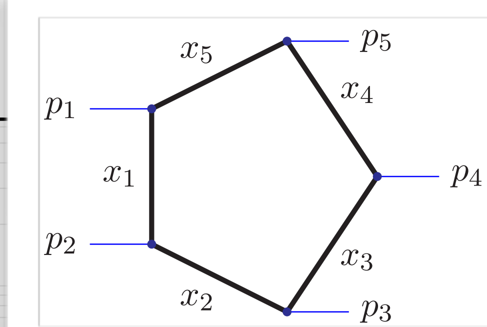
Numerical Performance

Goal: simplify/accelerate the numerical integration

Let's benchmark this using pySecDec (Sector Decomposition, Quasi Monte Carlo)

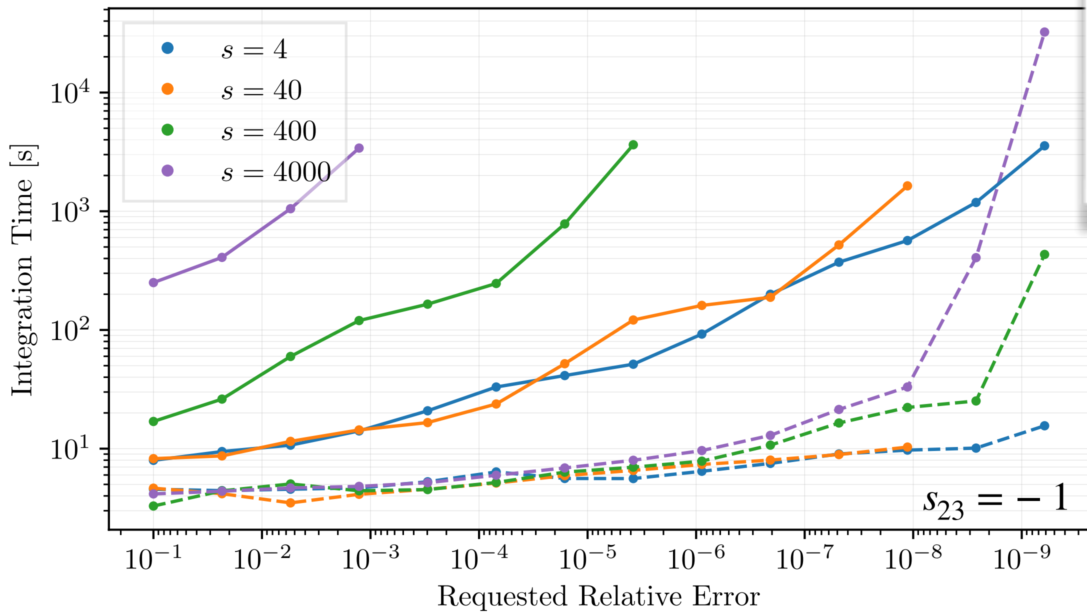
Note: code strongly optimised for contour deformed integrals, many possible optimisations possible for the type of integrals we obtain here (real, positive, compact)

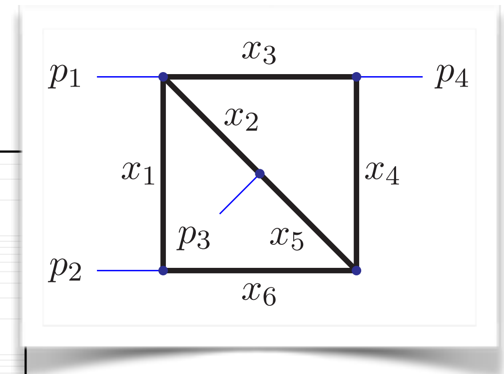




Speed-up of $> 10^4 \times$ esp. close to singular/pinched points, promising for >1-loop

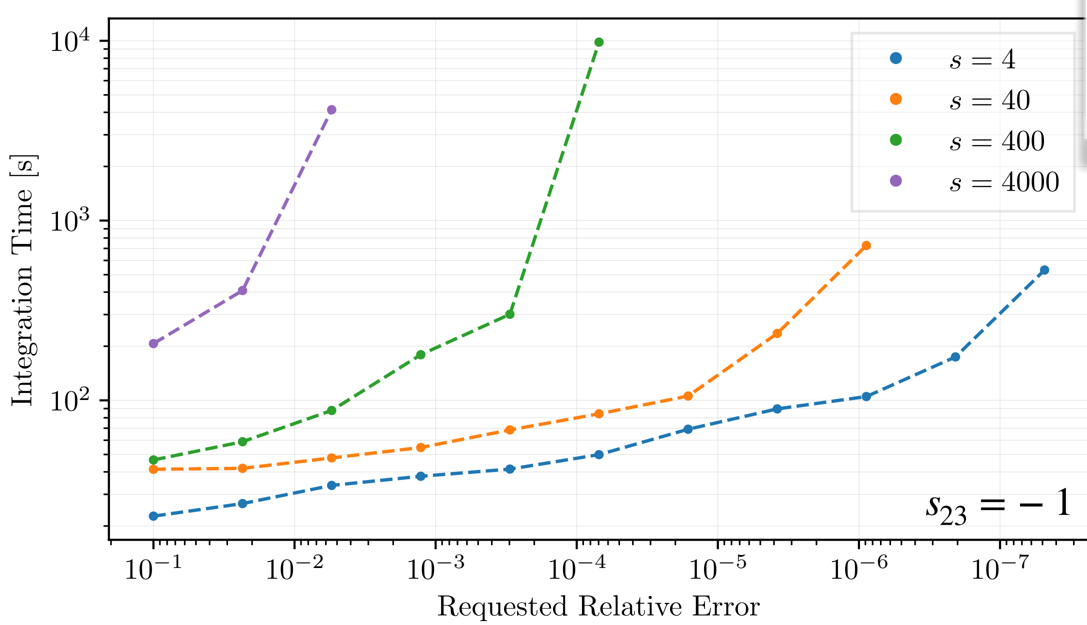
Numerical Performance (II)

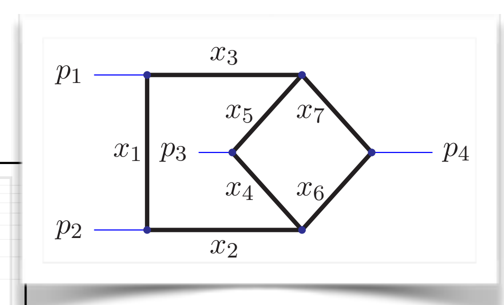




Fairly difficult to integrate with contour deformation (even after some tricks)

Speed-ups $> 10^2 \times$ for more challenging points

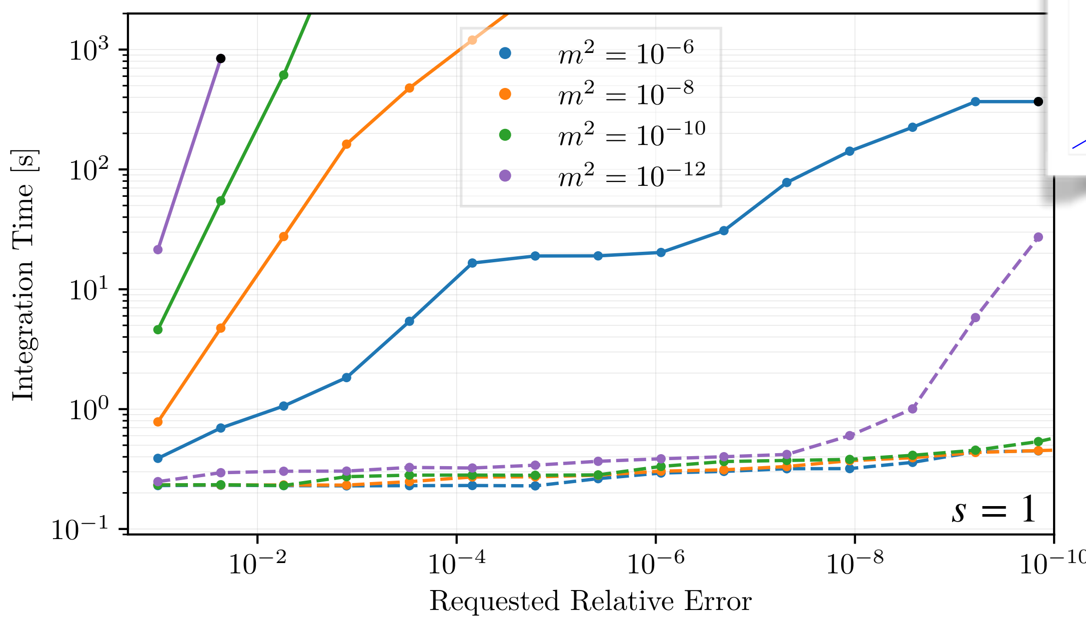


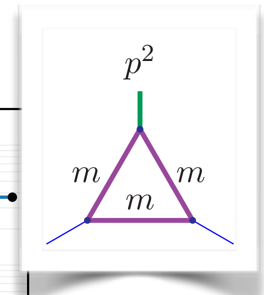


Although free of leading Landau singularities, extremely challenging to evaluate using contour deformation, much easier after resolution

Gardi, Herzog, SPJ, Ma, Schlenk 22

Numerical Performance (III)

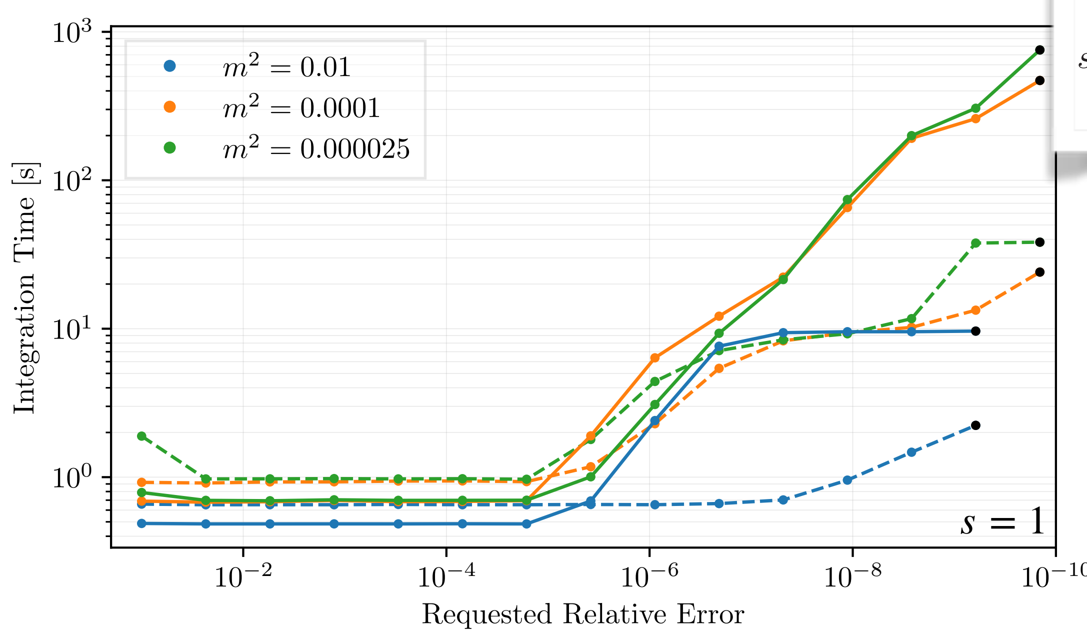


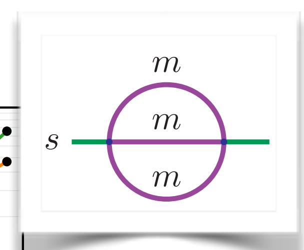


In $m^2 \to 0$ limit integral develops $1/\epsilon^2$ poles $\Longrightarrow$ end-point singularities

Contour is ``pinched'' against the boundary of integration

Speed-up $> 10^4 \times$





Proves that we can resolve and numerically evaluate elliptic integrals

By far not optimal implementation

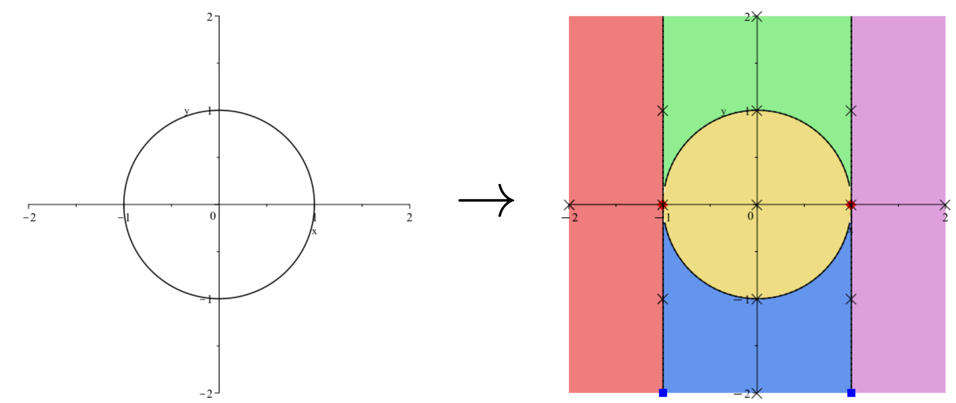
Speed-up of only $10 - 100 \times$

General Picture

(Work in Progress)

Can we always perform this procedure?

Trying to find sign-invariant decomposition of
$\{\mathscr{F}(\mathbf{x};\mathbf{s}) < 0\} \cup \{0 < \mathbf{x}\} \cup \mathbf{s}_R$,



Lee, del Río, Rahkooy 25

Borrow technique from real algebraic geometry
Generic **C**ylindrical **A**lgebraic **D**ecomposition

codim 0	projection of cells either disjoint or identical	cell boundaries are roots of polynomials	union of disjoint cells is $\mathbb{R}^N$

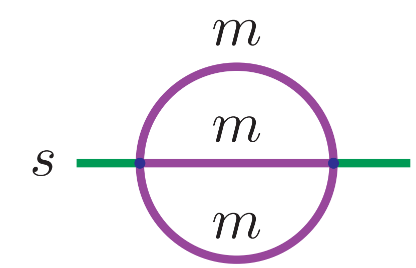
Collins 75; Davenport, Heintz 88; Lazard 94; McCallum 19

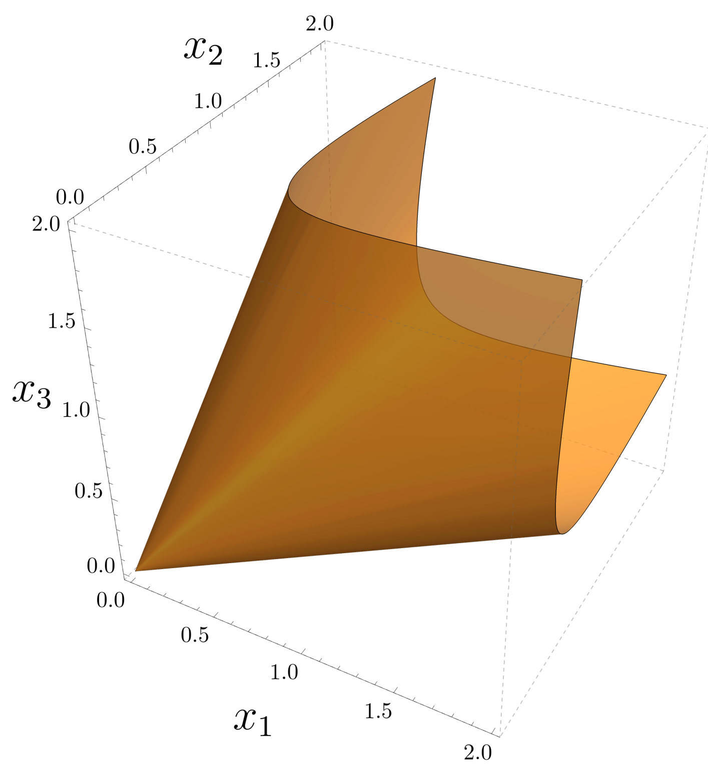
$$
\begin{array}{ccc}
\mathcal{F}_n & & \mathrm{CAD}(\mathcal{F}_n) \\
\big\downarrow \mathrm{proj} & & \big\uparrow \mathrm{Lift} \\
\mathcal{F}_{n-1} & & \mathrm{CAD}(\mathcal{F}_{n-1}) \\
\big\downarrow \mathrm{proj} & & \big\uparrow \mathrm{Lift} \\
\vdots & & \vdots \\
\big\downarrow \mathrm{proj} & & \big\uparrow \mathrm{Lift} \\
\mathcal{F}_2 & & \mathrm{CAD}(\mathcal{F}_2) \\
\big\downarrow \mathrm{proj} & & \big\uparrow \mathrm{Lift} \\
\mathcal{F}_1 & \xrightarrow[\text{split into cells}]{\text{find roots}} & \mathrm{CAD}(\mathcal{F}_1)
\end{array}
$$

Good: Guaranteed to work, constructive
Bad: Complexity doubly exponential in #vars, root degrees

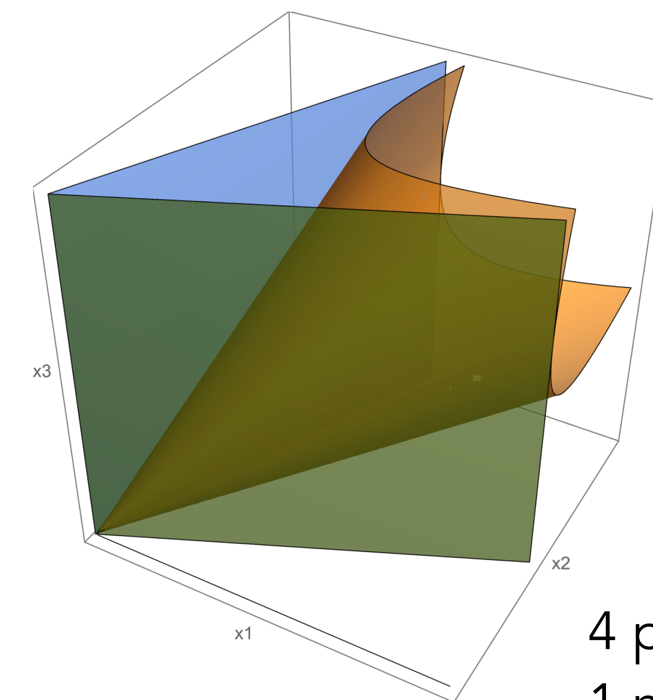
However, we are interested in a rather special case, $\mathscr{F}(\mathbf{x};\mathbf{s})$ homogeneous, linear (quadratic) in massless (massive) variables… in practice, often much simpler

Consider again the 2-loop sunrise before integrating δ-func





GCAD



4 positive cells +
1 negative cell

$$I_{\text{sun}}^{-} = 2^{3-3\epsilon} c_1^{3-4\epsilon} c_4 \int_{\mathbf{R}^3_{\geq 0}} x_3^{1-2\epsilon} R_1(x_1,x_2)^{\frac{3}{2}-2\epsilon} R_2(x_1,x_2,x_3)^{-1+\epsilon} R_3(x_1,x_2,x_3)^{-3+3\epsilon},$$

$$R_1(x_1,x_2) = x_1 x_2 \left(c_3 x_1 x_2 + 4c_2(x_1^2+x_2^2)\right),$$

$$R_2(x_1,x_2,x_3) = x_3 + (x_1+x_2)c_5,$$

$$R_3(x_1,x_2,x_3) = -\,c_1\sqrt{R_1(x_1,x_2)}\,\left(-x_3+(x_1+x_2)c_5\right) - R_2(x_1,x_2,x_3)\,\left((x_1^2+x_2^2)c_6 + x_1 x_2 c_7\right)$$

$c_1, \ldots, c_7$: algebraic functions of s/m (not depending on x_1, x_2, x_3)

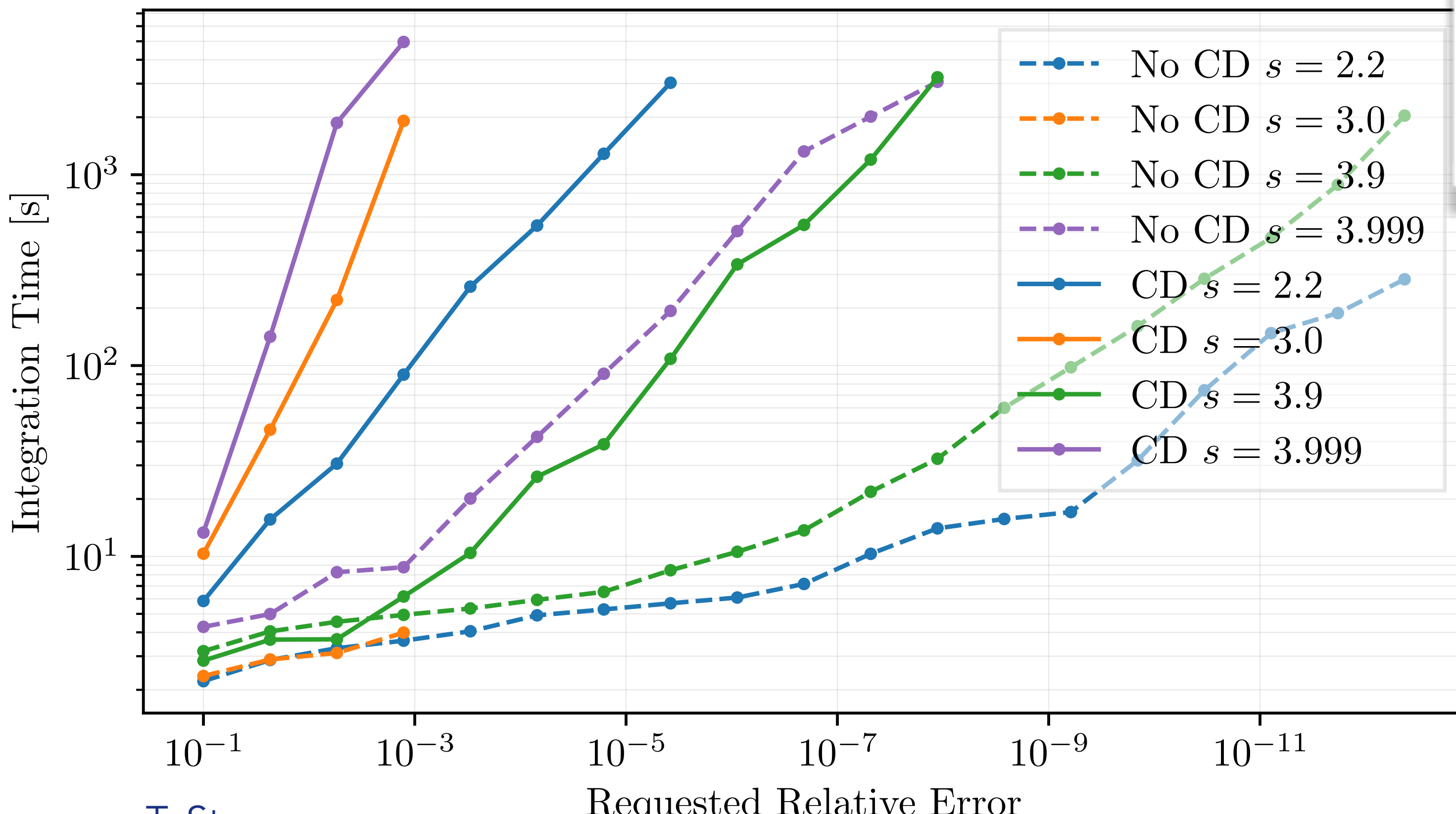
Dramatically simpler and more numerically stable than previous resolution

A Final Example

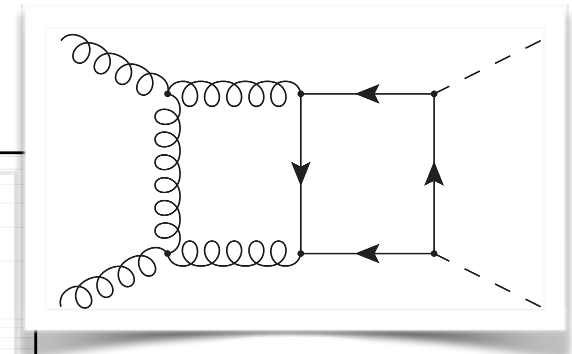
(Work in Progress)

Can apply these techniques to integrals of more phenomenological interest

Example



T. Stone



$$\mathbf{s}_R = \{4m_H^2 < s < 4m_t^2, \\ -t < s < 0, \\ m_H^2 > 0, m_t^2 > m_H^2\}$$

Speedup $10 - 10^3 \times$

$gg \to HH$ amplitude @ 2-loops (406 MI) pySecDec & Contour Def: ~90 seconds/PS

T. Stone

Contour deformation is (the?) bottleneck for $2 \to 3$ @ 2-loop, $2 \to 2$ @ 3-loop

What is possible once we avoid contour deformation?

Do These Integrals Have Any Meaning?

Interpretation

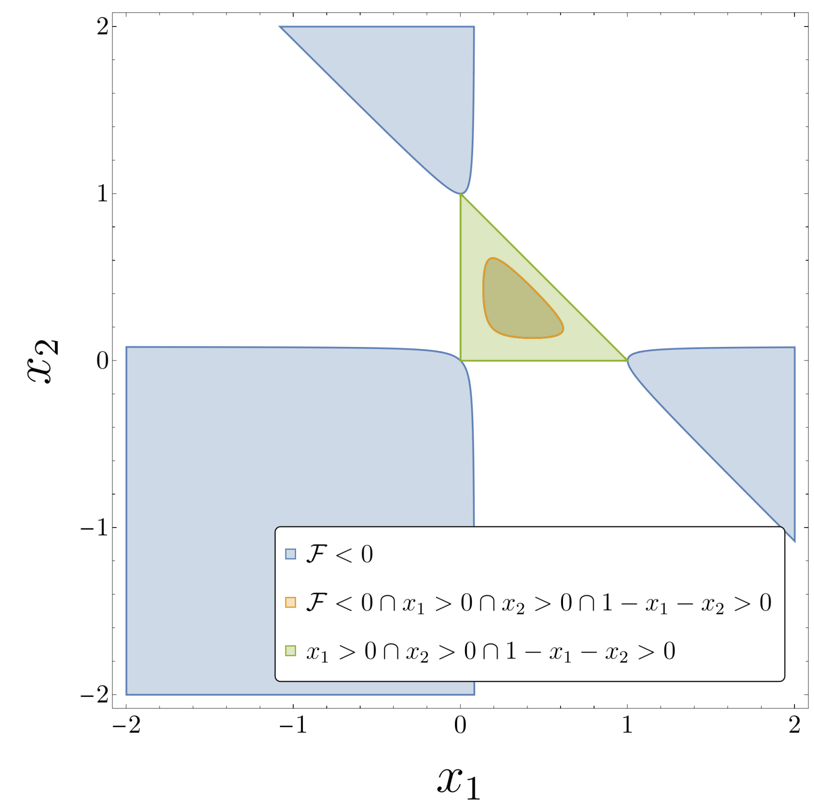
These integrals are related to discontinuities/ generalised cuts:

$\mathrm{Disc}_s[\mathscr{F}^\lambda] = (\mathscr{F} - i\delta)^\lambda - (\mathscr{F} + i\delta)^\lambda = -\theta[-\mathscr{F}][-\mathscr{F}]^\lambda 2i\sin(\pi\lambda)$

Maximal cuts: only $\mathscr{F} = 0$ as boundary

Non-maximal cuts: combination of $\mathscr{F} = 0$ and coordinate hyper-planes as boundary

Britto 23



Relations

Presumably, can express entire amplitudes as sums of these positive integrals

$$\mathscr{A} \sim \sum_i c_i\, I^+ + \sum_j \lim_{\delta\to 0} c_j(\delta)\, I^-$$

Would be very useful to have efficient IBPs available for parametrised integrals

Bitoun, Bogner, Klausen, Panzer 19; Chen 19, 19; Artico, Magnea 24;

Conclusion

Summary:

- For several massless integrals up to 3-loop 4-point it is rather straightforward to avoid contour deformation
- Massive integrals up to 3-loop 2-point (elliptic, hyperelliptic) can also be addressed, as well as integrals with a mix of massless/massive propagators
- Trivial algorithm works for many simple cases
- Generic Cylindrical Algebraic Decomposition seems to provide a general algorithm

Next Steps

- Implement algorithm for massless (UB and non-UB) in public codes
- Investigate use of and optimisation of Cylindrical Algebraic Decomposition for the special case of Feynman integrals

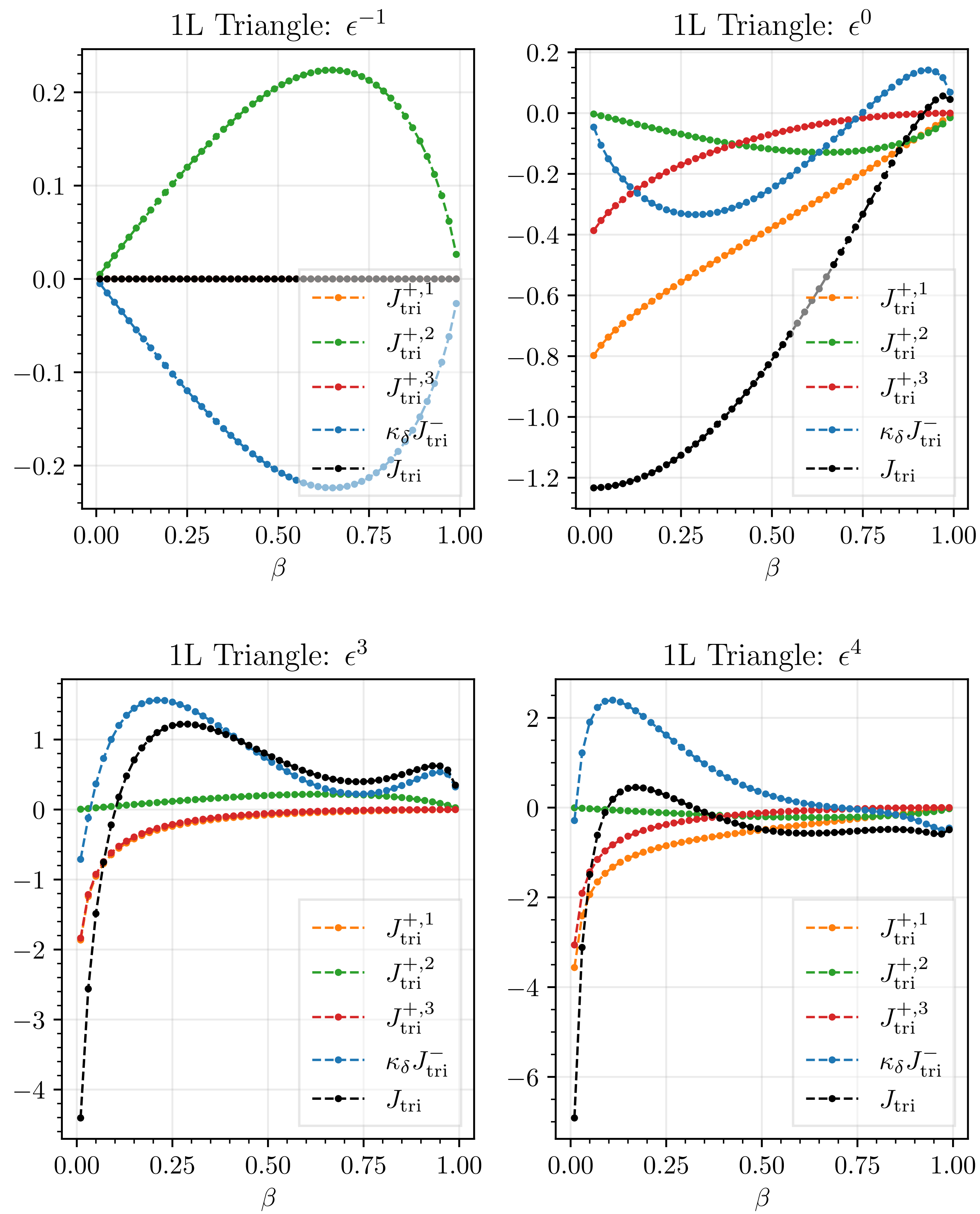
Outlook

- Can we automate and use this at scale for entire amplitudes?
- Can we further connect this picture to e.g. cuts or other approaches to studying amplitudes?

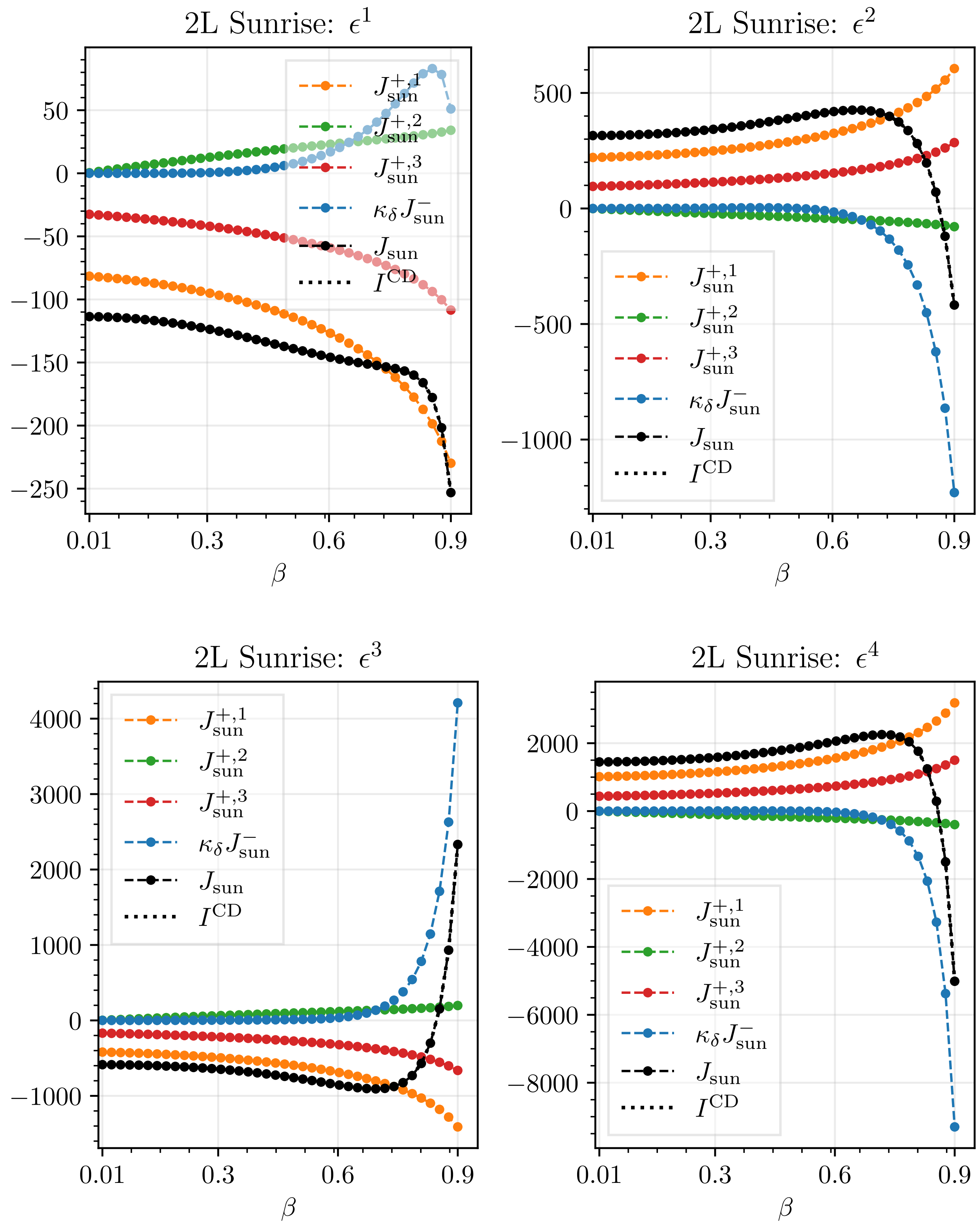
Thank you for listening!

Backup

Cancellations



Cancellations



Sector Decomposition

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t$$

$$\mathcal{N}(I) = \mathrm{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

Normal vectors incident to each extremal vertex define a local change of variables*

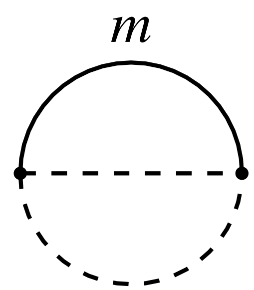
Kaneko, Ueda 10

$$x_i = \prod_{f \in S_j} y_f^{\langle \mathbf{n}_f, \mathbf{e}_i \rangle}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_0^1 \left[\mathrm{d}\mathbf{y}_f\right] \underbrace{\prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle - t a_f}}_{\textbf{Singularities}} \left(\underbrace{c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f}}_{\textbf{Finite}} \right)^t$$

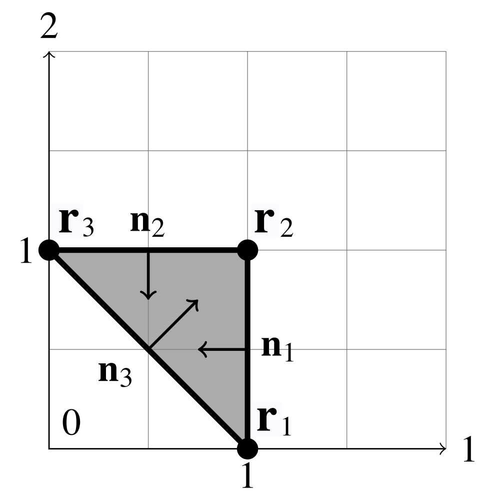
*If $|S_j| > N$, need triangulation to define variables (simplicial normal cones $\sigma \in \Delta^T_{\mathcal{N}}$)

Sector Decomposition in a Nutshell

$$I = $$ 

$$= -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon} \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\left(x_1^1 x_2^0 + x_1^1 x_2^1 + x_1^0 x_2^1\right)^{2-\varepsilon}}.$$

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathcal{N}(I) = $$ 

$$= \quad \mathbf{n}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \mathbf{n}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \mathbf{n}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$a_1 = 1 \quad a_2 = 1 \quad a_3 = -1$$

For each vertex make the local change of variables

e.g. $\mathbf{r}_1$: $x_1 = y_1^{-1} y_3^1$, $x_2 = y_1^0 y_3^1$, $\mathbf{r}_2 : x_1 = y_1^{-1} y_2^0$, $x_2 = y_1^0 y_2^{-1}$, $\mathbf{r}_3 : x_1 = y_2^0 y_3^1, x_2 = y_2^{-1} y_3^1$

$$I = -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon} \int_0^1 \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \, \frac{y_1^{-\varepsilon} y_2^{-\varepsilon} y_3^{-1+\varepsilon}}{(y_1+y_2+y_3)^{2-\varepsilon}} \left[\delta(1-y_2) + \delta(1-y_3) + \delta(1-y_1)\right]$$

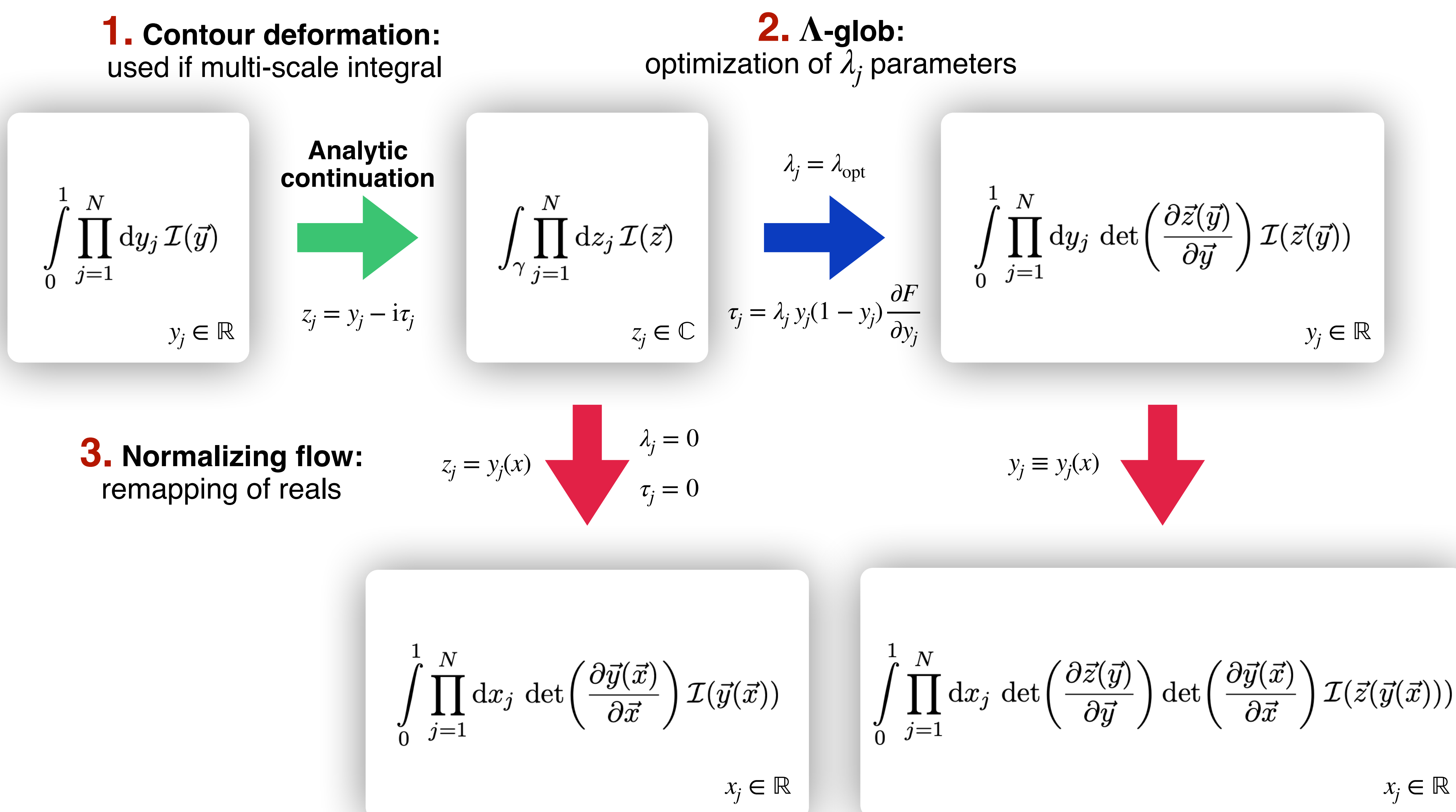
Schlenk 2016

Neural Network Contour Deformation

Neural Networks for Contour Deformation

Normalizing Flows consist of a series of (trainable) bijective mappings for which we can efficiently compute the Jacobian

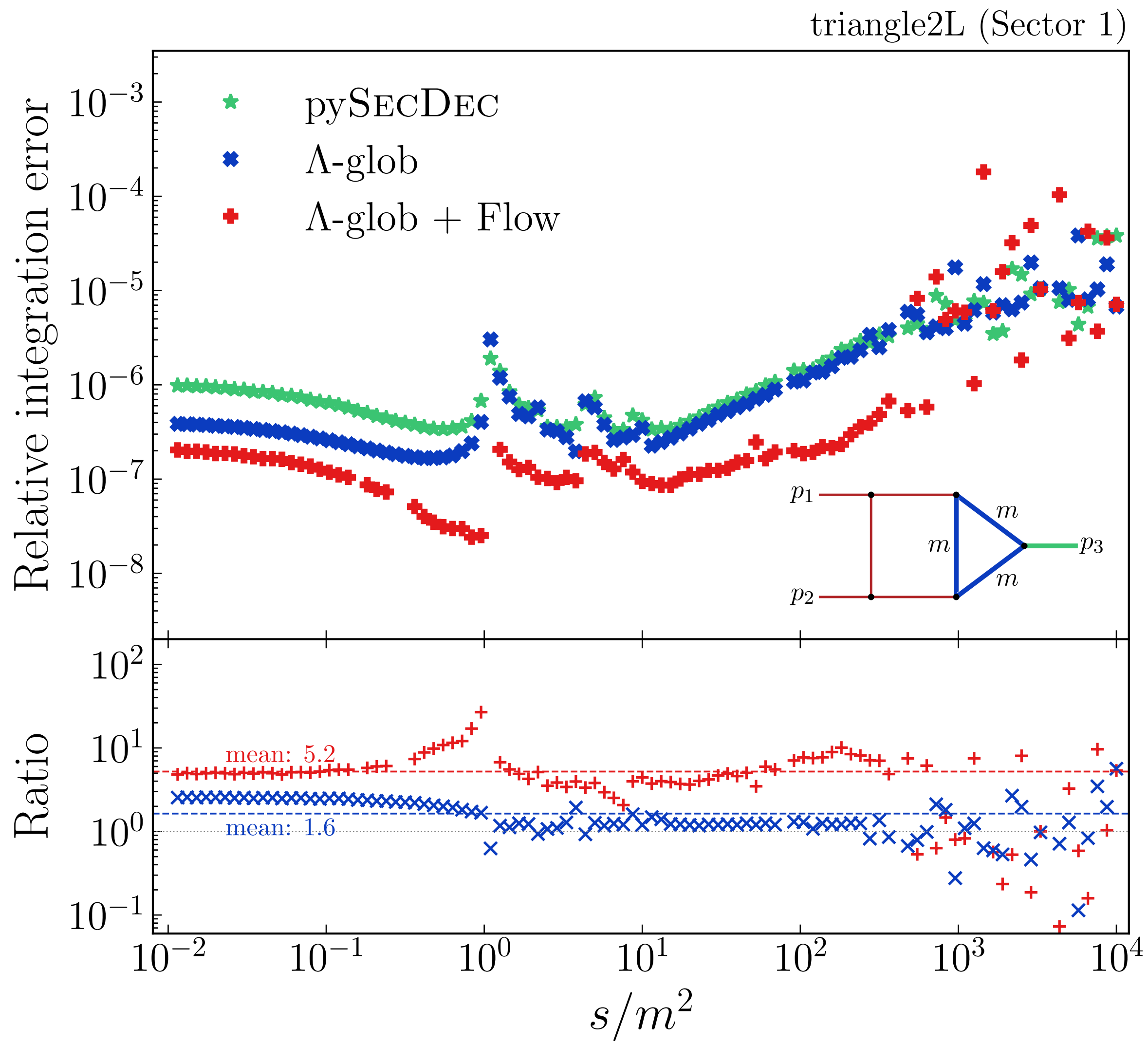
Procedure

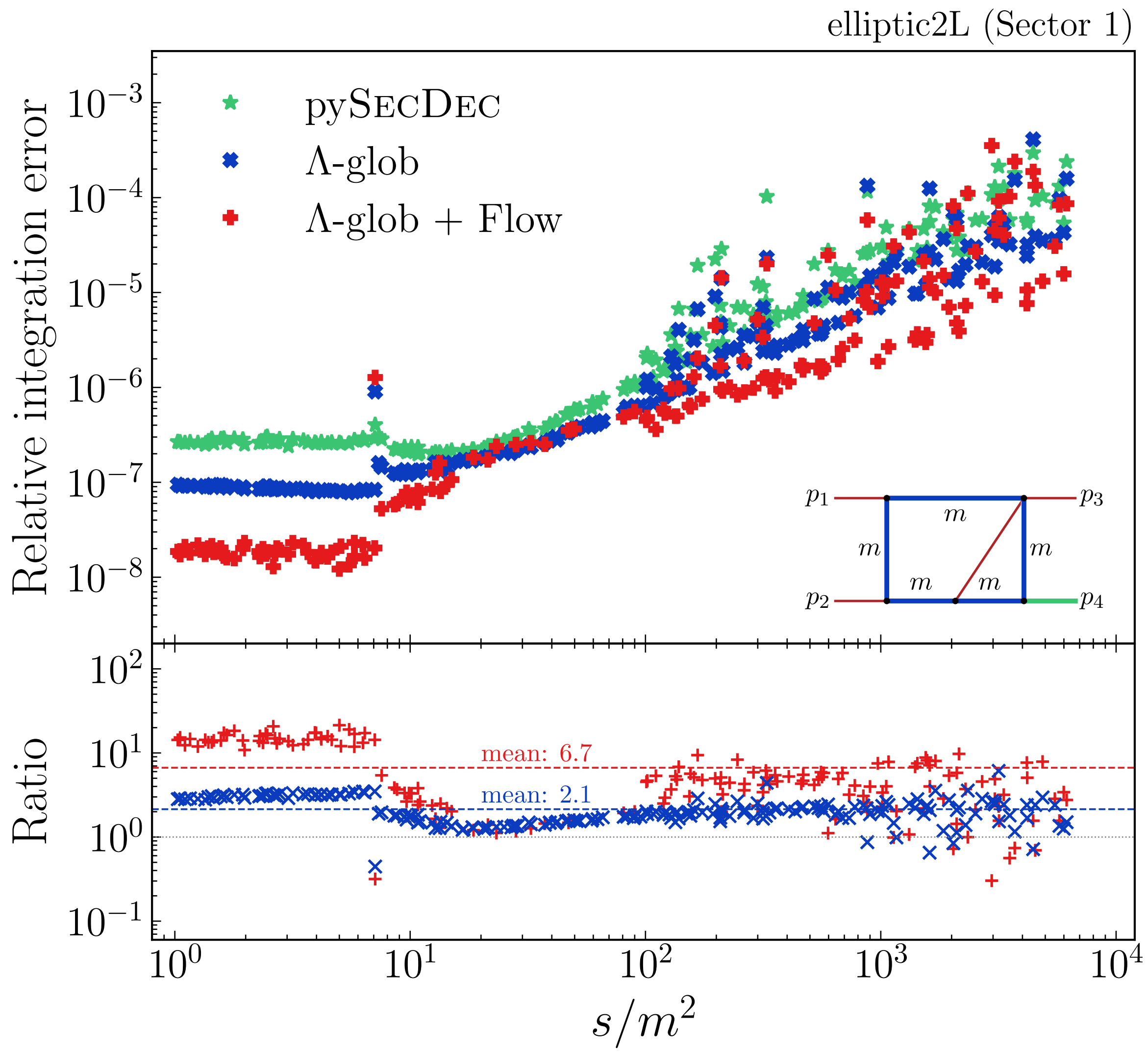


Loss: $L = L_{\mathrm{MC}} + L_{\mathrm{sign}}$ constructed to minimise variance without crossing poles

Neural Networks for Contour Deformation

Applied to several 1 & 2-loop Feynman Integrals with multiple masses/thresholds using tensorflow





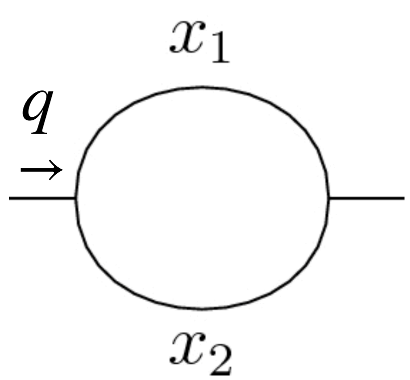
Proof of principle that Machine Learning can help to find improved contours and reduce variance, still a tradeoff between training time/ integrating time

Method of Regions

Regions due to Cancellation

What happens if c_i have different signs?

Consider a 1-loop massive bubble at *threshold* $y = m^2 - q^2/4 \to 0$



$$I = \Gamma(\epsilon) \int d\alpha_1 d\alpha_2 \frac{\delta(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2)^{-2+2\epsilon}}{\left(\mathscr{F}_{\text{bub}}(\alpha_1,\alpha_2;q^2,y)\right)^{\epsilon}}$$

$$\mathscr{F}_{\text{bub}} = \frac{q^2}{4}(\alpha_1 - \alpha_2)^2 + y(\alpha_1+\alpha_2)^2$$

Can split integral into two subdomains $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$ then remap

$$\begin{aligned} \alpha_1 &= \alpha_1'/2 \\ \alpha_2 &= \alpha_2' + \alpha_1'/2 \end{aligned} : \quad \mathscr{F}_{\text{bub},1} \to \frac{q^2}{4}\alpha'^2_2 + y(\alpha'_1 + \alpha'_2)^2 \quad \text{(for first domain)}$$

Jantzen, A. Smirnov, V. Smirnov 12

Before split: only **hard** region found ($\alpha_1 \sim y^0, \alpha_2 \sim y^0$)

After split: also **potential** region found ($\alpha_1 \sim y^0, \alpha_2 \sim y^{1/2}$)

Method of Regions

Consider expanding an integral about some limit:
$p_i^2 \sim \lambda Q^2$, $p_i \cdot p_j \to \lambda Q^2$ or $m^2 \sim \lambda Q^2$ for $\lambda \to 0$

Issue: integration and series expansion do not necessarily commute

Method of Regions

$$I(\mathbf{s}) = \sum_R I^{(R)}(\mathbf{s}) = \sum_R T_{\mathbf{t}}^{(R)} I(\mathbf{s})$$

1. Split integrand up into regions (R)
2. Series expand each region in λ
3. Integrate each expansion over the whole integration domain
4. Discard scaleless integrals (= 0 in dimensional regularisation)
5. Sum over all regions

Smirnov 91; Beneke, Smirnov 97; Smirnov, Rakhmetov 99; Pak, Smirnov 11; Jantzen 2011; …

Finding Regions

Assuming all c_i have the same sign we rescale $\boldsymbol{s} \to \lambda^{\boldsymbol{\omega}}\boldsymbol{s}$ $s_i \to \lambda^{\omega_i} s_i$

Newton Polytope

$$I \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t \to \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i} \lambda^{r_{i,N+1}}\right)^t \to \mathcal{N}^{N+1}$$

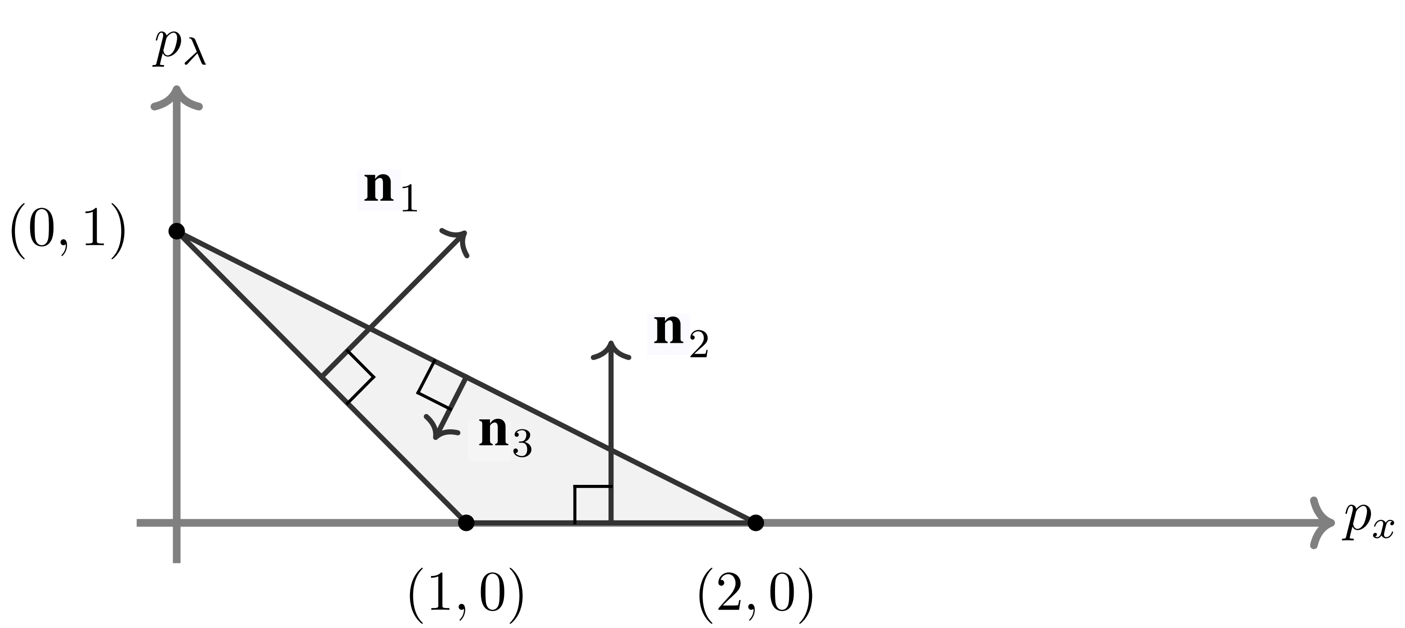
Normal vectors w/ positive λ component define change of variables $\mathbf{n}_f = (v_1, \ldots, v_N, 1)$

$$\boldsymbol{x} = \lambda^{\mathbf{n}_f}\, \mathbf{y}\,, \qquad \lambda \to \lambda$$

Pak, Smirnov 10; Semenova, A. Smirnov, V. Smirnov 18

Example

$p(x, \lambda) = \lambda + x + x^2$



$1,2 \in F^+$

$3 \notin F^+$

Original integral I may then be approximated as $I = \sum_{f \in F^+} I^{(f)} + \ldots$

Additional Regulators/ Rapidity Divergences

MoR subdivides $\mathcal{N}(I) \to \{\mathcal{N}(I^R)\} \implies$ new (internal) facets $F^{\text{int.}}$

New facets can introduce spurious singularities not regulated by dim reg

Lee Pomeransky Representation:

$$\mathcal{N}(I^{(R)}) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_{\mathbb{R}^N_{\geq 0}} \left[\mathrm{d}\mathbf{y}_f \right] \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle + \frac{D}{2} a_f} \left(c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f} \right)^{-\frac{D}{2}}$$

If $f \in F^{\text{int}}$ have $a_f = 0$ need analytic regulators $\boldsymbol{\nu} \to \boldsymbol{\nu} + \boldsymbol{\delta\nu}$

Heinrich, Jahn, SJ, Kerner, Langer, Magerya, Põldaru, Schlenk, Villa 21; Schlenk 16

Integrals with Pinch Singularities

Looking for Trouble: Algorithm

Generally, solutions of the Landau equations depend on $\mathbf{s}$.
Let us restrict our search to solutions with *generic* kinematics

$$\mathcal{F} = -\sum_i s_i \left[f_i(\boldsymbol{\alpha}) - g_i(\boldsymbol{\alpha})\right] = \sum_i \mathcal{F}_{i,-} + \mathcal{F}_{i,+}$$

$$\mathcal{F}_{i,-} = -\, s_i\, f_i(\boldsymbol{\alpha}), \quad \mathcal{F}_{i,+} = s_i\, g_i(\boldsymbol{\alpha}), \quad f_i(\boldsymbol{\alpha}), g_i(\boldsymbol{\alpha}) \geq 0$$

Algorithm (finds integrals which *potentially* have a pinch in the massless case)

For each s_i:

1) Compute $\mathcal{F}_{i,-}$, $\mathcal{F}_{i,+}$

2) If $\mathcal{F}_{i,-} = 0$ or $\mathcal{F}_{i,+} = 0 \rightarrow$ **Exit (no cancellation)**

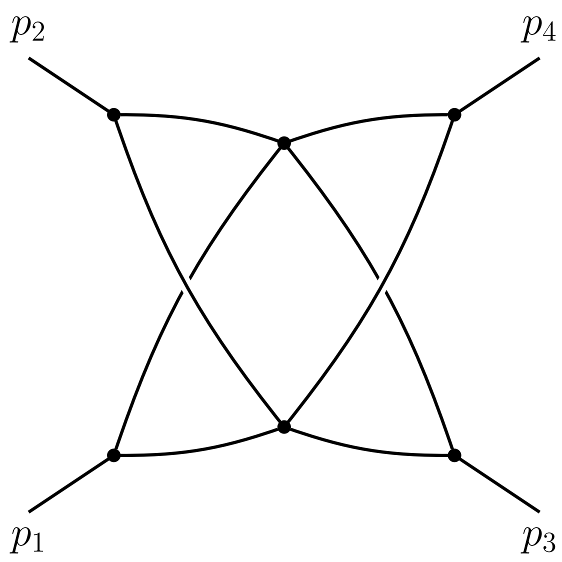
3) If $\partial\mathcal{F}_{i,-}/\partial\alpha_j = 0$ or $\partial\mathcal{F}_{i,+}/\partial\alpha_j = 0$ set $\alpha_j = 0 \rightarrow$ Goto 1

Else $\rightarrow$ **Exit (potential cancellation)**

Much more sophisticated algorithms for solving Landau equations exist

(E.g.) Mizera, Simon Telen 21; Fevola, Mizera, Telen 23
(See also) Gambuti, Kosower, Novichkov, Tancredi 23

Interesting Example



$$= \int_0^\infty \mathrm{d}x_0 \ldots \mathrm{d}x_7 \, \frac{\mathscr{U}(\mathbf{x})^{4\epsilon}}{\mathscr{F}(\mathbf{x};\mathbf{s})^{2+3\epsilon}} \delta(1-x_7)$$

$$\mathscr{U}(\alpha) = \alpha_0\alpha_2\alpha_4 + \alpha_0\alpha_2\alpha_5 + \alpha_0\alpha_2\alpha_6 + (29 \text{ terms})$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4 - \alpha_0\alpha_5)\,(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)\,(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_0} = s_{12}\;\alpha_5(\alpha_3\alpha_6 - \alpha_2\alpha_7) + s_{13}\;\alpha_3(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\vdots$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_7} = s_{12}\;\alpha_2(\alpha_1\alpha_4 - \alpha_0\alpha_5) + s_{13}\;\alpha_4(\alpha_1\alpha_2 - \alpha_0\alpha_3)$$

Can have a leading Landau singularity with *generic kinematics* (arbitrary s_{12}, s_{13}) when each factor of $\mathscr{F}$ vanishes!

Interesting Example

Let's try to compute this with sector decomposition (pySecDec)

Fails to find contour…

Contour Deformation

But for this class of examples $\mathscr{F}(\boldsymbol{\alpha})$ and all $\partial\mathscr{F}(\boldsymbol{\alpha})/\partial\alpha_i$ vanish at the same point inside the integration domain

→ *pinch* singularity

Example

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4-\alpha_0\alpha_5)\,(\alpha_3\alpha_6-\alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2-\alpha_0\alpha_3)\,(\alpha_5\alpha_6-\alpha_4\alpha_7),$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_0} = s_{12}\,\alpha_5(\alpha_3\alpha_6-\alpha_2\alpha_7) + s_{13}\,\alpha_3(\alpha_5\alpha_6-\alpha_4\alpha_7),$$

$$\vdots$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_7} = s_{12}\,\alpha_2(\alpha_1\alpha_4-\alpha_0\alpha_5) + s_{13}\,\alpha_4(\alpha_1\alpha_2-\alpha_0\alpha_3)$$

vanish for

$$\alpha_2 = \frac{\alpha_0\alpha_3}{\alpha_1}, \qquad \alpha_4 = \frac{\alpha_0\alpha_5}{\alpha_1}, \qquad \alpha_6 = \frac{\alpha_0\alpha_7}{\alpha_1}\,.$$

Resolution

The problem is that we have monomials with different signs…

Asy2.1 PreResolve->True

```
-bash                                                              ⌥⌘1
MACTHXJONES:fiesta sj$ cat diagram2636.m
Get["asy2.1.m"];

Print["Diagram2636"];
result = AlphaRepExpand[{k1,k2,k3}, {k1^2, k2^2,k3^2, (k1+p1)^2, (k2+p2)^2, (k3+p3)^2,
(k1+k2+k3)^2, (k1+k2+k3+p1+p2+p3)^2}, {p1^2->0, p2^2->0, p3^2->0, p1*p2->s12/2, p1*p3->
s13/2, p2*p3->-s12/2-s13/2}, {s12 -> 1, s13 -> -1/5}, PreResolve->True] (* 3-loop box *
)
Print[result];
Print["=========="];
Exit[];

MACTHXJONES:fiesta sj$ wolframscript -file diagram2636.m
Diagram2636
Asy2.1
Variables for UF: {k1, k2, k3, p1, p2, p3}
WARNING: preresolution failed
{}
==========
MACTHXJONES:fiesta sj$ 
```

Correctly identifies that iterated linear changes of variables are not sufficient to resolve the singularity and reports that pre-resolution has failed

Resolution

1) Rescale parameters to *linearise* singular surfaces

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4 - \alpha_0\alpha_5)\,(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)\,(\alpha_5\alpha_6 - \alpha_4\alpha_7)$$

$$\alpha_0 \to \alpha_0\alpha_1,\ \alpha_2 \to \alpha_2\alpha_3,\ \alpha_4 \to \alpha_4\alpha_5,\ \alpha_6 \to \alpha_6\alpha_7$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_4-\alpha_0)(\alpha_6-\alpha_2) - s_{13}(\alpha_2-\alpha_0)(\alpha_6-\alpha_4)\right]$$

2) Split the integral by imposing $\alpha_i \geq \alpha_j \geq \alpha_k \geq \alpha_l$

$$\begin{aligned}\alpha_0 &\to \alpha_0+\alpha_2+\alpha_4+\alpha_6,\\ \alpha_2 &\to \alpha_2+\alpha_4+\alpha_6,\\ \alpha_4 &\to \alpha_4+\alpha_6,\\ \alpha_6 &\to \alpha_6\end{aligned} \qquad +\text{perms}$$

$$\begin{aligned}\mathscr{F}_1(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_0+\alpha_2)(\alpha_2+\alpha_4) - s_{13}(\alpha_0)(\alpha_4)\right]\\ \mathscr{F}_2(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2)(\alpha_0+\alpha_2+\alpha_6) + s_{13}(\alpha_0)(\alpha_6)\right]\\ &\vdots\\ \mathscr{F}_{24}(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2+\alpha_4)(\alpha_4+\alpha_6) - s_{13}(\alpha_2)(\alpha_6)\right]\end{aligned}$$

All coefficients of s_{12}, s_{13} now have definite sign