Loop Summit 2 - new perturbative results and methods in precision physics: Quantum Field Theory and Collider Physics
Cadenabbia, Italy, July 24, 2025

Massive 3-loop Form Factors: analytic computation techniques

Carsten Schneider

RISC–DESY cooperation: A. De Freitas – J. Bluemlein, P. Marquard

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

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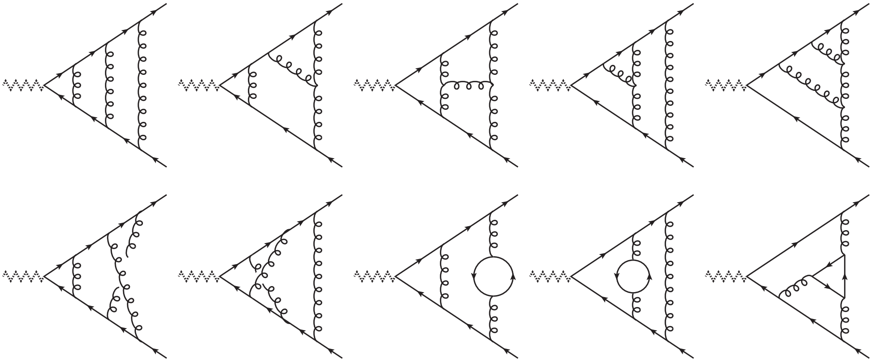
Der Wissenschaftsfonds.

Preperation step (for our RISC-software)

We use QGRAF [Nogueira, '93] to generate 337 diagrams; for instance,



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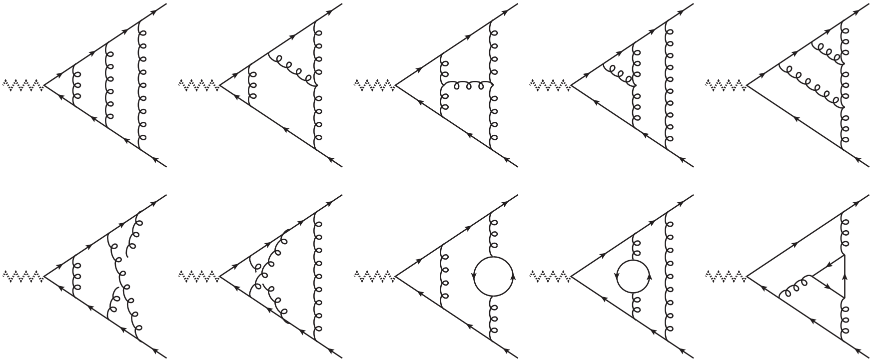

M. Fael, F. Lange, K. Schönwald, and M. Steinhauser, Phys. Rev. D 106, 034029 (2022)
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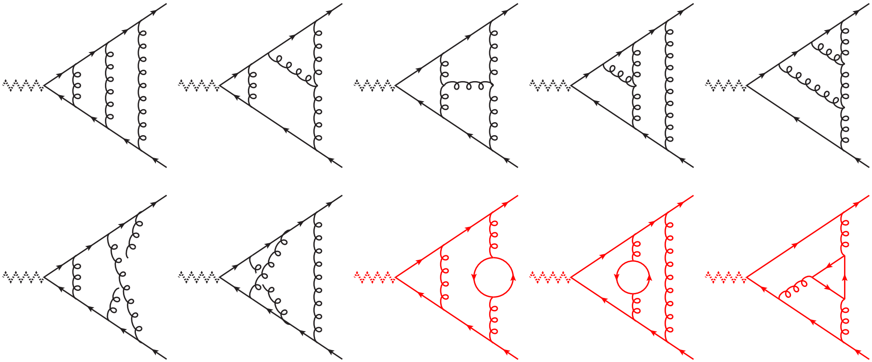

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J. Blümlein, A. De Freitas, P. Marquard, N. Rana, CS. Phys. Rev. D 108, 094003 (2023)

Preperation step (for our RISC-software)

It is convenient to introduce the variable x given by

$$x = \frac{\sqrt{4-s} - \sqrt{-s}}{\sqrt{4-s} + \sqrt{-s}}$$

or

$$s = \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

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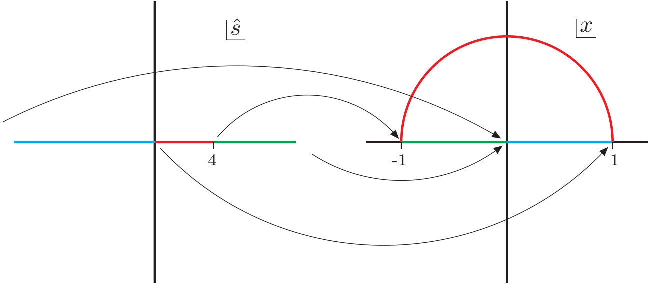

We work with both variables, x and s

Preperation step (for our RISC-software)

We use integration-by-parts identities (IBPs) [Chetyrkin, Tkachov, '81] to reduce to a basis of master integrals.

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and ontain a system of differential equations derived for the master integrals from the IBPs.

$$\frac{d}{dx}M(x,\varepsilon) = A(x,\varepsilon)M(x,\varepsilon)$$

Approach 1: Solve coupled systems of differential equations

[coming from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

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given

A whole industry (for solutions of ε-expansions) started with

[J. Henn. Multiloop integrals in dimen- sional regularization made simple. Phys. Rev. Lett., 110:251601, 2013.]

Here we follow another successful tactic.

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$$\Big\downarrow \text{ uncoupling algorithms (Zürcher, Abramov/Zima, Gauss, ...)}$$

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinComb}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinComb}(\dots, D^i\hat{R}_i(x), \dots)$$

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DE-solver

`HarmonicSums.m` can find all iterative integrals over hyperexpontial functions (in parts also Liouvillian solutions with Kovacic's algorithm)

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DE-solver
`HarmonicSums.m` can find all iterative integrals over hyperexpontial functions (in parts also Liouvillian solutions with Kovacic's algorithm)

RE-solver
Sigma.m can find all iterative sums over hypergeometric products

Approach 1: solving REs

DE system

$$\boxed{D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)}$$

Approach 1: solving REs

$$\boxed{\begin{array}{c}\text{DE system}\\ D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)\end{array}} \xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}} \boxed{\begin{array}{c}\text{uncoupled DE system}\\ \sum_i b_i(x) D^i \hat{I}_1(x) = r(x)\\ \hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1\end{array}}$$

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$$\xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}}$$

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holonomic closure prop.

linear recurrence
$$\sum_i a_i(n) I_1(n+i) = R(n)$$

$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

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OreSys package (S. Gerhold) $\longrightarrow$ uncoupling algorithm

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$\downarrow$ holonomic closure prop.

linear recurrence

$$\textstyle\sum_i a_i(n) I_1(n+i) = R(n)$$

`Sigma's` RE-solver $\longleftarrow$

closed form solutions of $I_1(n)$
in the class of nested sums and products
– if this is possible

Approach 1: solving REs (SolveCoupledSystem package)

DE system

$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$\sum_i b_i(x) D^i \hat{I}_1(x) = r(x)$

$\hat{\mathbf{I}}_{\mathbf{k}}(\mathbf{x}) = \text{expr}_{\mathbf{k}}(\hat{\mathbf{I}}_{\mathbf{1}}(\mathbf{x})), \mathbf{k} > \mathbf{1}$

extract coefficient

SumProduction package

holonomic closure prop.

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J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, 2016.
J. Ablinger, A. Behring, J. Bluemlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. Physical Review D 97(094022), 2018.

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

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Calculations based on Approach 1:

- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(n)$ of the Variable Flavor number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), pp. 1-19. 2016.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, pp. 33-112. 2016.
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, n. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D 97(094022), pp. 1-44. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, CS, K. Schönwald. The two-mass contribution to the three-loop pure singlet operator matrix element. Nucl. Phys. B(927), pp. 339-367. 2018. ISSn 0550-3213.
- J. Blümlein, A. De Freitas, CS, K. Schönwald. The Variable Flavor number Scheme at next-to-Leading Order. Physics Letters B 782, pp. 362-366. 2018.
- J. Ablinger, J. Blümlein, P. Marquard, n. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit. Physics Letters B 782, pp. 528-532. 2018.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, A. von Manteuffel, CS, K. Schönwald. The Unpolarized and Polarized Single-Mass Three-Loop Heavy Flavor Operator Matrix Elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$. Journal of High Energy Physics 2022(12), pp. 1-55. 2022.

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
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Approach 2: Compute large moments and guessing REs/DEs

[coming, e.g., from IBP methods]

Approach 1: solving REs (SolveCoupledSystem package)

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$$\sum_i b_i(x) D^i \hat{I}_1(x) = r(x)$$

$$\mathbf{\hat{I}_k(x)} = \text{expr}_{\mathbf{k}}(\mathbf{\hat{I}_1(x)}), \mathbf{k > 1}$$

extract coefficient

SumProduction package

holonomic closure prop.

linear recurrence

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closed form solutions of

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in the class of nested sums and products

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closed form solutions of $I_1(n)$ in the class of nested sums and products – if th

Given $R(0), R(1), \ldots, R(10000)$,
compute $I_1(0), I_1(1), \ldots, I_1(10000)$

J. Blümlein, C. Schneider. Physics Letters B 771, pp. 31-36. 2017.

General strategy: physical problem $\hat{P}(x)$

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$n = 0, 1, \dots, 10000$

$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(n)}_{\text{numbers}} + \dots$$

$n = 0, 1, \dots, 100000$

Example (quarkonic case of the Form factor calculation):

For $\propto n_h$ scalar we get

$$\hat{P}(x) = \sum_{n=0}^{\infty} \sum_{k=-3}^{\infty} P_k(n) \varepsilon^k (1-x)^n$$

with

$$
\begin{array}{llllllll}
P_0(0) = & \frac{6998000}{243} & - \frac{2750912}{81}\zeta_2 & - \frac{503744}{27}\zeta_3 & - \frac{7525}{42}\zeta_2^2 & - 320\,\zeta_5 & + \cdots \\
P_0(1) = & 0 & + 0\,\zeta_2 & + 0\,\zeta_3 & + 0\,\zeta_2^2 & + 0\,\zeta_5 & + \cdots \\
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P_0(4) = & \frac{819352033}{54675} & - \frac{243122252}{18225}\zeta_2 & - \frac{28654097}{12150}\zeta_3 & - \frac{857288}{2025}\zeta_2^2 & - \frac{88}{3}\zeta_5 & + \cdots \\
P_0(5) = & \frac{806455766}{54675} & - \frac{78210808}{6075}\zeta_2 & - \frac{13828097}{6075}\zeta_3 & - \frac{323632}{675}\zeta_2^2 & - 32\,\zeta_5 & + \cdots \\
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P_0(2) = & \frac{110996644}{729} & - & \frac{50322416}{3645} & \zeta_2 - & \frac{197680}{81} & \zeta_3 - & \frac{148736}{405} & \zeta_2^2 - & \frac{80}{3} & \zeta_5 + \cdots \\
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\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \ddots
\end{array}$$

$$\begin{array}{c} \text{coupled systems for } P(n) \\ \Big\downarrow{\scriptstyle \texttt{SolveCoupledSystem.m}} \\ \text{large no. of moments} \end{array}$$

coupled systems for $P(n)$

$\downarrow$ `SolveCoupledSystem.m`

large no. of moments

$\swarrow$

numerics

$$\begin{tikzcd}
& \text{coupled systems for } P(n) \arrow[d, "\texttt{SolveCoupledSystem.m}"'] \\
& \text{large no. of moments} \arrow[dl] \arrow[d, "\text{guessing (}\texttt{ore_algebra}\text{ in Sage)}"] \\
\text{numerics} & \text{recurrence}
\end{tikzcd}$$

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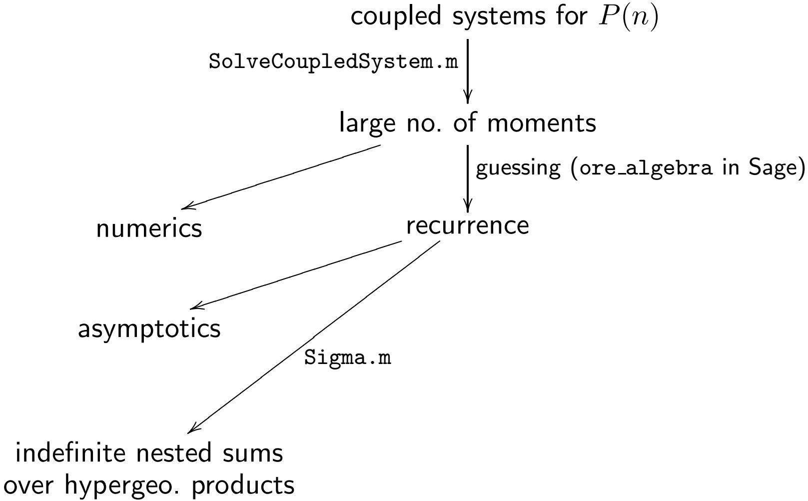
 & & \text{coupled systems for } P(n) \arrow[d, "\texttt{SolveCoupledSystem.m}"'] \\
 & & \text{large no. of moments} \arrow[dll] \arrow[d, "\text{guessing (}\texttt{ore_algebra}\text{ in Sage)}"] \\
\text{numerics} & & \text{recurrence} \arrow[dll] \\
\text{asymptotics} & &
\end{tikzcd}$$


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
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Sigma.m
indefinite nested sums
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coupled systems for $P(n)$

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asymptotics

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over hypergeo. products

indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

$$
\begin{tikzcd}
 & \text{coupled systems for } P(n) \arrow[d, "\texttt{SolveCoupledSystem.m}"'] \\
 & \text{large no. of moments} \arrow[d, "\text{guessing } (\texttt{ore\_algebra} \text{ in Sage})"] \\
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\begin{array}{c}\text{indefinite nested sums}\\ \text{over hypergeo. products}\end{array} &
\end{tikzcd}
$$

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial** $=<<$ **iFile16**

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In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial =<< iFile16**

Out[2]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

21648380901382517/328583783127600,

52869784323778576751/802218994536960000,

49422862094045523994231/753773992230616156800,

33131879832907935920726113/509557943985299969760000,

5209274721836755168448777/80949984111854180459136,

56143711997344769021041145213/882589266383586456384353664,

453500433353845628194790025124807/7217228048879468556886950000000,

14061543374120479886110159898869387/226643167590350326435656036000000,

715586522666491903324905785178619936571168370307700222807811495895030000000,

16286729046359273892841271257418854056836413/269396588055480390401343344736943104000000,

1428729642632302467951426905844691837805299/23940759575034122827861315961573673600000,

498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000, In[3]:= **rec =<< rFile16**

Out[3]=

$$
\begin{aligned}
&(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + \\
&2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + \\
&873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + \\
&1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + \\
&575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + \\
&69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + \\
&901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + \\
&7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + \\
&32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + \\
&86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + \\
&124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + \\
&87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + \\
&25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + \\
&1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]
\end{aligned}
$$

$$
\begin{aligned}
&-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 161315106167193 6n^{37} + \\
&101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
&2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
&9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
&7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
&356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
&9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
&170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
&1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + \\
&13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
&59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
&167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
&284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
&279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} + \\
&149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} + \\
&38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} + \\
&3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} + \\
&101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\mathrm{P}[n+1]
\end{aligned}
$$

$$
\begin{aligned}
&+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+\\
&274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+\\
&25180108291969215434326016n^{32}+284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+\\
&23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+1178515602115604757944201871360n^{27}+\\
&6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+\\
&722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+\\
&28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+\\
&424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+\\
&1499359936674956711935311062995422344n^{15}+2378007025570977662661938772843220240n^{14}+\\
&3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+\\
&4627675779563752366067861596232781096n^{11}+4473175960511956000526499430851993603n^{10}+\\
&3761696365025837909581516781307249585n^{9}+2726553473467254373993685951699145492n^{8}+\\
&1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+\\
&371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+330487623301456239690587044486973 13n^{3}+\\
&6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\big)P[n+2]
\end{aligned}
$$

$$
\begin{aligned}
&-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+\\
&276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+\\
&25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+\\
&24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+\\
&7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+\\
&745539218875020737621728364206080n^{23}+2824909633156578132652259733712896n^{22}+9618101958268071244680677589035520n^{21}+\\
&29441860528446423517613263360742912n^{20}+81033563306363873505877563416477312n^{19}+200454769103641040142838133702338304n^{18}+\\
&445286624972461749049425309485328992n^{17}+887028447418790661018847407251573152n^{16}+\\
&1581538101499869694224895701784875304n^{15}+2517550244392724509968791166585362672n^{14}+\\
&3566593026520465155504695877897282630n^{13}+4479066125207404898722179511912639638n^{12}+\\
&4962006990874351800791769650243464872n^{11}+4819992643914265990647887896664485209n^{10}+\\
&4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+\\
&1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+\\
&412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+\\
&7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)\mathsf{P}[n+3]
\end{aligned}
$$

$$
\begin{aligned}
&+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} + \\
&3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} + \\
&38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} + \\
&69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} + \\
&32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} + \\
&1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} + \\
&20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} + \\
&260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} + \\
&2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} + \\
&10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} + \\
&34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} + \\
&69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} + \\
&82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} + \\
&55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} + \\
&19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} + \\
&3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} + \\
&168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n + \\
&1274120732351744651125603886400\Big)\,\mathsf{P}[n+4]
\end{aligned}
$$

$$
\begin{aligned}
&-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} + \\
&1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + \\
&136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + \\
&83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + \\
&12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + \\
&510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + \\
&6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + \\
&18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + \\
&29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} + \\
&18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} + \\
&2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} + \\
&19313175036907229252501700n + 1248723341516324359641600\Big)\mathsf{P}[n+5]==0
\end{aligned}
$$

In[4]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]];}$

In[5]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[4]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]];}$

In[5]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

Out[5]=

$$\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-$$

$$\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+$$

$$\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+$$

$$\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-$$

$$\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}3(1+n)(1+2n)}{+}\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}$$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= $\mathbf{sol = TransformToSSums[sol];}$

In[8]:= $\mathbf{sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;}$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

Out[8]=
$$\frac{1}{3(1+n)^4(1+2n)^4}\left(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\right)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\left(147+985n+1871n^2+1268n^3+212n^4\right)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\Big)S[1,2n]-\frac{4(3+2n)\left(23+123n+114n^2\right)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\left(53+229n+190n^2\right)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\Big)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\left(1+8n+8n^2\right)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[9]:= **SExpansion[sol, n, 2]**

$$\text{Out[9]}= \ln 2^2\left(\frac{64\text{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+ \ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\text{LG}[n]+\frac{64\text{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+ \zeta_2\left(\frac{160\text{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\text{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\text{LG}[n]+\frac{64\text{LG}[n]^3}{3n}+ \frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

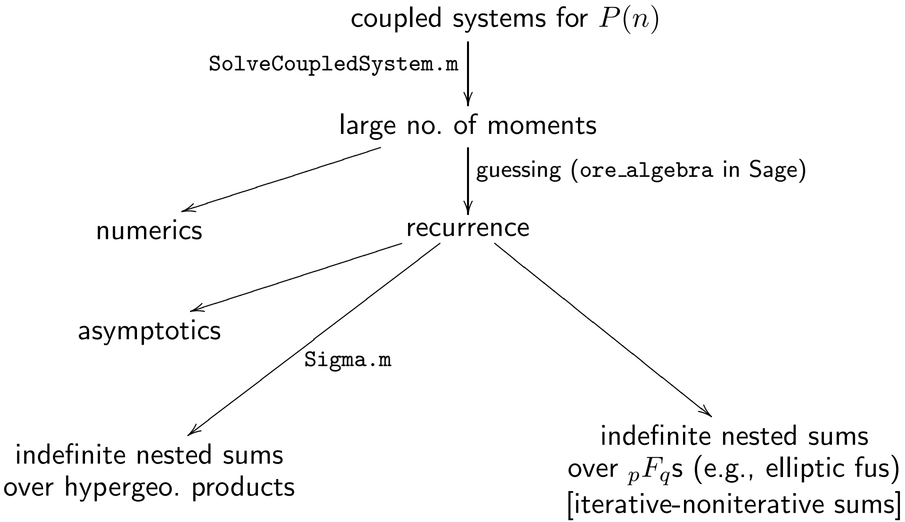
In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

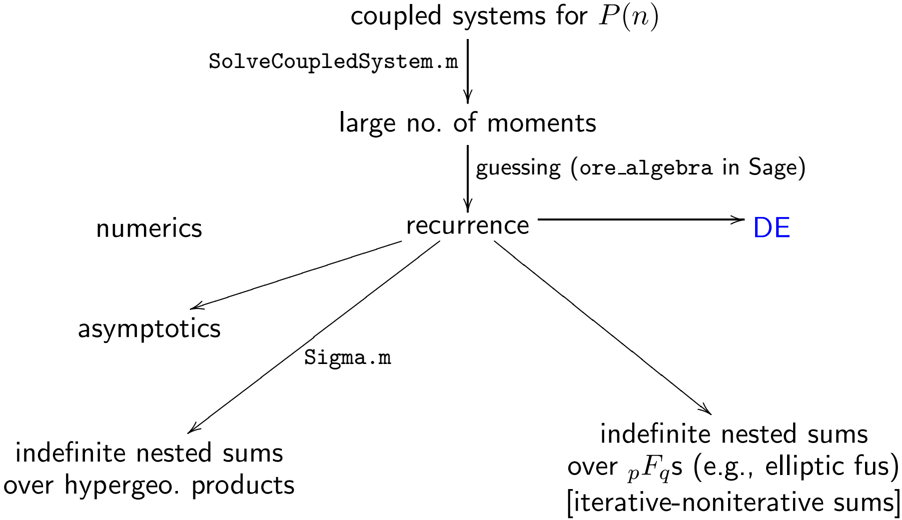
In[9]:= **SExpansion[sol, n, 2]**

Out[9]=
$$\ln 2^2\left(\frac{64\text{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\text{LG}[n]+\frac{64\text{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\text{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\text{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\text{LG}[n]+\frac{64\text{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Special function algorithms

- `HarmonicSums` package
  Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
  Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]
- `RICA` package
  Blümlein, Fadeev, CS. ACM Communications in Computer Algebra 57(2), pp. 31-34. 2023.


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]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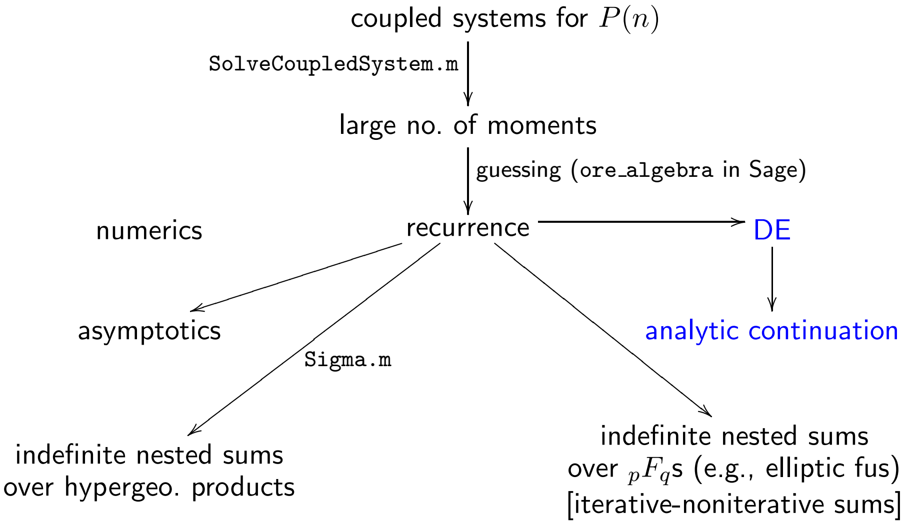
[iterative-noniterative sums]


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
recurrence
DE
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analytic continuation
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indefinite nested sums
over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

The easy (quarkonic) case

Evaluate beyond 0

convergency radius

$r = 1$

$$\sum_{n=0}^{\infty} f_n (1-x)^n$$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

convergency radius $\quad r = 1$

$0 \qquad x \qquad 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n \qquad\qquad \sum_{n=0}^{\infty} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency
radius

$r = 1$

$0 \qquad x \qquad 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{\infty} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency radius

$r = 1$

0

$x < 0.078$

1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{\infty} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency radius

$r = 1$

0 $\quad x < 0.078 \quad$ 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$ by a guessed rec of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency
radius

$r = 1$

0 $\quad x < 0.078 \quad$ 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency
radius

$r = 1$

0 $\quad x < 0.078 \quad$ 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48,
deg 2800)

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$
by a guessed rec
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J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency radius

$r = 1$

0 $\quad x = 0.005$ $\quad 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48, deg 2800)

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$
1400 digits precision

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency radius

$r = 1$

0 $\quad x = 0.005 \quad$ 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48, deg 2800)

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$
1400 digits precision

$\Big\downarrow$ PSLQ

$g_{j,n} \in \mathbb{Q}(\pi, \zeta_3, \ldots)$

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

The hard (gluonic) case: calculations for the constants:

$$\left\{1,\, \zeta_2,\, \zeta_3,\, \ln(2)\zeta_2,\, \zeta_2^2,\, \ln^2(2)\zeta_2,\, \mathrm{Li}_4\left(\tfrac{1}{2}\right),\, \ln^4(2),\, \zeta_2\zeta_3,\, \zeta_5\right\}$$

Challenge 1: Compute moments and guess REs/DEs

The hard (gluonic) case: calculations for the constants:

$$\left\{1,\, \zeta_2,\, \zeta_3,\, \ln(2)\zeta_2,\, \zeta_2^2,\, \ln^2(2)\zeta_2,\, \mathrm{Li}_4\left(\tfrac{1}{2}\right),\, \ln^4(2),\, \zeta_2\zeta_3,\, \zeta_5\right\}$$

Challenge 1: Compute moments and guess REs/DEs

- ζ_4, $\ln^4(2)$, $\mathrm{Li}_4\left(\frac{1}{2}\right)$, $\ln^2(2)\zeta_2$: 800 coefficients in x (500 in s) to generate the recursions and differential equations.

The hard (gluonic) case: calculations for the constants:

$$\left\{1,\, \zeta_2,\, \zeta_3,\, \ln(2)\zeta_2,\, \zeta_2^2,\, \ln^2(2)\zeta_2,\, \mathrm{Li}_4\left(\tfrac{1}{2}\right),\, \ln^4(2),\, \zeta_2\zeta_3,\, \zeta_5\right\}$$

Challenge 1: Compute moments and guess REs/DEs

- ζ_4, $\ln^4(2)$, $\mathrm{Li}_4\left(\frac{1}{2}\right)$, $\ln^2(2)\zeta_2$: 800 coefficients in x (500 in s) to generate the recursions and differential equations.
- $\ln(2)\zeta_2$: up to 6000 coefficients in s.

The hard (gluonic) case: calculations for the constants:

$$\left\{1,\, \zeta_2,\, \zeta_3,\, \ln(2)\zeta_2,\, \zeta_2^2,\, \ln^2(2)\zeta_2,\, \mathrm{Li}_4\left(\tfrac{1}{2}\right),\, \ln^4(2),\, \zeta_2\zeta_3,\, \zeta_5\right\}$$

Challenge 1: Compute moments and guess REs/DEs

- ζ_4, $\ln^4(2)$, $\mathrm{Li}_4\left(\frac{1}{2}\right)$, $\ln^2(2)\zeta_2$: 800 coefficients in x (500 in s) to generate the recursions and differential equations.
- $\ln(2)\zeta_2$: up to 6000 coefficients in s.
- ζ_3: 10000 coefficients in s.
  Time for MIs: 15 months CPU time [3 month by parallelization]

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$$

$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_{0}(n) + \dots}_{\text{only numbers in } \mathbb{Q}}$$

$n = 0, 1, \dots, 100000$

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$$

$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_{0}(n) + \dots}_{\text{only numbers in } \mathbb{Q}}$$

$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(n)}_{\text{numbers}} + \dots$$

$n = 0, 1, \dots, 100000$

The hard (gluonic) case: calculations for the constants:

$$\left\{1,\, \zeta_2,\, \zeta_3,\, \ln(2)\zeta_2,\, \zeta_2^2,\, \ln^2(2)\zeta_2,\, \mathrm{Li}_4\left(\tfrac{1}{2}\right),\, \ln^4(2),\, \zeta_2\zeta_3,\, \zeta_5\right\}$$

Challenge 1: Compute moments and guess REs/DEs

- ζ_4, $\ln^4(2)$, $\mathrm{Li}_4\left(\frac{1}{2}\right)$, $\ln^2(2)\zeta_2$: 800 coefficients in x (500 in s) to generate the recursions and differential equations.
- $\ln(2)\zeta_2$: up to 6000 coefficients in s.
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  Time for MIs: 15 months CPU time [3 month by parallelization]

  Time for combination: 2 month [12 days (d) by parallelization]

  Scalar: 12 d [2 d]; PS: 13 d [2 d]; Vector: 23 d [4 d]; AV: 24 d [4 d]

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RE orders for the scalar case

	ζ_2^2	$\ln^2(2)\zeta_2$	$\ln^4(2)$	$\mathrm{Li}_4\left(\frac{1}{2}\right)$	$\ln(2)\zeta_2$	ζ_3	ζ_2
C_F^3	14	14	14	14	38	58	59
$C_A C_F^2$	14	14	14	14	38	57	58
$C_A^2 C_F$	13	13	12	12	37	55	56

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DE orders for the scalar case

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$C_A C_F^2$	13	13	13	13	32	57	57
$C_A^2 C_F$	12	12	10	10	31	55	55

The easy (quarkonic) case

Matching evaluations at a common point x

$r = 0.078$

convergency
radius

$r = 1$

0 $\quad x = 0.005$ $\quad 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48, deg 2800)

$$\sum_{n=0}^{500000} f_n (1-x)^n$$

find $g_{j,n} \in \mathbb{R}$
1400 digits precision

given $f_n \in \mathbb{Q}$
by a guessed rec
of order 55

$\xrightarrow{\text{PSLQ}}$

$g_{j,n} \in \mathbb{Q}(\pi, \zeta_3, \ldots)$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

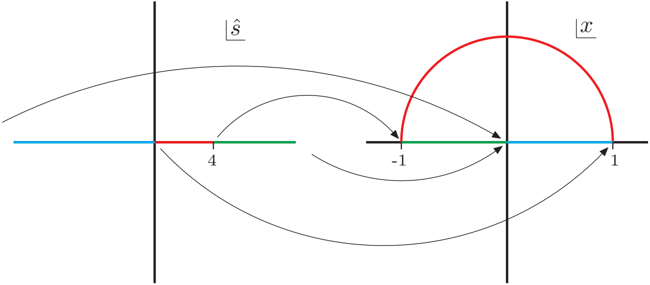
Preperation step (for our RISC-software)

It is convenient to introduce the variable x given by

$$x = \frac{\sqrt{4-s} - \sqrt{-s}}{\sqrt{4-s} + \sqrt{-s}}$$

or

$$s = \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$



We work with both variables, x and s

E.g., the differential equations for $\ln(2)\zeta_2$ exhibit poles at the following points

$$s \in \{-4, -1, -1/2, 1, 3, 4, 16\}.$$

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Only $s = 4$ and $s = 16$ will remain in the final expression

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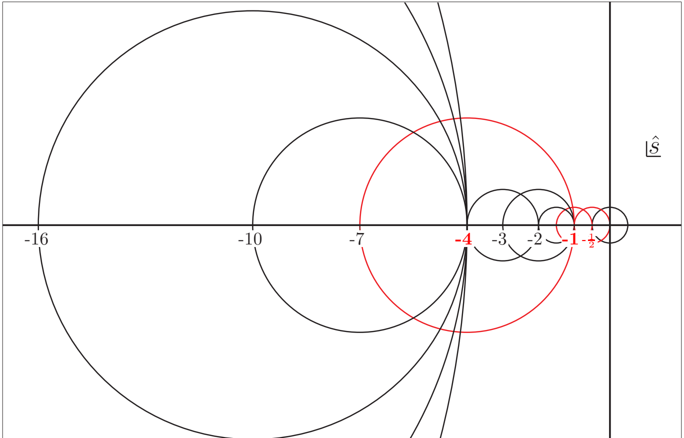
To obtain the expansion at $s = -\infty$ $(x = 0)$, we computed expansions, one after another, at the following points

$$s \in \{-196, -100, -52, -28, -16, -10, -7, -4, -3, -2, -3/2, -1, -1/2\}.$$



- $\ln(2)\zeta_2$: around 1400 digits precision; PSLQ produced all coefficients in terms of:

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- ζ_3: so far 1000 digits precision; PSLQ yield many coefficients in terms of:

$$\left\{1, \pi, \pi^3, \zeta_2, \zeta_3, \ln(2), \pi\ln(2), \zeta_2\ln(2), \ln^2(2), \frac{1}{\pi^2}, \frac{\zeta_3}{\pi^2}, \frac{\zeta_5}{\pi^2}, \frac{\ln^4(2)}{\pi^2}, \frac{\mathrm{Li}_4\left(\frac{1}{2}\right)}{\pi^2}\right\}.$$

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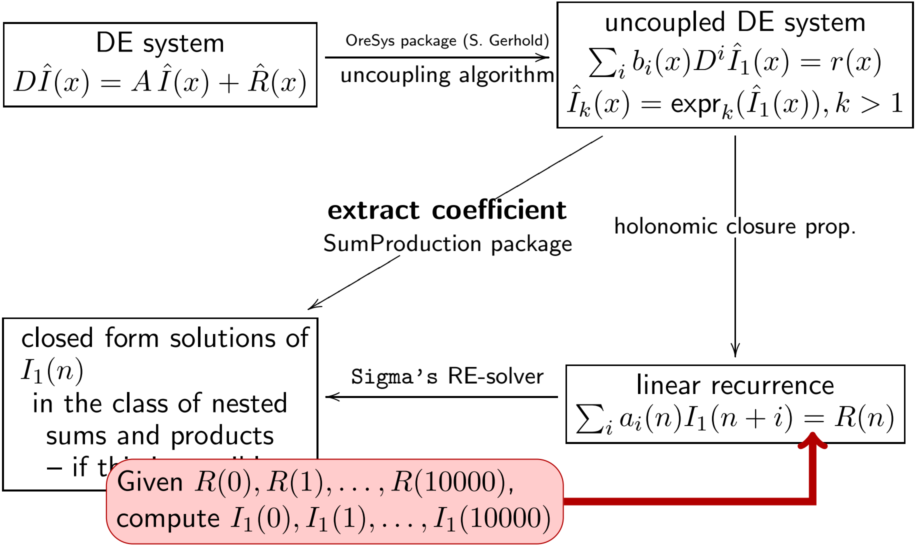
- 1: up to 25000 coefficients in s
  Time for MIs: 41 months [7 month]
  Time for combination: Scalar: 165 d [20 d]; Vector: 378 d [40 d]

Challenge 3: Enhancing the large moment method

- **Trading order versus simple right hand sides**

Approach 2: compute large moments (SolveCoupledSystem package)



J. Blümlein, C. Schneider. Physics Letters B 771, pp. 31-36. 2017.

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$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = r(x)$$

$$\downarrow$$

$$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = R(n).$$

where

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Tactic 1+2: produce more than 200 values with Tactic 1 and apply afterwards Tactic 2 to compute 25K values

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  - the 147 systems consist of 419 MIs
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  - different uncoupling strategies (Gauss, Zuercher, etc) required different (partially manually controlled) communication

Calculation of the recurrence from the 25K values

We obtained all REs for the scalar case

C_F^3
RE: order 85 and degree 3581; values needed: 22356 done
DE: order 96 and degree 3708; values needed: 33726

$C_A C_F^2$
RE: order 83 and degree 3417; values needed: 21216 done
DE: order 93 and degree 3544; values needed: 32060

$C_A^2 C_F$
RE: order 81 and degree 3127; values needed: 19908 done
DE: order 90 and degree 3248; values needed: 29565
and currently guess the DEs.

Calculation of the recurrence from the 25K values

Ongoing calculations of the REs for the vector (hardest) case

$g1C_F^3$:	order 90, degree 3900, 24716 values	done
$g1C_AC_F^2$:	order 90, degree 3841, 24420 values	done
$g2C_F^3$:	order 87, degree 3685, 23309 values	
$g2C_AC_F^2$:	order 86, degree 3633, 23004 values	
$g1C_A^2C_F$:	order 85, degree 3484, 21962 values	done
$g2C_A^2C_F$:	order 82, degree 3306, 20748 values	

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Currently, the DEs for the Vector case are guessed and the 25K moments for the AV and PS cases are computed

World record recurrence:

$$a_0(n)F(n) + a_1(n)F(n+1) + \cdots + \boxed{a_{90}(n)}\,F(n+90) = 0$$

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$$a_{90}(n) = \boxed{A_0} + A_1\, n + A_2\, n^2 + \cdots + A_{3900}\, n^{3900} \in \mathbb{Z}[n]$$

World record recurrence: $\sim$650K A4-pages

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$A_0 =$ 35084493216269801067044913119243756965706172885540510672024044834179236759652 89854752447143682982246272432008854616185695787450273097241682242188547343256 34517613303204671856994649391739373813911028268735799555
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(8084 decimal digits)

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5. large scale computers
   - 11 servers, operated in common with DESY
     (the newest with up to 2 TB memory)
   - the Supercomputer MACH-2 at JKU
     (1728 processor cores with 20 TB global shared memory)