Two-loop helicity amplitudes for $pp \to tt+j$ scattering

Simon Badger (University of Turin)

based on work with:

Becchetti, Brancaccio, Czakon, Hartanto, Poncelet, Zoia

Loop Summit 2, Cadenabbia

21st July 2025

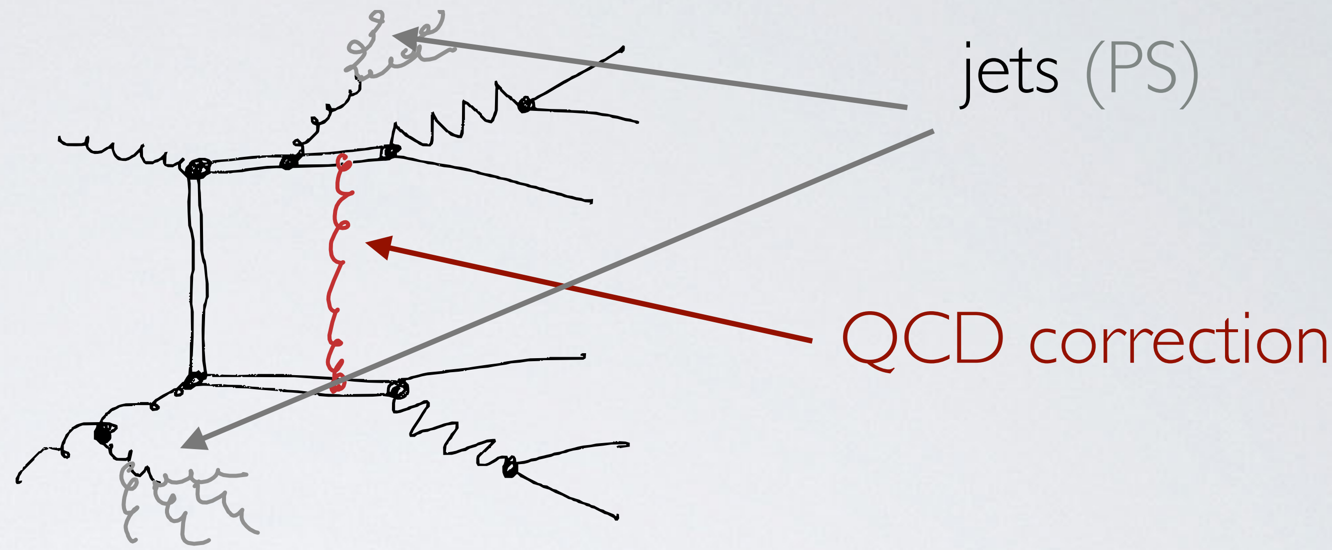
FARE
RICERCA IN ITALIA
FRAMEWORK PER L'ATTRAZIONE E IL RAFFORZAMENTO
DELLE ECCELLENZE PER LA RICERCA IN ITALIA

Precision top quark measurements

LHC is a top factory

main production: pp➝tt

~40% events associated with p_T>40GeV jet



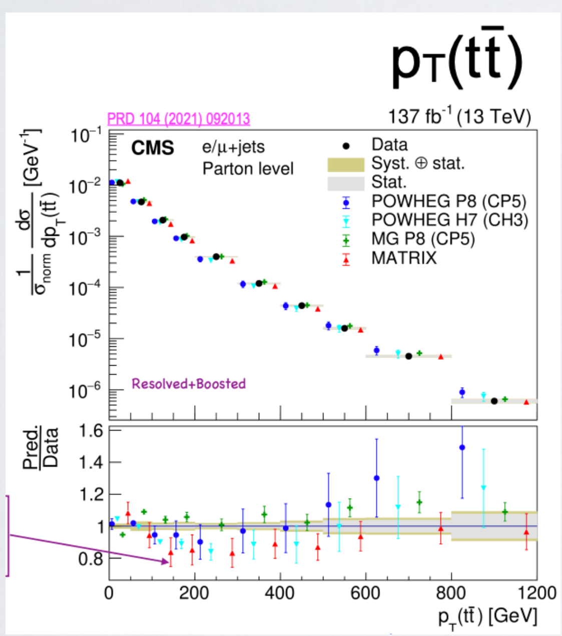
Current state-of-the-art for pp➝tt

NNLO QCD + NLO EW

Czakon et al. 1705.04105

+ resummations (e.g soft, threshold)

need NNLO pp➝tt+j



Behnke, talk at TOP24

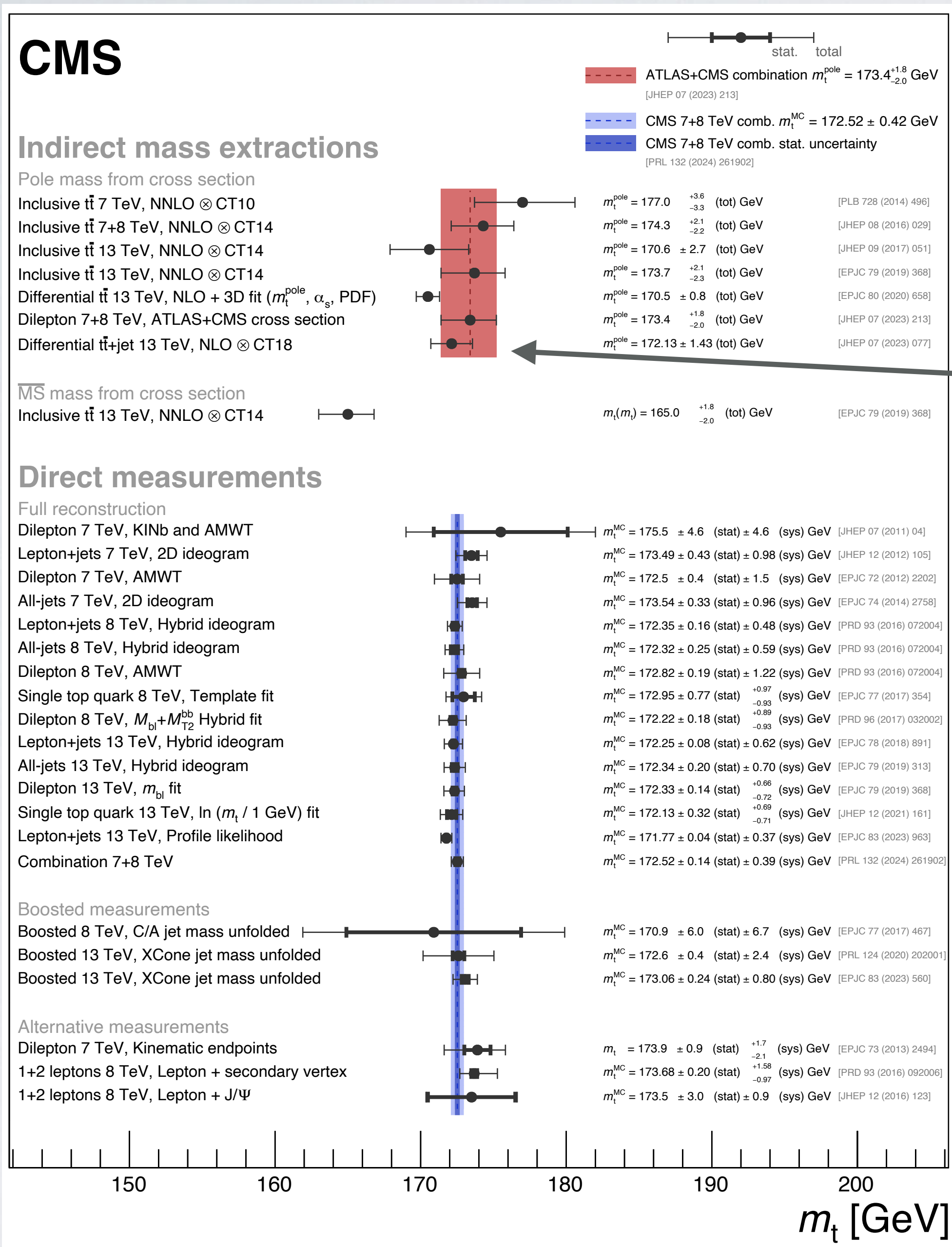
⇒ Both NNLO and POW exhibit some wiggles around the data

⇒ Need NNNLO

top mass and vacuum stability

$$\mathcal{R}(m_t^{\text{pole}},\rho_s) = \frac{1}{\sigma_{t\bar{t}+1\text{-jet}}}\frac{d\sigma_{t\bar{t}+1\text{-jet}}}{d\rho_s}(m_t^{\text{pole}},\rho_s) \qquad \rho_s = \frac{2m_0}{\sqrt{s_{t\bar{t}j}}}.$$

$$\mathcal{S}(\rho_s) = \sum_{\Delta=\pm5-10\text{ GeV}} \frac{|\mathcal{R}(170\text{ GeV},\rho_s) - \mathcal{R}(170\text{ GeV}+\Delta,\rho_s)|}{2|\Delta|\mathcal{R}(170\text{ GeV},\rho_s)}$$



mass extraction summary 2403.01313

m_t from normalised ttj distributions

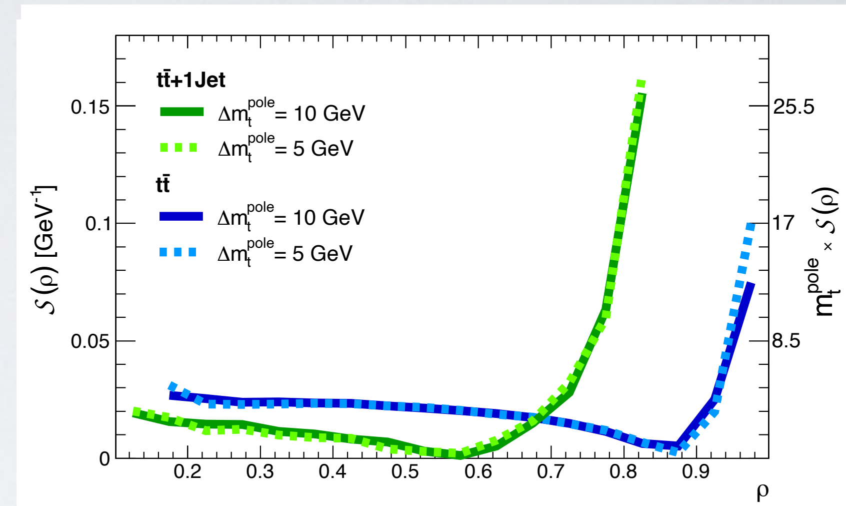
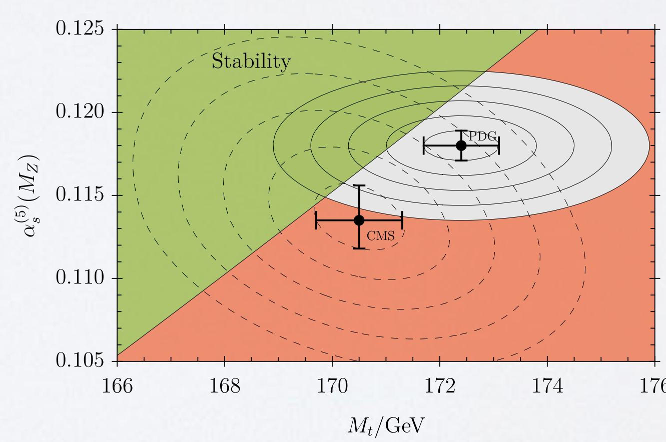


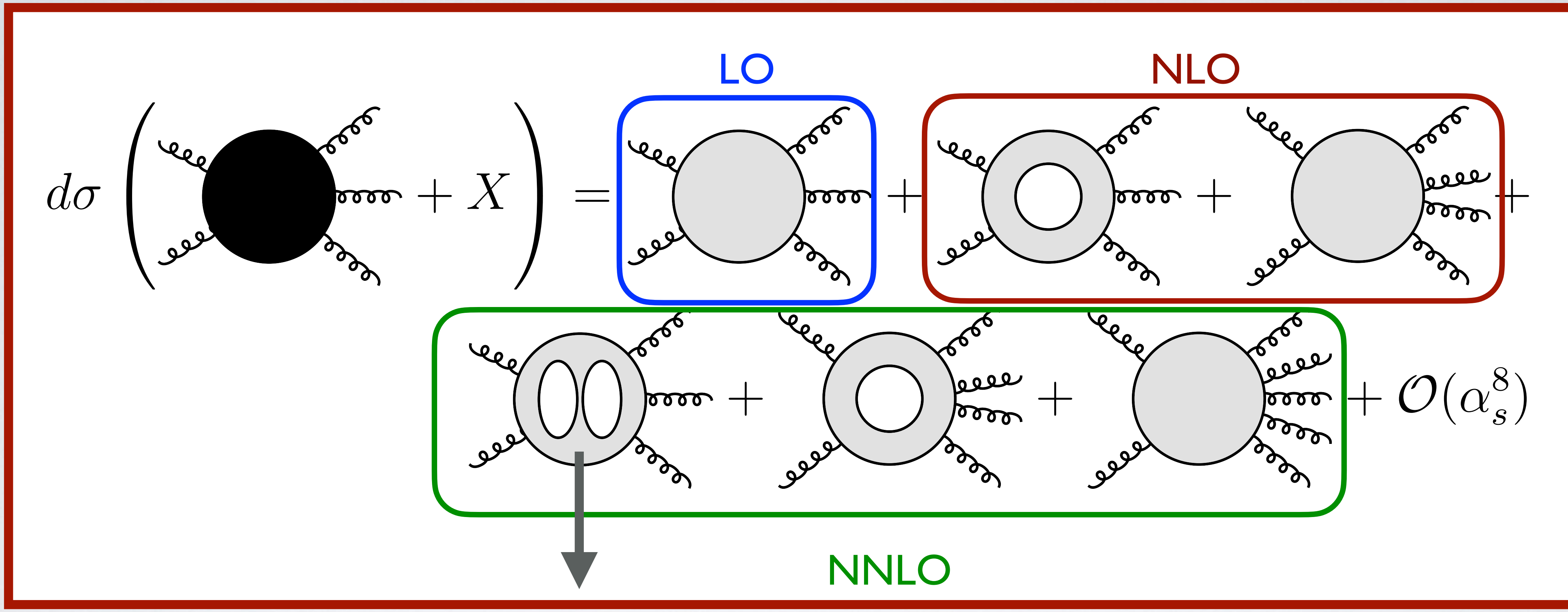
Fig. 5. The sensitivity $\mathcal{S}(\rho_s)$ of $\mathcal{R}$ with respect to the top-quark mass as defined in Eq. (5).

Alioli et al. 1303.6415



impact for vacuum stability studies

Hiller et al. 2401.08811



missing

Theory predictions for tt+(massless) signals (i.e. tt+j / tt+γ) present a **serious** challenge

⇒ we need to develop new calculational techniques

[parallel on-going theory efforts for ttH, ttW, ttZ] bare amplitudes

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

integrals/special functions

finite remainders

$$F^{(L)} = A^{(L),4-2\epsilon} - \sum_{k=1}^{L} I^{(k),4-2\epsilon} A^{(L-k),4-2\epsilon}$$

universal IR/UV poles

[Catani (1998)][Becher, Neubert (2009)]
[Magnea, Gardi (2009)]

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon,\{p\})\,\mathcal{F}_i(\epsilon,\{p\})$$

integrals/special functions

many ways to evaluate Feynman integrals:

Sector decomposition (numerical)	[Binoth, Heinrich….]
Differential equations	[Kotikov, Gehrmann, Remiddi, Henn…]
Mellin-Barnes	[Smirnov, Tausk, Czakon, Kosower,…]
Direct parametric integration	[Panzer, Borinsky…]

`secdec,Fiesta` `Hyperint` `mbtools, ambre` `feyntrop`

must have fast and stable numerical implementation

canonical form differential equations

Henn '13

$$\partial_x M_i = \epsilon A_{ij}(x) M_j$$

$d = 4 - 2\epsilon$

M_i integral basis usually called 'master integrals' (MIs)

A_{ij} matrix depends on kinematic invariants

if $dA = \sum_i d\log(W_i)$ it is (relatively) easy to define a special function basis from iterated integrals

e.g. 1M **pentagon functions**: Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia [2306.15431]

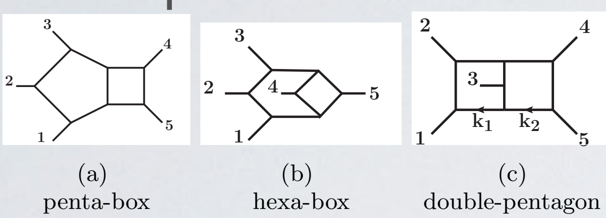
W_i alphabet

automated approaches to find canonical bases (Fuchsia, epsilon, initial, dlog…) often not sufficient to handle complicated kinematics

Analytic forms for DEQ matrices, A can be obtained via IBP reduction over finite fields

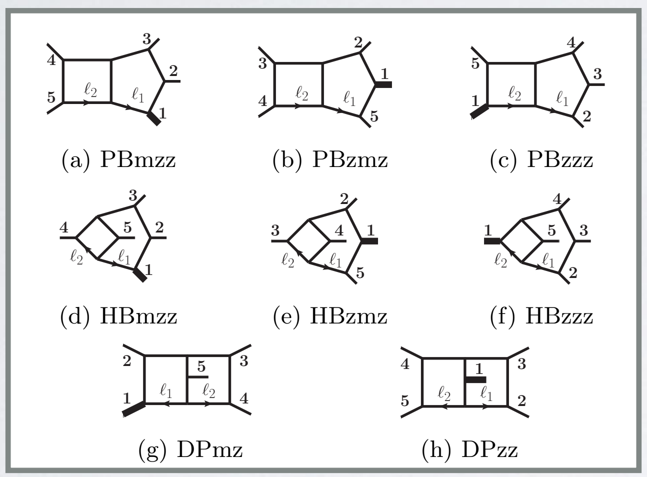
canonical form differential equations: 2→3 configurations

5pt all massless



[Abreu,Chicherin,Dixon,Gehrmann,Henn ,Herrmann,LoPresti,Papadopoulos,Page, Sotnikov,Tomassini,Wasser,Wever,Zeng, Zhang,Zoia(2015-2020)]

5pt 1-mass

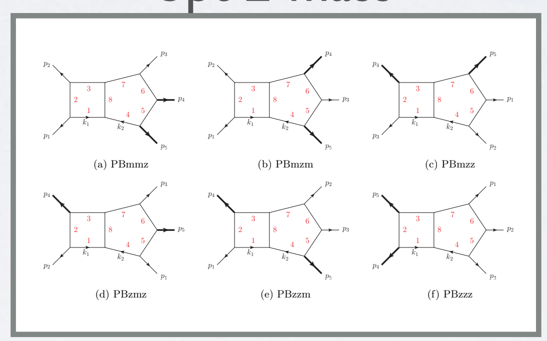


[Abreu,Canko,Chicherin,Ita,Kardos,Moriello, Page,Papadopoulos,Smirnov,Sotnikov,Syrrakos, Tomassini,Tschernow,Wever,Zeng,Zoia (2015-2023)]

fast and stable function basis: pentagon functions

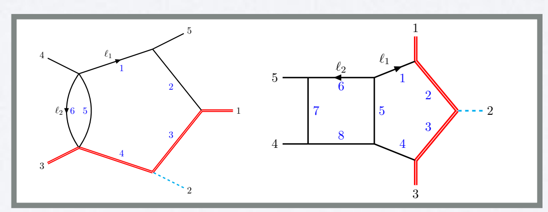
[Gehrmann,Henn,Lo Presti(2018)] [Chicherin,Sotnikov(2020)][Chicherin, Sotnikov,Zoia(2021)][Abreu,Chicherin,Ita, Page,Sotnikov,Tschernow,Zoia(2023)]

5pt 2-mass



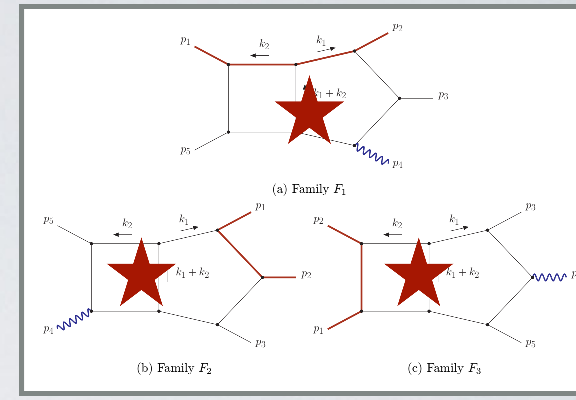
[Jiang,Liu,Xu,Yang(2024)] [Abreu,Chicherin,Sotnikov,Zoia(2024)]

ttH nf



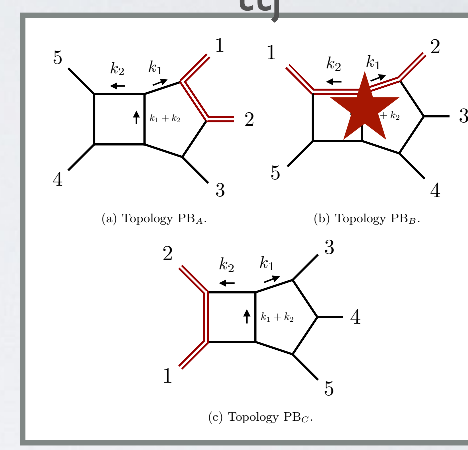
[Febres Cordero,Figueiredo,Kraus,Page,Reina(2023)]

ttW



[Becchetti,Canko,Chestnov ,Peraro,Pozzoli,Zoia(2025)]

ttj



[Badger,Becchetti,Chaubey,Marzucca(2022)] [Badger,Becchetti,Giraudo,Zoia(2024)] [Becchetti,Dlapa,Zoia(2025)]

dlog not always possible: nested square roots and elliptic structures

finite fields for IBPs/amplitudes

von Manteuffel, Schabinger '14

Peraro '16,'19

$$A^{(L),4-2\epsilon} = \sum_i \boxed{c_i(\epsilon, \{p\})} \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

avoid large intermediate algebraic expressions using modular arithmetic

great for solving IBP systems e.g.
FINRED [von Manteuffel],
KIRA+FIREFLY [Maierhoefer, Usovitsch, Uwer, Klappert, Lange]

useful features:

- reconstruct exact results using chinese remainder theorem
- extremely efficient solutions to large linear systems
- reconstruct rational functions using Newton/Thiele interpolation
- modular approach in FiniteFlow allows us to link different algorithms and avoid large intermediate steps

finite fields for IBPs/amplitudes

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon,\{p\}) \mathcal{F}_i(\epsilon,\{p\})$$

rational functions

our approach…

QGRAF + FORM/MATHEMATICA + rational phase-space (Momentum Twistors) —pre-processing→ colour ordered helicity amplitudes

$$M^{(2)}(\{p\},\epsilon) = \sum_i c_i(\{p\},\epsilon)\mathcal{F}_i(\{p\},\epsilon)$$

↓ IBPs

$$M^{(2)}(\{p\},\epsilon) = \sum_i d_i(\{p\},\epsilon)\mathrm{MI}_i(\{p\},\epsilon)$$

↓ IR/UV sub + expansion to function basis

$$F^{(2)}(\{p\}) = \sum_i e_i(\{p\})\mathrm{mon}_i(f_j^{(w)})$$

↓

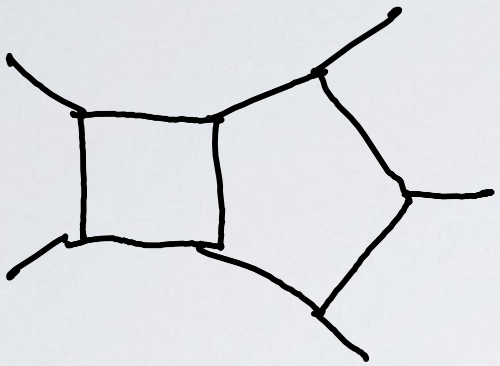
Q-linear relations, univariate apart, analytic reconstruction…

complete reduction setup implemented in FINITEFLOW [Peraro '19]

IBPs generated with help from LITERED [LEE 12',23'] +FINITEFLOW

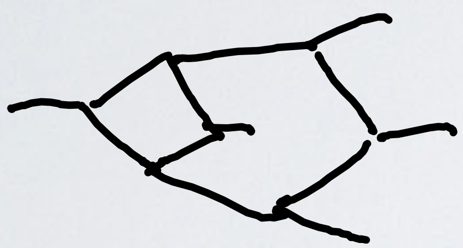
further reading: de Laurentis, Page [2203.04269], Liu [2306.12262], Chawdhry [2312.03672]

2→3 two-loop hard functions



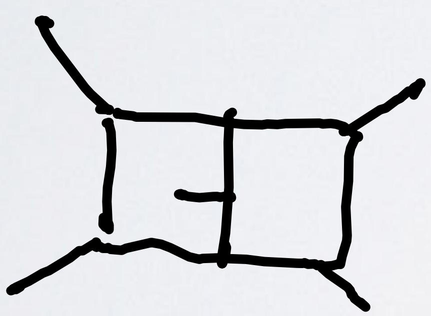
pp→γγγ

[Abreu,Page,Pascual,Sotnikov(2020)]
[Chawdhry,Czakon,Mitov,Poncelet(2021)]
[Abreu,De Laurentis,Ita,Klinkert,Page,Sotnikov(2023)]



pp→γγj

[Agarwal,Buccioni,von ManteuPel,Tancredi(2021)]
[Chawdhry,Czakon,Mitov,Poncelet(2021)]
[SB,Brønnum-Hansen,Chicherin,Gehrmann,Hartanto,Henn,Marcoli,Moodie,Peraro,Zoia(2021)]



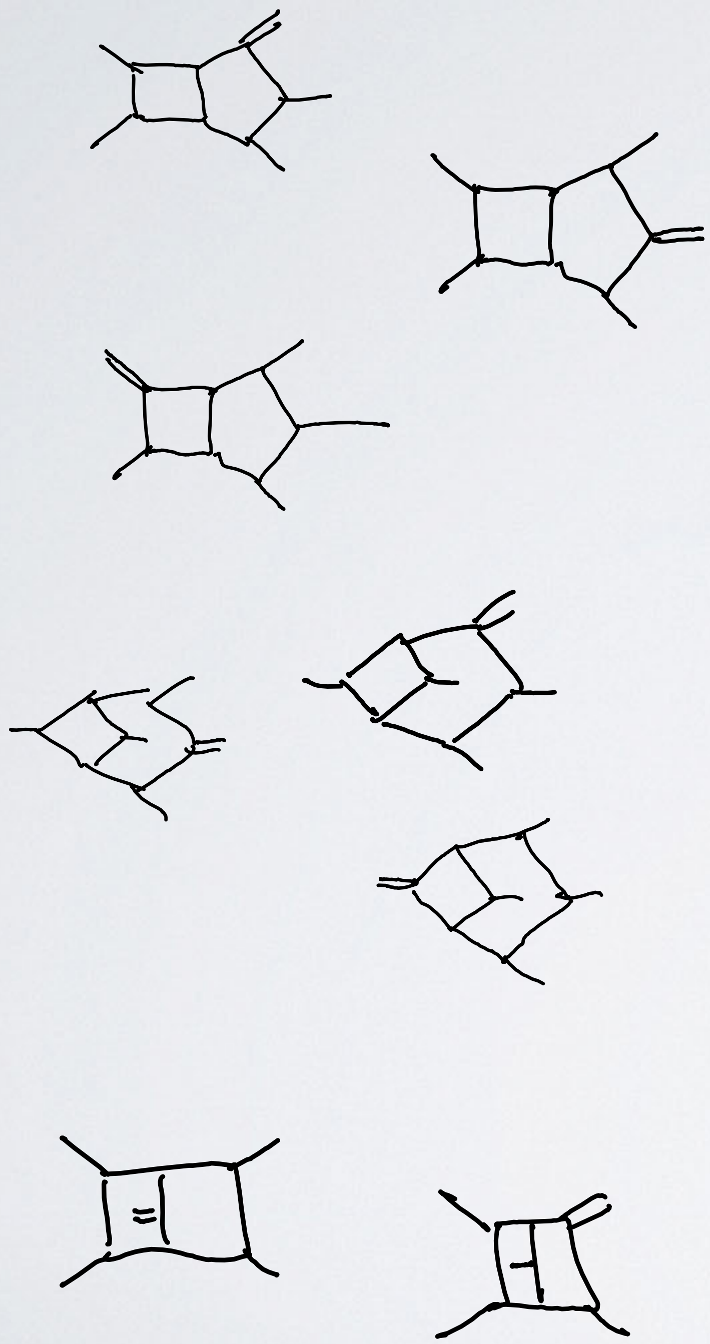
pp→γjj

[SB,Czakon,Hartanto,Moodie,Peraro,Poncelet,Zoia(2023)]

pp→jjj

[Abreu,Febres Cordero,Ita,Page,Sotnikov(2021)]
[De Laurentis,Ita,Klinkert,Sotnikov(2023)]
[Agarwal,Buccioni,Devoto,Gambuti,von Manteuffel,Tancredi(2023)]
[De Laurentis,Ita,Sotnikov(2023)]

2→3 two-loop hard functions



$pp \to Wbb$ (LC, $m_b=0$)

[SB,Hartanto,Zoia(2021)][Hartanto,Poncelet,Popescu,Zoia(2022)]

$pp \to Wjj$ (LC)

[Abreu,Febres Cordero,Ita,Klinkert,Page,Sotnikov(2022)]
[De Laurentis,Ita,Page,Sotnikov(2025)]

W/Z bb (LC, $\approx m_b$)

[Buonocore,Devoto,Kallweit,Mazzitelli,Rottoli,Savoini(2022)]
[Mazzitelli,Sotnikov,Wiesemann(2024)]

$pp \to W\gamma j$

[SB,Hartanto,Krys,Zoia(2022)]

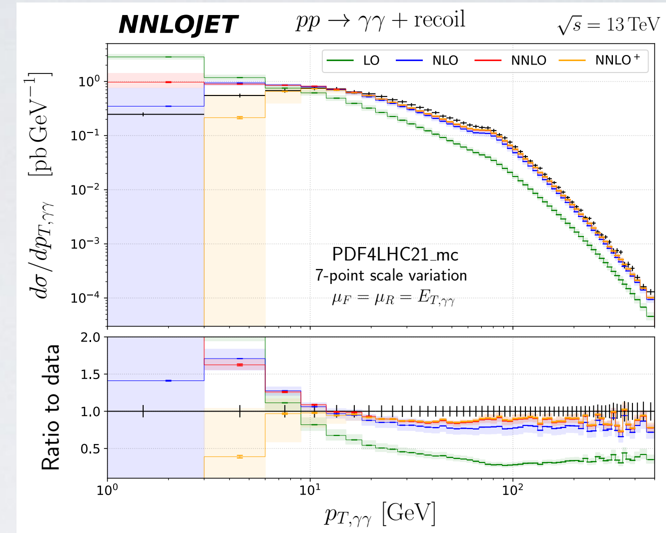
$pp \to W\gamma\gamma$

[SB,Hartanto,Wu,Zhang,Zoia(2024)]

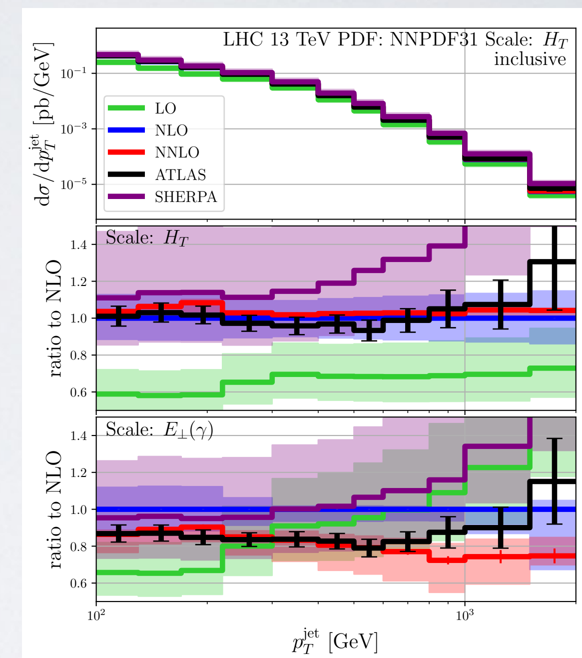
$pp \to Hbb$ ($m_b=0$)

[SB,Hartanto,Krys,Zoia(2021)]
[SB,Hartanto,Poncelet,Wu,Zhang,Zoia(2024)]

2→3 dσ

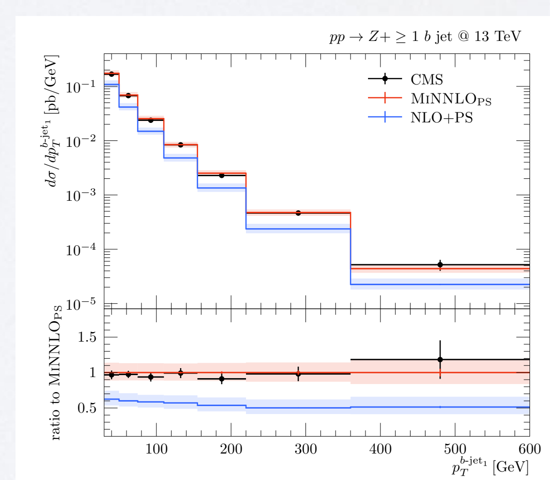


[γγγ Chawdhry,Czakon,Mitov,Poncelet(2019)]
[γγγ Kallweit,Sotnikov,Wiesemann(2020)]
[γγj Czakon,Mitov,Poncelet(2020)]
[* Htt Catani,Devoto,Grazini,Kallweit,Mazzitelli,Savoini(2020)]
[3j Czakon,Mitov,Poncelet(2021)]
[†Wbb Hartanto,Poncelet,Popescu,Zoia(2022)]
[*Wtt Buonocore,Devoto,Grazzini,Kallweit,Mazzitelli,Rottoli,Savoini(2023)]
[γjj SB,Czakon,Hartanto,Moodie,Peraro,Poncelet,Zoia(2023)]
[gg→ggg Chen,Gehrmann,Glover,Huss,Marcoli(2022)]
[jet event shapes Alvarez,Cantero,Czakon,Llorente,Mitov,Poncelet(2023)]
[* W/Zbb Mazzitelli,Sotnikov,Wiesemann(2024)]
[γγ event shapes Buccioni,Chen,Feng,Gehrmann,Huss,Marcolli(2025)]



* approximated double virtual

† massless bottom quarks

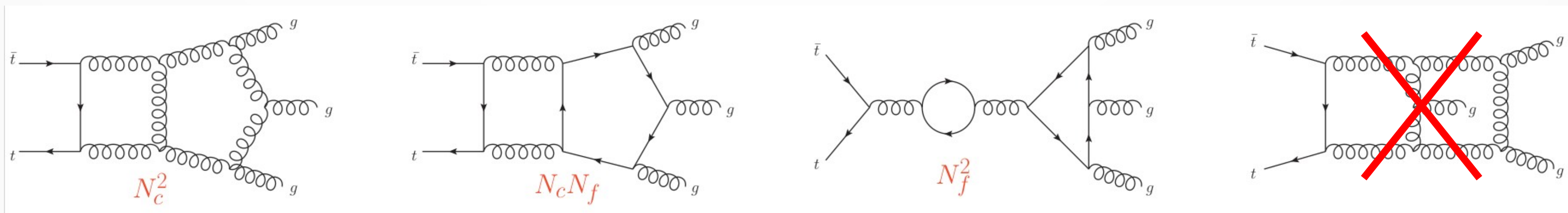


gg→ttg at 2-loops: helicity amplitudes

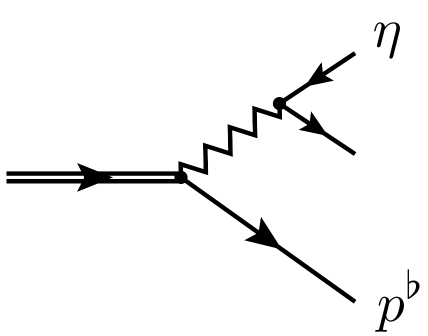
leading colour approximation

diagrams	tot	LC
tree-level	16	6
1-loop	384	77
2-loop	11370	1357

$$\mathcal{A}^{(L)}_{\rm LC}(1_{\bar t},2_t,3_g,4_g,5_g) = \sqrt{2}\,\bar{g}_s^3\, n^L \sum_{\sigma\in Z_3} \left(t^{a_{\sigma(3)}}t^{a_{\sigma(4)}}t^{a_{\sigma(5)}}\right)^{\bar{i}_1}_{\;i_2} A^{(L)}(1_{\bar t},\,2_t,\sigma(3)_g,\sigma(4)_g,\sigma(5)_g)$$



helicity amplitudes encode spin correlations in the narrow width approximation



$$\overline{u_\pm}(p,m;n) = \frac{\langle \eta \mp |(\not{p}+m)}{\langle \eta \mp |p^\flat \pm\rangle}$$

e.g. Melnikov, Schulze '09

4D projectors [Tancredi, Peraro '19,'20]

see also 1L ttH [Buccioni,Kreer,Liu,Tancredi '23]

$$\mathcal{A}^{(L)h_3h_4h_5} = \sum_{i=1}^{4} \Gamma_i\, \mathcal{G}_i^{(L)h_3h_4h_5}$$

$$\mathcal{G}_i^{(L)h_3h_4h_5} = \sum_{i=1}^{4} \left(\Gamma^\dagger\Gamma\right)^{-1}_{ij} \left[\Gamma_j^\dagger\, \mathcal{A}^{(L)}\right]^{h_3h_4h_5}$$

"contracted helicity amplitude"

$\Gamma_1 = m_t^2 \bar{u}_2 v_1$

$\Gamma_2 = m_t \bar{u}_2 \not{p}_3 v_1$

$\Gamma_3 = m_t \bar{u}_2 \not{p}_4 v_1$

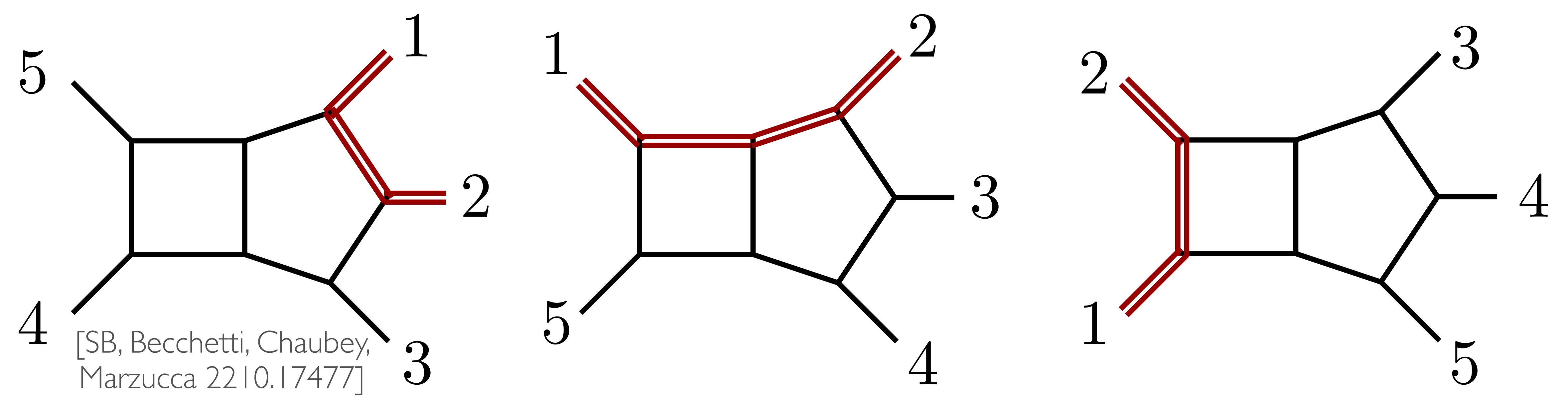
$\Gamma_4 = \bar{u}_2 \not{p}_3 \not{p}_4 v_1$

easy to square amplitude or attach decays

gg→ttg at 2-loops: differential equations

SB, Becchetti, Giraudo, Zoia [2404.12325]

$$4_p 5_p \to 1_t 2_t 3_j$$



88 MIs	121 MIs	109 MIs
dlog ✓	dlog ✗	dlog ✓
74 letters		79 letters

high precision boundary values with Auxiliary Mass Flow (AMFLOW Liu '22)

gg→ttg at 2-loops: differential equations

SB, Becchetti, Giraudo, Zoia [2404.12325]

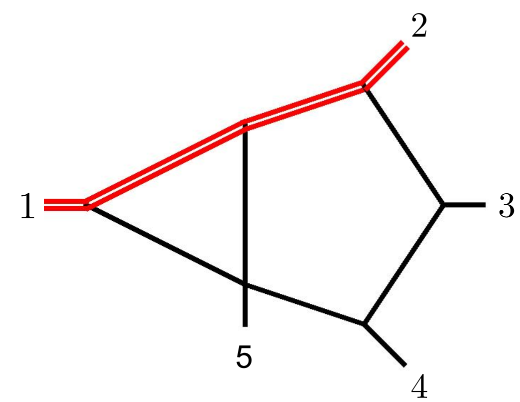
dlog candidates for most integrals follow patterns seen in previous examples:

- local and extra-dim. numerator insertions
- square roots (e.g. Gram determinants)
  8 square roots in total
- algebraic letters of the form

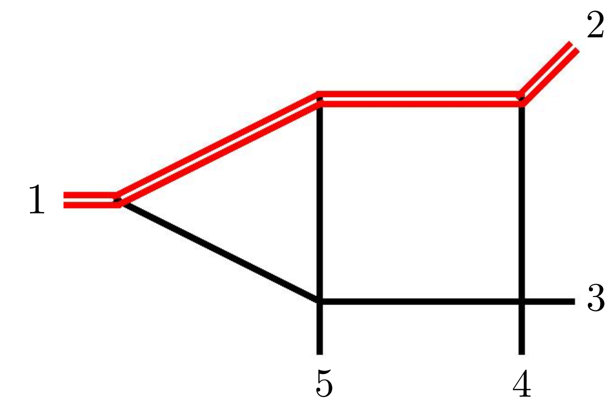
$$\frac{a+\sqrt{b}}{a-\sqrt{b}}$$

elliptic structure → resort to numerical evaluation of the DEQs

problem sectors



nested square root required to rotate into ε factorised form



Picard-Fuchs analysis confirms it to contain an elliptic curve

B topology basis chosen such that $k_{max}=2$

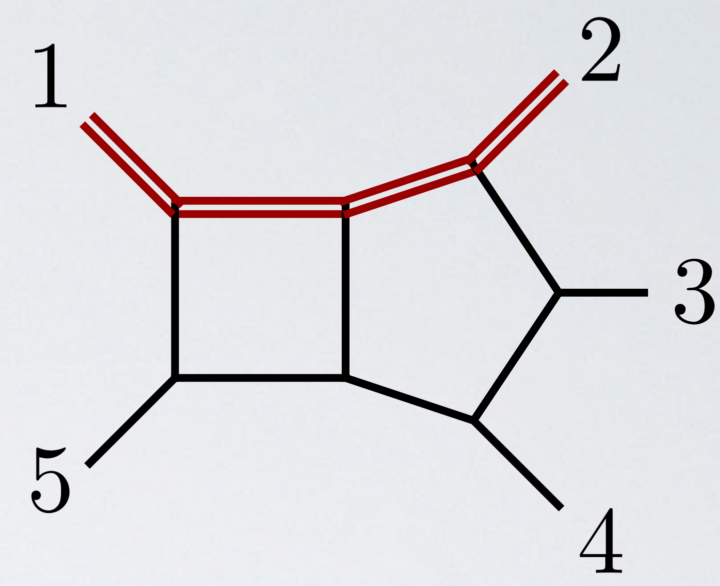
$$\mathrm{d}A^{(B)}(\vec{x},\varepsilon)=\sum_{k=0}^{k_{\max}}\sum_{i}\varepsilon^{k}\,\omega_i(\vec{x})\,c_{k,i}^{(B)}$$

ω are linearly independent one-forms (135 of which 72 are dlog)

compact analytic representation (for all topologies)

gg→ttg at 2-loops: differential equations

Becchetti, Dlapa, Zoia [2503.03603]



canonical form is possible

$$\mathrm{d}\vec{\mathcal{J}}(\vec{x},\epsilon) = \epsilon \sum_{i=1}^{112} M_i\, \omega_i(\vec{x}) \cdot \vec{\mathcal{J}}(\vec{x},\epsilon)$$

$$\sqrt{n_\pm} = \sqrt{d_{23}^2 \Delta_5 - 8 r_2 r_4 \pm 4 d_{23} r_3 \sqrt{\Delta_5}}$$

dlog form found for nested square root

$$y^2 = \mathcal{P}(z)$$

$$\mathcal{P}(z) = (z-e_1)(z-e_2)(z-e_3)(z-e_4)$$

complete description of elliptic sectors

further study required for efficient numerical evaluation

gg→ttg at 2-loops: special function basis

SB, Brancaccio, Becchetti, Hartanto Zoia [2412.13876]

while the DEQ is not in ε-factorised form - we can do suprisingly well with the expansion around d=4

- most of the DEQ is in dlog form
- elliptic sectors only appear at order 4 in ε [check with BCs]

$$d\vec{g} = (dA^{(0)}(x) + \epsilon dA^{(1)}(x) + \epsilon^2 dA^{(2)}(x))\vec{g} \qquad \vec{g} = \sum_{k\geq 0} \epsilon^k \vec{g}^{(k)}$$

$$d\vec{g}^{(w)} = dA^{(0)}\vec{g}^{(w)} + dA^{(1)}\vec{g}^{(w-1)} + dA^{(2)}\vec{g}^{(w-2)}$$

- $dA^{(0)}$ and $dA^{(2)}$ mostly zero's and up to ε^3 everything is dlog.
- elliptic sectors do not decouple at ε^4 so simply keep MI component as basis function (6x2 perms = 12 functions)

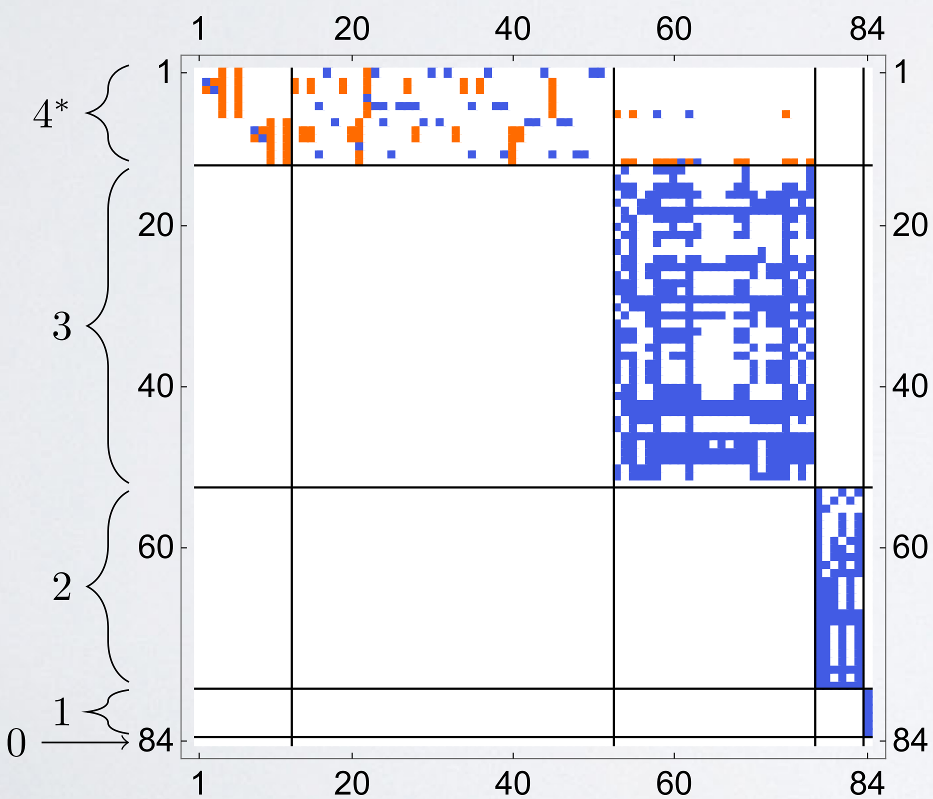
result is an (overcomplete) basis for defined through a differential equation

gg→ttg at 2-loops: special function basis

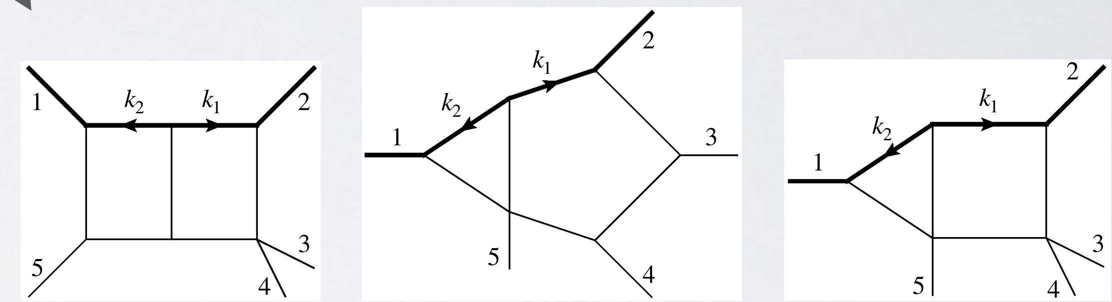
SB, Brancaccio, Becchetti, Hartanto Zoia [2412.13876]

transcendental weight	1	2	3	4	4*	all
# of functions	6	8	45	166	12	237

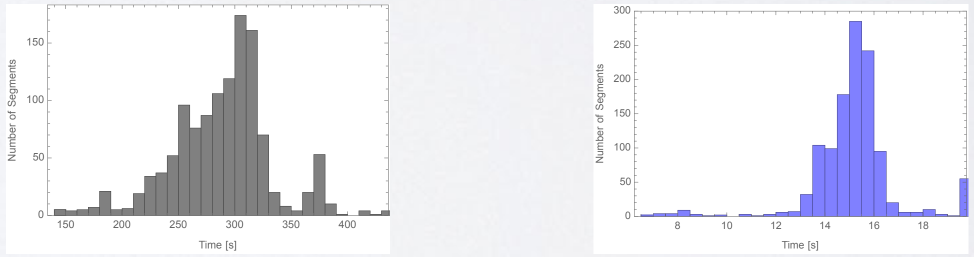
DEQ matrix



2x



(a) g_{15} (b) g_{19}, g_{20} (c) g_{35}, g_{36}, g_{37}



(a) All special functions (b) Non-polylogarithmic special functions

gg→ttg at 2-loops: benchmark evaluation

SB, Brancaccio, Becchetti, Hartanto Zoia [2412.13876]

optimzed IBP system with NeatIBP [Wu, Boehm, Ma, Xu, Zhang '23]

$$d_{12} = \frac{16177828451106651539}{15068333897971200000}, \quad d_{23} = \frac{335}{1232}, \quad d_{34} = -\frac{5}{32},$$
$$d_{45} = \frac{3665}{7328}, \quad d_{15} = -\frac{45}{1408}, \quad m_t^2 = \frac{376940175237098461}{15068333897971200000}$$

$$\mathrm{tr}_5 = \mathrm{i}\,\frac{\sqrt{582950030096630501}}{426229309440}$$

(physical region)

Helicity	$R^{(0),[1]}$	$R^{(0),[2]}$	$R^{(0),[3]}$	$R^{(0),[4]}$
$+++$	$0.26326 - 0.0097514\,\mathrm{i}$	0	0	0
$+-+$	$5.9619 - 0.16047\,\mathrm{i}$	0	0	$-0.31659 - 0.097935\,\mathrm{i}$
$++-$	$-5.9575 + 0.0089231\,\mathrm{i}$	$-12.606 - 0.067440\,\mathrm{i}$	$4.6564 + 0.024911\,\mathrm{i}$	$-1.9692 - 0.010535\,\mathrm{i}$
Helicity	$R^{(1),[1]}/R^{(0),[1]}$	$R^{(1),[2]}/R^{(0),[1]}$	$R^{(1),[3]}/R^{(0),[1]}$	$R^{(1),[4]}/R^{(0),[1]}$
$+++$	$38.396 - 5.8002\,\mathrm{i}$	$71.982 - 4.0653\,\mathrm{i}$	$-14.289 + 0.70866\,\mathrm{i}$	$17.909 - 0.39528\,\mathrm{i}$
$+-+$	$19.221 - 8.4151\,\mathrm{i}$	$-4.8506 + 4.8015\,\mathrm{i}$	$0.67096 - 0.09959\,\mathrm{i}$	$-1.2201 + 2.1594\,\mathrm{i}$
$++-$	$20.369 - 19.991\,\mathrm{i}$	$41.522 - 41.969\,\mathrm{i}$	$-15.990 + 15.739\,\mathrm{i}$	$6.2964 - 6.4584\,\mathrm{i}$
Helicity	$R^{(2),[1]}/R^{(0),[1]}$	$R^{(2),[2]}/R^{(0),[1]}$	$R^{(2),[3]}/R^{(0),[1]}$	$R^{(2),[4]}/R^{(0),[1]}$
$+++$	$882.48 - 91.619\,\mathrm{i}$	$2489.7 - 266.72\,\mathrm{i}$	$-492.28 + 8.1003\,\mathrm{i}$	$593.35 - 87.569\,\mathrm{i}$
$+-+$	$414.16 - 206.87\,\mathrm{i}$	$-171.78 + 189.69\,\mathrm{i}$	$25.226 - 1.5639\,\mathrm{i}$	$-54.820 + 95.716\,\mathrm{i}$
$++-$	$332.97 - 646.02\,\mathrm{i}$	$623.01 - 1325.1\,\mathrm{i}$	$-259.14 + 512.33\,\mathrm{i}$	$89.185 - 198.65\,\mathrm{i}$

gg→ttg at 2-loops: analytic reconstruction

SB, Czakon, Brancaccio, Becchetti, Hartanto, Poncelet, Zoia [on-going work]

	MIs	SFs	#1	#2	#3	#4	# of points
$[\Gamma_1^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	404/393	314/303	291/280	291/0	44/40	44/0	137076
$[\Gamma_2^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	398/389	305/296	287/278	287/0	55/51	54/0	89624
$[\Gamma_3^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	411/402	321/312	299/293	299/0	57/54	54/0	161482
$[\Gamma_4^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	420/411	326/317	304/299	304/0	58/54	56/0	179838

MIs: mass-renormalised amplitude, all-order ϵ dependence

SFs: finite remainder

#1: linear relations

#2: denominator guessing

#3: partial fraction in x_{5123} (no x_{5123} dependence)

#4: more denominator guessing

independent helicities evaluated simultaneously

$$\vec{h} = (+++,++-,+-+)$$

mom. twistor variables

$$(s_{34}, t_{12}, t_{23}, t_{45}, t_{51}, x_{5123})$$

$$s_{34} = (p_3+p_4)^2,$$
$$t_{12} = s_{12}/s_{34},$$
$$t_{23} = (s_{23}-m_t^2)/s_{34},$$
$$t_{45} = s_{45}/s_{34},$$
$$t_{15} = (s_{15}-m_t^2)/s_{34},$$
$$x_{5123} = -\frac{\langle 5|p_1 p_{45}|3\rangle}{\langle 53\rangle s_{12}}.$$

$$m_t^2 = \frac{s_{34}t_{12}}{t_{45}}\Big(t_{23}(t_{51}+(-1+t_{12})x_{5123}) + x_{5123}(t_{45}+t_{12}t_{51}-t_{45}t_{51}+t_{12}(-1+t_{12}-t_{45})x_{5123})\Big)$$

gg→ttg at 2-loops: hard function

SB, Czakon, Brancaccio, Becchetti, Hartanto,Poncelet, Zoia [on-going work]

colour sum requires 12 permutations of the partial colour amplitude

larger function basis:

transcendental weight	1	2	3	4	4*	all
# of functions	11	22	167	699	48	949

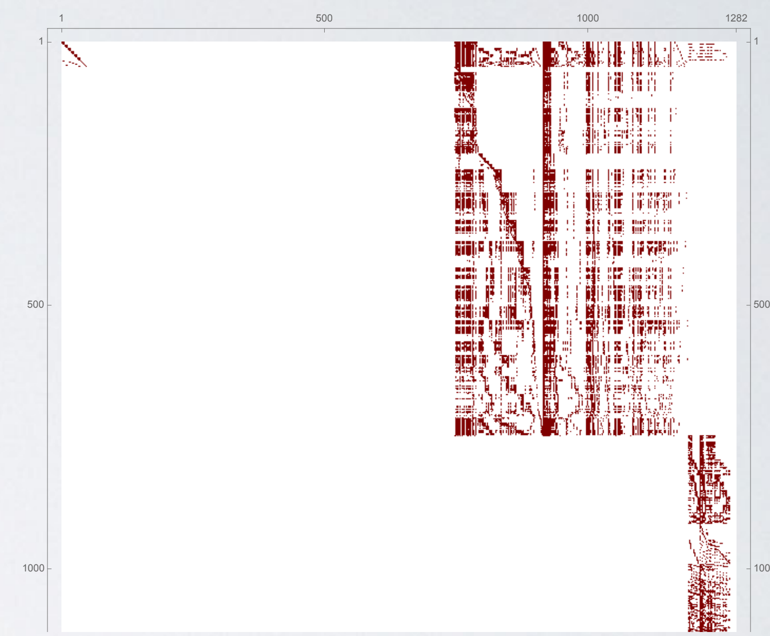
complete squared finite remainders for stable top (including light fermion loops)

$$\mathcal{H}^{(0)} = \overline{\sum} \mathcal{F}^{(0)*} \mathcal{F}^{(0)}$$

$$\mathcal{H}^{(1)} = 2\,\mathrm{Re} \overline{\sum} \mathcal{F}^{(0)*} \mathcal{F}^{(1)}$$

$$\mathcal{H}^{(2)} = 2\,\mathrm{Re} \overline{\sum} \mathcal{F}^{(0)*} \mathcal{F}^{(2)} + \overline{\sum} \mathcal{F}^{(1)*} \mathcal{F}^{(1)}$$

full DEQ 1282x1282



sparse, block diagonal form

gg→ttg at 2-loops: hard function

SB, Czakon, Brancaccio, Becchetti,
Hartanto,Poncelet, Zoia [on-going work]

numerical evaluation via direct solution to the differential equations

1) generalized series expansion [Moriello '20]

e.g. DiffExp (Hidding '20), SeaSyde ('22), LINE (Prisco et al. '25)

2) Runge-Kutta (Cash-Karp/Dormand-Prince) or Bulirsch–Stoer methods

e.g. 2-loop sunrise (Caffo et al. '02), 2-loop pp→tt (Czakon '08)

new elements: path parametrisation, branch cuts in square roots

new implementation for the ttj extremely promising for phenomenological applications

new work: Rosàs, Bobadilla 2507.12548

outlook

- complete stabilty checks for hard function over a realistic phase-space
- complete analytic reconstruction of qq→ttg
- combine with real radiation for NNLO dσ

- general method for numerical integration of DEQs
- is this the limit of analytic reconstruction?