New developments of the Local Analytic Sector subtraction

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Loop Summit 2025

Automation of subtraction at NLO based on

- Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer
- Catani-Seymour (CS) Dipole subtraction — Catani, Seymour, Dittmaier, et al.
- Nagy-Soper subtraction — Bevilacqua, Czakon, Kubocz, Worek

The long way to automation of subtraction at NNLO ...

- Antenna subtraction — Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, et al.
- CoLoRFul subtraction — Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.
- STRIPPER — Czakon et al.
- Nested soft-collinear subtraction — Melnikov et al.
- Local analytic sector subtraction — Magnea, Maina, Torrielli, U. et al.
- qT-slicing — Catani, Grazzini, et al.
- N-jettiness slicing — Boughezal, Petriello, et al.
- Projection to Born — Cacciari, Salam, Zanderighi, et al.
- Sector decomposition — Anastasiou, Binoth, et al.
- ε-prescription — Frixione, Grazzini
- Unsubtraction — Rodrigo et al.
- Geometric — Herzog

Features of the
Local Analytic Sector Subtraction

- Universal subtraction formula
- Sector functions to disentangle singularities
- Local counterterms defined by means of nested singular limits
- Phase-space mappings to achieve a complete analytic integration of counterterms

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Inspired by Catani et al. and Antenna Subtraction

State-of-the-art of the Local Analytic Sector Subtraction

- Initial and final state radiation in the massless case at NLO with damping factors
- Partial treatment of the massive cases at NLO
- Final state radiation in the massless case at NNLO
- Partial implementation in MADNkLO

TO-DO list

- Complete treatment of the massive cases at NLO
- Initial state radiation in the massless case at NNLO
- Treatment of the massive cases at NNLO
- Complete the implementation in MADNkLO

Today's talk is about ...

- Phase-space mappings to achieve a complete analytic integration of counterterms

with the goals ...

- Complete treatment of the massive cases at NLO
- Initial state radiation in the massless case at NNLO

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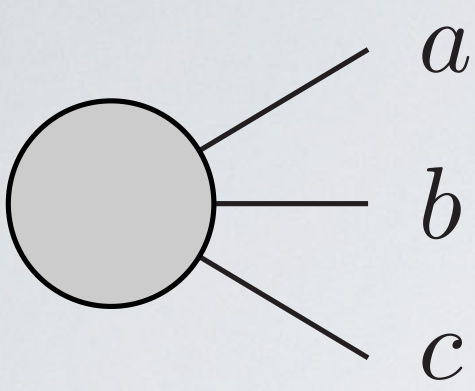
Let's start with ...

phase-space symmetries

Three-particle phase space

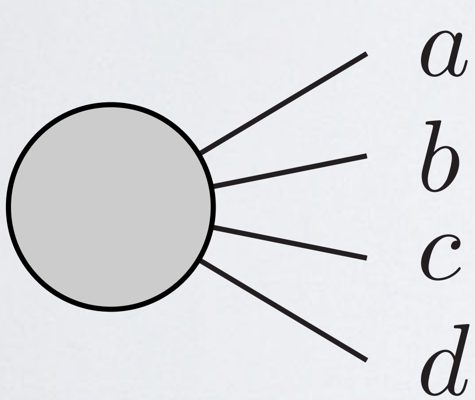


$$d\Phi_3 = (2\pi)^{3-2d}\,2^{-1-d}\,(q^2)^{\frac{2-d}{2}}\,d\Omega_{d-1}\,d\Omega_{d-2}$$
$$ds_{ab}\,ds_{ac}\,ds_{bc}\,(s_{ab}s_{ac}s_{bc})^{\frac{d-4}{2}}\,\delta(q^2-s_{ab}-s_{ac}-s_{bc})$$

Symmetric under:

- **permutations of** $k_a,\ k_b,\ k_c$

Four-particle phase space



$$d\Phi_4 = (2\pi)^{4-3d}\,2^{-1-2d}\,(q^2)^{\frac{6-d}{2}}$$
$$d\Omega_{d-1}\,d\Omega_{d-2}\,d\Omega_{d-3}\,ds_{ab}\,ds_{ac}\,ds_{bc}\,ds_{ad}\,ds_{bd}\,ds_{cd}$$
$$\lambda^{\frac{d-4}{2}}\,\Theta(\lambda)\,\delta(q^2-s_{ab}-s_{ac}-s_{bc}-s_{ad}-s_{bd}-s_{cd})$$

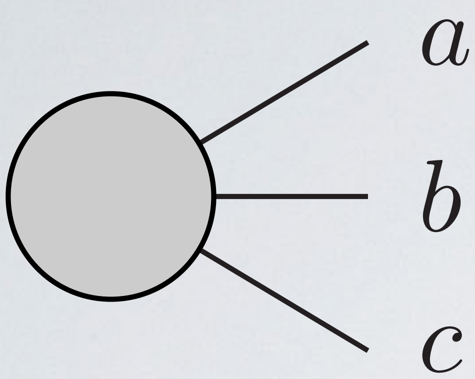
$$\lambda = \lambda(s_{ab}s_{cd}, s_{ac}s_{bd}, s_{ad}s_{bc})$$

Gehrmann-De Ridder, Gehrmann, Heinrich

Symmetric under:

- **permutations of** $k_a,\ k_b,\ k_c,\ k_d$
- $s_{ab} \leftrightarrow s_{cd},\ \ s_{ac} \leftrightarrow s_{bd},\ \ s_{ad} \leftrightarrow s_{bc},$

Catani's mapping
a,b,c finals



$$\int d\Phi_{n+1}\big(\{k\}\big) = \frac{\varsigma_{n+1}}{\varsigma_n} \int d\Phi_n\big(\{\bar{k}\}^{(abc)}\big) \int d\Phi_{\rm rad}^{(abc)}$$

$s_{ab} = y\, s$ $\quad s_{ac} = z(1-y)s$ $\quad s_{bc} = (1-z)(1-y)s$

$$\int d\Phi_{rad}^{(abc)} = G_1(\epsilon)\, s^{1-\epsilon} \int_0^1 dy \int_0^1 dz \left[y(1-y)^2 z(1-z)\right]^{-\epsilon} (1-y)$$

$s = \bar{s}_{bc}^{(abc)}$

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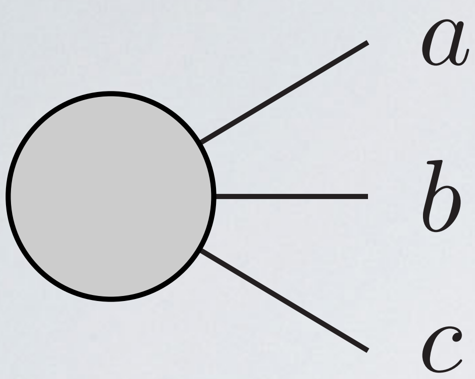

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permutations of $k_a,\ k_b,\ k_c$

$\boxed{k_a \leftrightarrow k_b}$ $\quad \boxed{k_b \leftrightarrow k_c}$ $\quad \boxed{k_a \leftrightarrow k_c}$

$z \leftrightarrow 1-z$ $\quad y \leftrightarrow z(1-y)$ $\quad y \leftrightarrow (1-z)(1-y)$

$\int d\Phi_{\rm rad}^{(abc)}$ is invariant $\Longrightarrow \int d\Phi_{\rm rad}^{(abc)} \equiv \int d\Phi_{\rm rad}^{\rm inv}$

Apply phase-space symmetries to not-symmetric phase spaces

Apply phase-space symmetries to not-symmetric phase spaces

or

"Reinvent the wheel"

Apply phase-space symmetries to not-symmetric phase spaces

or

"Reinvent the wheel"

- Start with a symmetric radiative phase-space

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- Start with a symmetric radiative phase-space
- Introduce an explicit mapping of momenta

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- Start with a symmetric radiative phase-space
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- Pick up a symmetry which simplifies phase-space integration

Apply phase-space symmetries to not-symmetric phase spaces

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“Reinvent the wheel”

- Start with a symmetric radiative phase-space
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- Translate the symmetry into transformation of integration variables

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- Pick up a symmetry which simplifies phase-space integration
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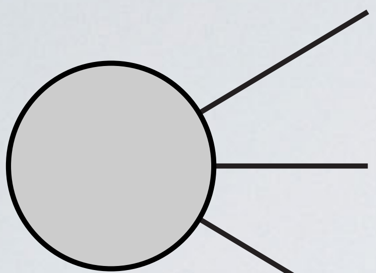
- Introduce an explicit mapping for the non-symmetric phase space
- Apply the variable transformation to the non-symmetric phase space

Catani's mapping
a,b,c finals - b,c massive



$$\int d\Phi_{n+1}(\{k\}) = \frac{\varsigma_{n+1}}{\varsigma_n} \int d\Phi_n(\{\bar{k}\}^{(abc)}) \int d\Phi^{(abc)}_{\mathrm{rad,MM}}$$

$$s_{ab} = y\, s \qquad s_{ac} = z(1-y)s \qquad s_{bc} = (1-z)(1-y)s$$

$$\int d\Phi^{(abc)}_{\mathrm{rad,MM}} = G_1(\epsilon)\left(\frac{s}{\sqrt{\lambda}}\right)^{1-2\epsilon} s\int_0^{y_+} dy \int_{z_-}^{z_+} dz \left[(z_+-z)(z-z_-)(1-y)^2(sy+m_b^2)\right]^{-\epsilon}(1-y)$$

$$y_+ = \frac{(Q-m_c)^2-m_b^2}{s} \qquad z_\pm = \frac{sy}{2(sy+m_b^2)}\left(1 \pm \frac{\sqrt{s^2(1-y)^2-4m_c^2(sy+m_b^2)}}{s(1-y)}\right)$$

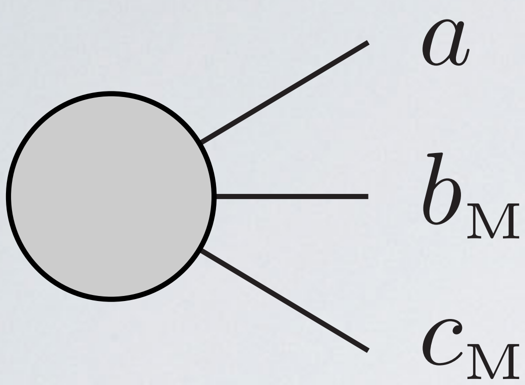
$$Q^2 = s + m_b^2 + m_c^2 \qquad \lambda = s^2 - 4m_b^2 m_c^2 \qquad s = \bar{s}^{(abc)}_{bc}$$

- Soft massive kernel is symmetric for $k \leftrightarrow l$ $(k,l = b,c)$

$$\mathcal{E}^{(i)}_{kl} = f_i^g\left(\frac{s_{kl}}{s_{ik}s_{il}} - \frac{m_k^2}{s_{ik}^2} - \frac{m_l^2}{s_{il}^2}\right)$$

- Phase-space not symmetric for permutations of k_a, k_b, k_c
- But we can use variable transformations

Catani's mapping
a,b,c finals - b,c massive



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$$Q^2 = s + m_b^2 + m_c^2 \qquad \lambda = s^2 - 4m_b^2 m_c^2 \qquad s = \bar{s}^{(abc)}_{bc}$$

$$\boxed{k_a \leftrightarrow k_c} \qquad y \leftrightarrow (1-z)(1-y)$$

$$\int d\Phi^{(abc)}_{\rm rad,MM} = G_1(\epsilon)\left(\frac{s}{\sqrt{\lambda}}\right)^{1-2\epsilon} s\int_{y_-}^{1} dy \int_{z_-}^{z_+} dz \left[(z_+-z)(z-z_-)(1-y)^2(sy+m_b^2+m_c^2)\right]^{-\epsilon}(1-y)$$

$$y_- = \frac{2m_b m_c}{s} \qquad z_\pm = \frac{sy+2m_c^2 \pm \sqrt{s^2y^2-4m_b^2m_c^2}}{2(sy+m_b^2+m_c^2)}$$

Soft massive counterterm

$$I^{ikl}_{\rm s,MM} = \mathcal{N}_1 \int d\Phi^{(ikl)}_{\rm rad,MM}\, \mathcal{E}^{(i)}_{kl} = f^g_i\, \mathcal{N}_1 \int d\Phi^{(ikl)}_{\rm rad,MM} \left(\frac{s_{kl}}{s_{ik}s_{il}} - \frac{m_k^2}{s_{ik}^2} - \frac{m_l^2}{s_{il}^2} \right)$$

Basic mapping

- By hand computation in 8 pages
- Cumbersome result

$$\begin{aligned} I^{ikl}_{\rm s,MM} = \; & f^g_i \frac{\alpha_{\rm s}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\Bigg\{\left(\frac{1}{\epsilon} - \ln\frac{\lambda^2}{m_k^2 m_l^2 Q^2 s}\right)\left(\frac{s}{2\sqrt{\lambda}}\ln\eta + 1\right) - \frac{s+2m_k^2}{2\sqrt{\lambda}}\ln\eta + 2 \\ & + \frac{s}{\sqrt{\lambda}}\Bigg[-\mathbf{Li}_2(1-\eta) - \frac{\ln^2\eta}{4} + 2\frac{\sqrt{\lambda}}{s} + 2\,\frac{m_k^2 - m_l^2}{s}\ln\frac{1-\beta}{1+\beta} \\ & + \mathbf{Li}_2\left(\frac{-2\beta}{1-\beta}\right) - \mathbf{Li}_2\left(\frac{2\beta}{1+\beta}\right) + \mathbf{Li}_2\left(\frac{1+\beta}{2}\right) - \mathbf{Li}_2\left(\frac{1-\beta}{2}\right) \\ & - \mathbf{Li}_2\left(\frac{1-\alpha}{1-\beta}\right) + \mathbf{Li}_2\left(\frac{1-\alpha}{1+\beta}\right) - \mathbf{Li}_2\left(\frac{1-\beta}{1+\alpha}\right) + \mathbf{Li}_2\left(\frac{1+\beta}{1+\alpha}\right) \\ & + \mathbf{Li}_2\left(\frac{-2\beta}{\alpha-\beta}\right) - \mathbf{Li}_2\left(\frac{2\beta}{\alpha+\beta}\right) - \ln\frac{1-\beta^2}{2(1+\alpha)}\ln\frac{1-\beta}{1+\beta}\Bigg]\Bigg\} \end{aligned}$$

$$\alpha = \frac{Q - m_l}{Q + m_l} \qquad \beta = \sqrt{\frac{(Q-m_l)^2 - m_k^2}{(Q+m_l)^2 - m_k^2}} \qquad \eta = \frac{s - \sqrt{\lambda}}{s + \sqrt{\lambda}}$$

$$Q^2 = s + m_b^2 + m_c^2 \qquad \lambda = s^2 - 4m_b^2 m_c^2 \qquad s = \bar{s}^{(abc)}_{bc}$$

Soft massive counterterm

$$I^{ikl}_{\mathrm{s,MM}} = \mathcal{N}_1 \int d\Phi^{(ikl)}_{\mathrm{rad,MM}}\, \mathcal{E}^{(i)}_{kl} = f^g_i\, \mathcal{N}_1 \int d\Phi^{(ikl)}_{\mathrm{rad,MM}} \left(\frac{s_{kl}}{s_{ik} s_{il}} - \frac{m_k^2}{s_{ik}^2} - \frac{m_l^2}{s_{il}^2} \right)$$

"Reinvented" mapping

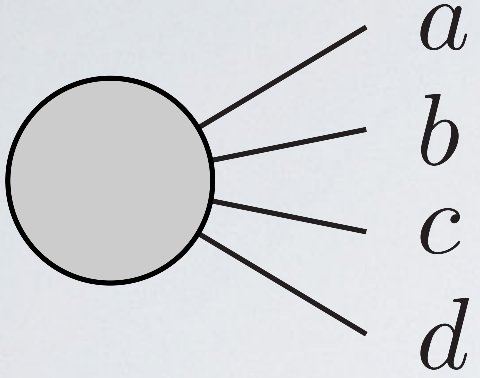
- By hand computation in 5 pages
- Compact result

$$\begin{aligned} I^{ikl}_{\mathrm{s,MM}} &= f^g_i \frac{\alpha_s}{2\pi} \left(\frac{s}{\mu^2}\right)^{-\epsilon} \Bigg\{ \left[\frac{1}{\epsilon} + \ln \frac{Q^2\, s}{(s-2m_k m_l)^2}\right] \left[\frac{s}{2\sqrt{\lambda}} \ln \eta + 1\right] + 2\, \frac{\sqrt{\lambda}}{s} \left[2 + \ln \frac{m_k m_l}{s+2m_k m_l}\right] \\ &\quad - \frac{s}{\sqrt{\lambda}} \left[\mathbf{Li}_2(1-\eta_k) + \mathbf{Li}_2(1-\eta_l) + \frac{1}{8} \ln^2 \frac{\eta_k}{\eta_l} + \frac{1}{8} \ln^2 \eta\right] - \frac{m_k^2 - m_l^2}{2\sqrt{\lambda}} \ln \frac{\eta_k}{\eta_l} \\ &\quad + 2\, \mathbf{Li}_2(\eta) + \frac{1}{2} \ln^2 \eta - \ln \frac{m_k m_l}{s-2m_k m_l} \ln \eta - \frac{Q^2}{2\sqrt{\lambda}} \ln \eta - 2\,\zeta_2 + \mathcal{O}(\epsilon) \Bigg\} \end{aligned}$$

$$\eta_{k,l} = \frac{s + 2m_{k,l}^2 - \sqrt{\lambda}}{s + 2m_{k,l}^2 + \sqrt{\lambda}} \qquad \eta = \frac{s - \sqrt{\lambda}}{s + \sqrt{\lambda}}$$

$$Q^2 = s + m_b^2 + m_c^2 \qquad \lambda = s^2 - 4 m_b^2 m_c^2 \qquad s = \bar{s}^{(abc)}_{bc}$$

Double Catani's mapping
a,b,c,d finals



$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n} \int d\Phi_n(\{\bar{k}\}^{(abc;bcd)}) \int d\Phi^{(abc;bcd)}_{rad,2}$$

$$s_{ab} = y\,y'\,s \qquad s_{ac} = y\,(1-y')\,z'\,s \qquad s_{bc} = y\,(1-y')\,(1-z')\,s$$

$$s_{ad} = (1-y)\Big[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

$$s_{bd} = (1-y)\Big[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

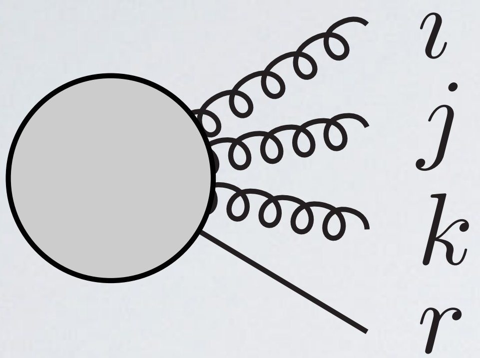
$$s_{cd} = (1-y)\,(1-y')\,(1-z)\,s$$

$$\int d\Phi^{(abc,bcd)}_{rad,2} = G_2(\epsilon)\,s^{2-2\epsilon}\int_0^1 dy \int_0^1 dz \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \Big[w'(1-w')\Big]^{-\epsilon-\frac{1}{2}} \qquad s = \bar{s}^{(abc,bcd)}_{cd}$$

$$\times \Big[y'(1-y')^2 z'(1-z')y^2(1-y)^2 z(1-z)\Big]^{-\epsilon}(1-y')y(1-y)$$

Final-state radiation
i,j,k,r finals

Sample integrand: Collinear 3-gluon kernel



$$\frac{P^{(3g)}_{ijk}}{s^2_{ijk}} = C_A^2 \left\{ \frac{1-\epsilon}{4s^2_{ij}} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{3}{4}\frac{1-\epsilon}{s^2_{ijk}} \right.$$
$$+ \frac{1}{2s_{ij}s_{ik}} \left[\frac{2z_i z_j z_{ik}(1-2z_k)}{z_k z_{ij}} + \frac{1+2z_i+2z_i^2}{z_{ik}z_{ij}} + \frac{1-2z_i z_{jk}}{z_j z_k} + 2z_j z_k + z_i(1+2z_i) - 4 \right]$$
$$\left. + \frac{1}{s_{ijk}s_{ij}} \left[4\frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1-z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}z_k + \frac{3}{2} \right] \right\} + (\text{ 5 perm. })$$

$$s_{ijk} = s_{ij} + s_{ik} + s_{jk} \qquad z_h = \frac{s_{hr}}{s_{ir}+s_{jr}+s_{kr}} \qquad z_{hl} = \frac{s_{hr}+s_{lr}}{s_{ir}+s_{jr}+s_{kr}} \qquad h, l = i, j, k$$

Integrands are rational functions of invariants

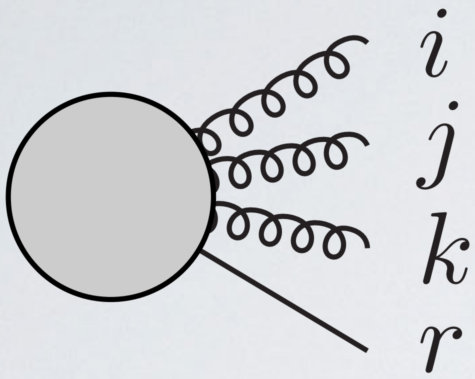
- Numerators: harmless for integration
- Denominators:

Invariants without r : $s_{ij}, \quad s_{ik}, \quad s_{jk}, \quad s_{ij}+s_{ik}+s_{jk}$

Invariants with r : $s_{ir}, \quad s_{jr}, \quad s_{kr}, \quad s_{ir}+s_{jr}+s_{kr}, \quad s_{ir}+s_{jr}, \quad s_{ir}+s_{kr}, \quad s_{jr}+s_{kr}$

Final-state radiation
i,j,k,r finals

Sample integrand: Collinear 3-gluon kernel



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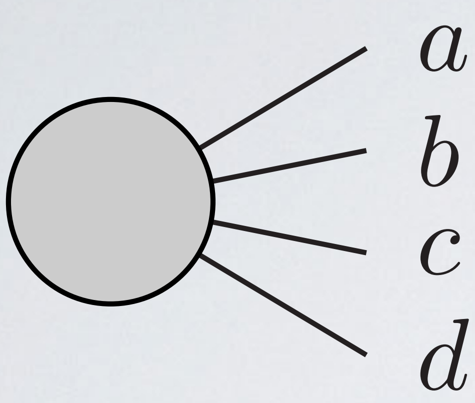
Final-state radiation
a,b,c,d finals

General case



- Denominators:

Invariants without d **:** $s_{ab}, \quad s_{ac}, \quad s_{bc}, \quad s_{ab}+s_{ac}+s_{bc}$

Invariants with d **:** $s_{ad}, \quad s_{bd}, \quad s_{cd}, \quad s_{ad}+s_{bd}+s_{cd},$

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Double Catani's mapping
a,b,c,d finals

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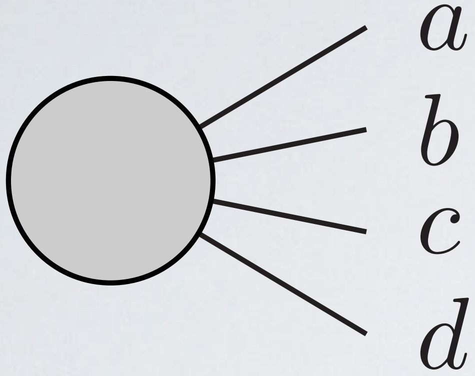

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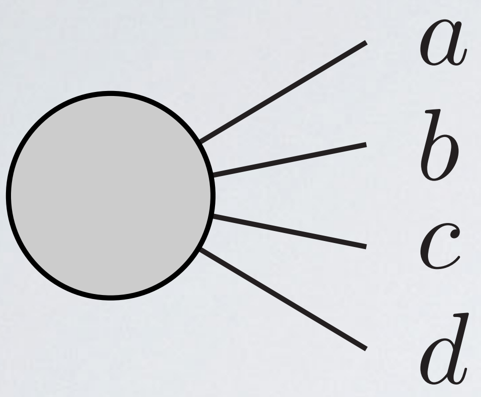
$$s_{cd} = (1-y)\,(1-y')\,(1-z)\, s$$

Double Catani's mapping a,b,c,d finals

Simple factors

General case



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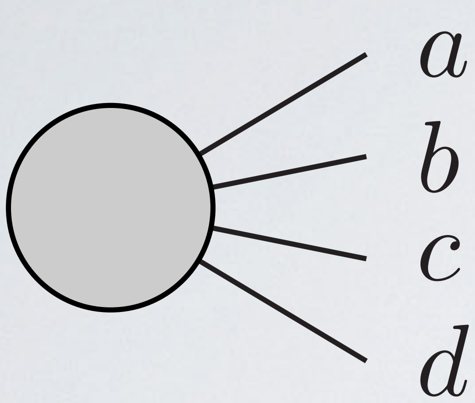
$$s_{cd} = (1-y)\,(1-y')\,(1-z)\,s$$

$$s_{ab}+s_{ac}+s_{bc} = y\,s \qquad s_{cd}+s_{ad}+s_{bd} = (1-y)s$$

Double Catani's mapping
a,b,c,d finals

Simple factors

General case



- Denominators:

Invariants without d: $s_{ab}, \quad s_{ac}, \quad s_{bc}, \quad s_{ab}+s_{ac}+s_{bc}$

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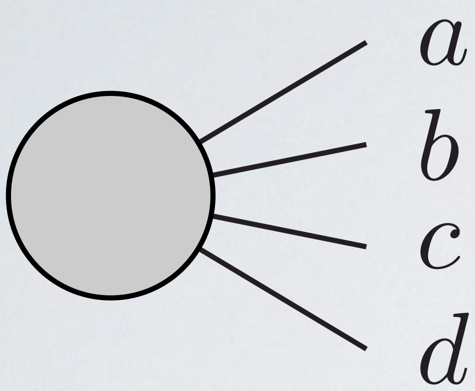
Double Catani's mapping
a,b,c,d finals

Simple factors

Simple hypergeometric integrals

General case



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a,b,c,d finals

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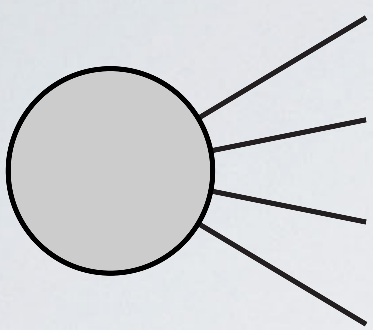

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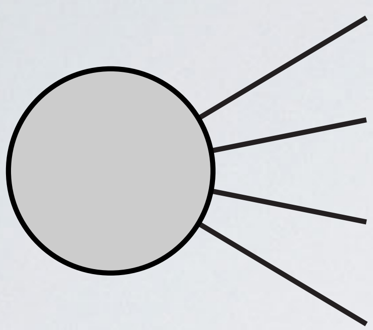
Double Catani's mapping
a,b,c,d finals

Simple factors

Simple hypergeometric integrals

Not-obvious hypergeometric integrals

General case



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Inspired by Gehrmann-De Ridder PhD thesis

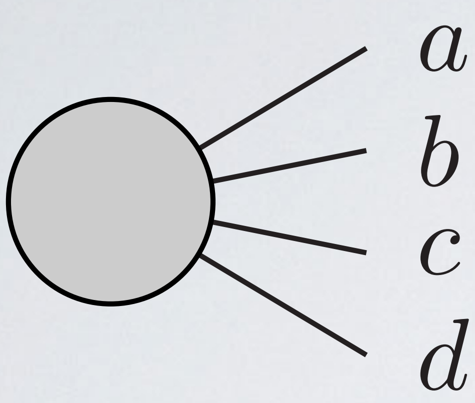
Double Catani's mapping
a,b,c,d finals

Simple factors

Simple hypergeometric integrals

Not-obvious hypergeometric integrals

General case



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Double Catani's mapping
a,b,c,d finals

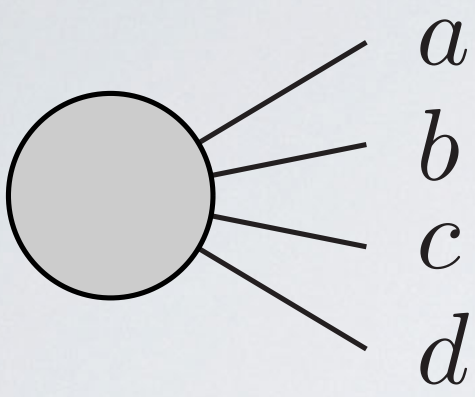
Simple factors

Simple hypergeometric integrals

Not-obvious hypergeometric integrals

hard integrals

General case



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Double Catani's mapping
a,b,c,d finals

Simple factors

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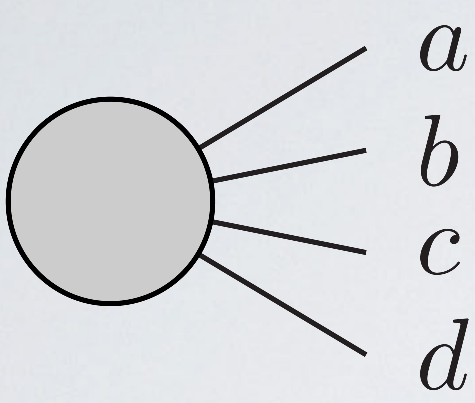
hard integrals

General case



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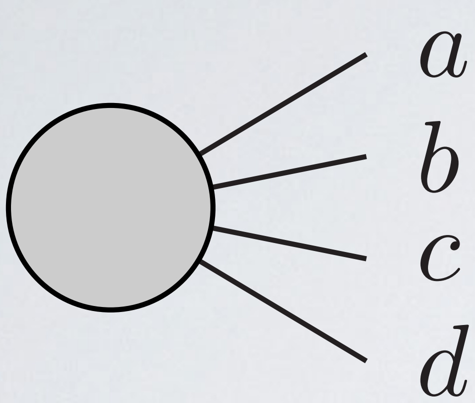
Don't solve hard integrals, when you have symmetries

Double Catani's mapping
a,b,c,d finals

General case



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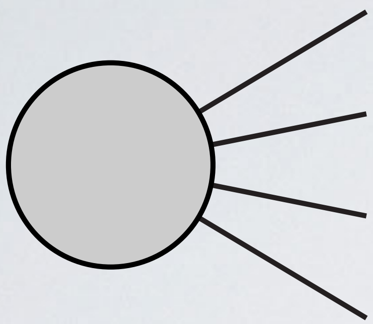
Double Catani's mapping
a,b,c,d finals

Simple factors

General case



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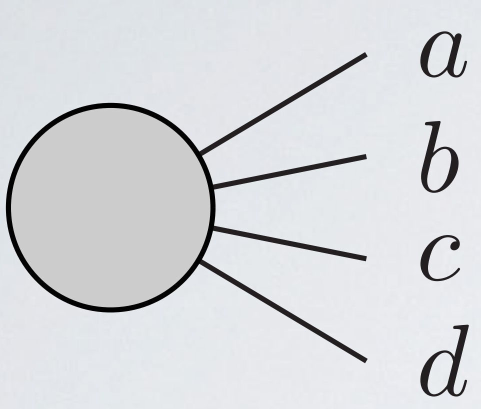
$$s_{ab}+s_{ac}+s_{bc} = (1-y)s \qquad s_{ad}+s_{bd}+s_{cd} = y\,s \qquad s_{ad}+s_{bd} = y(1-y')s$$

Double Catani's mapping a,b,c,d finals

Simple factors

General case



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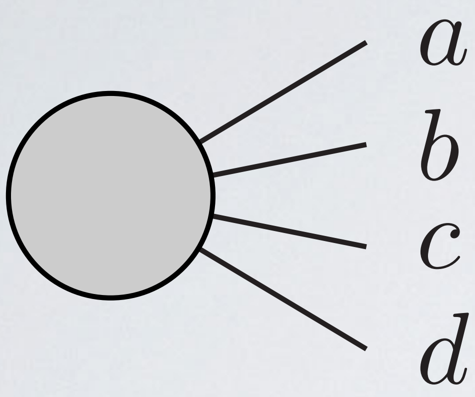
Double Catani's mapping
a,b,c,d finals

Simple factors

Simple hypergeometric integrals

General case



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$$\int_0^1 dz \frac{(z')^{l-1-\epsilon}(1-z')^{n-1-\epsilon}}{1-z'+y'z'} = \frac{\Gamma(l-\epsilon)\Gamma(n-\epsilon)}{\Gamma(l+n-2\epsilon)} \, {}_2F_1\left(1, l-\epsilon, l+n-2\epsilon, 1-y'\right)$$

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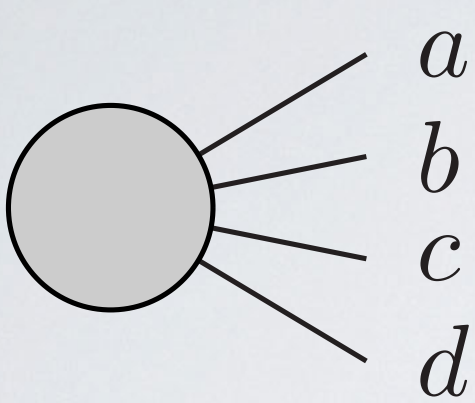

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a,b,c,d finals

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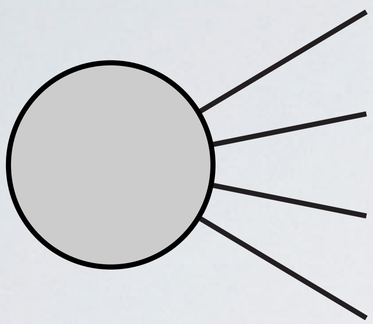

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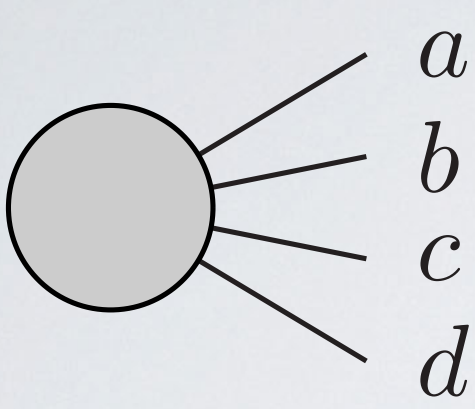

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a,b,c,d finals

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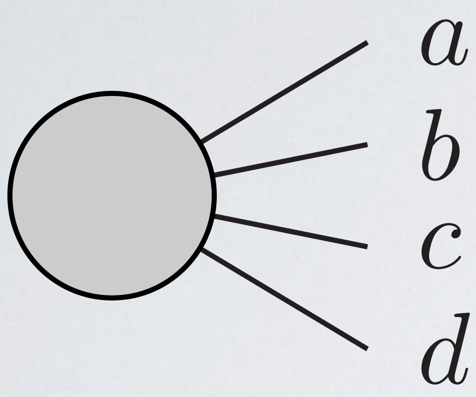
hypergeometric integrals ?

General case



- Denominators:

Basic mapping

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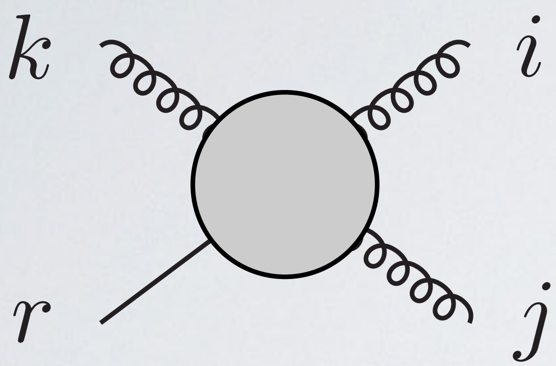
Initial-state radiation
i,j finals - k,r initials

Sample integrand: Collinear 3-gluon kernel



$$\frac{P^{(3g)}_{ij,k}}{s^2_{ij,k}} = \left.\frac{P^{(3g)}_{ijk}}{s^2_{ijk}}\right|_{\substack{k_k \to -k_k \\ k_r \to -k_r}}$$

Initial-state radiation

i,j finals - k,r initials

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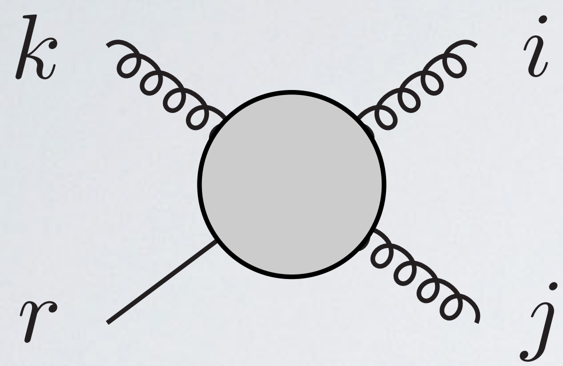

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Integrands are rational functions of invariants

- Numerators: harmless for integration
- Denominators:

Invariants without r : $\quad s_{ij}, \quad s_{ik}, \quad s_{jk}, \quad s_{ik}+s_{jk}-s_{ij}$

Invariants with r : $\quad s_{ir}, \quad s_{jr}, \quad s_{kr}, \quad s_{kr}-s_{ir}-s_{jr}, \quad s_{ir}+s_{jr}, \quad s_{kr}-s_{ir}, \quad s_{kr}+s_{jr}$

Initial-state radiation
i,j finals - k,r initials

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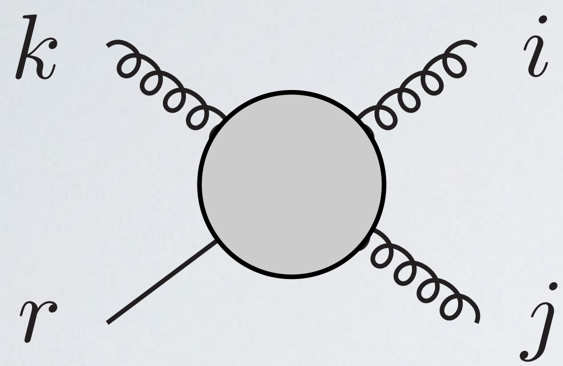


- Denominators:

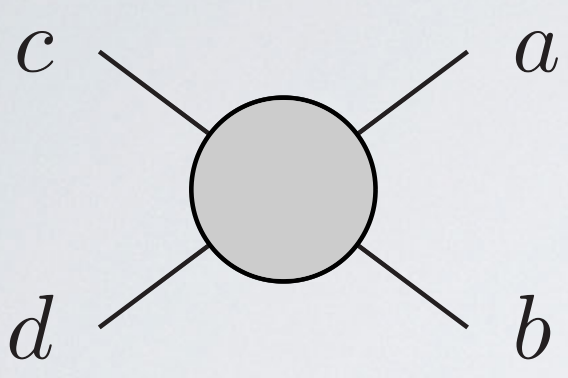
Invariants without r : $s_{ij}, \quad s_{ik}, \quad s_{jk}, \quad s_{ik}+s_{jk}-s_{ij}$

Invariants with r : $s_{ir}, \quad s_{jr}, \quad s_{kr}, \quad s_{kr}-s_{ir}-s_{jr},$

$s_{ir}+s_{jr}, \quad s_{kr}-s_{ir}, \quad s_{kr}-s_{jr}$

Initial-state radiation
a,b finals - c,d initials

General case



- Denominators:

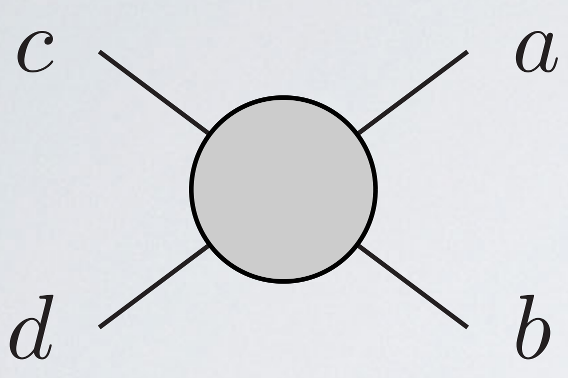
Invariants without d **:** s_{ab}, s_{ac}, s_{bc}, $s_{ac}+s_{bc}-s_{ab}$

Invariants with d **:** s_{ad}, s_{bd}, s_{cd}, $s_{cd}-s_{ad}-s_{bd}$,

$s_{ad}+s_{bd}$, $s_{cd}-s_{ad}$, $s_{cd}-s_{bd}$

Initial-state radiation
a,b finals - c,d initials

General case



- Denominators:

Invariants without d : $s_{ab},\quad s_{ac},\quad s_{bc},\quad s_{ac}+s_{bc}-s_{ab}$

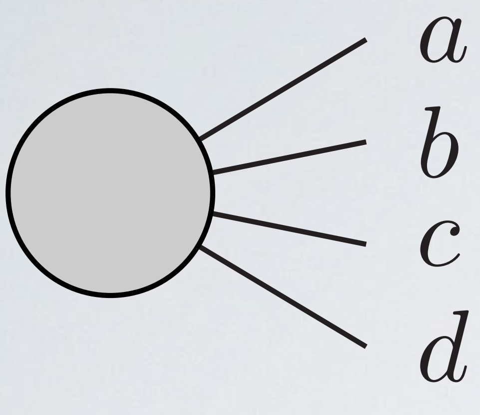
Invariants with d : $s_{ad},\quad s_{bd},\quad s_{cd},\quad s_{cd}-s_{ad}-s_{bd},$

$s_{ad}+s_{bd},\quad s_{cd}-s_{ad},\quad s_{cd}-s_{bd}$

- The phase-space is not symmetric for $s_{ab} \leftrightarrow s_{cd}$, $s_{ac} \leftrightarrow s_{bd}$, $s_{bc} \leftrightarrow s_{ad}$
- But we can use variable transformations

Go back to

Double Catani's mapping
a,b,c,d finals



$$s_{ab} = y\,y'\,s \qquad s_{ac} = y\,(1-y')\,z'\,s \qquad s_{bc} = y\,(1-y')\,(1-z')\,s$$

$$s_{ad} = (1-y)\Big[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

$$s_{bd} = (1-y)\Big[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

$$s_{cd} = (1-y)\,(1-y')\,(1-z)\,s$$

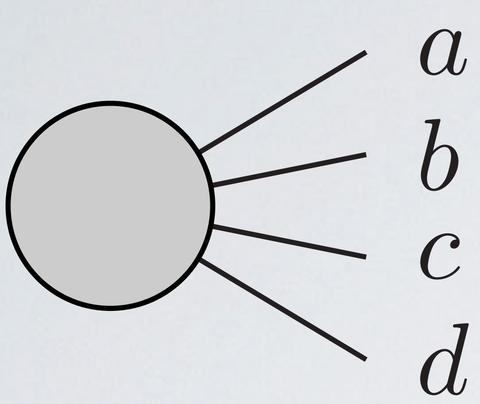
$$s = \bar{s}_{cd}^{(abc,bcd)}$$

$$\int d\Phi_{rad,2}^{(abc,bcd)} = G_2(\epsilon)\,s^{2-2\epsilon}\int_0^1 dy \int_0^1 dz \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \,\Big[w'(1-w')\Big]^{-\epsilon-\frac{1}{2}}$$

$$\times \Big[y'(1-y')^2 z'(1-z')y^2(1-y)^2 z(1-z)\Big]^{-\epsilon}(1-y')y(1-y)$$

Go back to

Double Catani's mapping a,b,c,d finals



$$s_{ab} = y\, y'\, s \qquad s_{ac} = y\,(1-y')\,z'\,s \qquad s_{bc} = y\,(1-y')\,(1-z')\,s$$

$$s_{ad} = (1-y)\Big[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\, s$$

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$$s_{cd} = (1-y)\,(1-y')\,(1-z)\,s$$

$$s = \bar{s}_{cd}^{(abc,bcd)}$$

$$\int d\Phi_{rad,2}^{(abc,bcd)} = G_2(\epsilon)\, s^{2-2\epsilon} \int_0^1 dy \int_0^1 dz \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \Big[w'(1-w')\Big]^{-\epsilon-\frac{1}{2}}$$
$$\times \Big[y'(1-y')^2 z'(1-z') y^2 (1-y)^2 z(1-z)\Big]^{-\epsilon} (1-y')y(1-y)$$

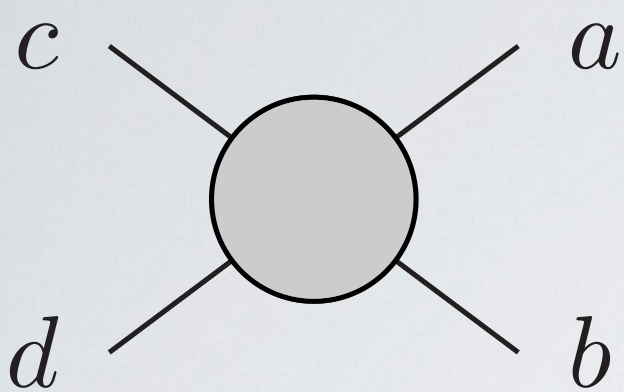
$$s_{ab} \leftrightarrow s_{cd}, \qquad s_{ac} \leftrightarrow s_{bd}, \qquad s_{ad} \leftrightarrow s_{bc}$$

Derive variable transformation

$$\begin{cases} y \leftrightarrow 1-y \\ y' \leftrightarrow (1-y')(1-z) \\ (1-y')z' \leftrightarrow y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)} \end{cases}$$

$$\int d\Phi_{\mathrm{rad},2}^{(abc,bcd)} \text{ is invariant} \Longrightarrow \int d\Phi_{\mathrm{rad},2}^{(abc,bcd)} \equiv \int d\Phi_{\mathrm{rad},2}^{\mathrm{inv}}$$

Double Catani's mapping
a,b finals - c,d initials



$$\int d\Phi_{n+2}(\{k\}) \;=\; \frac{\varsigma_{n+2}}{\varsigma_n}\int\!\!\int d\Phi_n\big(x\,k_c,k_d;\{\bar{k}\}^{(ab,c;b,cd)}\big)\,d\Phi^{(ab,c;b,cd)}_{rad,2}$$

$$x \;=\; \frac{s_{ab,cd}}{s} \qquad\qquad s_{ab,cd} \;=\; s_{ab}-s_{ac}-s_{bc}+s_{cd}-s_{ad}-s_{bd}$$

$$s_{ab}=(1-y)(1-y')z\,s \qquad s_{ac}=(1-y)(1-y')(1-z')s \qquad s_{bc}=(1-y)(1-y')z's$$

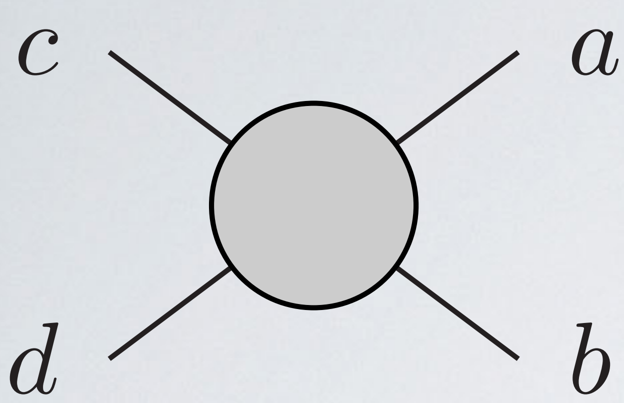
$$s_{ad} \;=\; \left[y'(1-z')(1-z)+z'z-2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{bd} \;=\; \left[y'z'(1-z)+(1-z')z+2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{cd} \;=\; s$$

$$\int d\Phi^{(ab,c;b,cd)}_{rad,2} \;=\; \int d\Phi^{\text{inv}}_{rad,2}\left(\frac{s_{cd}-s_{ad}-s_{bd}}{s_{ab,cd}}\right)^{1-2\epsilon}\frac{s_{cd}-s_{ad}-s_{bd}}{s}$$

Double Catani's mapping
a,b finals - c,d initials



$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n} \int\int d\Phi_n\left(x\,k_c, k_d; \{\bar{k}\}^{(ab,c;b,cd)}\right) d\Phi_{rad,2}^{(ab,c;b,cd)}$$

$$x = \frac{s_{ab,cd}}{s}$$

$$s_{ab,cd} = s_{ab} - s_{ac} - s_{bc} + s_{cd} - s_{ad} - s_{bd}$$

$$s_{ab} = (1-y)(1-y')z\,s \qquad s_{ac} = (1-y)(1-y')(1-z')s \qquad s_{bc} = (1-y)(1-y')z's$$

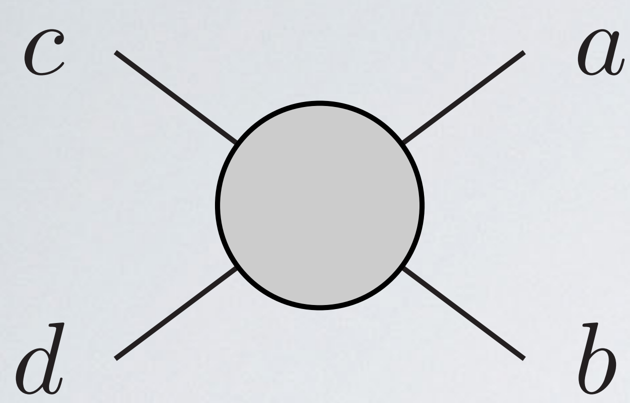
$$s_{ad} = \left[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{bd} = \left[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{cd} = s$$

$$\int d\Phi_{rad,2}^{(ab,c;b,cd)} = \int d\Phi_{rad,2}^{\text{inv}} \left(\frac{s_{cd} - s_{ad} - s_{bd}}{s_{ab,cd}}\right)^{1-2\epsilon} \frac{s_{cd} - s_{ad} - s_{bd}}{s}$$

Double Catani's mapping
a,b finals - c,d initials



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$$x \;=\; \frac{s_{ab,cd}}{s} \qquad\qquad s_{ab,cd} \;=\; s_{ab}-s_{ac}-s_{bc}+s_{cd}-s_{ad}-s_{bd}$$

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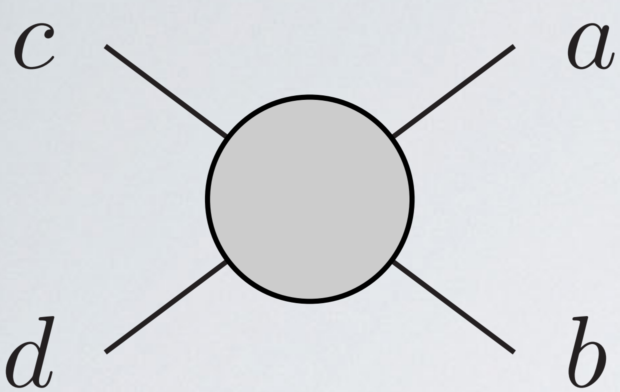
$$s_{cd} \;=\; s$$

$$\int d\Phi^{(ab,c;b,cd)}_{rad,2} \;=\; \int d\Phi^{\rm inv}_{rad,2}\left(\frac{s_{cd}-s_{ad}-s_{bd}}{s_{ab,cd}}\right)^{1-2\epsilon}\frac{s_{cd}-s_{ad}-s_{bd}}{s}$$

Double Catani's mapping
a,b finals - c,d initials



$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n} \int\int d\Phi_n\left(x\, k_c, k_d; \{\bar{k}\}^{(ab,c;b,cd)}\right) d\Phi_{rad,2}^{(ab,c;b,cd)}$$

$$x = \frac{s_{ab,cd}}{s} \qquad s_{ab,cd} = s_{ab} - s_{ac} - s_{bc} + s_{cd} - s_{ad} - s_{bd}$$

$$s_{ab} = (1-y)(1-y')z\, s \qquad s_{ac} = (1-y)(1-y')(1-z')s \qquad s_{bc} = (1-y)(1-y')z's$$

$$s_{ad} = \left[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{bd} = \left[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{cd} = s$$

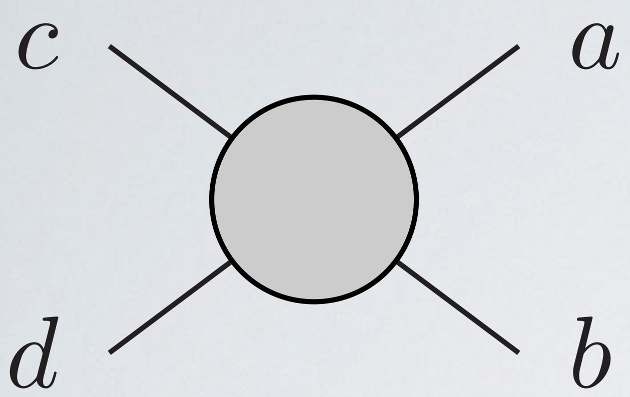
$$\int d\Phi_{rad,2}^{(ab,c;b,cd)} = \int d\Phi_{rad,2}^{\text{inv}} \left(\frac{s_{cd} - s_{ad} - s_{bd}}{s_{ab,cd}}\right)^{1-2\epsilon} \frac{s_{cd} - s_{ad} - s_{bd}}{s}$$

Apply variable transformation

$$\begin{cases} y' \leftrightarrow (1-y')(1-z) \\ (1-y')z' \leftrightarrow y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)} \end{cases}$$

and get ...

Double Catani's mapping
a,b finals - c,d initials



"Reinvented" mapping

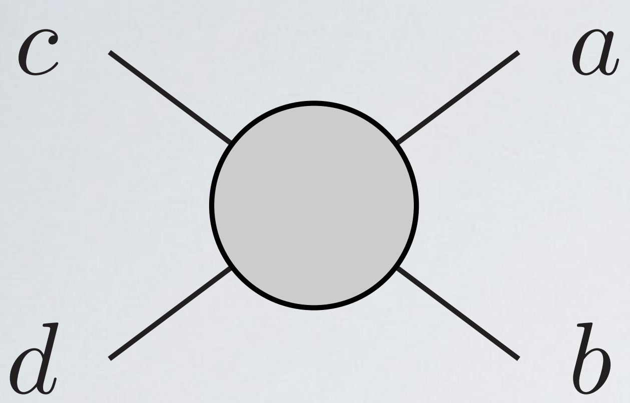
$$s_{ab} = (1-y)\,(1-y')\,z\,s$$

$$s_{ac} = (1-y)\left[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right] s$$

$$s_{bc} = (1-y)\left[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right] s$$

$$s_{ad} = (1-y')\,(1-z')\,s \qquad s_{bd} = (1-y')\,z'\,s \qquad s_{cd} = s$$

Double Catani's mapping
a,b finals - c,d initials



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$$s_{bc} = (1-y)\Big[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

$$s_{ad} = (1-y')\,(1-z')\,s \qquad s_{bd} = (1-y')\,z'\,s \qquad s_{cd} = s$$

Very similar to the final-state one

$$s_{ab} = (1-y)\,(1-y')\,(1-z)\,s$$

$$s_{ac} = (1-y)\Big[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

$$s_{bc} = (1-y)\Big[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

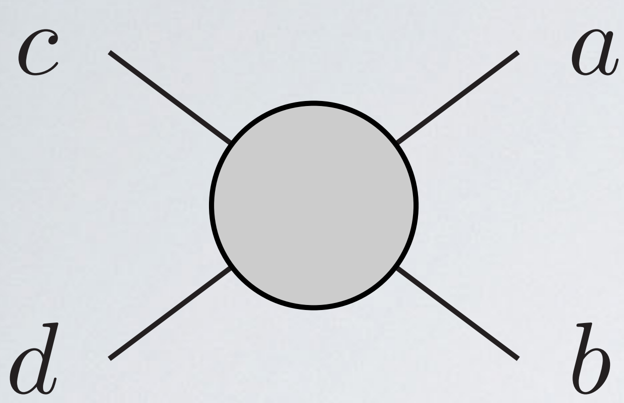
$$s_{ad} = y\,(1-y')\,(1-z')\,s \qquad s_{bd} = y\,(1-y')\,z'\,s \qquad s_{cd} = y\,y'\,s$$

Double Catani's mapping
a,b finals - c,d initials

Simple factors

Simple hypergeometric integrals

Not-obvious hypergeometric integrals



"Reinvented" mapping

$$s_{ab} = (1-y)\,(1-y')\,z\,s$$

$$s_{ac} = (1-y)\left[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right] s$$

$$s_{bc} = (1-y)\left[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right] s$$

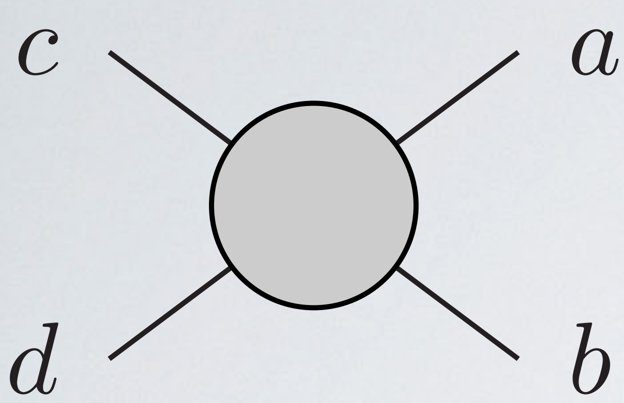
$$s_{ad} = (1-y')\,(1-z')\,s \qquad s_{bd} = (1-y')\,z'\,s \qquad s_{cd} = s$$

- Denominators:

Invariants without d **:** s_{ab}, s_{ac}, s_{bc}, $s_{ac}+s_{bc}-s_{ab}$

Invariants with d **:** s_{ad}, s_{bd}, s_{cd}, $s_{cd}-s_{ad}-s_{bd}$, $s_{ad}+s_{bd}$, $s_{cd}-s_{ad}$, $s_{cd}-s_{bd}$

Double Catani's mapping
a,b finals - c,d initials



$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n} \int\int d\Phi_n\left(x\,k_c, k_d; \{\bar{k}\}^{(ab,c;b,cd)}\right) d\Phi_{rad,2}^{(ab,c;b,cd)}$$

"Reinvented" mapping

$$x = \frac{s_{ab,cd}}{s}$$

$$s_{ab,cd} = s_{ab} - s_{ac} - s_{bc} + s_{cd} - s_{ad} - s_{bd}$$

$$s_{ab} = (1-y)\,(1-y')\,z\,s$$

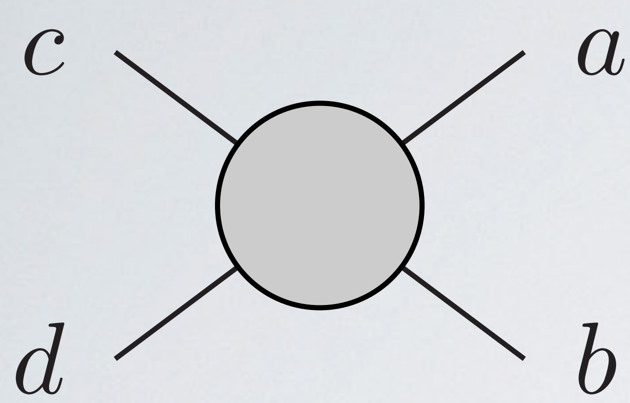
$$s_{ac} = (1-y)\Big[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

$$s_{bc} = (1-y)\Big[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\,s$$

$$s_{ad} = (1-y')\,(1-z')\,s \qquad s_{bd} = (1-y')\,z'\,s \qquad s_{cd} = s$$

$$\int d\Phi_{rad,2}^{(ab,c;b,cd)} = \int d\Phi_{rad,2}^{\text{inv}} \left(\frac{s_{cd} - s_{ad} - s_{bd}}{s_{ab,cd}}\right)^{1-2\epsilon} \frac{s_{cd} - s_{ad} - s_{bd}}{s}$$

Double Catani's mapping
a,b finals - c,d initials



$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n}\int\!\!\int d\Phi_n\left(x\,k_c, k_d; \{\bar{k}\}^{(ab,c;b,cd)}\right) d\Phi^{(ab,c;b,cd)}_{rad,2}$$

"Reinvented" mapping

$$s_{ab} = (1-y)\,(1-y')\,z\,s \qquad s_{ab,cd} = s_{ab}-s_{ac}-s_{bc}+s_{cd}-s_{ad}-s_{bd}$$

$$s_{ac} = (1-y)\left[y'(1-z')(1-z)+z'z-2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{bc} = (1-y)\left[y'z'(1-z)+(1-z')z+2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\right]s$$

$$s_{ad} = (1-y')\,(1-z')\,s \qquad s_{bd} = (1-y')\,z'\,s \qquad s_{cd} = s$$

$$\int d\Phi^{(ab,c;b,cd)}_{rad,2} = \int d\Phi^{\rm inv}_{rad,2}\left(\frac{s_{cd}-s_{ad}-s_{bd}}{s_{ab,cd}}\right)^{1-2\epsilon}\frac{s_{cd}-s_{ad}-s_{bd}}{s}$$

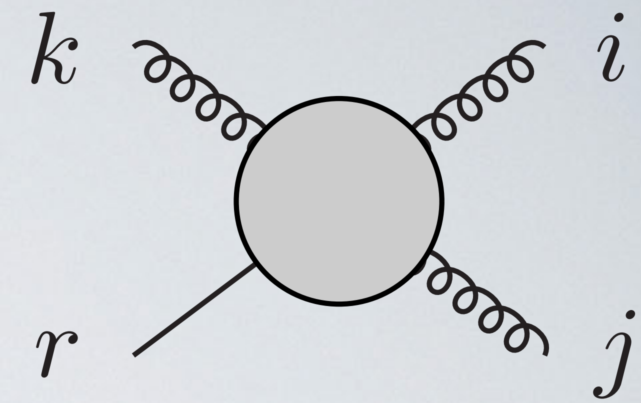
Simple expression for

$$x = \frac{s_{ab,cd}}{s} = y\,y'$$

Last variable transformation

$$\begin{cases} y = \dfrac{x}{1-(1-x)x'} \\ y' = 1-(1-x)x' \end{cases}$$

Initial-state radiation



(1 permutation)

Sample integral: Collinear 3-gluon kernel

$$\mathcal{N}_1^2 \int d\Phi^{(ij,k;j,kr)}_{rad,2} \, \frac{P^{(3g)}_{ij,k}}{s^2_{ij,k}} = \left(\frac{\alpha_S}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2 \left[I^{(3g)}_{\rm cc,II} + \int_0^1 \frac{dx}{x} \, J^{(3g)}_{\rm cc,II}(x) \right]$$

$$I^{(3g)}_{\rm cc,II} = \frac{3}{4}\frac{1}{\epsilon^4} + \frac{11}{12}\frac{1}{\epsilon^3} + \left(\frac{67}{36} - \frac{11}{12}\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{101}{27} - \frac{77}{72}\pi^2 - \frac{67}{4}\zeta_3\right)\frac{1}{\epsilon} + \frac{607}{81} - \frac{469}{216}\pi^2 - \frac{341}{18}\zeta_3 - \frac{19}{1440}\pi^4 + \mathcal{O}(\epsilon)$$

$$\begin{aligned}
J^{(3g)}_{\rm cc,II} = {} & \frac{1}{\epsilon^3}\left\{ -3\left[\frac{x}{1-x}\right]_+ - \frac{1-x}{x} - \frac{3}{2}x(1-x) \right\} \\
& + \frac{1}{\epsilon^2}\left\{ 12\left[\frac{x\ln(1-x)}{1-x}\right]_+ - \left[\frac{x}{1-x}\right]_+\left[\frac{11}{3} + \ln x\right] - \frac{1-x}{x}\left[\frac{25}{6} - 4\ln(1-x) - 2\ln x\right] - x(1-x)\left[\frac{19}{4} - 6\ln(1-x) + \ln x\right] \right\} \\
& + \frac{1}{\epsilon}\left\{ -24\left[\frac{x\ln^2(1-x)}{1-x}\right]_+ + \left[\frac{x\ln(1-x)}{1-x}\right]_+\left[\frac{44}{3} + 4\ln x\right] - \left[\frac{x}{1-x}\right]_+\left[\frac{67}{9} - \frac{11}{3}\pi^2 + \ln x - \ln^2 x\right] \right. \\
& \quad + \frac{1-x}{x}\left[-\frac{167}{18} + \frac{7}{6}\pi^2 + \frac{50}{3}\ln(1-x) + \frac{25}{3}\ln x - 8\ln^2(1-x) + 3\ln^2 x - 8\ln x\ln(1-x) + 10\,\mathrm{Li}_2\left(\frac{x-1}{x}\right) \right] \\
& \quad \left. - x(1-x)\left[\frac{205}{12} - \frac{11}{6}\pi^2 - 19\ln(1-x) + \frac{15}{4}\ln x + 12\ln^2(1-x) - 4\ln x\ln(1-x) + 2\,\mathrm{Li}_2\left(\frac{x-1}{x}\right)\right] - \frac{1}{6} + 3(3+x)\ln x \right\} \\
& + 32\left[\frac{x\ln^3(1-x)}{1-x}\right]_+ - \left[\frac{x\ln^2(1-x)}{1-x}\right]_+\left[\frac{88}{3} + 8\ln x\right] + \left[\frac{x\ln(1-x)}{1-x}\right]_+\left[\frac{268}{9} - \frac{44}{3}\pi^2 + 4\ln x - 4\ln^2 x\right] \\
& + \left[\frac{x}{1-x}\right]_+\left[-\frac{404}{27} + \frac{77}{18}\pi^2 + 67\zeta_3 - \left(2 - \frac{7}{6}\pi^2\right)\ln x + \frac{4}{3}\ln^2 x + \frac{17}{3}\ln^3 x + \left(\frac{5}{3} + 9\ln x\right)\mathrm{Li}_2\left(\frac{x-1}{x}\right) - \mathrm{Li}_3\left(\frac{x-1}{x}\right) + 10\,\mathrm{S}_{12}\left(\frac{x-1}{x}\right) \right] \\
& + \frac{1-x}{x}\left[-\frac{1117}{54} + \frac{175}{36}\pi^2 + \frac{62}{3}\zeta_3 + \left(\frac{334}{9} - \frac{14}{3}\pi^2\right)\ln(1-x) + \left(\frac{167}{9} - \frac{7}{3}\pi^2\right)\ln x - \frac{100}{3}\ln^2(1-x) + \frac{25}{2}\ln^2 x - \frac{100}{3}\ln x\ln(1-x) \right. \\
& \quad + \frac{32}{3}\ln^3(1-x) + \frac{31}{3}\ln^3 x + 16\ln x\ln^2(1-x) - 12\ln^2 x\ln(1-x) \\
& \quad \left. + \left(\frac{125}{3} - 40\ln(1-x) + 20\ln x\right)\mathrm{Li}_2\left(\frac{x-1}{x}\right) + 40\,\mathrm{Li}_3\left(\frac{x-1}{x}\right) + 34\,\mathrm{S}_{12}\left(\frac{x-1}{x}\right) \right] \\
& + x(1-x)\left[-\frac{512}{9} + \frac{137}{24}\pi^2 + \frac{67}{2}\zeta_3 + \left(\frac{205}{3} - \frac{22}{3}\pi^2\right)\ln(1-x) - \left(\frac{181}{12} - \frac{7}{6}\pi^2\right)\ln x - 38\ln^2(1-x) - \frac{7}{8}\ln^2 x + 15\ln x\ln(1-x) \right. \\
& \quad \left. + 16\ln^3(1-x) + \ln^3 x - 8\ln x\ln^2(1-x) - \left(\frac{15}{2} - 8\ln(1-x) - \ln x\right)\mathrm{Li}_2\left(\frac{x-1}{x}\right) - 9\,\mathrm{Li}_3\left(\frac{x-1}{x}\right) - 2\,\mathrm{S}_{12}\left(\frac{x-1}{x}\right) \right] \\
& - \frac{11}{18} + \frac{2}{3}\ln(1-x) + \left(\frac{59}{3} + \frac{27}{2}x\right)\ln x - 12(3+x)\ln x\ln(1-x) + \frac{7}{2}(3+x)\ln^2 x + 12(3+x)\mathrm{Li}_2\left(\frac{x-1}{x}\right) + \mathcal{O}(\epsilon)
\end{aligned}$$

Preliminar

State-of-the-art of the Local Analytic Sector Subtraction

- Initial and final state radiation in the massless case at NLO with damping factors
- Partial treatment of the massive cases at NLO
- Final state radiation in the massless case at NNLO
- Partial implementation in MADNkLO

TO-DO list

- Complete treatment of the massive cases at NLO
- Initial state radiation in the massless case at NNLO
- Treatment of the massive cases at NNLO
- Complete the implementation in MADNkLO

State-of-the-art of the Local Analytic Sector Subtraction

- Initial and final state radiation in the massless case at NLO with damping factors
- Partial treatment of the massive cases at NLO
- Final state radiation in the massless case at NNLO
- Partial implementation in MADNkLO

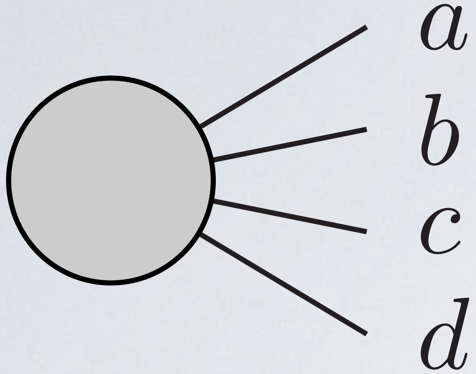
TO-DO list

- Complete treatment of the massive cases at NLO — DONE
- Initial state radiation in the massless case at NNLO — ALMOST DONE
TO DO: K_{RV} and checks
- Treatment of the massive cases at NNLO — THINKABLE
- Complete the implementation in MADNkLO

Thanks for your attention

Back-up slides

Double Catani's mapping
a,b,c,d finals



$$s_{ab} = y\, y'\, s \qquad s_{ac} = y(1-y')z'\, s \qquad s_{bc} = y(1-y')(1-z')\, s$$

$$s_{ad} = (1-y)\Big[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\, s$$

$$s_{bd} = (1-y)\Big[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)}\Big]\, s$$

$$s_{cd} = (1-y)(1-y')(1-z)\, s$$

$$\int d\Phi^{(abc,bcd)}_{rad,2} = G_2(\epsilon)\int_0^1 dy \int_0^1 dz \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \Big[w'(1-w')\Big]^{-\epsilon-\frac{1}{2}}$$
$$\times \Big[y'(1-y')^2 z'(1-z')y^2(1-y)^2 z(1-z)\Big]^{-\epsilon}(1-y')y(1-y)$$

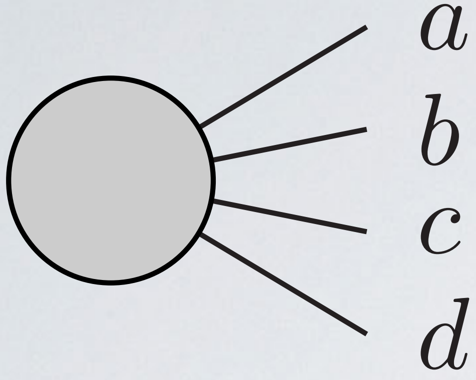
- **permutations of** $k_a,\ k_b,\ k_c,\ k_d$

$$\boxed{k_a \leftrightarrow k_b}$$

$$z' \leftrightarrow 1-z'$$

$$\int d\Phi^{(abc,bcd)}_{\mathrm{rad},2} \quad \text{is invariant}$$

Double Catani's mapping
a,b,c,d finals



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$$\times\;\Big[y'(1-y')^2 z'(1-z')y^2(1-y)^2 z(1-z)\Big]^{-\epsilon}(1-y')y(1-y)$$

- **permutations of** $k_a,\ k_b,\ k_c,\ k_d$

$$\boxed{k_b \leftrightarrow k_c}$$

$$\begin{cases} y' \;\leftrightarrow\; (1-y')z' \\ (1-y')(1-z) \;\leftrightarrow\; y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)} \end{cases}$$

$$\int d\Phi^{(abc,bcd)}_{\mathrm{rad},2} \quad \text{is invariant}$$