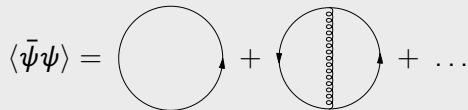


Five loop QCD corrections to zero scale quantities

Andreas Maier Peter Marquard York Schröder



Heavy-quark condensate

$$\langle \bar{\psi}\psi \rangle = \text{[vacuum bubble]} + \text{[vacuum bubble with gluon loop]} + \dots$$


- Leading non-analytic contribution in
 - ▶ Operator Product Expansion ($m \sim \Lambda_{\text{QCD}}$)
 - ▶ Asymptotic small-mass expansion ($m \gg \Lambda_{\text{QCD}}$)

Extrapolation from heavy to light quarks via
Renormalisation Group Optimised Perturbation Theory

[Kneur, Neveu 2010-2020]

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Extrapolation from heavy to light quarks via
Renormalisation Group Optimised Perturbation Theory

[Kneur, Neveu 2010-2020]

- Direct relation to vacuum anomalous dimension:

[Spiridonov, Chetyrkin 1988]

$$\mu^2 \frac{d}{d\mu^2} m \langle \bar{\psi}\psi \rangle = -4m^4 \gamma_0$$

$\hookrightarrow$ independent check of five-loop result [Baikov, Chetyrkin 2018]

Decoupling

Top quarks are problematic for QCD processes with $E < m_t$:

- Diagrams with massive internal lines $\Rightarrow$ hard to calculate
- Large logarithms $\ln\left(\frac{E}{m_t}\right)$ spoil perturbative convergence

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Solution: effective 5-flavour theory with coupling

$$\alpha_s^{(5)} = \alpha_s^{(6)} \times \left[\frac{\left(\text{triangle diagram} \right)^2}{\left(\text{bubble diagram} \right)^2 \cdot \text{gluon bubble}} \right]$$

$q=0$
 $m_t \neq 0$
 $m_Q=0$

Relation known to four loops [Schröder, Steinhauser 2005]

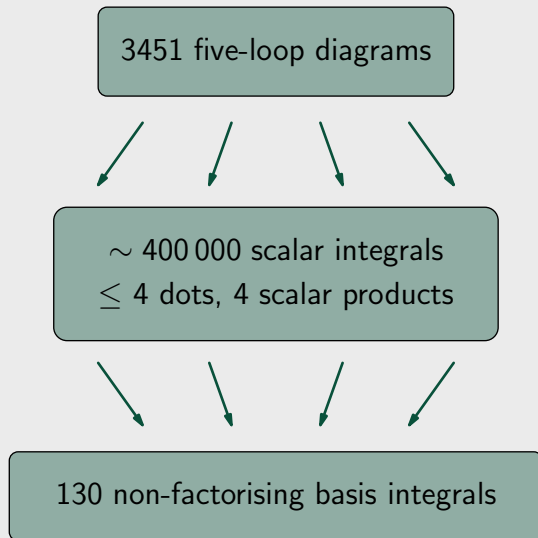
Calculation

Setup

- ✓ Generate diagrams: QGRAF [Nogueira 1991]
- ✓ Identify diagram families: autopsy or dynast
based on nauty and Traces [McKay, Piperno 2014]
↪ 34 mass colourings of 
- ✓ Express diagrams in terms of scalar integrals: FORM [Vermaseren et al.]
- ✓ Reduce to basis integrals: crusher + tinbox
based on Fermat, GiNaC, NTL
- ? Compute and insert basis integrals

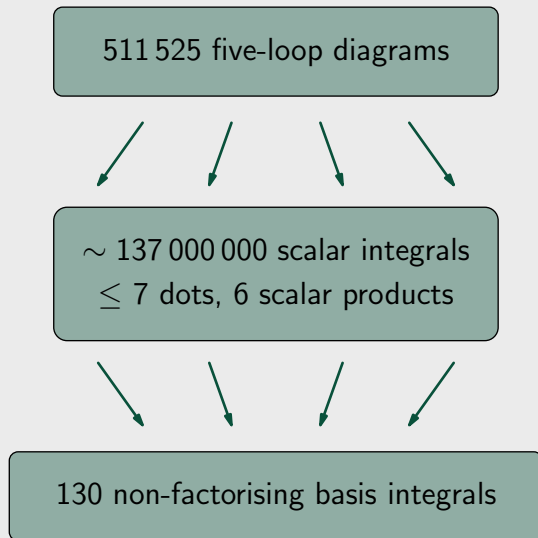
Calculation

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Calculation

Decoupling



Basis integrals

Sector decomposition [Hepp 1966; Speer 1968; Binoth, Heinrich 2000]

Compute basis integrals with FIESTA:

[Smirnov et al. 2008-2021]

$$\langle \bar{\psi} \psi \rangle \Big|_{\left(\frac{\alpha_s}{\pi}\right)^4} = \frac{m^3}{16\pi^2} \left[\frac{(-3.5 \pm 3.0) \times 10^{-8}}{\epsilon^{11}} + \frac{(-2.1 \pm 7.6) \times 10^{-6}}{\epsilon^{10}} \right. \\ \left. + \frac{(0.2 \pm 2.1) \times 10^{-4}}{\epsilon^9} + \frac{(-0.5 \pm 7.5) \times 10^{-2}}{\epsilon^8} + \dots \right]$$

Basis integrals

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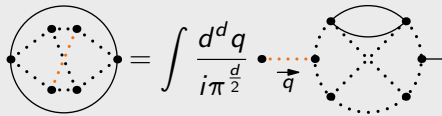
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- ~ 2 digits lost per order in $\epsilon \Rightarrow$ **need better than double precision**
- Found no significant improvement with
 - ▶ Quasi-Monte Carlo methods [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk 2018]
 - ▶ Quasi-finite integral basis [von Manteuffel, Panzer, Schabinger 2014]
 - ▶ Above plus direct Feynman parameter integration
 - ▶ Tropical integration [Borinsky 2020]
 - ▶ pySecDec [Heinrich, Jones, Kerner, Magerya, Olsson, Schlenk 2023]

Basis integrals

Direct integration c.f. [Faisst, Chetyrkin, Kühn 2004]



The diagram shows an equation between two Feynman diagrams. On the left is a circle with eight dots on its circumference and eight dots inside arranged in a square pattern. A dashed orange line connects two of the internal dots. This is followed by an equals sign, then the integral $\int \frac{d^d q}{i\pi^{\frac{d}{2}}}$. To the right of the integral is a diagram consisting of a dashed orange line with an arrow pointing right, labeled $\vec{q}$, which connects to a more complex diagram. This second diagram has eight dots on its outer boundary and several internal dots. It features a loop at the top and a dashed orange line connecting two internal dots, similar to the first diagram.

Basis integrals

Direct integration c.f. [Faisst, Chetyrkin, Kühn 2004]

$$\text{Diagram 1} = \int \frac{d^d q}{i\pi^{\frac{d}{2}}} \text{Diagram 2}$$

With $z = -q^2$:

$$\text{Diagram 1} = \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \text{Diagram 2}$$

Basis integrals

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With $z = -q^2$:

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q does not need to be a line momentum:

$$\text{Diagram} = \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \text{Diagram}_q$$

Expand in ϵ , integrate numerically

Basis integrals

Infrared & ultraviolet divergences

Problem: integrand is **infrared** & **ultraviolet** divergent

Basis integrals

Infrared & ultraviolet divergences

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Initial idea:

- Choose q as **massive** line momentum
 - All heavy fermion lines are closed
 - ⇒ No massless cuts in integrand
 - ⇒ No **infrared** divergences
- Take propagator with momentum q to higher power
 - ⇒ No **ultraviolet** divergence

Basis integrals

Infrared & ultraviolet divergences

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Drawbacks:

- Needs five-loop basis change
- Needs closed massive lines
- Needs **unknown** ingredients → next slides

Alternative: **infrared** and **ultraviolet** subtraction

Basis integrals

Infrared subtraction

Subtract **infrared** divergences:

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \left[\text{Diagram} - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\text{IR}}} \Delta_{k,l}^{\text{IR}} \left(\text{Diagram} \right)^{k+l\epsilon} \right]$$

The diagram inside the brackets is a Feynman diagram with a dashed line on the left and a solid line on the right, connected by a loop of solid lines. The diagram is enclosed in a dashed box.

Basis integrals

Infrared subtraction

Subtract **infrared** divergences:

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \left[\text{Diagram 1} - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{lR}} \Delta_{k,l}^{\text{IR}} \left(\text{Diagram 2} \right)^{k+l\epsilon} \right]$$

=

$$\text{Diagram 3} - \underbrace{\text{Diagram 4}}_{=0}^{k+l\epsilon}$$

Translate **infrared** to **ultraviolet** divergences

Basis integrals

Ultraviolet subtraction

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \left[\text{Diagram} - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\text{IR}}} \Delta_{k,l}^{\text{IR}} \left(\text{Diagram}_1 \right)^{k+l\epsilon} - \sum_{l=0}^{L-1} \sum_{k=k_{\text{UV}}}^2 \Delta_{k,l}^{\text{UV}} \left(\text{Diagram}_2 \right)^{l\epsilon} \left(\text{Diagram}_3 \right)^k \right]$$

The diagram in the first term is a loop diagram with a bubble and a tadpole. The diagrams in the second and third terms are represented by dots and arrows, indicating their structure.

Basis integrals

Ultraviolet subtraction

$$\begin{aligned}
 & \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \left[\text{Diagram} - \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\text{IR}}} \Delta_{k,l}^{\text{IR}} \left(\text{Diagram} \right)^{k+l\epsilon} \right. \\
 & \quad \left. - \sum_{l=0}^{L-1} \sum_{k=k_{\text{UV}}}^2 \Delta_{k,l}^{\text{UV}} \left(\text{Diagram} \right)^{l\epsilon} \left(\text{Diagram} \right)^k \right] \\
 &= \text{Diagram} - \sum_{l=0}^{L-1} \sum_{k=1}^2 \Delta_{k,l}^{\text{UV}} \frac{\Gamma((l+1)\epsilon + k - 2) \Gamma(2 - (l+1)\epsilon)}{\Gamma(2-\epsilon) \Gamma(k)}
 \end{aligned}$$

Differential Equations

Generalised power series ansatz

$P_0 =$  from differential equations for propagator basis integrals:

$$z \frac{\partial}{\partial z} P_i = Q_{ij} P_j$$

[Kotikov 1991, Remiddi 1997,...]

Infrared and ultraviolet counterterms from series ansätze:

$$P_i = \sum_{l=0}^{L-1} \sum_{k=k_{IR}}^{N-1} c_{ikl} z^{k-l\epsilon} + \mathcal{O}(z^N), \quad P_i = \sum_{l=0}^{L-1} \sum_{k=k_{UV}}^{N-1} d_{ikl} \frac{1}{z^{k-l\epsilon}} + \mathcal{O}\left(\frac{1}{z^N}\right)$$

Boundary conditions $c_{ik_{IR}l}$, $d_{ik_{UV}l}$ from asymptotic expansion:
known massless propagators and massive vacuum diagrams with ≤ 4 loops

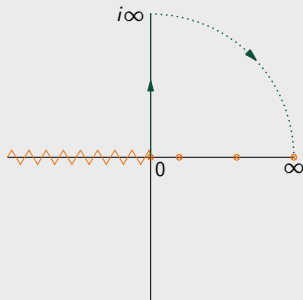
[Laporta 2002; Czakon 2004, Schröder, Vuorinen 2005; Lee, Terekhov 2010; Baikov, Chetyrkin 2010; Lee, Smirnov, Smirnov 2011]

Differential Equations

Direct numeric integration

Spurious poles $\otimes$ for $z > 0$

- Deform contour:



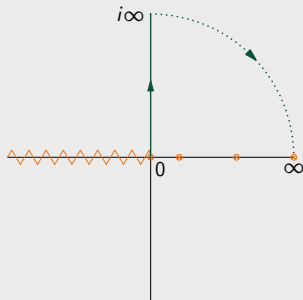
- Integrate analytically for $z \approx 0$, $z \approx \infty$

Differential Equations

Direct numeric integration

Spurious poles $\otimes$ for $z > 0$

- Deform contour:



- Integrate analytically for $z \approx 0$, $z \approx \infty$


$$= -1.03692775514337\epsilon^{-2} - 2.2715175939186296\epsilon^{-1} - 51.677219826124914 + \mathcal{O}(\epsilon)$$

Differential Equations

Padé approximation

Idea: rational function approximation from

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- Subtract logarithms from expansion of $z^{-\epsilon}$

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- Conformal mapping $z \rightarrow \frac{4\omega}{(1-\omega)^2}$

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- Combine into expansion around single point
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$$P_i \approx \frac{a_0 + a_1\omega + \cdots + a_n\omega^n}{1 + b_1\omega + \cdots + b_m\omega^m} z(\omega)^{1-N} + P_i^{\text{subtr}}(\omega)$$

Differential Equations

Padé approximation

Result for $N = 25$:


$$\begin{aligned} &= -1.03692775514337\epsilon^{-2} - 2.2715175939186296\epsilon^{-1} \\ &\quad - (51.677219826125416 \pm 10^{-12}) \\ &\quad - (74.11363065494112 \pm 10^{-10})\epsilon \\ &\quad - (1534.5823448304886 \pm 10^{-9})\epsilon^2 \\ &\quad - (997.9241152040024 \pm 10^{-8})\epsilon^3 \\ &\quad - (69346.26975774963 \pm 10^{-7})\epsilon^4 \\ &\quad + \mathcal{O}(\epsilon^5) \end{aligned}$$

Quark condensate

Status

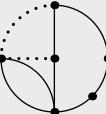
New results for basis integrals:

$$\langle \bar{\psi}\psi \rangle \Big|_{\left(\frac{\alpha_s}{\pi}\right)^4} = \frac{m^3}{16\pi^2} \left[\frac{10^{\cancel{8}-95}}{\epsilon^{11}} + \frac{10^{\cancel{6}-36}}{\epsilon^{10}} + \frac{10^{\cancel{4}-28}}{\epsilon^9} + \frac{10^{\cancel{2}-14}}{\epsilon^8} + \frac{10^{-3}}{\epsilon^7} + \frac{10^{-2}}{\epsilon^6} + \dots \right]$$

What is happening here?

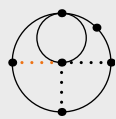
Padé approximation

Limitations


$$= \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon} \dots$$

$$= \frac{1}{120\epsilon^5} + \frac{1}{12\epsilon^4} - \frac{1.1 \pm 2.7}{\epsilon^3} + \mathcal{O}(\epsilon^{-2})$$

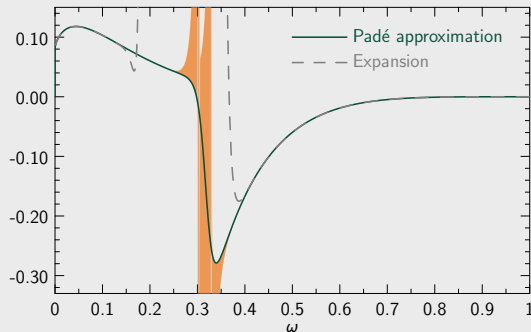
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$$= \frac{1}{120\epsilon^5} + \frac{1}{12\epsilon^4} - \frac{1.1 \pm 2.7}{\epsilon^3} + \mathcal{O}(\epsilon^{-2})$$

Padé approximation at order ϵ^{-3} :



Conclusions

- Ongoing five-loop QCD calculations: quark condensate and decoupling
- Reduction to basis integrals complete
- Need higher precision for basis integrals:
 - ▶ Direct integration
 - ▶ Differential equations
 - ▶ Padé approximation