Direct calculation of time-like N3LO splitting and coefficient functions

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Motivation

- One of the cleanest process to probe parton hadronization is the semi-inclusive electron-positron annihilation.
- Experimental uncertainties have reached the scale of theoretical uncertainties.
- Thus, in order to draw meaningful conclusions from data, we must reduce the theoretical uncertainties.
- This would open up not just the possibility of reanalyzing data from LEP, Belle, BaBar; but also lay the theoretical groundwork for the new colliders, like the FCC-ee.

Theoretical background

-Mass-refactorization: hard cross section

$$
\begin{aligned}
\sigma_p = {} & C_p^{(0)} + \\
& \alpha_s \left[-\frac{1}{\varepsilon} \boxed{P^{(0)}_{pp'}} \otimes C^{(0)}_{p'} + \boxed{C^{(1)}_{p}} + \varepsilon \boxed{A^{(1)}_{p}} + \varepsilon^2 \boxed{B^{(1)}_{p}} \right] + \\
& \alpha_s^2 \Bigg[\frac{1}{\varepsilon^2} \left(\frac{1}{2} P^{(0)}_{pi} \otimes P^{(0)}_{ip'} + \frac{\beta_0}{2} P^{(0)}_{pp'} \right) \otimes C^{(0)}_{p'} + \\
& \qquad \frac{1}{\varepsilon} \left(-\frac{1}{2} \boxed{P^{(1)}_{pp'}} \otimes C^{(0)}_{p'} - P^{(0)}_{pp'} \otimes C^{(1)}_{p'} \right) + \\
& \qquad \left(\boxed{C^{(2)}_{p}} - P^{(0)}_{pp'} \otimes A^{(1)}_{p'} \right) + \varepsilon \left(\boxed{A^{(2)}_{p}} - P^{(0)}_{pp'} \otimes B^{(1)}_{p'} \right) \Bigg] + \\
& \alpha_s^3 \Bigg[\frac{1}{\varepsilon^3} \left(-\frac{\beta_0^2}{3} P^{(0)}_{pp'} - \frac{\beta_0}{2} P^{(0)}_{pi} \otimes P^{(0)}_{ip'} - \frac{1}{6} P^{(0)}_{pi} \otimes P^{(0)}_{ij} \otimes P^{(0)}_{jp'} \right) \otimes C^{(0)}_{p'} + \\
& \qquad \frac{1}{\varepsilon^2} \Bigg\{ \left(\frac{\beta_0}{2} P^{(0)}_{pp'} + \frac{1}{2} P^{(0)}_{pi} \otimes P^{(0)}_{ip'} \right) \otimes C^{(1)}_{p'} + \\
& \qquad\qquad \left(\frac{\beta_1}{3} P^{(0)}_{pp'} + \frac{\beta_0}{2} P^{(1)}_{pp'} + \frac{1}{3} P^{(0)}_{pi} \otimes P^{(1)}_{ip'} + \frac{1}{6} P^{(1)}_{pi} \otimes P^{(0)}_{ip'} \right) \otimes C^{(0)}_{p'} \Bigg\} + \\
& \qquad \frac{1}{\varepsilon} \Bigg\{ -\frac{1}{3} \boxed{P^{(2)}_{pp'}} \otimes C^{(0)}_{p'} - \frac{1}{3} P^{(1)}_{pp'} \otimes C^{(1)}_{p'} - P^{(0)}_{pp'} \otimes C^{(2)}_{p'} + \left(\frac{\beta_0}{2} P^{(0)}_{pp'} + \frac{1}{2} P^{(0)}_{pi} \otimes P^{(0)}_{ip'} \right) \otimes A^{(1)}_{p'} \Bigg\} + \\
& \qquad\qquad \left(\boxed{C^{(3)}_{p}} - P^{(0)}_{pp'} \otimes A^{(2)}_{p'} - \frac{1}{2} P^{(1)}_{pp'} A^{(1)}_{p'} + \left(\frac{\beta_0}{2} P^{(0)}_{pp'} + \frac{1}{2} P^{(0)}_{pi} \otimes P^{(0)}_{ip'} \right) \otimes B^{(1)}_{p'} \right) \Bigg]
\end{aligned}
$$

- The calculation is pretty straight forward. We have to:
  - calculate the relevant amplitudes,
  - reduce the integrals with an IBP software,
  - calculate the integrals,
  - do the renormalization of the amplitudes,
  - iteratively determine the splitting and coefficient functions.

- The 2-,3-,4- and 5-particle-cut semi-inclusive integrals were calculated at 4 loops by dr. Vitaly Magerya during his PhD.
- He calculated the integrals with the exclusion method, meaning he restricted the phase space; put the final state particles on mass-shell and introduced the tag as a mass.

$$I = \int \prod_i \frac{d^d l_i}{(2\pi)^d} \mathrm{dPS}_n(q) \prod_j \frac{1}{D_j^{\nu_j}} \delta\left(x - 2\frac{qp}{q^2}\right)$$

Theoretical background

-Calculation of the integrals

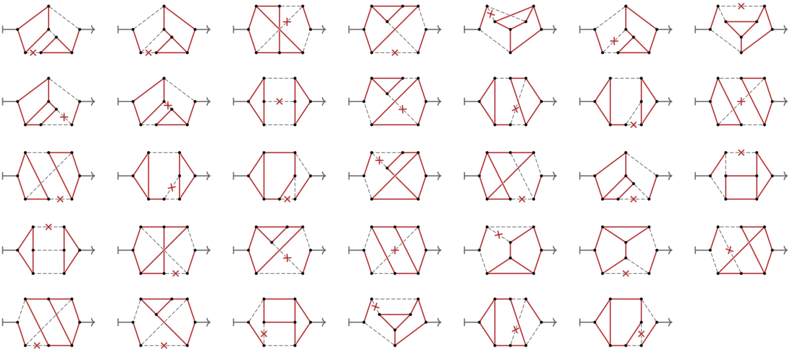

Figure: 3-particle-cut semi-inclusive integral families; source [1]

- The quark and gluon form factors contribute the part proportional to $\delta(1-x)$. Although, these can be found in the existing literature we have recalculated them. [2]

$$\Bigg[\Bigg\{\left(-\frac{\beta_0^3}{\varepsilon^3}-\frac{\beta_0\beta_1}{\varepsilon^2}\right)\langle M^{(0)}|M^{(0)}\rangle_{gg}+\left(\frac{\beta_0^2}{\varepsilon^2}-\frac{\beta_1}{\varepsilon}\right)\langle M^{(1)}|M^{(0)}\rangle_{gg}-\frac{\beta_0}{\varepsilon}\langle M^{(2)}|M^{(0)}\rangle_{gg}-\frac{2\beta_0}{\varepsilon}\langle M^{(1)}|M^{(1)}\rangle_{gg}+\langle M^{(2)}|M^{(1)}\rangle_{gg}\Bigg\}_{[2\times1]}+$$

$$\Bigg\{\left(-\frac{\beta_0^3}{\varepsilon^3}-\frac{2\beta_0\beta_1}{\varepsilon^2}-\frac{\beta_2}{\varepsilon}\right)\langle M^{(0)}|M^{(0)}\rangle_{gg}+\left(\frac{3\beta_0^2}{\varepsilon^2}-\frac{3\beta_1}{2\varepsilon}\right)\langle M^{(1)}|M^{(0)}\rangle_{gg}-\frac{3\beta_0}{\varepsilon}\langle M^{(2)}|M^{(0)}\rangle_{gg}+\langle M^{(3)}|M^{(0)}\rangle_{gg}\Bigg\}_{[3\times0]}\Bigg]_{\underline{2\ \mathrm{cut}}}$$

Challenges

-Exploding complexity: the hard cross-section

- The 3[1],4 and 5 cut cases were never calculated at N3LO order, thus these are completely new.

$$2\left[\left\{\frac{9\beta_0^2}{4\varepsilon^2}\langle M^{(0)}|M^{(0)}\rangle_{q\bar{q}g} - \frac{3\beta_0}{\varepsilon}\langle M^{(1)}|M^{(0)}\rangle_{q\bar{q}g} + \langle M^{(1)}|M^{(1)}\rangle_{q\bar{q}g}\right\}_{[1\times1]} + \left\{\left(\frac{15\beta_0^2}{8\varepsilon^2} - \frac{5\beta_0}{4\varepsilon}\right) - \frac{5\beta_0}{2\varepsilon}\langle M^{(1)}|M^{(0)}\rangle_{q\bar{q}g} + \langle M^{(2)}|M^{(0)}\rangle_{q\bar{q}g}\right\}_{[2\times1]}\right]_{\underline{3\ \text{cut}}} +$$

$$2\left[\left\{\frac{9\beta_0^2}{4\varepsilon^2}\langle M^{(0)}|M^{(0)}\rangle_{ggg} - \frac{3\beta_0}{\varepsilon}\langle M^{(1)}|M^{(0)}\rangle_{ggg} + \langle M^{(1)}|M^{(1)}\rangle_{ggg}\right\}_{[1\times1]} + \left\{\left(\frac{15\beta_0^2}{8\varepsilon^2} - \frac{5\beta_0}{4\varepsilon}\right) - \frac{5\beta_0}{2\varepsilon}\langle M^{(1)}|M^{(0)}\rangle_{ggg} + \langle M^{(2)}|M^{(0)}\rangle_{ggg}\right\}_{[2\times1]}\right]_{\underline{3\ \text{cut}}}$$

[2] During our calculation an independent group has published an article, where they calculate these parts [3]

- From the computational perspective these diagrams pose no problem.
- The longest calculation is the $H \rightarrow ggggg$, which has $230^2 = 52900$ diagrams. However, since they are tree level they can be calculated in less than 4 days, with 4 cores per diagram and running 100 calculation parallel.

$$\left[\left\{ -\frac{2\beta_0}{\varepsilon} \langle M^{(0)}|M^{(0)}\rangle_{q\bar{q}gg} + \langle M^{(1)}|M^{(0)}\rangle_{q\bar{q}gg} \right\}_{[1\times 0]}\right]_{4\ \mathit{cut}} +$$

$$\left[\left\{ -\frac{2\beta_0}{\varepsilon} \langle M^{(0)}|M^{(0)}\rangle_{gggg} + \langle M^{(1)}|M^{(0)}\rangle_{gggg} \right\}_{[1\times 0]}\right]_{\underline{4\ \mathrm{cut}}} +$$

$$\left[\left\{ \langle M^{(0)}|M^{(0)}\rangle_{ggggg} + \langle M^{(0)}|M^{(0)}\rangle_{q\bar{q}ggg} + \langle M^{(0)}|M^{(0)}\rangle_{q\bar{q}q\bar{q}g} \right\}_{[0\times 0]}\right]_{\underline{5\ \mathrm{cut}}}$$

Challanges

-Bottlenecks the technical side

UHH ELTE

- The number of families at four loops in the case of five final state particles reach above a hundred and the reduction of even one family requires great resources.
- The size and complexity of the results of the IBP reduction also increases;
  - expressions are not particularly huge, IBP results are at the order of $10^6 - 10^7$ fully expanded and maximum are a few tens of Gbs in Mathematica's own binary dump formats (.mx or .wxf).
  - however, even basic operations like series expansion and partial fractioning pose a challenge.
- We overcome these by two methods:
  - optimizing already existing algorithm,
  - heavy parallelization.

Challenges
-Bottlenecks the technical side

- Optimizing already existing algorithm:
  - LinApart for partial fractioning the IBP output for faster gathering and series expansion [4] ,
  - alibrary for harmonic polylogarithms and streamlining the calculation.
- heavy parallelization:
  - with KIRA2 some families require 1Tb+ memory and days of runtime on 30 cores (main bottleneck is the equation generation).
  - in order to get a result in a reasonable time-frame, we run the families parallel; this requires up to 300-400 cores and up to 10Tbs of memory.
  - due to the size of the IBP results, further operations must also be run on 100s of cores. We run the particularly time-consuming operations on 500 cores with basic load balancing in Wolfram Mathematica.

Challenges

-Nuances: plus-distribution with higher multiplicities

- In the case of higher multiplicities during the construction of the plus-distribution we must subtract the Laurent series of the singularity

$$\left[(1-x)^{-n+k\varepsilon}\right]_+ = \int_0^1 dx(1-x)^{-n+k\varepsilon}\left(F(x) - \sum_{i=0}^{n-1}(-1)^i(1-x)^i \lim_{x\to 1}\left[F(x)^{(i)}\right]\right) = \sum_{i=0}^{\infty}\varepsilon^i \frac{k^i}{i!}\left[\frac{ln^i(1-x)}{(1-x)^n}\right]_+$$

- But in this case even though $\lim_{x\to 1}\left[F(x)\right] = F(1)$, the derivatives can diverge in the limit; $\lim_{x\to 1}\left[F(x)^{(i)}\right] \to \infty$.

Challenges

-Nuances: plus-distribution with higher multiplicities

- Although this mathematical possibility exists, it would mean higher order terms in ε influence lower order terms, since:

$$\int_0^1 dx \ln(1-x)^a (1-x)^{-n+k\varepsilon} = -\frac{\Gamma(1+a)}{(-1+n-k\varepsilon)^{1+a}}$$

- For example a double pole, would mean that we still have double singularities like collinear-collinear, soft-soft or collinear-soft in our expression.
- We think that, these are spurious and in the end have to cancel; in the processes we have so far checked, these are indeed absent.

- According to the KNL theorem UV and IR singularities cancel if one sums over ***all*** degenerate initial and final states.
- But since we tag particles we leave out some processes from the sum. In the N2LO, tagged g, Higgs mediated case, we have:

$\lvert\mathcal{M}_{gg}^{[1x1]}\rvert^2$	$\lvert\mathcal{M}_{gg}^{[2x0]}\rvert^2$	$\lvert\mathcal{M}_{ggq}^{[1x0]}\rvert^2$	$\lvert\mathcal{M}_{qqqq}^{[0x0]}\rvert^2$
$\frac{1}{9}\frac{Nf^2}{\varepsilon^2}$	$\frac{2}{9}\frac{Nf^2}{\varepsilon^2}$	$-\frac{4}{9}\frac{Nf^2}{\varepsilon^2}$	$\frac{1}{9}\frac{Nf^2}{\varepsilon^2}$

- Since we leave out the *qqqq* case, the Nf^2 parts do not cancel in the lower orders like $\frac{1}{\varepsilon^2}$.

Status of our research

-What are we aiming for

Process	γ		Z	H
Projection	Transversal	Longitudinal	Axial	
Tagged particle	$q\ g\ \gamma$	$q\ g\ \gamma$	$q\ g\ \gamma$	$q\ g\ \gamma$

- Our aim is plain and simple, we want to directly calculate the time-like N2LO splitting and N3LO coefficient functions.
- A direct calculation has never been done before, the existing results were acquired by the means of analytic continuation.
- With our results, we would confirm the already existing results and deliver all the other yet missing pieces.
- We also plan to publish our code base in order to facilitate future research in this direction.

Status of our research

-What do we have right now

- We have successfully recalculated all of the relevant fully-inclusive results at N3LO in terms of Ca, Cf, Nf. [5, 6]
- We have calculated the amplitudes for the tagged quarks and gluons in case of a mediating photon (both transversal and longitudinal), Z- and Higgs-boson.
- The IBP reduction for the 3- and 4-cut-particle cases have finished, the 5-cut is running.

Status of our research
-What do we have right now

- We have started the construction of the plus-distribution and the cross checks with the inclusive results.
- So far we have checked the following amplitudes:

Loops	Process	Tagged
1 x 1	$H \to q\bar{q}g$	q
1 x 1	$H \to q\bar{q}g$	g
1 x 1	$H \to ggg$	g
2 x 0	$H \to q\bar{q}g$	q
2 x 0	$H \to q\bar{q}g$	g

- We intend to publish the N3LO Higgs tagged quark coefficient functions this year; along with the recalculated inclusive results.

Summary

- A number of different coefficient functions are yet to be calculated at N3LO.
- These are accessible with a direct calculation, which would also serve as a strong check of the already existing results acquired by the means of analytic continuation.
- Our calculation is already in an advanced stage. However along the way we faced harsh, mostly technical difficulties.
- We overcome these, by writing new, faster software solutions and fully utilizing the available resources.

Thank you!

Thank you for your attention!

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