Five loop QCD corrections to zero scale quantities

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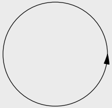
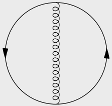
Institut de Física d'Altes Energies

Heavy-quark condensate

$\langle \bar{\psi}\psi \rangle =$  $+$  $+ \ \ldots$

- Leading non-analytic contribution in
  - Operator Product Expansion ($m \sim \Lambda_{\text{QCD}}$)
  - Asymptotic small-mass expansion ($m \gg \Lambda_{\text{QCD}}$)

  Extrapolation from heavy to light quarks via
  Renormalisation Group Optimised Perturbation Theory

[Kneur, Neveu 2010-2020]

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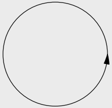
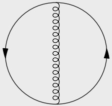
$\langle\bar{\psi}\psi\rangle =$  $+$  $+ \dots$

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  Extrapolation from heavy to light quarks via
  Renormalisation Group Optimised Perturbation Theory [Kneur, Neveu 2010-2020]
- Direct relation to vacuum anomalous dimension: [Spiridonov, Chetyrkin 1988]

$$\mu^2 \frac{d}{d\mu^2} m\langle\bar{\psi}\psi\rangle = -4m^4\gamma_0$$

$\hookrightarrow$ independent check of five-loop result [Baikov, Chetyrkin 2018]

Decoupling

Top quarks are problematic for QCD processes with $E < m_t$:

- Diagrams with massive internal lines $\Rightarrow$ hard to calculate
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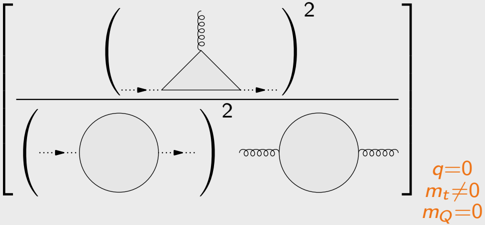
Solution: effective 5-flavour theory with coupling

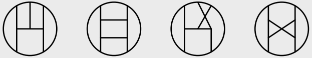
$$\alpha_s^{(5)} = \alpha_s^{(6)} \times$$



Relation known to four loops [Schröder, Steinhauser 2005]

Calculation

Setup

- ✓ Generate diagrams: `QGRAF` [Nogueira 1991]
- ✓ Identify diagram families: `autopsy` or `dynast`
  based on `nauty` and `Traces` [McKay, Piperno 2014]
  ↪ 34 mass colourings of 
- ✓ Express diagrams in terms of scalar integrals: `FORM` [Vermaseren et al.]
- ✓ Reduce to basis integrals: `crusher` + `tinbox`
  based on `Fermat`, `GiNaC`, `NTL`
- ? Compute and insert basis integrals

Calculation

Heavy-quark condensate

3451 five-loop diagrams

↓

$\sim$ 400 000 scalar integrals

$\leq$ 4 dots, 4 scalar products

↓

130 non-factorising basis integrals

Calculation

Decoupling

511 525 five-loop diagrams

$\sim$ 137 000 000 scalar integrals

$\leq$ 7 dots, 6 scalar products

130 non-factorising basis integrals

Basis integrals

Sector decomposition [Hepp 1966; Speer 1968; Binoth, Heinrich 2000]

Compute basis integrals with `FIESTA`: [Smirnov et al. 2008-2021]

$$\langle\bar{\psi}\psi\rangle\Big|_{\left(\frac{\alpha_s}{\pi}\right)^4} = \frac{m^3}{16\pi^2}\left[\frac{(-3.5\pm 3.0)\times 10^{-8}}{\epsilon^{11}} + \frac{(-2.1\pm 7.6)\times 10^{-6}}{\epsilon^{10}} + \frac{(0.2\pm 2.1)\times 10^{-4}}{\epsilon^{9}} + \frac{(-0.5\pm 7.5)\times 10^{-2}}{\epsilon^{8}} + \ldots\right]$$

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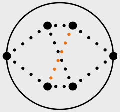
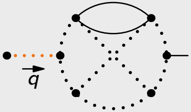
- ~ 2 digits lost per order in $\epsilon \Rightarrow$ need better than double precision
- Found no significant improvement with
  - Quasi-Monte Carlo methods [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk 2018]
  - Quasi-finite integral basis [von Manteuffel, Panzer, Schabinger 2014]
  - Above plus direct Feynman parameter integration
  - Tropical integration [Borinsky 2020]
  - `pySecDec` [Heinrich, Jones, Kerner, Magerya, Olsson, Schlenk 2023]

Basis integrals

Direct integration c.f. [Faisst, Chetyrkin, Kühn 2004]



$$= \int \frac{d^d q}{i\pi^{\frac{d}{2}}}$$



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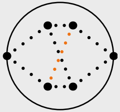

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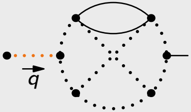
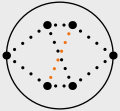
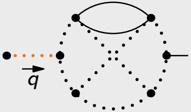


With $z = -q^2$:



$$= \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)}\, z^{1-\epsilon}$$



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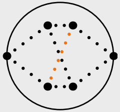
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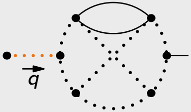

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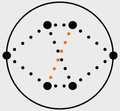
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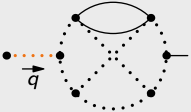
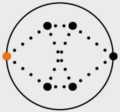


q does not need to be a line momentum: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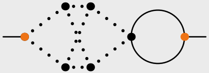

Expand in ϵ, integrate numerically

Basis integrals

Infrared & ultraviolet divergences

Problem: integrand is infrared & ultraviolet divergent

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Infrared & ultraviolet divergences

Problem: integrand is infrared & ultraviolet divergent

Initial idea:

- Choose q as massive line momentum
  - All heavy fermion lines are closed
  - $\Rightarrow$ No massless cuts in integrand
  - $\Rightarrow$ No infrared divergences
- Take propagator with momentum q to higher power

  $\Rightarrow$ No ultraviolet divergence

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Drawbacks:

- Needs five-loop basis change
- Needs closed massive lines
- Needs unknown ingredients $\to$ next slides

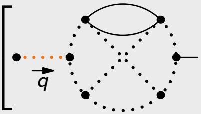
Alternative: infrared and ultraviolet subtraction

Basis integrals

Infrared subtraction

Subtract infrared divergences:

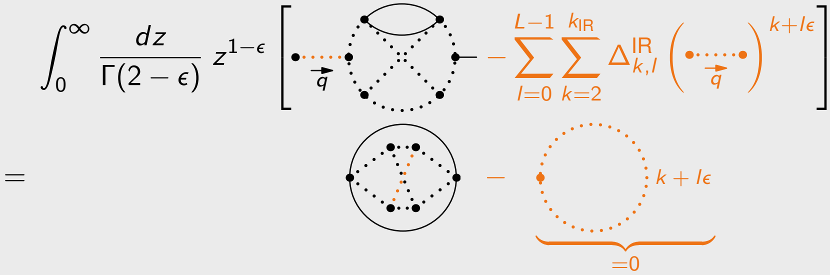
$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)}\, z^{1-\epsilon} \Bigg[$$



$$- \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\mathrm{IR}}} \Delta^{\mathrm{IR}}_{k,l} \Bigg($$

$$\Bigg)^{k+l\epsilon} \Bigg]$$

Subtract infrared divergences:

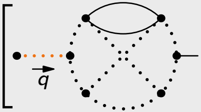


Translate infrared to ultraviolet divergences

Basis integrals

Ultraviolet subtraction

$$\int_0^\infty \frac{dz}{\Gamma(2-\epsilon)}\, z^{1-\epsilon} \Bigg[$$



$$- \sum_{l=0}^{L-1} \sum_{k=2}^{k_{\mathrm{IR}}} \Delta^{\mathrm{IR}}_{k,l} \Bigg($$

$$\Bigg)^{k+l\epsilon}$$

$$- \sum_{l=0}^{L-1} \sum_{k=k_{\mathrm{UV}}}^{2} \Delta^{\mathrm{UV}}_{k,l} \Bigg($$

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$$\Bigg)^{k} \Bigg]$$

Basis integrals

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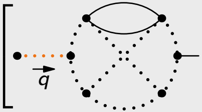

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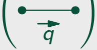

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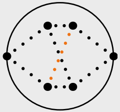
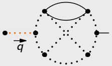
$$=$$



$$- \sum_{l=0}^{L-1} \sum_{k=1}^{2} \Delta^{\mathrm{UV}}_{k,l} \frac{\Gamma((l+1)\epsilon + k - 2)\Gamma(2-(l+1)\epsilon)}{\Gamma(2-\epsilon)\Gamma(k)}$$

Differential Equations

Generalised power series ansatz

$P_0 =$ 

from differential equations for propagator basis integrals:

$$z\frac{\partial}{\partial_z}P_i = Q_{ij}P_j$$

[Kotikov 1991, Remiddi 1997,...]

Infrared and ultraviolet counterterms from series ansätze:

$$P_i = \sum_{l=0}^{L-1}\sum_{k=k_{\text{IR}}}^{N-1} c_{ikl}z^{k-l\epsilon} + \mathcal{O}(z^N), \qquad P_i = \sum_{l=0}^{L-1}\sum_{k=k_{\text{UV}}}^{N-1} d_{ikl}\frac{1}{z^{k-l\epsilon}} + \mathcal{O}\left(\frac{1}{z^N}\right)$$

Boundary conditions $c_{ik_{\text{IR}}l}$, $d_{ik_{\text{UV}}l}$ from asymptotic expansion:
known massless propagators and massive vacuum diagams with ≤ 4 loops

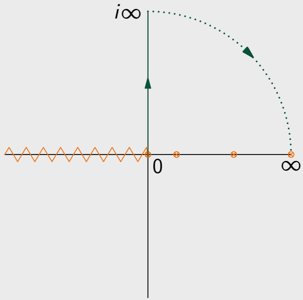
[Laporta 2002; Czakon 2004, Schröder, Vuorinen 2005; Lee, Terekhov 2010; Baikov, Chetyrkin 2010; Lee, Smirnov, Smirnov 2011]

Differential Equations

Direct numeric integration

Spurious poles ⊗ for $z > 0$

- Deform contour:



- Integrate analytically for $z \approx 0$, $z \approx \infty$

Differential Equations

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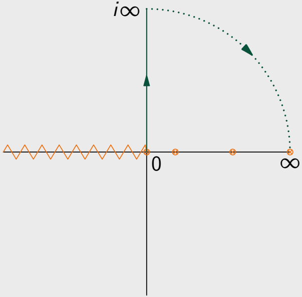

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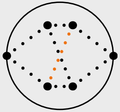 $= -1.03692775514337\epsilon^{-2} - 2.27151759391862\,96\epsilon^{-1} - 51.677219826124914 + \mathcal{O}(\epsilon)$

Idea: rational function approximation from

$$P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\text{IR}}}^{N-1} c_{ikl} z^{k-l\epsilon} + \mathcal{O}(z^N), \qquad P_i = \sum_{l=0}^{L-1} \sum_{k=k_{\text{UV}}}^{N-1} d_{ikl} \frac{1}{z^{k-l\epsilon}} + \mathcal{O}\left(\frac{1}{z^N}\right)$$

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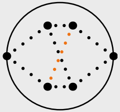
$$P_i \approx \frac{a_0 + a_1\omega + \cdots + a_n\omega^n}{1 + b_1\omega + \cdots + b_m\omega^m} z(\omega)^{1-N} + P_i^{\text{subtr}}(\omega)$$

Differential Equations

Padé approximation

Result for $N = 25$:



$$= -1.03692775514337\epsilon^{-2} - 2.27151759391862 96\epsilon^{-1}$$

$$- (51.6772198261254 16 \pm 10^{-12})$$

$$- (74.1136306549411 2 \pm 10^{-10})\,\epsilon$$

$$- (1534.5823448304886 \pm 10^{-9})\,\epsilon^2$$

$$- (997.9241152040024 \pm 10^{-8})\,\epsilon^3$$

$$- (69346.26975774963 \pm 10^{-7})\,\epsilon^4$$

$$+ \mathcal{O}(\epsilon^5)$$

Quark condensate
Status

New results for basis integrals:

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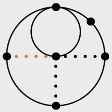
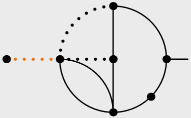
What is happening here?



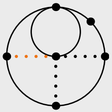
$$= \int_0^\infty \frac{dz}{\Gamma(2-\epsilon)} z^{1-\epsilon}$$



$$= \frac{1}{120\epsilon^5} + \frac{1}{12\epsilon^4} - \frac{1.1 \pm 2.7}{\epsilon^3} + \mathcal{O}(\epsilon^{-2})$$

Padé approximation

Limitations



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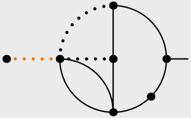

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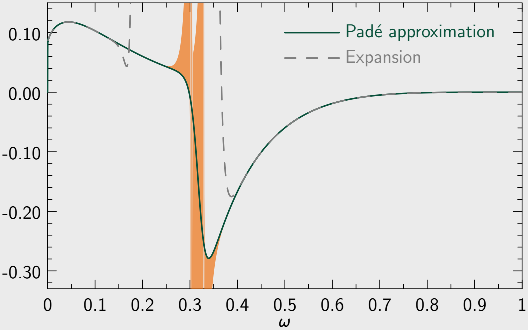
Padé approximation at order ϵ^{-3}:



Conclusions

- Ongoing five-loop QCD calculations: quark condensate and decoupling
- Reduction to basis integrals complete
- Need higher precision for basis integrals:
  - Direct integration
  - Differential equations
  - Padé approximation