FOUR-LOOP ANOMALOUS DIMENSIONS IN QCD

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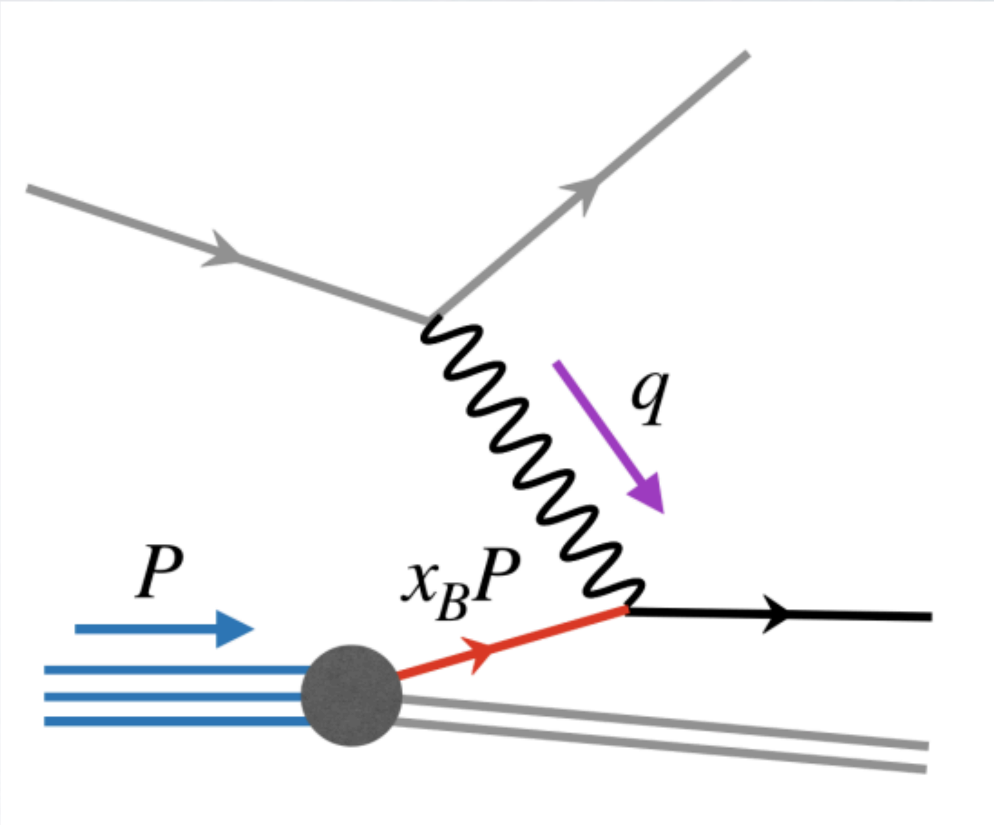
based on work with Thomas Gehrmann, Vasily Sotnikov, Tong-Zhi Yang

Loop Summit 2, Cadenabbia, 20-25 July 2025

SPLITTING FUNCTIONS

- Factorization of hadronic cross section:

$$\sigma \sim \sum_k f_{k|N}(x) \otimes \sigma_k(x) \quad \text{with } x = -\frac{q^2}{2P \cdot q}$$



[Image credit: Tong-Zhi Yang]

- Splitting functions P_{ik} govern DGLAP evaluations of PDFs:

$$\frac{df_{i|N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k|N}$$

- Consistent N3LO cross section requires 4-loop splitting functions, only partially known

More on splitting functions in talks by Johannes Blümlein, Fekésházy Levente, Sven Moch, Kay Schönwald

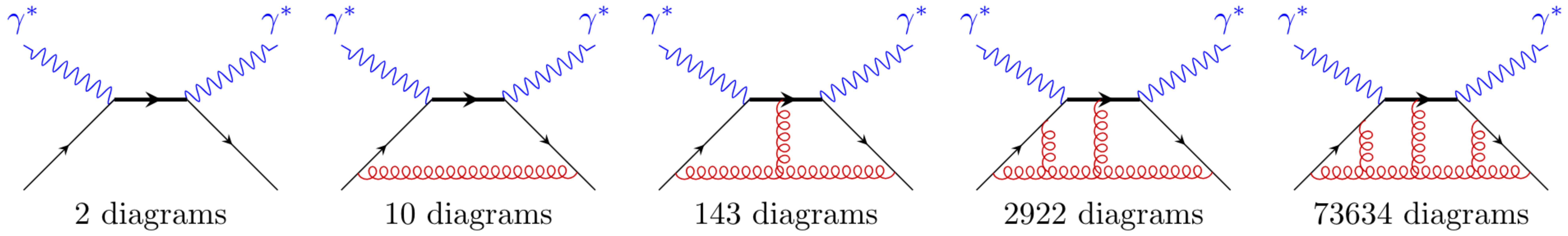
PARTIAL RESULTS @ 4 LOOPS

- Large n_f limit *[Gracey '94,'96; Davies, Vogt, Ruijl, Ueda, Vermageren '16]*
- Non-singlet $n \leq 16$ from off-shell OMEs *[Moch, Ruijl, Ueda, Vermaseren, Vogt '17]*
- Singlet $n \leq 8$ from DIS *[Moch, Ruijl, Ueda, Vermaseren, Vogt '21]*
- All channels $n \leq 10$, pure singlet $n \leq 12$, gq/qg $n \leq 20$ from off-shell OMEs *[Falcioni, Herzog, Moch, Vogt '23, '23, '24]*
- Pure-singlet all-n n_f^2, non-singlet all-n $n_f C_F^3$ *[Gehrmann, AvM, Sotnikov, Yang '24, '24]*
- Non-singlet fermionic all-n *[Kniehl, Moch, Velizhanin, Vogt '25]*
- **Our goal:** complete all-n results for pure-singlet and non-singlet splitting functions at 4 loops

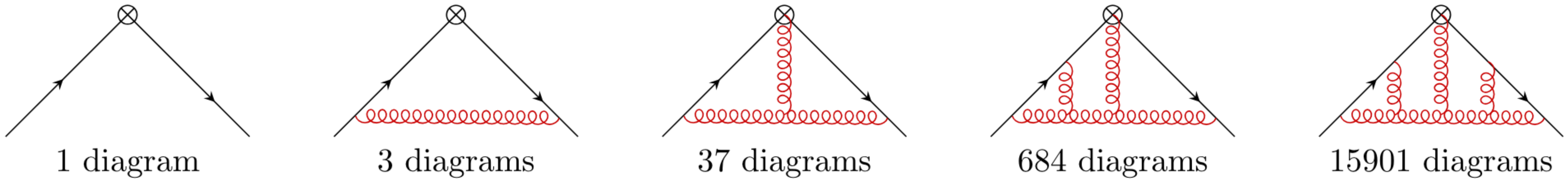
THE OFF-SHELL OME METHOD,
RENORMALIZATION OF SINGLET CONTRIBUTIONS

DIS VS OFF-SHELL OME

Forward DIS (gauge invariant)



Partonic off-shell OME (fewer diagrams, easier integrals)



[Image credit: Tong-Zhi Yang]

SPLITTING FUNCTIONS FROM OPERATORS

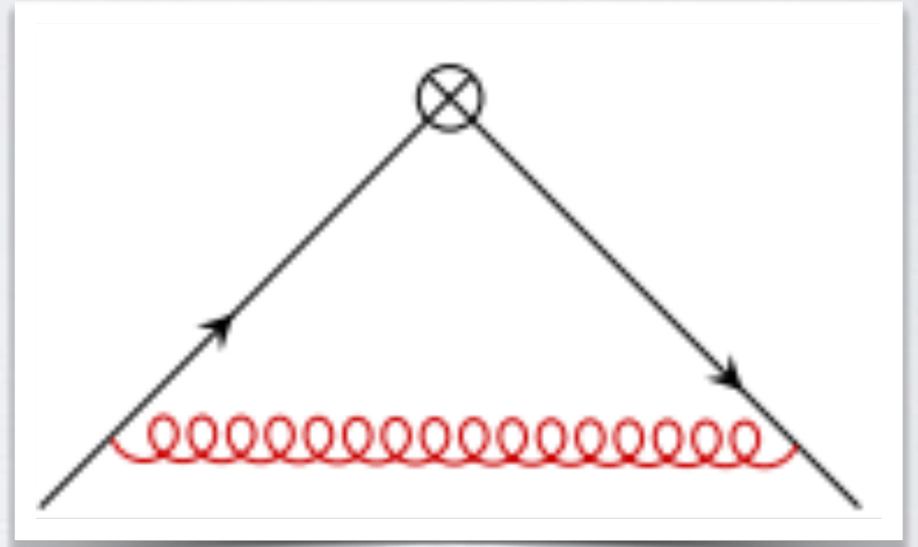
- With Mellin transform $f_q(n) = -\int_0^1 dx\, x^{n-1} f_q(x)\,,\quad \gamma_{ij}(n) = -\int_0^1 dx\, x^{n-1} P_{ij}(x)$

  DGLAP becomes $\dfrac{df_i(n,\mu)}{d\ln\mu} = -2\sum_j \gamma_{ij}(n) f_j(n,\mu)$

- The $\gamma_{ij}(n)$ appear as **anomalous dimensions** of **twist-two operators**,

  e.g. flavor non-singlet: $O_{q,k} = \dfrac{i^{n-1}}{2}\left[\overline{\psi}\Delta_\mu\gamma^\mu(\Delta\cdot D)^{n-1}\dfrac{\lambda_k}{2}\psi\right]$

  with multiplicative renormalization $O^R_{q,k} = Z^{ns} O^B_{q,k}$ where $\dfrac{dZ^{ns}}{d\ln\mu} = -2\gamma^{ns}Z^{ns}$



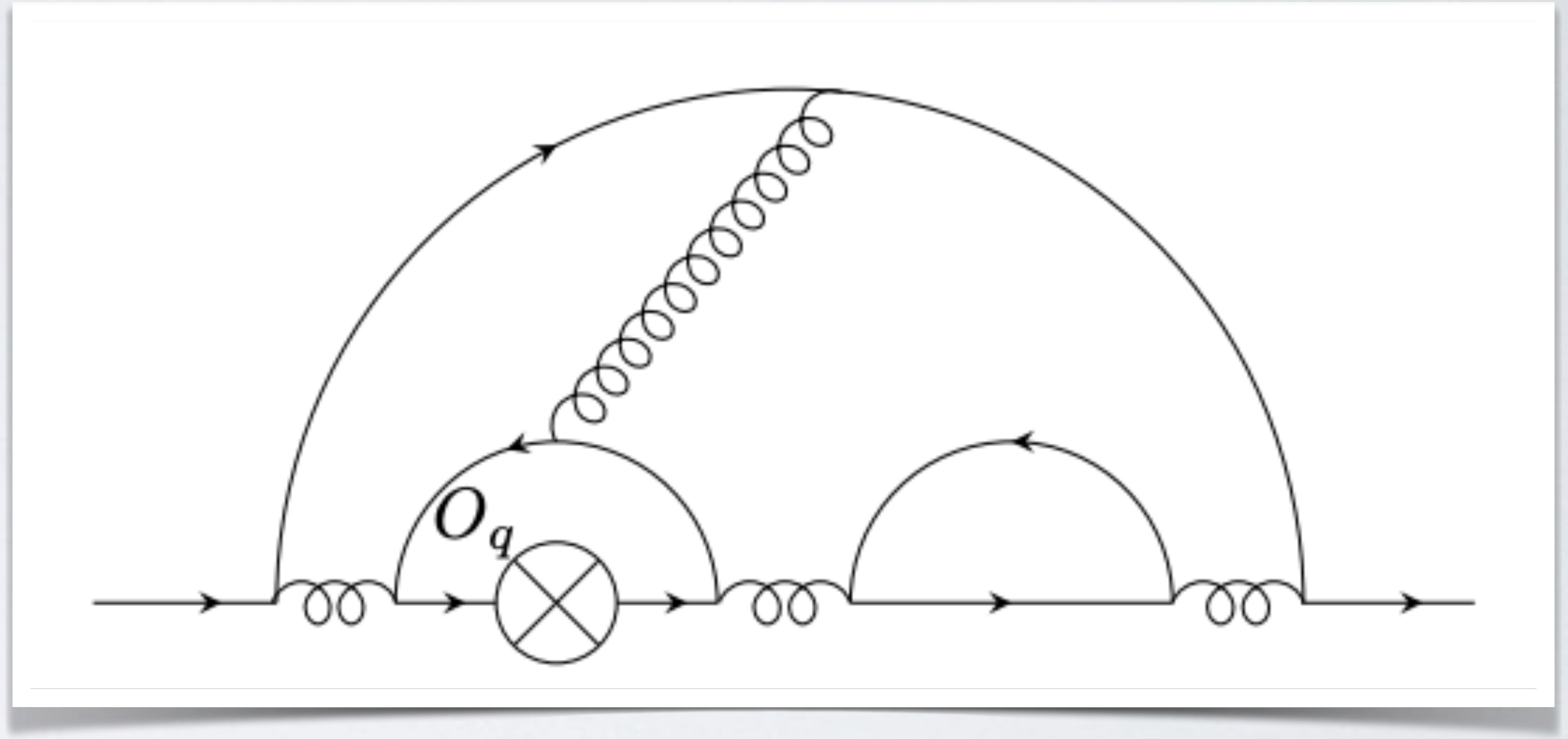
- Mellin moment n related to spin of operator, twist = dimension - spin, leading twist dominates short distance
- Poles of (off-shell) operator matrix elements: **efficient** way to find $f_q(n)$

SINGLET CASE AND OPERATOR MIXING

- Singlet twist-two operators:

$$O_q = \frac{i^{n-1}}{2}\left[\overline{\psi}\Delta_\mu\gamma^\mu(\Delta\cdot D)^{n-1}\psi\right]$$

$$O_g = -\frac{i^{n-2}}{2}\left[\Delta_\mu G^{a\,\mu}{}_{\nu}(\Delta\cdot D)^{n-2}_{ab}\Delta_\kappa G_b{}^{\kappa\nu}\right]$$



- Singlet operators **mix under renormalization**, naively:

$$\begin{pmatrix} O_q \\ O_g \end{pmatrix}^R_{\text{naive}} = \begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix}\begin{pmatrix} O_q \\ O_g \end{pmatrix}^B$$

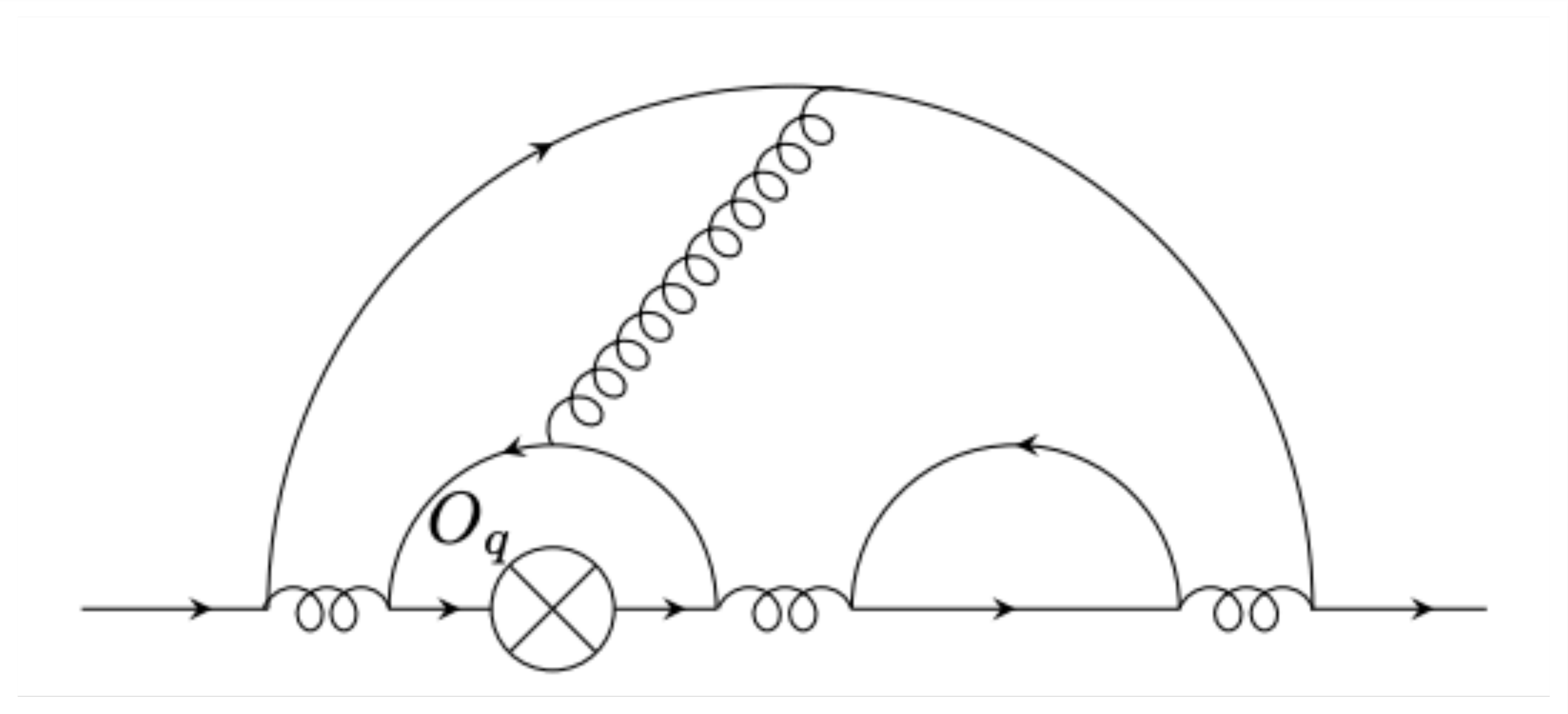
- For off-shell OME, also **new, unknown gauge-variant / e.o.m. operators** contribute
- Caused some confusion when first encountered

SINGLET CASE AND OPERATOR MIXING

- Singlet twist-two operators:

$$O_q = \frac{i^{n-1}}{2}\left[\overline{\psi}\Delta_\mu \gamma^\mu (\Delta \cdot D)^{n-1}\psi\right]$$

$$O_g = -\frac{i^{n-2}}{2}\left[\Delta_\mu G^{a\,\mu}{}_{\nu}(\Delta \cdot D)^{n-2}_{ab}\Delta_\kappa G_b{}^{\kappa\nu}\right]$$



- Correct renormalization:

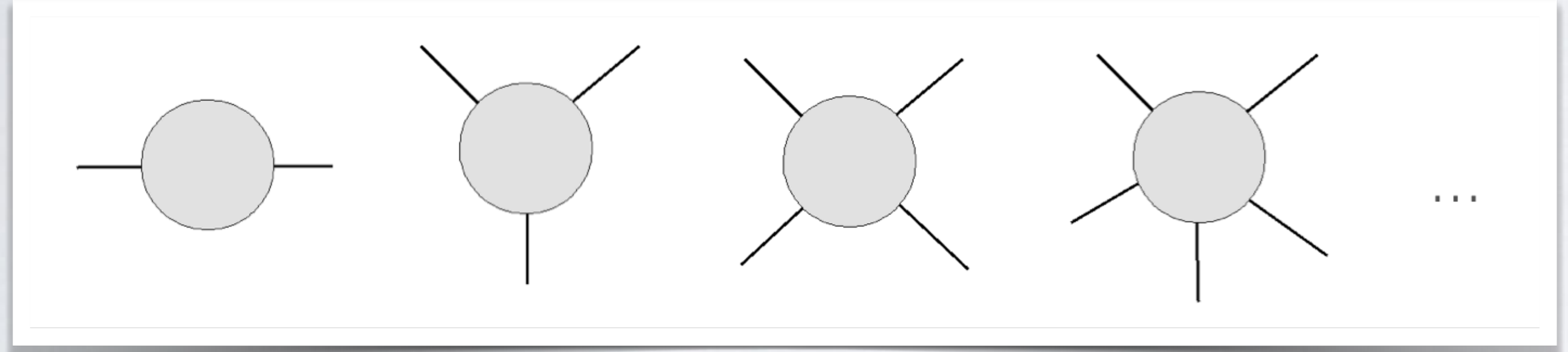
$$\begin{pmatrix} O_q \\ O_g \\ O_A \\ \cdots \end{pmatrix}^R = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} & \cdots \\ Z_{gq} & Z_{gg} & Z_{gA} & \cdots \\ Z_{Aq} & Z_{Ag} & Z_{AA} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_A \\ \cdots \end{pmatrix}^B$$

with gauge-variant operators O_A , …

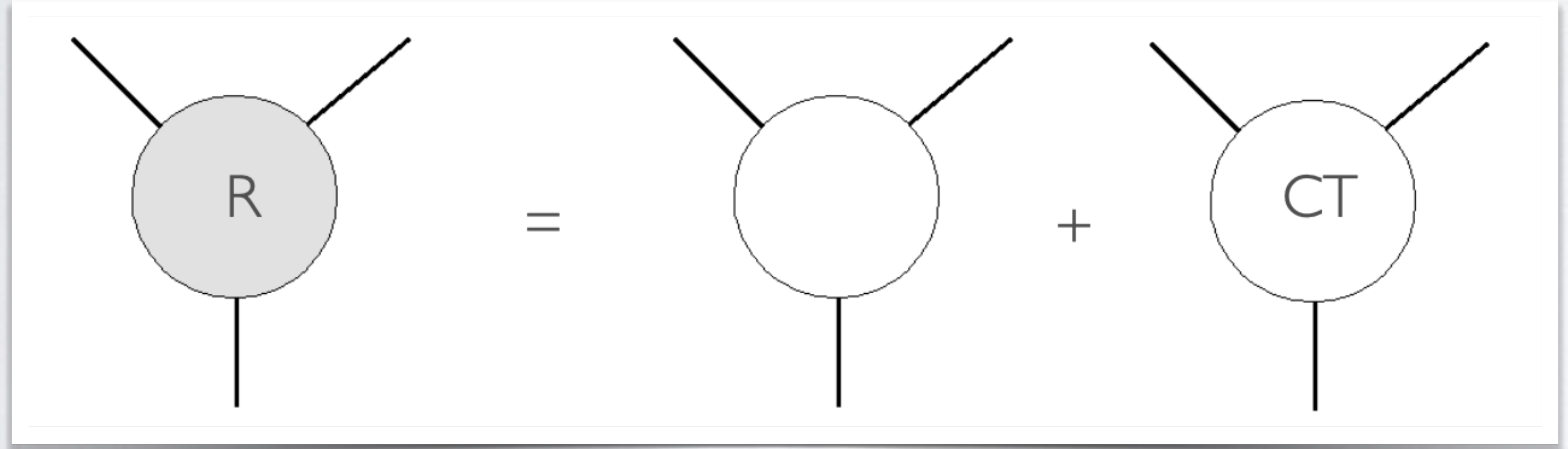
- Construction of operators from **generalized BRST**: *[Falcioni, Herzog '22; Falcioni, Herzog, Moch, van Thurenhout '24]*
- Our method: directly compute **counter term Feynman rules** from multi-leg off-shell OMEs, goal: **all-n results** *[Gehrmann, AvM, Yang '23, '24]*

GENERIC APPROACH TO RENORMALIZATION

- Consider arbitrary theory
- May be "non-renormalizable" due to higher-dimensional operators
- Theory can be renormalized by imposing renormalization conditions on 1-PI vertex functions $\Gamma_R^{(n)}(p_1, \ldots, p_n)$



- Here, we use a counter-term approach, which cancels only the divergence ($\overline{MS}$ renormalization)



- In order to perform renormalization of Greens functions, we do not need to know which operators generated some CT, **effective Feynman rules** are sufficient
- Compute CT from **off-shell multi-leg** matrix elements to avoid IR poles

COUNTER TERMS FROM MULTI-LEG OMES

- Renormalization: $\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^R = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ Z_{Aq} & Z_{Ag} & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^B + \begin{pmatrix} [ZO]_q^{GV} \\ [ZO]_g^{GV} \\ [ZO]_A^{GV} \end{pmatrix}^B$

- Take OMEs according to $\langle j | O | j + mg \rangle$ with $j = q, g, c$ and m **additional gluons**
- Expand $[ZO]^{GV} = \sum_l [ZO]^{GV,(l)} \alpha_s^l$ etc., orders:
- $Z_{qA} = \mathcal{O}(a_s^2), \quad Z_{gA} = \mathcal{O}(a_s), \quad [ZO]_q^{GV} = \mathcal{O}(a_s^3), \quad [ZO]_g^{GV} = \mathcal{O}(a_s^2), \quad Z_{AA} = \mathcal{O}(a_s^0), \quad [ZO]_A^{GV} = \mathcal{O}(a_s)$
- Determine **counter terms** from OMEs with extra legs, e.g.:

$$\langle j | O_g | j + m\, g \rangle_{\text{1PI}}^{\mu_1 \cdots \mu_m, \text{R}} = Z_j (\sqrt{Z_g})^m \Big[\langle j | (Z_{gq} O_q + Z_{gg} O_g) | j + m\, g \rangle_{\text{1PI}}^{\mu_1 \cdots \mu_m, \text{B}} \Big] + Z_j (\sqrt{Z_g})^m \Big[Z_{gA} \langle j | O_{ABC} | j + m\, g \rangle_{\text{1PI}}^{\mu_1 \cdots \mu_m, \text{B}} + \langle j | \left[ZO \right]_g^{\text{GV}} | j + m\, g \rangle_{\text{1PI}}^{\mu_1 \cdots \mu_m, \text{B}} \Big]$$

- For ghost-ghost-gluon CT vertex due to O_C:

$$\langle c | O_C | c + m\, g \rangle^{\mu_1 \cdots \mu_m, (0), (m)} = \frac{-1}{Z_{gA}^{(1)}} \Big[\langle c | O_g | c + m\, g \rangle_{\text{1PI}}^{\mu_1 \cdots \mu_m, (1), (m), \text{B}} \Big]_{\text{div}},$$

this gives $Z_{gA}^{(1)} = \frac{1}{\epsilon} \frac{C_A}{n(n-1)}$

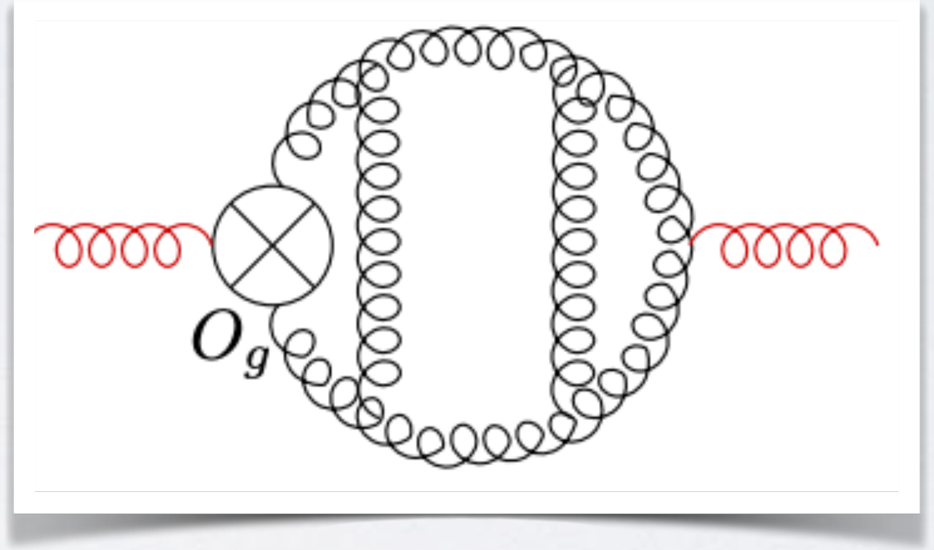
COUNTER TERMS FOR 3-LOOP RENORMALIZATION

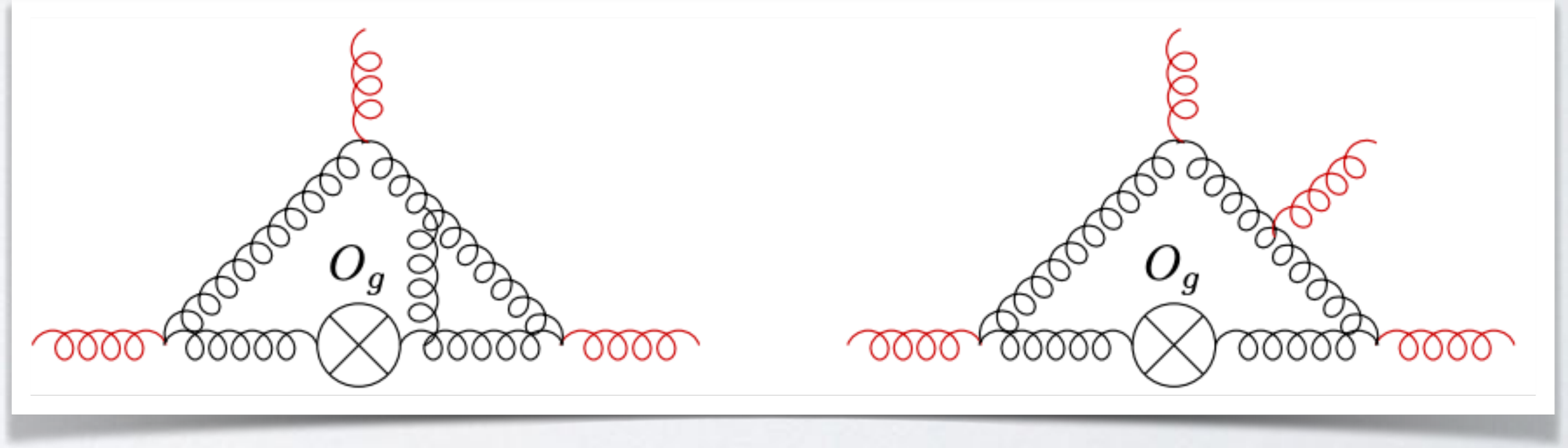
- Renormalization: $$\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^R = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ Z_{Aq} & Z_{Ag} & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^B + \begin{pmatrix} [ZO]_q^{GV} \\ [ZO]_g^{GV} \\ [ZO]_A^{GV} \end{pmatrix}^B$$

- For 3-loop splitting functions we need to calculate:

Loops \ Legs	2	3	4	5
0		$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g	
2	O_{ABC}	O_g		
3	O_q, O_g			

[Gehrmann, AvM, Yang ‘23]





COMPUTATION OF
OPERATOR MATRIX ELEMENTS

TRACING PARAMETER

- Generate Feynman diagrams with Qgraf
- Operator insertions introduce n-dependent powers of scalar products
- **Tracing parameter** t maps to linear propagators *[Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12]*

$$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\,\Delta \cdot p}$$

which allows to use standard IBP technology

- At end of calculation, **re-expand in** t to find Mellin moments, e.g. for harmonic polylog

$$H(a_1, a_2, \ldots; t) = \sum_{t=0}^{\infty} b_n t^n$$

compute b_n e.g. with HarmonicSums package in terms of harmonic sums

$$S_{\pm m_1, m_2, \ldots}(n) = \sum_{j=1}^{n} (\pm 1)^j \frac{1}{j^{m_1}} S_{m_2, \ldots}(j), \quad S_{\emptyset}(j) = 1$$

INTEGRATION-BY-PART IDENTITIES

- IBP identities in dimensional regularization since integrals over total derivatives vanish:

$$\int d^d k_1 \cdots d^d k_L \, \frac{\partial}{\partial k_i^\mu} \left(v^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right) = 0 \,, \qquad D_j = q_j^2 \text{ or } 1 - x\Delta \cdot q_j$$

  v^μ loop or ext. mom.

- Gives linear relations between loop integrals *[Chetyrkin, Tkachov '81]*
- **Systematic reduction** to finite basis of master integrals *[Laporta '00]*
- Plenty of **anomalous relations** (generated in super sectors), discrete shift symmetries
- Many top sectors, many master integrals

FINITE FIELDS AND RATIONAL RECONSTRUCTION

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

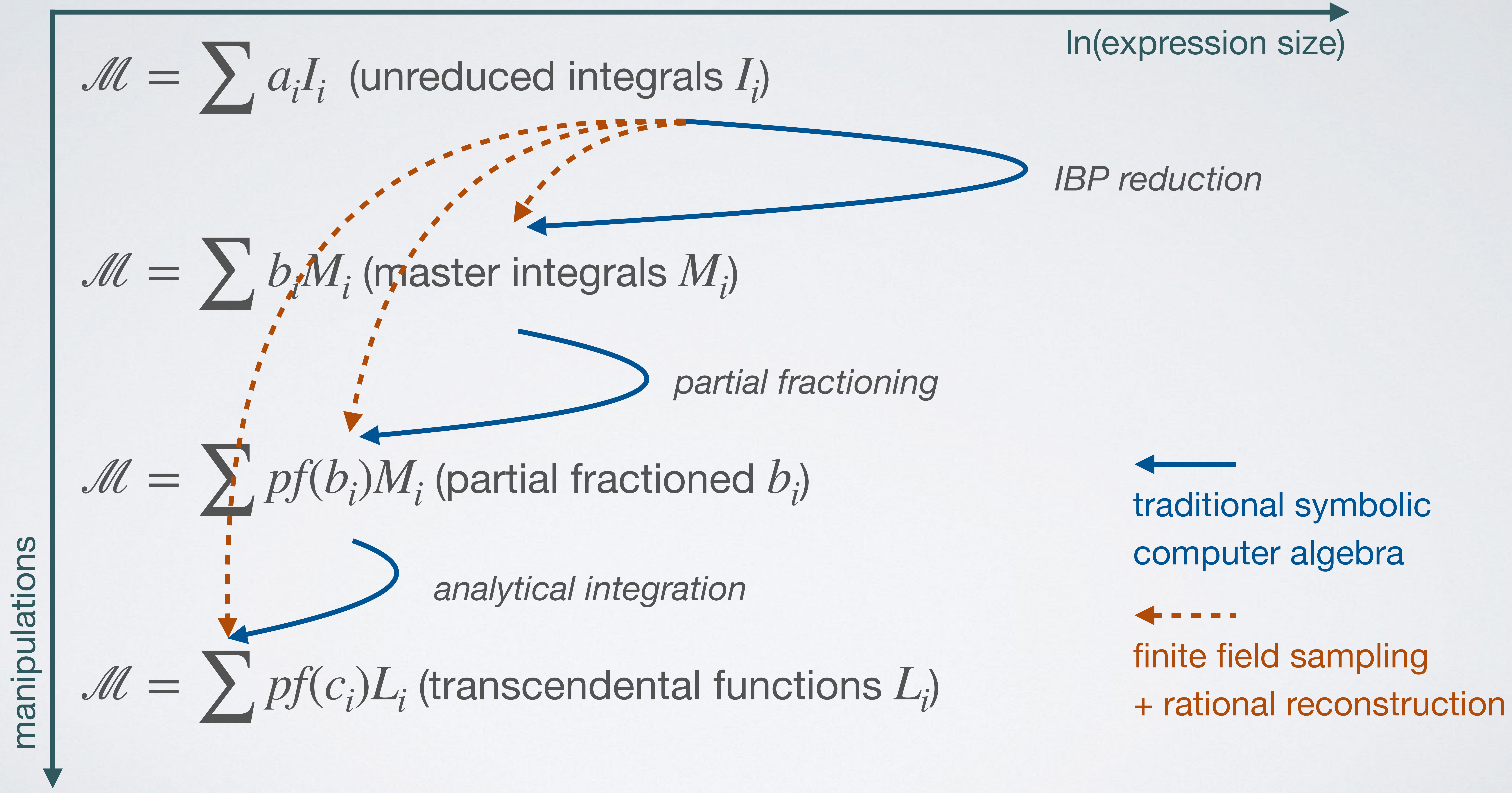
$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p[x]} & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p[x]} & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_p, x_2} & \longrightarrow & O_{\mathbb{Z}_p, x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p, x3} & \longrightarrow & O_{\mathbb{Z}_p, x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

[AvM, Schabinger ’14; Peraro ’16; …], use implementation in Finred

BY-PASSING COMPLEXITY



Syzygy Based IBPs Without Dots

[Gluza, Kajda, Kosower '11; Schabinger '11; Ita '15; Larsen, Zhang '15; Böhm, Georgoudis, Larsen, Schulze, Zhang '18; Agarwal, Jones, AvM '20, ...]

Baikov's parametric representation of Feynman integrals:

$$I(\nu_1, \ldots, \nu_N) = \mathcal{N} \int \mathrm{d}z_1 \cdots \mathrm{d}z_m P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}}$$

[Böhm, Georgoudis, Larsen, Schulze, Zhang '18]: useful for IBPs without dots

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{m} \frac{\partial}{\partial z_i} \left(a_i P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}} \right)$$

$$= \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{N} \left(\frac{\partial a_i}{\partial z_i} + \frac{d-L-E-1}{2P} a_i \frac{\partial P}{\partial z_i} - \frac{\nu_i a_i}{z_i} \right) P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}}$$

explicit solutions to constraint:

$$\left(\sum_{i=1}^{N} a_i \frac{\partial P}{\partial z_i} \right) + bP = 0 \qquad \text{(absence of dim. shifts)}$$

in addition, require for denominators of sector:

$$a_i = b_i z_i \qquad \text{(absence of dots)}$$

need intersection of two syzygy modules

DIFFERENTIAL EQUATIONS

- Solve **master integrals** with method of differential equations *[Kotikov '91, Remiddi '97]*:

$$\partial_t \vec{I}(t,\epsilon) = A(t,\epsilon)\vec{I}(t,\epsilon)$$

where t is the tracing parameter.

Talks: Samuel Abreu, Simon Badger, Arnd Behring, Carsten Schneider, Lorenzo Tancredi, Stefan Weinzierl, …

- Homogeneous solutions for $\epsilon \equiv (4-d)/2 = 0$ (leading singularities):
  - **Rational number**
  - **Rational functions**
  - **Algebraic functions**
  - **Elliptic integrals, …**

- **Basis change** involving homogenous solutions allows to find ϵ-form:

$$\mathrm{d}\vec{m} = \epsilon\, \mathrm{d}A(t)\, \vec{m}$$

*[Kotikov '10, **Henn '13**, Remiddi, Tancredi '16, Adams, Weinzierl '18]*

ANALYTICAL SOLUTIONS

- For all 3-loop splitting functions and a number of 4-loop splitting functions: **canonical basis** possible with algebraic basis change
- Used Canonica and Libra, integrate diff. eqs. in terms of **GPLs**
- Consider only solutions which are **regular** for tracing parameter $t \to 0$
- **Boundary values** for $t \to 0$ from standard self-energy integrals, known
- Apply MultivariateApart to t-space results
- IBP reduce directly to canonical basis

MULTI-LEG OPERATOR MATRIX ELEMENTS

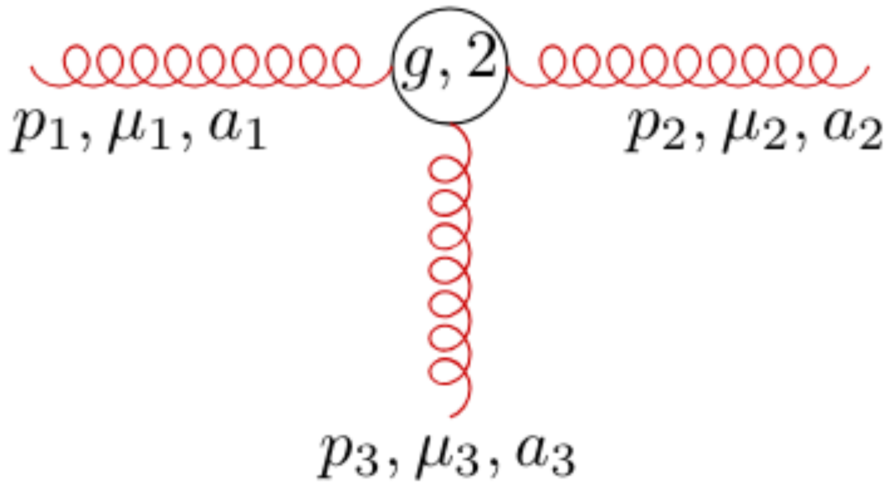
- Singlet counter terms required computation of multi-leg OMEs in our approach
- Use **tensor decomposition** of off-shell OMEs for computation
- **Dimensional analysis**, symmetries reduce # Lorentz structures:
  e.g. for 5-gluon counter-term only 31 instead of naively 4400 tensors:

$$
\begin{aligned}
\left[\langle g|O_A|gggg\rangle^{\mu_1\mu_2\mu_3\mu_4\mu_5,(0),(3)}\right] &= a_1\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4}\Delta^{\mu_5} \\
&+ \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4}(a_2 p_1^{\mu_5} + a_3 p_2^{\mu_5} + a_4 p_3^{\mu_5} + a_5 p_4^{\mu_5}) \\
&+ \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_5}(a_6 p_1^{\mu_4} + a_7 p_2^{\mu_4} + a_8 p_3^{\mu_4} + a_9 p_4^{\mu_4}) \\
&+ \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_4}\Delta^{\mu_5}(a_{10} p_1^{\mu_3} + a_{11} p_2^{\mu_3} + a_{12} p_3^{\mu_3} + a_{13} p_4^{\mu_3}) \\
&+ \Delta^{\mu_1}\Delta^{\mu_3}\Delta^{\mu_4}\Delta^{\mu_5}(a_{14} p_1^{\mu_2} + a_{15} p_2^{\mu_2} + a_{16} p_3^{\mu_2} + a_{17} p_4^{\mu_2}) \\
&+ \Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4}\Delta^{\mu_5}(a_{18} p_1^{\mu_1} + a_{19} p_2^{\mu_1} + a_{20} p_3^{\mu_1} + a_{21} p_4^{\mu_1}) \\
&+ a_{22}\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3} g^{\mu_4\mu_5} + a_{23}\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_4} g^{\mu_3\mu_5} + a_{24}\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_5} g^{\mu_3\mu_4} + a_{25}\Delta^{\mu_1}\Delta^{\mu_3}\Delta^{\mu_4} g^{\mu_2\mu_5} \\
&+ a_{26}\Delta^{\mu_1}\Delta^{\mu_3}\Delta^{\mu_5} g^{\mu_2\mu_4} + a_{27}\Delta^{\mu_1}\Delta^{\mu_4}\Delta^{\mu_5} g^{\mu_2\mu_3} + a_{28}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} g^{\mu_1\mu_5} \\
&+ a_{29}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_5} g^{\mu_1\mu_4} + a_{30}\Delta^{\mu_2}\Delta^{\mu_4}\Delta^{\mu_5} g^{\mu_1\mu_3} + a_{31}\Delta^{\mu_3}\Delta^{\mu_4}\Delta^{\mu_5} g^{\mu_1\mu_2} \,,
\end{aligned}
$$

- **Avoid scales**: set $p_1^2,\, p_2^2,\, \ldots$ to numbers, $\Delta \cdot p_1 = 1,\ \Delta \cdot p_2 = z_1$ etc.

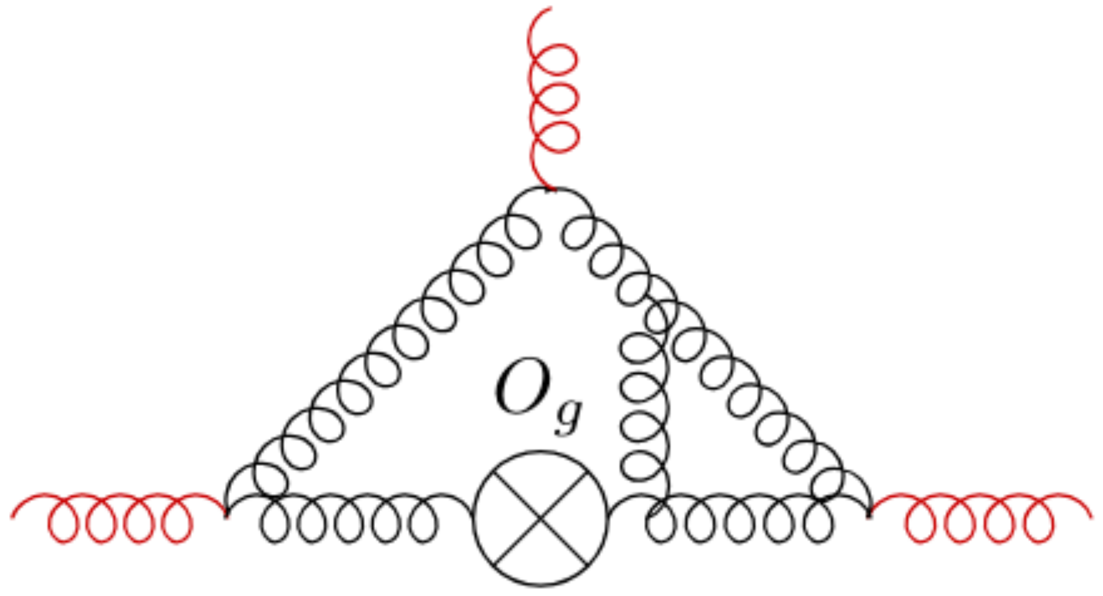
RESULTS FOR
3-LOOP SPLITTING FUNCTIONS

EXAMPLE: 2-LOOP CT CONTRIBUTIONS FOR 3-LOOP SPLITTING FUNCTIONS



$$\rightarrow 2ig_s C_A^2 f^{a_1 a_2 a_3} \frac{1+(-1)^n}{256n(n-1)} \frac{(\Delta\cdot p_1)^{n-2}}{\Delta\cdot p_2} \Big(\Delta^{\mu_2}\Delta^{\mu_3} p_1^{\mu_1} \Delta\cdot p_1 + \cdots \Big) \left\{ \frac{F_{-2,0}(\xi, z_1, n)}{\epsilon^2} + \frac{F_{-1,0}(\xi, z_1, n)}{\epsilon} \right\}, \; z_1 = \frac{\Delta\cdot p_2}{\Delta\cdot p_1}$$

extracted from



- Coefficients contain regular and generalized harmonic sums, e.g.:

$$S_1\Big(\frac{z_1+1}{z_1};n\Big),\; S_{-2}(n),\; S_2(n),\; S_{1,1}(n),\; S_2\Big(-z_1-1;n\Big),\; S_2\Big(-\frac{1}{z_1};n\Big),\; S_2\Big(\frac{1}{z_1};n\Big),$$
$$S_2\Big(-z_1;n\Big),\; S_2\Big(z_1;n\Big),\; S_2\Big(-\frac{1}{z_1+1};n\Big),\; S_2\Big(\frac{1}{z_1+1};n\Big),\; S_2\Big(-\frac{z_1}{z_1+1};n\Big),$$

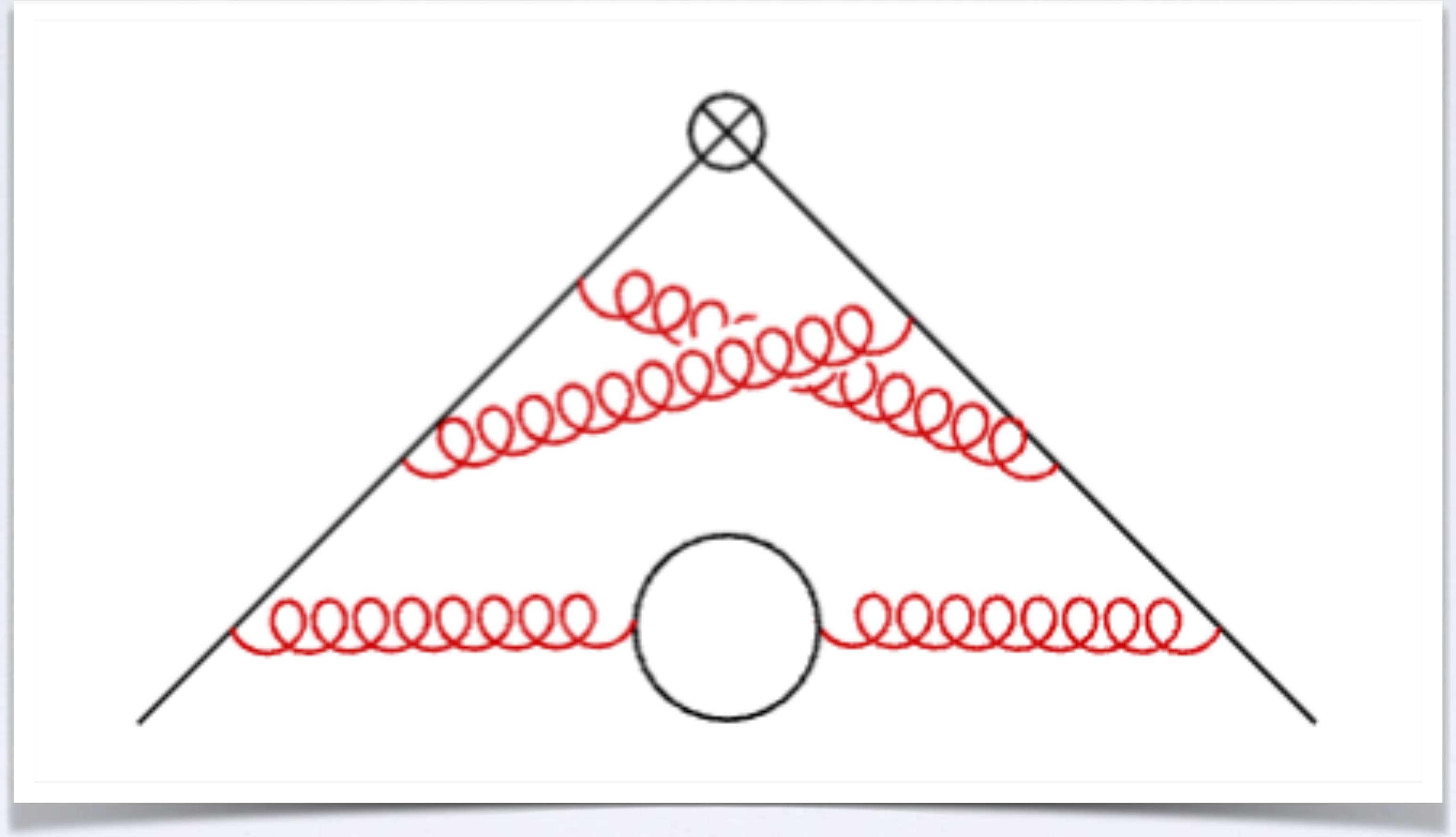
[Gehrmann, AvM, Yang ’23]

SPLITTING FUNCTIONS @ 3-LOOPS

- We applied our method to compute all **3-loop splitting functions** (non-singlet + singlet) *[Gehrmann, AvM, Yang ’23]*
- Our calculation was performed in a **general** R_ξ **gauge**
- The gauge parameter ξ drops out
- We find **full agreement** with *[Moch, Vermaseren, Vogt ’04, ’04]*

RESULTS FOR
4-LOOP SPLITTING FUNCTIONS

NON-SINGLET @ 4-LOOPS: $n_f C_F^3$



- Calculated $n_f C_F^3$ non-singlet, four-loop splitting functions with exact x dependence *[Gehrmann, AvM, Sotnikov, Yang '23]*
- fixed moments $n \leq 16$ *[Moch, Ruijl, Ueda, Vermaseren, Vogt '17]*
- ζ_4, ζ_5 terms conjectured *[Davies, Vogt '17]*

NON-SINGLET @ 4-LOOPS: $n_f C_F^3$

- limit $x \to 0$: $\ln^4(x), \ln^5(x)$ for $P_{ns}^{(3),+}(x)$ *[Davies, Kom, Moch, Vogt '22]*
- limit $x \to 1$: $P_{ns}^{(3),+} \approx P_{ns}^{(3),-} \approx A_4 \left[\frac{1}{1-x}\right]_+ + B_4 \delta(1-x) + C_4 \ln(1-x) + D_4 - A_4 + \mathcal{O}(1-x)$

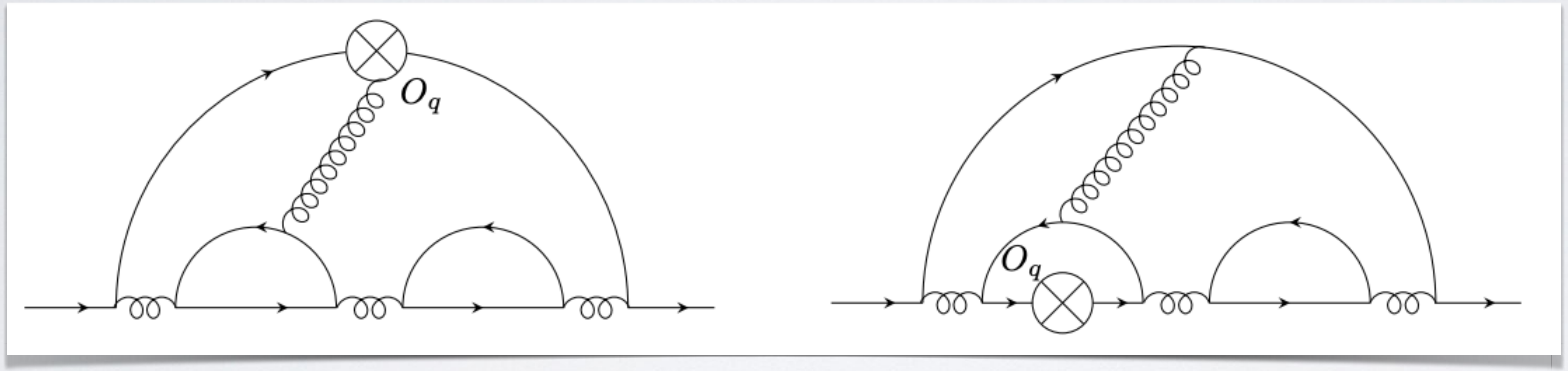
$$A_4\Big|_{N_f C_F^3} = \frac{592}{3}\zeta_3 - 320\zeta_5 + \frac{572}{9}, \quad B_4\Big|_{N_f C_F^3} = 224\zeta_3^2 - \frac{256}{3}\zeta_2\zeta_3 - 308\zeta_3 + 162\zeta_2 - 204\zeta_4 + 912\zeta_5 - \frac{6434}{9}\zeta_6 + 32 \simeq 80.779482\,,$$

$$C_4\Big|_{N_f C_F^3} = 256\zeta_3 - \frac{880}{3}, \quad D_4\Big|_{N_f C_F^3} = 80\zeta_2 - 192\zeta_3 + \frac{464\zeta_4}{3} - \frac{638}{3}$$

  - **Cusp anomalous dimension** A_4 analytic *[Grozin '18; Henn, Korchemsky, Mistlberger, '19; AvM, Panzer, Schabinger '20]*
  - **Virtual anomalous dimension** B_4 numeric *[Das, Mach, Vogt '19]*
  - **Collinear anomalous dimension** analytic *[Agarwal, AvM, Panzer, Schabinger '21,'22]*
  - From B_4 and coll. anom. dim. find **rapidity anomalous dimension** *[Moult, Zhu, Zhu '21; Duhr, Mistlberger, Vita '21]*
  - $C = A^2$, $D = A(B + \beta/(2a_s))$ *[Dokshitzer, Marchesini, Salam '05; Basso, Korchemsky '06]* confirmed for $n_f C_F^3$
- Recently confirmed by *[Kniehl, Moch, Velizhanin, Vogt '25]*

PURE SINGLET @ 4-LOOPS: N_f^2

- Calculated four-loop contributions for quark, with two or three closed fermion loops *[Gehrmann, AvM, Sotnikov, Yang '23]*

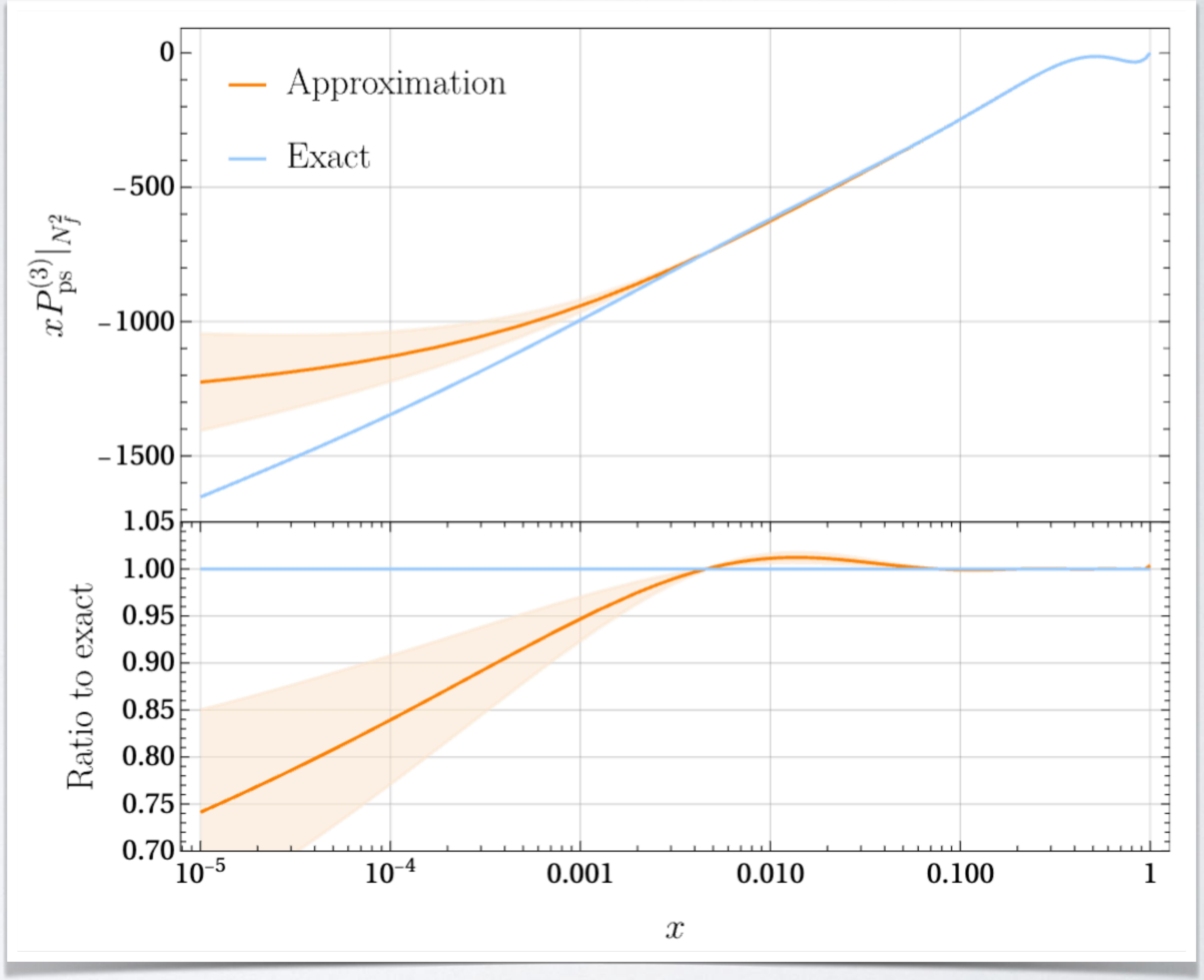


(singlet and non-singlet, also non-planar)

- Simple analytical result for splitting functions in terms of HPLs and powers of x

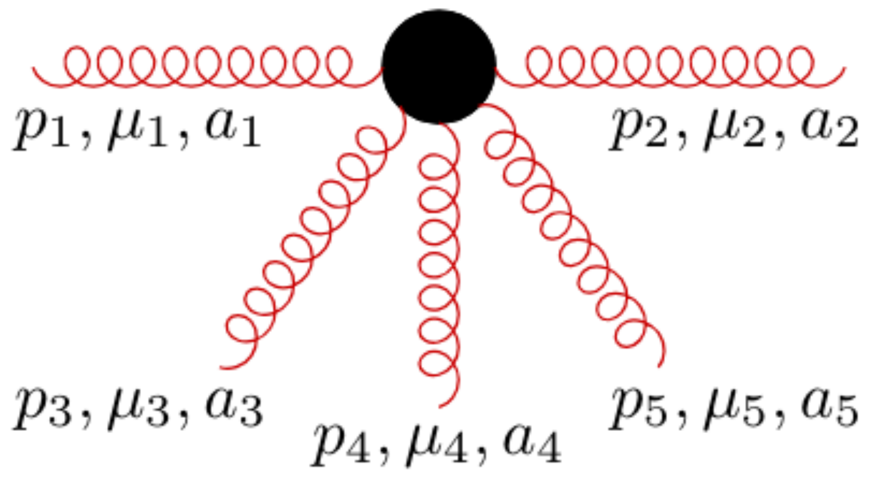
PURE SINGLET @ 4-LOOPS: N_f^2

- $n \leq 20$ by *[Falcioni, Herzog, Moch, Vogt '23]*
- partial information for $x \to 0$: *[Catani, Hautmann '94; Davies, Kom, Moch, Vogt '22]*
- leading terms for $x \to 1$: *[Soar, Moch, Vermaseren, Vogt '09]*
- Generate fit similar to *[Falcioni, Herzog, Moch, Vogt '23]*, compare to all-n result
- All-n result improves small x knowledge



[Gehrmann, AvM, Sotnikov, Yang '23]

1-LOOP CONTRIBUTION TO 5-GLUON CT



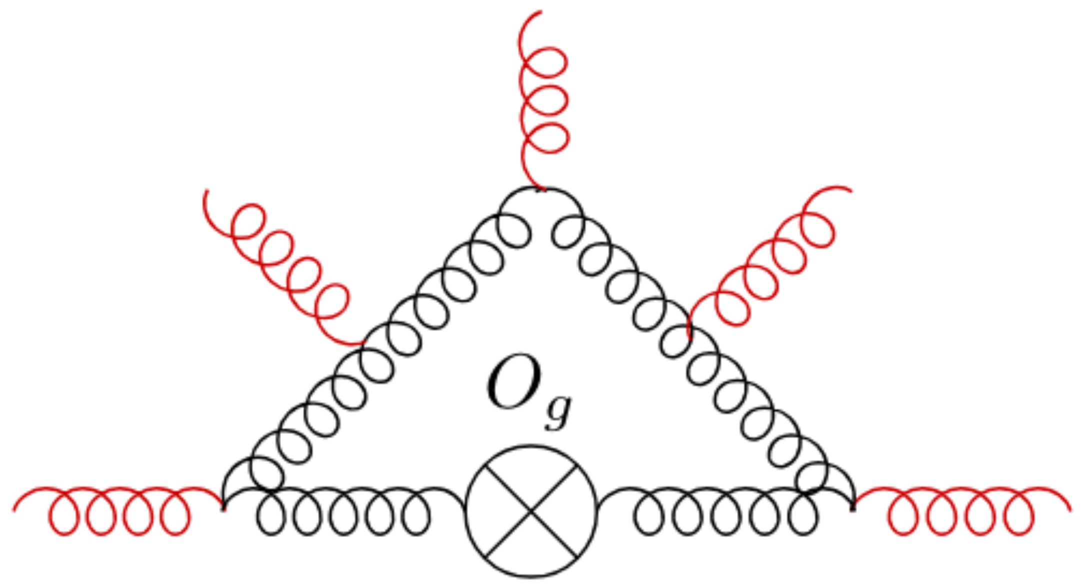
$$\to \frac{1+(-1)^n}{2}\, i\, g_s^3\, \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} p_1^{\mu_5}\Bigg[\frac{1}{C_A} f^{aa_1a_2} d_4^{aa_3a_4a_5}\Bigg\{$$

$$\frac{3}{32}\sum_{j_1=0}^{n-4}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}(-\Delta\cdot p_1)^{n-4-j_1}(-\Delta\cdot(p_1+p_2))^{j_1-j_2}$$

$$\times(\Delta\cdot(p_4+p_5))^{j_2-j_3}(\Delta\cdot p_5)^{j_3}+\cdots\Bigg\}$$

$$+\,11\text{ color structures}\Bigg] + 30\text{ Lorentz Structures} \qquad 17074\text{ lines}$$

extracted from



[Gehrmann, AvM, Yang '24]

see also: [Falcioni, Herzog, Moch, van Thurenhout '24]

SUMMARY

- **Four-loop splitting functions:**
  - Significant number of Mellin moments known
  - Ongoing efforts to compute exact x dependence
  - Off-shell OME conceptually challenging for singlet contributions, but computationally advantageous
  - Tracing parameter allows to obtain all-order n results using standard IBP + DEQs