Status and perspectives for Mellin-Barnes integrals

Janusz Gluza*,
University of Silesia in Katowice
*Thanks to Ievgen Dubovyk, Gabor Somogyi, Szymon Zieba

Loop Summit 2
New perturbative results and methods in precision physics:
Quantum Field Theory and Collider Physics

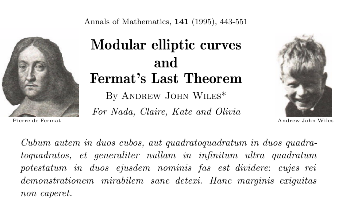
24 July 2025, Cadenabbia, Italy

NATIONAL SCIENCE CENTRE
POLAND

Collider physics ... magic of the math world!

$$T = 2\pi\sqrt{\frac{l}{g}} \times {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; \sin^2\theta\right]$$

Analytic solutions for massive multiloop integrals, which describe scattering processes/decays, go beyound elliptic functions - how far? 



Annals of Mathematics, **141** (1995), 443-551

Modular elliptic curves and Fermat's Last Theorem

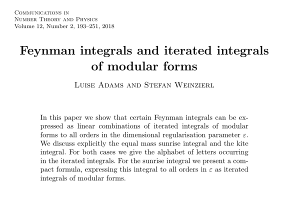
By Andrew John Wiles*

For Nada, Claire, Kate and Olivia

Pierre de Fermat

Andrew John Wiles

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatum in duos ejusdem nominis fas est dividere: cujes rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.



Communications in Number Theory and Physics
Volume 12, Number 2, 193–251, 2018

Feynman integrals and iterated integrals of modular forms

Luise Adams and Stefan Weinzierl

In this paper we show that certain Feynman integrals can be expressed as linear combinations of iterated integrals of modular forms to all orders in the dimensional regularisation parameter ε. We discuss explicitly the equal mass sunrise integral and the kite integral. For both cases we give the alphabet of letters occurring in the iterated integrals. For the sunrise integral we present a compact formula, expressing this integral to all orders in ε as iterated integrals of modular forms.

MIs with high accuracy: AMFlow, CERN 2022,

https://indico.cern.ch/event/1140580/

Summary and Outlook

➢ What we have

- Auxiliary mass flow method fully automized the computation of boundary conditions for differential equations.
- → AMFlow is the first public tool which can compute arbitrary Feynman loop integrals, at arbitrary kinematic point, to arbitrary precision.

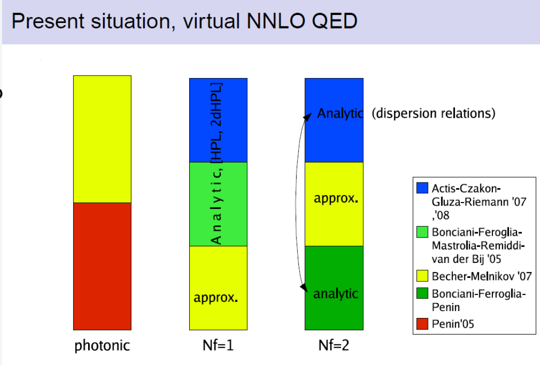
➢ What we need

- Powerful reduction techniques are urgently needed to construct differential equations, both for η and for dynamical variables.
- A guide for choosing better master integrals in general cases is needed, which may strongly simplify the differential equations.

17 June 2022 *Xiao Liu, University of Oxford*

Waves of changes (in methods efficiency)

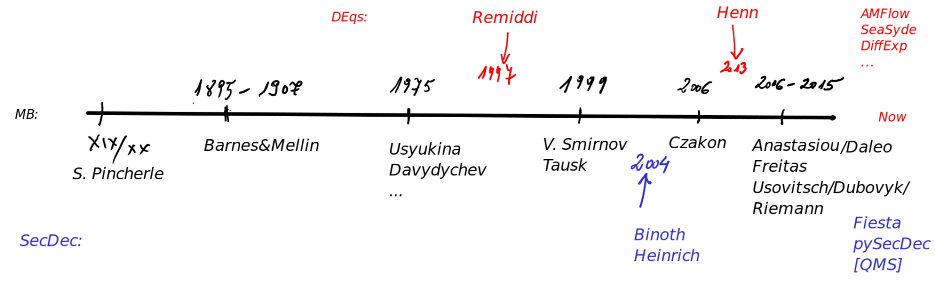
$e^+e^- \to e^+e^-$, ~15 years ago



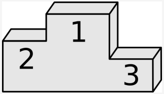
+ J. Henn, V. Smirnov, 2013 - analytic solutions for planar cases.

It is reasonable to keep developing different methods, complementarity, cross-checks etc.

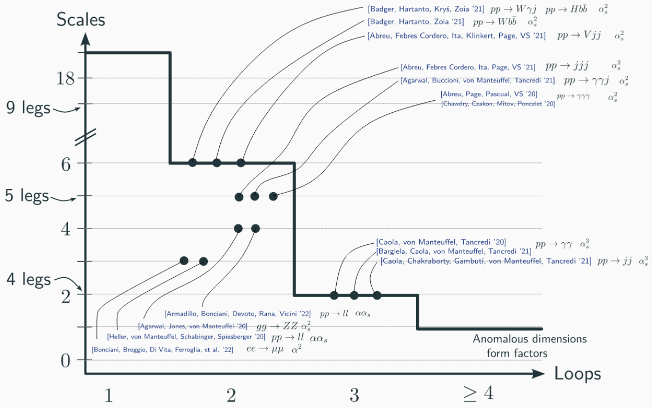
Ups and Downs in multiloop methods (not complete/subjective)



At the moment the most robust method for precision calculations seems to be **DEqs**;



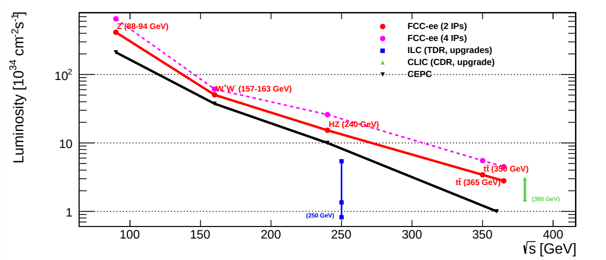
Precise methods and tools are needed at LHC and future e^+e^- colliders



Warning: a biased selection of references!

Workshop: Precision calculations for future e^+e^- colliders: targets and tools, CERN 2022 TH workshop, link, from a talk by V. Sotnikov.

Motivation For Precision Studies: Z,W,H,t and flavour electroweak factories



https://arxiv.org/abs/2203.06520 (from a summary for the US Snowmass Process)

Phase	Run duration (years)	Center-of-mass Energies (GeV)	Integrated Luminosity (ab^{-1})	Event Statistics
FCC-ee-Z	4	88-94	150	$5 \cdot 10^{12}$ **Z decays**
FCC-ee-W	2	157-163	10	10^8 WW events
FCC-ee-H	3	240	5	10^6 ZH events 25k WW $\to H$
FCC-ee-tt	5	340-365	$0.2 \div 1.5$	10^6 $t\bar{t}$ even ts 200k ZH 50k WW $\to H$

FCC-hh, optimal for 84 TeV pdf (ML Mangano talk)

To get to the experimental precision, we must improve very much!

$\Longrightarrow$ 1-2 orders of magnitude improved determination of measured observables!

Expected precision in 2040

J. Gluza

- **Conclusion of the 2018 Workshop**

> **"We anticipate that, at the beginning of the FCC-ee campaign of precision measurements, the theory will be precise enough not to limit their physics interpretation. This statement is however conditional to sufficiently strong support by the physics community and the funding agencies, including strong training programmes".**

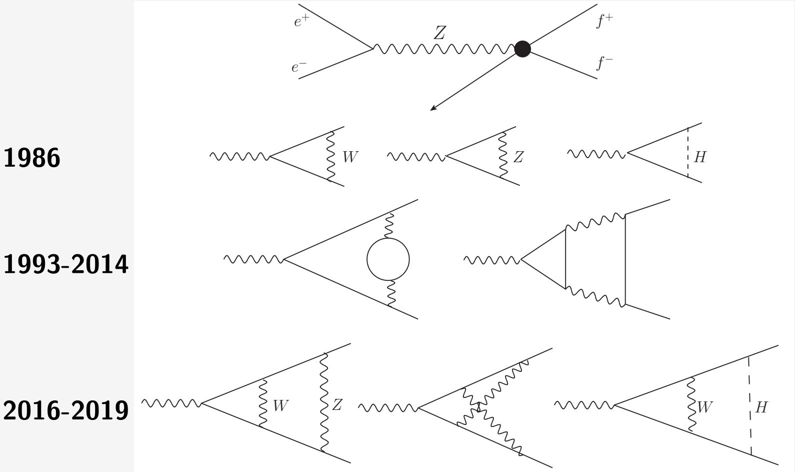
- **Numerical evaluation with three-loops calculations:** arXiv:1901.02648

	$\delta\Gamma_Z$ [MeV]	δR_l $[10^{-4}]$	δR_b $[10^{-5}]$	$\delta \sin^{2,l}_{eff}\theta$ $[10^{-6}]$
Present EWPO theoretical uncertainties				
EXP-2018	2.3	250	66	160
TH-2018	0.4	60	10	45
EWPO theoretical uncertainties when FCC-ee will start				
EXP-FCC-ee	~~0.1~~ 0.025	10	$2 \div 6$	~~6~~ 3
TH-FCC-ee	0.07	7	3	7

$0.5 \rightarrow 0.4$
Five years!

- **500 person-years needed over 20 years – Recognized as strategic priority.**

Precision context of collider studies: $Zf\bar{f}$ vertex and EW corrections

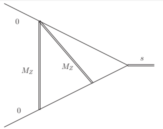


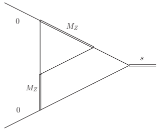
I. Dubovyk, A. Freitas, JG, T. Riemann, J. Usovitsch,
https://doi.org/10.1016/j.physletb.2016.09.012
https://doi.org/10.1016/j.physletb.2018.06.037
https://doi.org/10.1007/JHEP08(2019)113

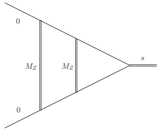
MB and SD methods are very much complementary!

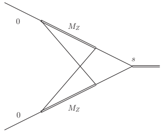
- MB works well for hard threshold, on-shell cases, not many internal masses (more IR);
  SD more useful for integrals with many internal masses

10^{-8} accuracy achieved for **any** self-energy and vertex Feynman integral with one of the methods - in **Minkowskian region**.

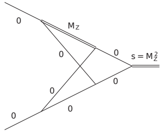








Substantial progress for critical cases



Euclidean results (constant part, $(p_1 + p_2)^2 = m^2 = 1$:):

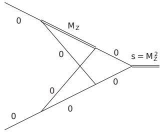
Analytical :	$-0.49661983060570 21$
MB(Vegas) :	-0.4969417442183914
MB(Cuhre) :	-0.4966198313219404
FIESTA :	-0.4966184488196595
SecDec :	-0.4966192150541896

Minkowskian results (constant part, $-(p_1 + p_2)^2 = m^2 = 1$:):

Analytical :	$-0.778599608979684 - 4.123512593396311 \cdot i$
MBnumerics :	$-0.778599608324769 - 4.123512600516016 \cdot i$
MB + thresholds :	$-0.7785242512636401 - 4.123512600516016 \cdot i$
SecDec :	big error [**2016**], $-0.77 - \mathrm{i} \cdot 4.1$ [**2017**], $-0.778 - \mathrm{i} \cdot 4.123$ [**2019**]
pySecDec + rescaling :	$-0.778598 - i \cdot 4.123512$ [**2020**]
pySecDec + NCD :	$? - i \cdot ?$ [**2025**]

SD and MB are independent of IBPs (at 2-loops SM we haven't used IBPs)

pySecDec 2025 $\longrightarrow$ see Monday's S. Jones talk



– **Split** – 4 mins 20 seconds

$$+ \quad (+0.778599584323023 90 + i \cdot 4.1235125801703916)$$
$$\pm \quad (+2.3248718074632117 + i \cdot 2.4896373214813187) \cdot 10^{-8}$$

running longer... in progress

– **Rescaling** – 4 mins 57 seconds

$$+ \quad (+0.77859958598318513 + i \cdot 4.1235125394269252)$$
$$\pm \quad (+2.6109756763268707 + i \cdot 2.4674307281565548) \cdot 10^{-7}$$

running longer... 54 mins 53 seconds

$$+ \quad (+0.77859960864456634 + i \cdot 4.1235125932779120e + 00j)$$
$$\pm \quad (+4.206351271221679 + i \cdot 4.4781421069263670) \cdot 10^{-10}$$

– **NoCD** – 8.4 seconds overhead + 1.8 seconds to evaluate

$$+ \quad +0.77859960898646108 + i \cdot 4.1235125933366232)$$
$$\pm \quad (+3.6966843561201167 \cdot 10^{-11} + i \cdot 1.2914986997032820 \cdot 10^{-13})$$

Direct numerical approaches (beyond 1-loop)

- Sector decomposition (SD) method:
  - FIESTA [2016], [A.V.Smirnov]
  - pySecDec [2022], Expansion by regions with pySecDec],
- The Mellin-Barnes (MB) method:
  - MB.m [M.Czakon, 2006],

    [C. Anastasiou, M. Daleo, 2006] (not public code)
  - MBnumerics [J.Usovitsch, I.Dubovyk, T.Riemann, 2015] – Minkowskian kinematics
- Differential equations (DEs) method:
  - `DiffExp` [F. Moriello, 2019; M. Hidding, 2021],
  - `AMFlow` [X. Liu, Y.-Q. Ma, 2022],
  - `SeaSyde` [T. Armadillo, R. Bonciani, S. Devoto, N. Rana, A. Vi, 2022] ,

SE: `TVID 2` (S. Bauberger, A. Freitas, D. Wiegand), `BOKASUN` (M. Caffo, H. Czyz, M. Gunia, E. Remiddi),

$+$ `DREAM` (dimensional recurrence relations solutions, R. Lee, K. Mingulov), αLoop loop-tree duality, PL, GPL, MPL, eMPL, integrand subtraction (≤ 2 loops: `NICODEMOS` - A. Freitas, Four-Dimensionally Regularized/Renormalized (FDR) integrals (R. Pittau), dispersion relations,

$\rightarrow$. Badger, S. Jones talks

The MB method still has a potential for efficient development

MB was crucial for completion of 2-loop Z-decay EWPOs (we demanded 8 digits accuracy)

One of the key features of the MB approach is focus on an individual Feynman integral, which can become advantageous.

MB applications cover basic research areas:

- Real (soft, collinear) and virtual corrections
- Analytical and numerical solutions

MB used so far, some examples

methods

- Evaluation of MIs (Tausk, Smirnov, ...) (MB & expansions),
- "On the Numerical Evaluation of Loop Integrals With Mellin-Barnes Representations", A. Freitas, Yi-Cheng Huang, JHEP, 2010
- "Evaluating multi-loop Feynman diagrams with infrared and threshold singularities numerically",
  C. Anastasiou, S. Beerli, A. Daleo, JHEP, 2007

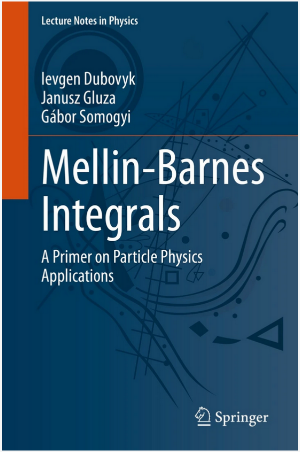
real, soft, collinear

- "Angular integrals in d dimensions", Gabor Somogyi, J.Math.Phys, 2011
- "A subtraction scheme for computing QCD jet cross sections at NNLO: integrating the doubly unresolved subtraction terms Gabor Somogyi, JHEP 04 (2013) 010
- "Soft triple-real radiation for Higgs production at N3LO",
  C. Anastasiou, C. Duhr, F. Dulat, B. Mistlberger, JHEP, 2013

virtual

- Bhabha massive QED 2-loop (M. Czakon, JG, T. Riemann, S. Actis) (MB & dispersion relations)
- High-Energy Expansion of Two-Loop Massive Four-Point Diagrams,
  G. Mishima, JHEP 02 (2019) 08, (Higgs pair production cross section)

Latest works connected with MB ($\geq$ 2023)



https://arxiv.org/abs/2211.13733
https://doi.org/10.1007/978-3-031-14272-7

在这两部分中, 分别做了变量替换$y \to tx$和$x \to ty$. 区域分解如图4所示, 可以看到, 对每一部分, 被积函数的奇点都被分离开了, 于是可以用通常办法进行减除, 而后做数值计算.

上述方法对多重积分也是有效的, 故可以用来计算费曼积分. 在费曼参数表示下, 流程与上述例子类似: 先对积分区域进行分解, 使得被积函数的奇点全分离开; 然后在奇点处做减除; 再对每一部分做数值计算, 即可得到最终结果. 区域分解的标准流程为[68]: 首先生成主区域(Primary Sector), 然后迭代地生成子区域(Sub Sector), 而后在每一个区域做ϵ的展开, 如下式所示:

$$I = (-1)^{\nu}\Gamma(\nu - LD/2)\sum_{i=1}^{N}\sum_{j=1}^{\Lambda_i} I_{ij}, \tag{14}$$

$$I_{ij} = \sum_{-2L}^{r} C_{ijk}\epsilon^k + O(\epsilon^{r+1}), \tag{15}$$

其中ν为完备的传播子幂次之和. 最后数值地计算有限积分, 即系数C_{ijk}通过蒙特卡罗方法得到.

当运动学参数取在欧式区域时, 区域分解法是很强大的, 因为此时费曼参数表示的被积函数是半正定的, 积分的数值稳定性很高. 但对于物理区域(也即闵氏区域)的取值, 人们需要设计非常精妙的积分围道来绕开奇点以提高数值稳定性, 而复杂的积分围道会降低计算效率. 所以区域分解对于复杂问题往往十分耗时且难以得到高精度的结果. 区域分解法计算费曼积分已经通过编程实现了自动化, 目前市面上可用的程序有FIESTA[69–72], SecDec[73–76], pySecDec[77]等.

4.2 Mellin-Barnes表示

Mellin-Barnes表示[78]通过将有质量传播子展开为无质量传播子来简化费曼积分的计算, 其运用的基

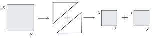


图 4 区域分解示意图
Figure 4 The diagrammatic sketch of sector decomposition.

本公式如下:

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)}\frac{Y^z}{X^{\lambda+z}}, \tag{16}$$

其中z是引入的参数. 上式右边的积分被称为Mellin-Barnes积分, 可将其运用于带质量的传播子

$$\frac{1}{(q^2-m^2)^{\nu}} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(\nu+z)\Gamma(-z)}{\Gamma(\nu)}\frac{(-m^2)^z}{(q^2)^{\nu+z}}. \tag{17}$$

对单圈单质量积分运用这种技巧可得

$$\begin{aligned} I[1,1] &= \int \frac{d^D\ell}{i\pi^{D/2}}\frac{1}{(\ell^2-m^2)(\ell+p)^2} \\ &= \frac{\Gamma(1-\epsilon)}{2\pi i}\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(-z)\Gamma(1-z-\epsilon)\Gamma(z+\epsilon)}{\Gamma(2-z-2\epsilon)}, \end{aligned} \tag{18}$$

为了简单这里已取$p^2 = -1, m^2 = 1$并积掉了圈动量. 根据Mellin-Barnes方法, 积分围道要处于所有$\Gamma(-z+\cdots)$奇点的左侧和所有$\Gamma(z+\cdots)$奇点的右侧. 然而对于上式积分, 当$\epsilon \to 0$时, $\Gamma(z+\epsilon)$和$\Gamma(-z)$的奇点会夹击(Pinch)积分围道, 使得无法选取一个符合要求的围道, 这种现象源于积分存在紫外发散. 有两种方案解决这个问题, 第一种是将积分围道进行变形(Deform)使其符合要求, 称为策略A, 市面上的自动化程序MBresolve.m[79]就是通过这种方式进行计算的; 另一种方案是在ϵ有限大时固定一条直线围道, 在$\epsilon \to 0$的过程中让极点穿过该围道并附加相应的留数, 称为策略B, 程序MB.m[80]是采用的这种方式. 处理完奇点后, 便可以对有限项进行数值积分了, 不过和区域分解类似, 数值积分通常难以得到高精度结果. 这套流程也比较系统化, 除上述两个程序外, 目前市面上使用Mellin-Barnes表示计算费曼积分的计算机程序还有AMBRE.m[81]和MBsums.m[82]等.

4.3 差分方程法

差分方程法是利用积分约化建立主积分关于时空维数D [83–87]或者传播子幂次[42]的差分方程, 然后求解差分方程得到主积分的结果. 目前差分方程的计算能力有限, 通常不如后面将要介绍的微分方程和辅助质量流方法.

4.4 微分方程法

微分方程法是一种间接计算主积分的系统方法[88], 它通过求解主积分满足的微分方程来得到积分

https://www.sciengine.com/SSPMA/doi/10.1360/SSPMA-2022-0454

Latest works connected with MB ($\geq$ 2023)

Functions involved

1. Hypergeometric structures in Feynman integrals, J. Blümlein, M. Saragnese, C. Schneider, Blumlein:2021hbq
   [Hurwitz harmonic sums, generalized hypergeometric functions, Appell-, Kamp ´e de F ´eriet-, Horn-, Lauricella-Saran-, Srivasta-, and Exton–type functions]

2. MultiHypExp: A Mathematica package for expanding multivariate hypergeometric functions in terms of multiple polylogarithms, Souvik Bera, Bera:2023pyz

3. Multiple Mellin-Barnes integrals and triangulations of point configurations, Sumit Banik, Samuel Friot, Banik:2023rrz

4. Feynman integrals and Fox functions, Giampiero Passarino, Passarino:2024ugq
   [see talk by Giampiero]

   "*New directions in science are launched by new tools* much more often than by new concepts".
   – Freeman Dyson

5. Multipoint conformal integrals in D dimensions. Part I: Bipartite Mellin-Barnes representation and reconstruction, K.B. Alkalaev, Semyon Mandrygin Alkalaev:2025fgn
   [conformal integrals in terms of MB]

6. IBIS: Inverse BInomial sum Solver, Paul A.J.W. van Hoegaerden, Coenraad B. Marinissen, Wouter J. Waalewijn, vanHoegaerden:2025nmh
   [in FORM, sums up to weight 6]
   $\rightarrow$ see also Kay Schönwald's talk

Latest works connected with MB ($\geq$ 2023)

Real corrections, soft, collinear

1. N-jettiness soft function at next-to-next-to-leading order in perturbative QCD, Prem Agarwal, Kirill Melnikov, Ivan Pedron, Agarwal:2024gws
   [triple-collinear limit of the soft-subtracted eikonal function]

2. Expansion by regions meets angular integrals, Vladimir A. Smirnov, Fabian Wunder, Smirnov:2024pbj

3. NNLO phase-space integrals for semi-inclusive deep-inelastic scattering, Taushif Ahmed, Saurav Goyal, Syed Mehedi Hasan, Roman N. Lee, Sven-Olaf Moch Ahmed:2024owh
   [the angular parts, radial integrals]

Latest works connected with MB ($\geq$ 2023)

Multiloop

1. Integrated unitarity for scattering amplitudes and the four-loop ladder Feynman integral, Piotr Bargiela, Bargiela:2024hgt
   [cuts]

2. Massive two-loop four-point Feynman integrals at high energies with AsyInt, Hantian Zhang Zhang:2024fcu
   [Analytic Expand&Fit method with 1 or 2 scales]

3. Three-loop jet function for boosted heavy quarks, Alberto M. Clavero, Robin Brüser, Vicent Mateu, Maximilian Stahlhofen, Clavero:2024yav
   [analytic 2-dim MB]

4. Application of the Meijer theorem in calculation of three-loop massive vacuum Feynman integrals and beyond, Jian Wang, Dongyu Yang, Wang:2024son
   [The Meijer theorem and its corollary are used to perform the in- tegration over the Gamma functions, exponential functions, and hypergeometric functions.]

5. Analytical Calculations of Contributions to the Anomalous Magnetic Moments of Leptons from Vacuum Polarization by Lepton Loops within the Mellin–Barnes Representation, O.P. Solovtsova, V.I. Lashkevich, L.P. Kaptari Solovtsova:2025gmj
   [tenth-order correction for the anomalous magnetic moment of lepton L]

Latest works connected with MB ($\geq$ 2023)

Exotics: gravity, ...

1. Scalar-graviton amplitudes and celestial holography, Adam Ball, Shounak De, Akshay Yelleshpur Srikant, Anastasia Volovich, Ball:2023ukj
   [amplitudes for scalars with up to 4 gravitons]
2. Loop corrections in the separate universe picture, Laura Iacconi, David Mulryne, David Seery, Iacconi:2023ggt
   [field correlators]
3. On the Asymptotic Nature of Cosmological Effective Theories Carlos Duaso Pueyo, Harry Goodhew, Ciaran McCulloch, Enrico Pajer, DuasoPueyo:2025lmq
4. Celestial string integrands their expansions, Daniel Bockisch Bockisch:2024bia
   [integration over the open-string worldsheet moduli space]

Mellin-Barnes representations in HEP - the method

- "Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895), "The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$mathematics \longrightarrow \frac{1}{(A+B)^\lambda} \quad = \quad \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^z}{A^{\lambda+z}}$$

$$physics \longrightarrow \frac{1}{(p^2-m^2)^a} \quad = \quad \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z) \frac{(m^2)^z}{(p^2)^{a+z}}$$

It is recursive $\Longrightarrow$ multidimensional complex integrals.

$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i} \; A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1} \Gamma(-z_i) \; \Gamma(\lambda+z_1+\ldots+z_{n-1})$$

Mellin and Barnes

To be honest, I do not know for sure who was the first to use the names of Mellin and Barnes together. In the end of 1980s, when I wrote my first papers, it was a sort of common to use both names, I just adopted it from my advisor Boris Arbuzov and his pupils (Eduard Boos, etc.). It is true, that in many old books they are called just Barnes integrals. Nevertheless, you may find Mellin–Barnes, e.g., in books by L.J. Slater ("Confluent hypergeometric functions", 1960), A.W. Babister ("Transcendental functions satisfying nonhomogeneous linear differential equations", Macmillan, 1967), etc.
Best regards,
Andrei

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} g(s) ds \longrightarrow g(s) = \int_0^\infty x^{s-1} f(x) dx.$$

$$I_C = \int_C \frac{dz}{2\pi i}\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(c+z)}(-s)^z$$

LIPS, subtractions and MB (simple example)

All momenta are massless, i.e., $p_1^2 = p_2^2 = p_3^2 = 0$

$$\int d\phi_3(p_1, p_2, p_3, Q) \frac{1}{(p_1 \cdot p_2)(p_1 \cdot p_2 + p_1 \cdot p_3)} ,$$

In $d = 4 - 2\epsilon$ dimensions, this integral is proportional to [Gehrmann-DeRidder:2003pne]

$$\int_0^1 dx \int_0^1 dy\, x^{-1-\epsilon}(1-x)^{1-2\epsilon} y^{-\epsilon}(1-y)^{-\epsilon} \frac{1}{x + (1-x)y} .$$

with MB represenation

$$\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(1-)\Gamma\left(-z_1\right)\Gamma\left(z_1+1\right)\Gamma\left(--z_1\right)\Gamma\left(z_1-\right)}{\Gamma(1-3)} ,$$

which can be further manipulated (e.g. expanded in or evaluated analytically and numerically [Dubovyk:2022obc])

LIPS, subtractions and MB (more complicated example, from Gabor Somogyi)

Integrating triple collinear subtraction term for hadron-initiated processes at NNLO
($s_{ij} = 2p_i \cdot p_j, P^2 = \xi_a \xi_b s_{ab}, 0 \le \xi_{a,b} \le 1, i, j = a, b, \mathbf{r}, \mathbf{s}$)

$$
\begin{aligned}
I &= \int d\phi \frac{1}{s_{as}(s_{ar}+s_{br})(s_{ar}+s_{as}+s_{br}+s_{bs})}\,, \\
d\phi &= \mathrm{LIPS}_3(p_r, p_s, P; p_a+p_b)\delta[\xi_b(s_{ab}-s_{br}-s_{bs})-\xi_a(s_{ab}-s_{ar}-s_{as})]\,,
\end{aligned}
$$

With expansion by regions, one piece

$$
\begin{aligned}
\lim_{\xi_b \to 1} I &= L_b I_4 = 2^{-1-3\epsilon}\xi_a^{-1-3\epsilon}(1-\xi_a)^{-1-3\epsilon}(1+\xi_a)^{\epsilon} \\
&\times \int_0^1 dx_1 \int_0^1 dx_2\, x_1^{-\epsilon}\,(1-x_1)^{-1-3\epsilon} x_2^{-1-2\epsilon}\,(1-x_2)^{-1+\epsilon}\,(2\xi_a + x_1 - x_1\xi_a)^{\epsilon} \\
&\times \left(2x_1x_2\xi_a^2 - 2x_2\xi_a^2 - x_1^2\xi_a + 2x_1\xi_a + x_1^2x_2\xi_a - 4x_1x_2\xi_a + 2x_2\xi_a + x_1^2 - x_1^2x_2\right)^{\epsilon}.
\end{aligned}
$$

Adding regulator $(1-x_2)^{\delta}$, 2-dim MB can be constructed, with $\delta, \epsilon \to 0$ 0-dim MB integral gives after ϵ expansion

$$
L_b I_4 = -\frac{6 + 6\epsilon\log\left(\frac{1+\xi_a}{2\xi_a(1-\xi_a)}\right) + 3\epsilon^2\log^2\left(\frac{1+\xi_a}{2\xi_a(1-\xi_a)}\right) + \epsilon^3\left[\log^3\left(\frac{1+\xi_a}{2\xi_a(1-\xi_a)}\right) - 24\zeta_3\right]}{24\xi_a(1-\xi_a)\epsilon^2} + \mathcal{O}(\epsilon^2)\,.
$$

e.g. NNLOCAL, V. Del Duca, C. Duhr, L. Fekésházy, F. Guadagni, P. Mukherjee, G. Somogyi, F. Tramontano, S. Van Thurenhout. 2412.21028

Multiloops: Modern Smirnov-Tausk scenery

$$G_L[1] = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \dots d^d k_L}{(q_1^2 - m_1^2)^{n_1} \dots (q_i^2 - m_i^2)^{n_j} \dots (q_N^2 - m_N^2)^{n_N}}$$

$$D_i = q_i^2 - m_i^2 + i\delta = \left[\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e\right]^2 - m_i^2 + i\delta,‘$$

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} = \frac{\Gamma(n_1 + \dots + n_N)}{\Gamma(n_1) \dots \Gamma(n_N)} \int_0^1 dx_1 \dots \int_0^1 dx_N \frac{x_1^{n_1-1} \dots x_N^{n_N-1} \delta(1 - x_1 - \dots - x_m)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}}$$

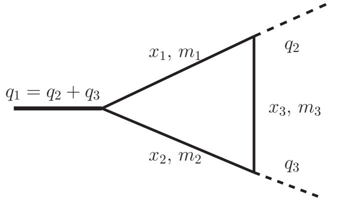
$$m^2(\vec{x}) = x_1 D_1 + \dots + x_i D_i + \dots + x_N D_N = k_i M_{ij} k_j - 2 Q_j k_j + J$$

$$m^2(\vec{x}) = kMk - 2Qk + J \Leftrightarrow U = \det M,$$

$$F = -\det M \; J + Q M^T Q$$

$$\boxed{G_L[1] = \frac{(-1)^{N_\nu} \Gamma\left(N_\nu - \frac{d}{2} L\right)}{\prod_{i=1}^{N} \Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}}$$

"One-loop" example:



$$U = x_1 + x_2 + x_3 \equiv 1$$
$$F_0 = -(q_2 + q_3)^2 x_1 x_2 - q_2^2 x_1 x_3 - q_3^2 x_2 x_3$$
$$F = F_0 + U(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2)$$

$$G(X) \sim \int dz_1 dz_2 dz_3 \, (-s x_1 x_2)^{z_1} (-q_2^2 x_1 x_3)^{z_2} (-q_3^2 x_2 x_3)^{z_3} \times \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)^{-z_1 - z_2 - z_3 - N_\nu + d/2}$$

Beyond one-loop:

- $U(\vec{x}) \neq 1$
- complexity/dimensionality starts to depend on $U(\vec{x})$ structure
- nontrivial simplification of graph polynomials is needed

$x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$	3-dim representation
$(x_1 + x_2)(x_3 + x_4)$	2-dim representation
$(x_1 + x_2)(x_3 + x_4) \to$ $[x_1 \to v_1\xi_{11}, x_2 \to v_1\xi_{12}, \delta(1 - \xi_{11} - \xi_{12});$ $x_3 \to v_2\xi_{21}, \ldots] \to v_1v_2$	0-dim representation
$(x_1 + x_2)(x_3 + x_4)$ + BL	0-dim representation *)

*)

$$(x_1 + x_2)^p \to \int dx_1\ dx_2\ dz_1\ \delta(1 - x_1 - x_2) x_1^{z_1} x_2^{p-z_1} \Gamma(-z_1)\Gamma(-p + z_1)$$

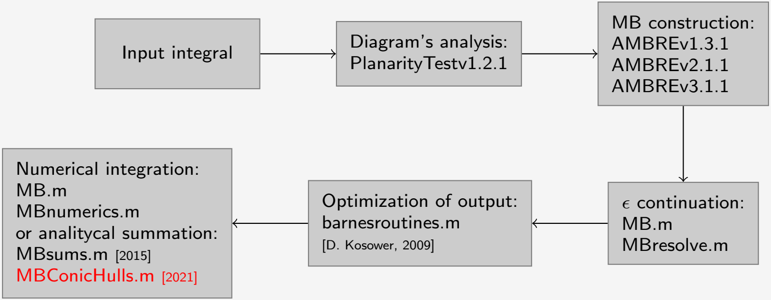
$$\to \int dz_1 \Gamma(-z_1)\Gamma(-p + z_1)\Gamma(z_1 + 1)\Gamma(p - z_1 + 1)/\Gamma(p + 2)$$

BL can be also applied without factorization, but this requires special transformation of z_i variables, see e.g., barnesroutines.m [D. Kosower, 2009]

$$\int_{-i\infty}^{i\infty} dz \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

Computation of Feynman integrals with Mellin-Barnes (MB) method

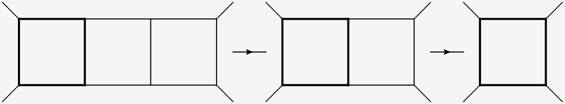
Operational sequence of the MB-suite:



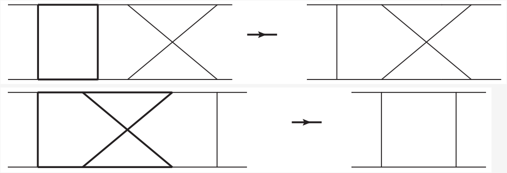
MBcreate.m [2023]: MB tools reloaded, A.V. Belitsky, A.V. Smirnov, V.A. Smirnov Belitsky:2022gba

AMBRE versions overview:

- iterativelly to each subloop – loop-by-loop approach (LA): mostly for planar (AMBREv1.3.1 & AMBREv2.1.1)



- in one step to the complete U and F polynomials – global approach (GA): general (AMBREv3.1.1), usually works for Length[U] $\leq$ Length[F]
- combination of the above methods – Hybrid approach (HA) (AMBREv4, not automated and not public)



Examples, description, links to basic tools and literature:
https://jgluza.us.edu.pl/ambre/

Limitations of GA approach

U polynomial for non-planar 3-loop box (64 terms) - *How to deal with that?*

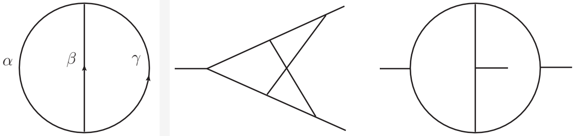
```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

F-polynomial: $\Gamma[0]$ issue, Cheng-Wu theorem

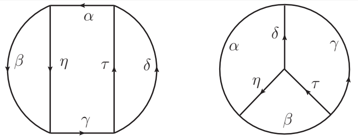
`AMBREv3.m`:

- topology based factorization - chain diagrams, Kinoshita ’74

2-loop:



3-loop:



Cheng-Wu theorem.

Transformation/rescaling of Feynman parameters:

$$\{\vec{x}\}_i: \;\; x_k \to v_i \xi_{ik} \;\; \times \delta\left(1 - \sum_{k=1}^{\eta_i} \xi_{ik}\right),$$

where i denotes chain index and $k \in [1, \eta_i]$, with η_i - number of propagators in chain. δ-function keeps number of variables unchanged.

For any 2-loop diagram:

$$U_{\text{2-loop}} = v_1 v_2 + v_2 v_3 + v_1 v_3$$

For any "ladder" 3-loop diagram (7-dim):

$$U_{\text{3-loop(I)}} = v_1v_2v_3+v_1v_2v_4+v_2v_3v_4+v_1v_2v_5+v_1v_3v_5+v_2v_3v_5+v_1v_4v_5+v_3v_4v_5$$

For any "mercedes" 3-loop diagram (15-dim):

$$U_{\text{3-loop(II)}} = v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_1v_2v_5+v_1v_3v_5+v_2v_3v_5+v_2v_4v_5+v_3v_4v_5 +v_1v_2v_6+v_2v_3v_6+v_1v_4v_6+v_2v_4v_6+v_3v_4v_6+v_1v_5v_6+v_3v_5v_6+v_4v_5v_6$$

- 2-loop: $\delta(1-v_1-v_2)$, $U(\vec{v}) = v_3 + v_1 v_2$
  no additional MB integrations from U, similar to 1-loop cases
- 3-loop: $\delta(1-v_1-v_2-v_3)$
  - "ladder" - 2 additional MB integrations *64-dim → 2-dim (!)*
  - "mercedes" - 4 additional MB integrations

To get minimal dimensionality:

- 1-loop: $U(\vec{x}) \equiv 1$ whenever it's possible
- 2- and 3-loop: expression for F polynomial is not expanded

$$F = F_0 + U \sum_{i=1}^{N} x_i m_i^2$$

The second term can be expanded (generating thresholds, no need for contour deformation) or not (lower dimensionality at price of worse integrated convergence)

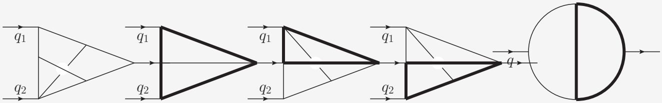
$$G(X) \sim \frac{U(x)^{N_\nu - d(L+1)/2}}{\left(F_0(x) + U(x) \sum_i m_i^2 x_i\right)^{N_\nu - dL/2}} \sim \prod_i \left(m_i^2 x_i\right)^{z_i} \frac{U(x)^{N_\nu - d(L+1)/2 + \sum_i z_i}}{F_0(x)^{N_\nu - dL/2 + \sum_i z_i}}.$$

e.g. $\mathtt{m}^2 x[1]x[4]x[5] - s x[1]x[4]x[5] | \rightarrow$ `m2s x[1] x[4] x[5]|`

- `Barnes lemmas`

Methods of brackets

M. Prausa, Mellin-Barnes meets Method of Brackets: a novel approach to Mellin-Barnes representations of Feynman integrals, Eur. Phys. J. C77 (9) (2017) 594. arXiv:1706.0985



(a) (b) (c) (d) (e)

diagram	Method of Brackets	AMBRE	planarity	AMBRE 4*/method
fig.(a)	**7**	13	NP	**4** H$(2 \to 1)$
fig.(b)	**1**	2	P	**1**
fig.(c)	**7**	9	NP	**5**, H$(1 \to 2)$
fig.(d)	**7**	8	NP	8
fig.(e)	5	**3**	P	**3**

The number of MB integrations of the representation constructed by the Method of Brackets and AMBRE

Based on Schwinger parametrization with generalized Ramanujan's master theorem and the solution of the linear system for vanishing of brackets

Smirnov new MB construction package MBcreate (2023)

MB:

$$\int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_n} \frac{d^n z}{(2\pi i)^n} \frac{\prod_i \left(\alpha_i + \beta_i + \sum_j \gamma_{ij} z_j\right)}{\prod_i \left(\alpha'_i + \beta'_i + \sum_j \gamma'_{ij} z_j\right)} \prod_k s_k^{d_k} \ .$$

1. The application of expansion by regions yields s_k -independent reduced Symanzik polynomials in parametric integrands.
2. A Feynman parametric integral independent of kinematical invariants

$$\int_0^\infty d^N x \prod_i x_i^{a_i - 1} \delta\left(\sum_{i=1}^N x_i - 1\right) \prod_j F_j^{p_j}\left(\{x\}\right) \ .$$

Integration over an x-parameter, if possible, making use of the integral

$$\int_0^\infty dx \, x^p (ax+b)^q = \frac{\Gamma(p+1)\Gamma(-p-q-1)}{\Gamma(-q)} a^{-p-1} b^{p+q+1} \ ,$$

The algorithm is built with variables transformations and grouping, to make possible for the above integration.

Asymptotic behavior: $\Gamma(z)_{|z|\to\infty} = \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\right]$

- core: ("smooth" function)

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \xrightarrow{|z|\to\infty} e^{z(\ln z - \ln(-z)) + \frac{1}{2}\ln z - \frac{5}{2}\ln(-z)}.$$

$\ln z - \ln(-z) = i\pi \,\mathsf{sign}(\Im m\, z)$

$z = z_0 + i\,t,\;\; t \in (-\infty, \infty),\; |z| \to \infty \Leftrightarrow t \to \pm\infty$

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \longrightarrow \mathbf{e}^{-\pi|\mathbf{t}|} \frac{1}{|t|^2} \;\text{(nice suppression)}$$

- kinematics: (oscillations)
  in Minkowskian case $s \to s + i\delta\,(s > 0)$ $\quad\to \frac{1}{\pm p^2 - m^2 + i\delta}$

$$\left(\frac{M_Z^2}{-s}\right)^z = e^{z\ln(-\frac{M_Z^2}{s} + i\,\delta)} \longrightarrow e^{i\,t\ln\frac{M_Z^3}{s}} \mathbf{e}^{-\pi\mathbf{t}}, s > 0$$

$e^{-\pi|t|}$ and $e^{-\pi t}$ cancel each other when $t \to -\infty$ and oscillations are **NOT** damped any more by an exponential factor

Contour shifts (MBnumerics)

PhD thesis by Johann Usovitsch,

https://edoc.hu-berlin.de/handle/18452/20256

Related and auxiliary Software

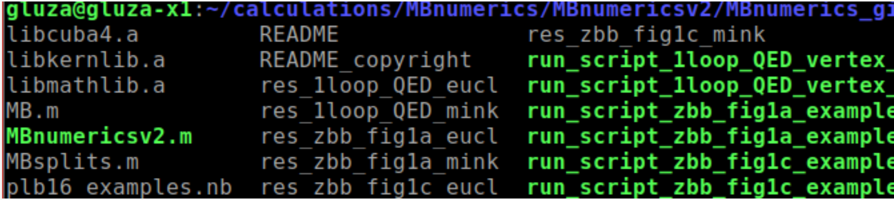
MBnumerics

Project: I. Dubovyk, T. Riemann, J. Usovitsch (jusovitsch,googlemail.com)
Software: Johann Usovitsch
Publications: https://doi.org/10.18452/19484 , https://doi.org/10.1016/j.cpc.2006.07.002, https://doi.org/10.
To be cited by users in publications, for details see README_copyright in the downloaded tarball.
Features: MBnumerics is a software for evaluation of MB integrals in the Minkowski kinematics
Download: http://us.edu.pl/~gluza/ambre/packages/mbnumerics.tgz

- gives high accuracy results up to certain dimensionality of MB integrals
- can produce huge cascade of lower-dimensional integrals

https://jgluza.us.edu.pl/ambre

The MBnumerics.m package

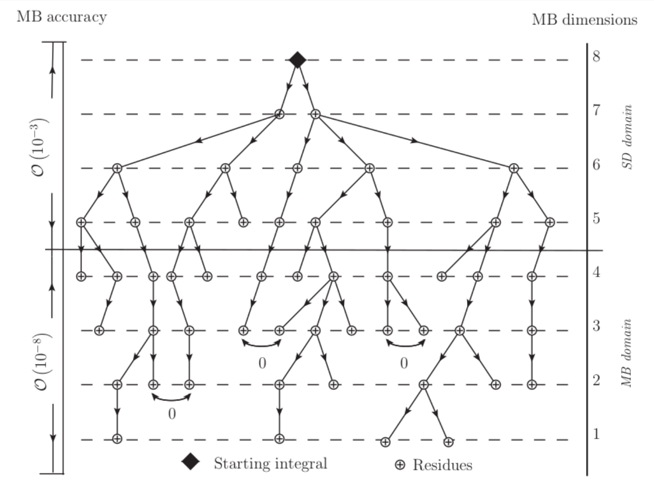


Needs:

1. MB.m
2. Cuba/Cuhre library
3. CERNlib hands-on examples on-line.

Top-bottom approach to evaluation of multidimensional MB integrals

MBnumerics.m - I. Dubovyk, J. Usovitsch, T. Riemann



Possible improvements

- Decoupling of Feynman variables

$$M_\Gamma Z \quad = \quad \begin{bmatrix} \alpha_{ij}(\text{numerator}) \\ \dots\dots\dots\dots\dots\dots\dots \\ \alpha_{ij}(\text{denominator}) \end{bmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}.$$

$$\Gamma[\sum_j \alpha_{ij} z_j + \beta_i]$$

Any linear variable transformation can be represented as

$$M_\Gamma Z \quad = \quad M_\Gamma U U^{-1} Z = M_\Gamma U Z', \;\; Z' = U^{-1} Z,$$

U - non-singular $r \times r$ transformation matrix . M_Γ encodes a new z structure of gamma functions for applying BL or decoupling:

$$M_\Gamma \longrightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

A very broad subject. We can do exactly (functions of any sort) or using expansions (in kinematic variables, regions) getting effectively simpler (massless-like) functions.
Recently progress with two ideas:

1. Conic hulls (a linear combination of vectors with non-negative coefficients) PRL127(2021)15,151601

i	Γ function	$\mathbf{e}_i$	$s_i(\mathbf{z})$
1	$\Gamma(-z_1)$	$(-1,0)$	$-z_1$
2	$\Gamma(-z_2)$	$(0,-1)$	$-z_2$
3	$\Gamma(a+z_1+z_2)$	$(1,1)$	z_1+z_2
4	$\Gamma(b_1+z_1)$	$(1,0)$	z_1
5	$\Gamma(b_2+z_2)$	$(0,1)$	z_2

Key point: building series representations of the N -fold MB integral without any convergence analysis, master series, ...

2. Triangulation (e.g. found simpler hypergeometric solutions, new convergent series representation for HPLs), used TOPCOM software, see in 2407.20120.

Limitations: No scale (boundary conditions), ..., example: mbCH.pdf
S. Friot, D. Greynat, S. Banik, B. Ananthanarayan, S. Bera, S. Datta, S. Ghosh, ...

MB series (≪ MB.m; ≪ MBsums.v1.0.3.m; ≪ MBConicHulls.wl)

```
In[-]:= (*2 fold integral from non-planar two-loop vertex integral with one massive line - MBsums and MBConicHulls - equivalent results*)
In[-]:= dim2int = MBint[(x^z1 Gamma[-z1] × Gamma[1 + z1] × Gamma[-1 - z1 - z2] × Gamma[-z2] × Gamma[1 + z2]^3) / Gamma[1 - z1 + z2], {{eps → 0}, {z1 → -0.42644, z2 → -0.826119}}];
In[-]:= dim2intCH = MBRep[{1}, {z1, z2}, {x, 1}, {{-z1, 1 + z1, -1 - z1 - z2, -z2, 1 + z2, 1 + z2, 1 + z2}, {1 - z1 + z2}}];
        Non-Straight Contours.
        Time Taken 0.055891 seconds
In[-]:= kin2dint = {x → -2};
In[-]:= dim2sumLR = MBIntToSum[dim2int, kin2dint, {z1 → L, z2 → R}]
        z1->L ( Re z1 < -10661/25000 )
        z2->R ( Re z2 > -0.826119 )
```

Out[-]=

$$\left\{\text{MBsum}\left[\frac{(-1)^{2\,n1+2\,n2}\,x^{-1-n1}\,((n1+n2)!)^2\,(\text{HarmonicNumber}[n2]-2\,\text{HarmonicNumber}[n1+n2]+\text{HarmonicNumber}[1+2\,n1+n2])}{n2!\,(1+2\,n1+n2)!},\ n1\geq 0\,\&\&\,n2\geq 0,\ \{n1,n2\}\right],\ \text{MBsum}\left[\frac{(-1)^{n1+n2}\,x^{-1-n1}\,(-1+n1-n2)!\,(n2!)^2}{(1+n1+n2)!},\ n1>0\,\&\&\,n2\geq 0\,\&\&\,1+n2\leq n1,\ \{n1,n2\}\right]\right\}$$

```
In[-]:= DoAllMBSums[dim2sumLR, 20, kin2dint] // N
Out[-]= -0.920411
```

```
In[-]:= dim2ResolveCH = ResolveMB[dim2intCH];
        Degenerate case with 14 conic hulls
        Found 5 series solutions.
        Cardinality 2:: Solution found 1.
        Cardinality 3:: Solution found 2.
        Cardinality 5:: Solution found 1.
        Cardinality 6:: Solution found 1.
```

Series Solution 1 :: Cardinality 2. Intersecting Conic Hulls $\{C_{2,3}, C_{3,4}\}$. Set of poles :: $\{\{-1-n_2, n_1\}, \{-1-n_1, -n_1+n_2\}\}$ with master series characteristic list and variables $\{\{n_1-n_2, n_2, n_2, -n_1-n_2, n_1, n_2\}, \{\frac{1}{x}, 1\}\}$.

Series Solution 2 :: Cardinality 3. Intersecting Conic Hulls $\{C_{1,4}, C_{2,3}, C_{2,4}\}$. Set of poles :: $\{\{n_1, 1+n_1+n_2\}, \{-1-n_2, n_1\}, \{-1+n_1-n_2, n_1\}\}$ with master series characteristic list and variables $\{\{n_1, n_1, -2n_1+n_2, n_1\}, \{\frac{1}{x}, x\}\}$.

Series Solution 3 :: Cardinality 3. Intersecting Conic Hulls $\{C_{3,5}, C_{3,6}, C_{3,7}\}$. Set of poles ::
$\{\{-1-n_1, -1-n_1-n_2\}, \{-1-n_1, -1-n_1-n_2\}, \{-1-n_1, -1-n_1-n_2\}\}$ with master series characteristic list and variables $\{\{n_1+n_2, -n_2, -n_1+n_2, n_1\}, \{\frac{1}{x}, 1\}\}$.

Series Solution 4 :: Cardinality 5. Intersecting Conic Hulls $\{C_{1,2}, C_{1,4}, C_{2,5}, C_{2,6}, C_{2,7}\}$. Set of poles
:: $\{\{n_1, n_2\}, \{n_1, 1+n_1+n_2\}, \{n_1+n_2, n_1\}, \{n_1+n_2, n_1\}, \{n_1+n_2, n_1\}\}$ with master series characteristic list and variables $\{\{n_1-n_2, n_1-n_2, -n_1+2n_2, n_2\}, \{1, x\}\}$.

Series Solution 5 :: Cardinality 6. Intersecting Conic Hulls $\{C_{1,5}, C_{1,6}, C_{1,7}, C_{2,5}, C_{2,6}, C_{2,7}\}$. Set of poles ::
$\{\{n_1, n_1-n_2\}, \{n_1, n_1-n_2\}, \{n_1, n_1-n_2\}, \{n_1+n_2, n_1\}, \{n_1+n_2, n_1\}, \{n_1+n_2, n_1\}\}$ with master series characteristic list and variables $\{\{n_1-n_2, -n_1, n_1+n_2, n_2\}, \{1, x\}\}$.

```
        Time Taken 0.857234 seconds
In[-]:= SumAllSeries [EvaluateSeries[dim2ResolveCH, {}, 1], kin2dint, 20,  RunInParallel → True];
        The series solution is a sum of the following 2 series.
```

Series Number 1 :: $\left\{\frac{(-1)^{2(n_1+n_2)}\,x^{-1-n_2}\,\text{Gamma}[1+n_1+n_2]^2\,(\text{PolyGamma}[0,1+n_1]-2\,\text{PolyGamma}[0,1+n_1+n_2]+\text{PolyGamma}[0,2+n_1+2n_2])}{\text{Gamma}[1+n_1]\,\text{Gamma}[2+n_1+2n_2]}\right\}$ valid for $n_1\geq 0\,\&\&\,n_2\geq 0$

Series Number 2 :: $\left\{\frac{(-1)^{n_1+n_2}\,x^{-1-n_1}\,\text{Gamma}[n_1-n_2]\,\text{Gamma}[1+n_2]^2}{\text{Gamma}[2+n_1+n_2]}\right\}$ valid for $n_1>n_2\,\&\&\,n_2\geq 0$

```
        Time Taken 2.12677 seconds
        Numerical Result: {-0.920411}
        Time Taken 0.603632 seconds
```

Conclusions

- MB method - can be complementary to DEqs/SD, can be applied to fix DEqs boundary conditions
- Alternative when DEs or SD cannot be used or face their own vices
- Particularly suitable for the analysis of asymptotic behavior of Feynman graphs for small/large values of kinematical variables
- Still unexplored areas:
  - analytical solutions (e.g. going beyond 1st and 2nd Barnes lemmas),
  - integrations (e.g. steepest decent contours with Lefschetz thimbles),
  - summations (analytical, numerical),

and more $\rightarrow$ Giampiero's talk.

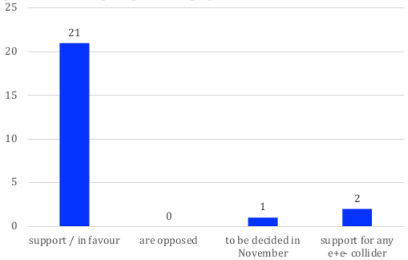
Backup slides

ESPPU in Venice, June 2025

Talk "National Inputs" by Calin Alexa, https://agenda.infn.it/event/44943/

a) Which is the preferred next major/flagship collider project for CERN?

- **Broad consensus** among CERN Member States in support of the Future Circular Collider (FCC) as a key long-term project to maintain Europe's leadership in particle physics.



Support for FCC	Belgium, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Israel, Italy, Norway, Poland, Portugal, Romania, Serbia, Slovak Republic, Spain, Sweden, Switzerland, United Kingdom
Opposed	None
To be finalized in November	Netherlands
Support for any e^+e^- collider	Austria, Bulgaria

Horizon Europe 2028-2034

16 July 2025, Commission President Ursula von der Leyen presented the Commission's proposal for the next Multiannual Financial Framework (MFF) 2028–34 and the European Competitiveness Fund (ECF), link - pdf

Based on orientation of the steering mechanism for the next MFF including the Competitiveness Coordination Tool, the Horizon Europe programme and the European Competitiveness Fund could finance a coherent sequence between research and innovation, demonstration, development and deployment, focusing efforts and funding, from the EU and national, public and private sector of 'moonshots' projects with a strong scientific component, boosting EU-wide value creation and strategic autonomy (see examples below) .
Possible 'moonshots':

- Investing in the European Organization for Nuclear Research's (CERN) Future Circular Collider, alongside other CERN's participating countries. The objective is to maintain Europe's leadership in particle physics research. The funding (up to 20% of the overall cost) could come from Horizon Europe.
- Developing Smart and Clean Aviation and European leadership in the next generation CO_2-free aircraft and automated air traffic management: It would require a partnership with industry, together with a strong scientific and engineering capacity, supported by Horizon Europe, but also a robust industrial deployment component from the Competitiveness Fund.

Moonshots

Objective: ambitious technology driven projects that boost the EU's strategic autonomy through research, development and deployment.

▸ the future circular collider ▸ clean aviation ▸ quantum computing ▸ next-generation AI ▸ data sovereignty ▸ automated transport and mobility ▸ regenerative therapies ▸ fusion energy ▸ space economy ▸ zero water pollution ▸ ocean observation

Jorgen D'Hondt, "Strategies and plans for particle physics in Europe",

Epiphany 2021, https://indico.cern.ch/event/934666

---

e^+e^- Higgs Factories (incl. B/c/τ/EW/top factories)

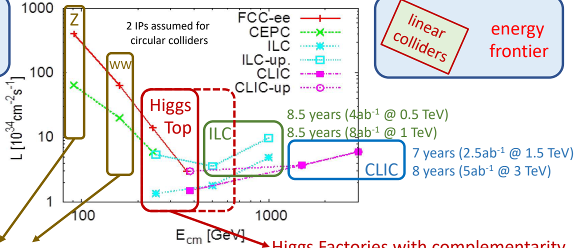
precision frontier — circular colliders

synchrotron radiation

↓

for the same power, less luminosity at higher E_{cm}

(Energy Recovery Linac technology might mitigate this & allow to go to higher E_{cm})



linear colliders — energy frontier

B/c/τ/EW Factories

per detector in e⁺e⁻	# Z	# B	# τ	# charm	# WW
LEP	4 x 10^6	1 x 10^6	3 x 10^5	1 x 10^6	2 x 10^4
SuperKEKB	-	10^{11}	10^{11}	10^{11}	-
FCC-ee	2.5 x 10^{12}	7.5 x 10^{11}	2 x 10^{11}	6 x 10^{11}	1.5 x 10^8

Higgs Factories with complementarity

- g_{HZZ} (250GeV) versus g_{HWW} (380GeV)
- top quark physics
- beam polarization for EW precision tests

(transverse polarization in circular e⁺e⁻ colliders only at lower E_{cm} while longitudinal polarization at linear colliders)

A. Blondel, P. Janot, 2106.13885

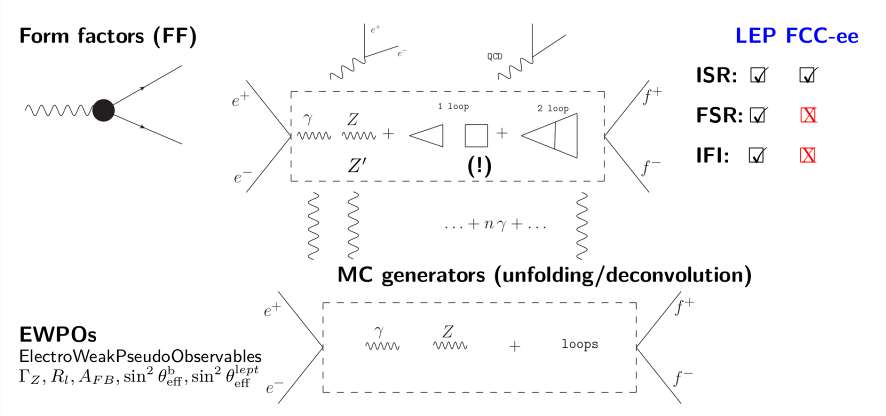
Observable	present value ± error	FCC-ee Stat.	FCC-ee Syst.	Comment and leading exp. error
m_Z (keV)	91186700 ± 2200	**4**	100	From Z line shape scan Beam energy calibration
Γ_Z (keV)	2495200 ± 2300	**4**	25	From Z line shape scan Beam energy calibration
$\sin^2\theta_W^{eff}(\times 10^6)$	231480 ± 160	**2**	2.4	from $A_{FB}^{\mu\mu}$ at Z peak Beam energy calibration
$1/\alpha_{QED}(m_Z^2)(\times 10^3)$	128952 ± 14	**3**	small	from $A_{FB}^{\mu\mu}$ off peak QED&EW errors dominate
R_ℓ^Z $(\times 10^3)$	20767 ± 25	**0.06**	0.2-1	ratio of hadrons to leptons **acceptance for leptons**
$\alpha_s(m_Z^2)$ $(\times 10^4)$	1196 ± 30	**0.1**	0.4-1.6	from R_ℓ^Z above
σ_{had}^0 $(\times 10^3)$ (nb)	41541 ± 37	**0.1**	4	peak hadronic cross section luminosity measurement
$N_\nu(\times 10^3)$	2996 ± 7	**0.005**	1	Z peak cross sections Luminosity measurement
R_b $(\times 10^6)$	216290 ± 660	**0.3**	< 60	ratio of $b\bar{b}$ to hadrons stat. extrapol. from SLD
$A_{FB,0}^b$ $(\times 10^4)$	992 ± 16	**0.02**	1-3	b-quark asymmetry at Z pole from jet charge
$A_{FB}^{pol,\tau}$ $(\times 10^4)$	1498 ± 49	**0.15**	<2	τ polarization asymmetry τ decay physics
τ lifetime (fs)	290.3 ± 0.5	**0.001**	0.04	radial alignment
τ mass (MeV)	1776.86 ± 0.12	**0.004**	0.04	momentum scale
τ leptonic $(\mu\nu_\mu\nu_\tau)$ B.R. (%)	17.38 ± 0.04	**0.0001**	0.003	e/μ/hadron separation
m_W (MeV)	80350 ± 15	**0.25**	0.3	From WW threshold scan Beam energy calibration

An excellent measurement of the centre of mass energy by the resonant depolarisation of pilot bunches

Gluza

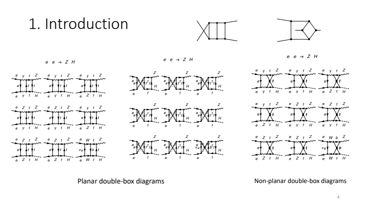
EWPOs at the Z-pole

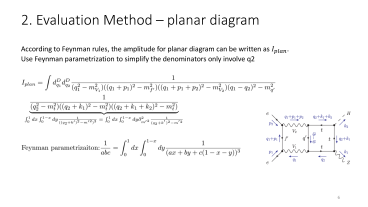
Experimental measurements at the Z-pole: after unfolding



One example - Qian Song, "NNLO EW corrections in HZ production",

https://indico.cern.ch/event/995644/





FCC-ee [exp]: 0.3%, present: $\delta_{TH} \sim 1\%$, full 2-loop $\sim 0.3\%$
6-dim $\rightarrow$ 3-dim integrals

To get to the experimental precision, we must improve very much!

I. Dubovyk et al JHEP 2019 (1906.08815)

Present theory uncertainty for Γ_Z: 0.4 MeV

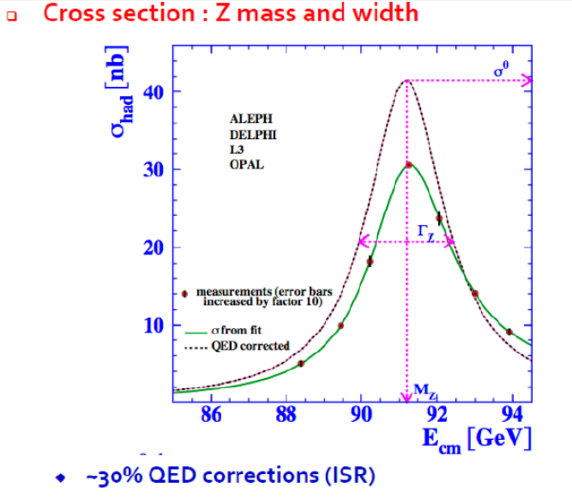
Observable	max. dev.	EXP now	FCC-ee	CEPC	GigaZ
Γ_Z [MeV]	0.04	2.3	0.1	0.5	0.8
$\sin^2\theta^{\ell}_{\rm eff} \times 10^4$	0.056	1.6	0.06	0.23	0.1
$\sin^2\theta^{b}_{\rm eff} \times 10^4$	0.025	160	9	9	15

Table 4. Goodness of fit for some chosen EWPOs, compared with the envisaged precision measurements for Γ_Z and $\sin^2\theta^{\ell}_{\rm eff}$ (statistical errors), and $\sin^2\theta^{b}_{\rm eff}$ (systematic errors) at the collider projects FCC-ee Tera-Z [84], CEPC [85] and ILC/GigaZ [86]. The values of maximal deviations are taken from tables 1 and 3. The entry "EXP now" gives the present experimental precision, as known since LEP 1 [44].

Present theoretical calculations are far behind the FCC-ee Tera-Z demands.

And enough for GigaZ LC.

To get to the experimental precision, we must improve very much!



$$\frac{\delta\Gamma_Z}{\Gamma_Z} = \frac{.025}{2495} \simeq 10^{-5}!$$

A few sample precision quantities of interest for the FCC-ee program

Quantity	Current precision	FCC-ee target precision	Required theory input	Available calc.	Needed theory improvement*
m_Z	2.1 MeV	0.1 MeV **0.1 MeV**	non-resonant $e^+e^- \to f\bar{f}$, initial-state radiation (ISR)	NLO, ISR logs up to 6th order	NNLO for $e^+e^- \to f\bar{f}$
Γ_Z	2.3 MeV	0.1 MeV **0.4** MeV			
$\sin^2 \theta^{\ell}_{\text{eff}}$	1.6×10^{-4}	0.6×10^{-5} $\mathbf{4.5} \times 10^{-5}$			
m_W	12 MeV	0.4 MeV **4** MeV	lineshape of $e^+e^- \to WW$ near threshold	NLO ($ee \to 4f$ or EFT framework)	NNLO for $ee \to WW$, $W \to ff$ in EFT setup
HZZ coupling	—	0.2% **3 %**	cross-sect. for $e^+e^- \to HZ$	NLO + NNLO QCD	NNLO electroweak
m_t	>100 MeV	17 MeV **50** MeV	threshold scan $e^+e^- \to t\bar{t}$	N^3LO QCD, NNLO EW, resummations up to NNLL	Matching fixed orders with resummations, merging with MC, α_s (input)

Theory: 1906.05379, 2106.11802

Quantity	Required theory input	Available calc.	Needed theory improvement*
Γ_Z $\sin^2\theta^{\ell}_{\rm eff}$	vertex corrections for $Z \to f\bar{f}$	NNLO + partial higher orders	$\mathrm{N^3LO}$ EW + partial higher orders
m_W	SM corrections to the muon decay rate	NNLO + partial higher orders	$\mathrm{N^3LO}$ EW + partial higher orders

1956, 1-loop, Behrends, Finkelstein Sirlin

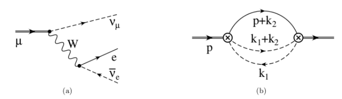
43 y ↓

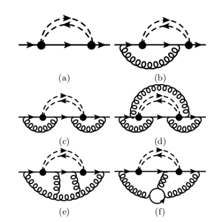
1999, 2-loops, van Ritbergen, Stuart

22 y ↓

2021, 3-loops

? ↓





Important (3-loop) step since 1999 (van Ritbergen & Stuart).

$$\Delta\tau_\mu(\alpha^3) \quad = \quad (9 \pm 1) \times 10^8 \;\; \mu s,$$

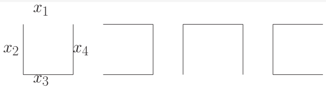
$$\tau_\mu^{exp} \quad = \quad 2.1969811 \pm 0.0000022 \; \mu s.$$

M. Fael, K. Schönwald, and M. Steinhauser, Third order corrections to the semileptonic $b \to c$ and the muon decays, PRD'2021, arXiv:2011.13654
M. Czakon, A. Czarnecki, and M. Dowling, Three-loop corrections to the muon and heavy quark decay rates, PRD'2021, arXiv:2104.05804

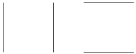
Multiloop Feynman diagrams, general MB integrals

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} \to \int \prod_{j=1}^{N} dx_j \; x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

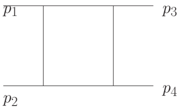
$$N_\nu = n_1 + \dots + n_N$$



Trees contributing to the polynomial U for the square diagram



$2-$ trees contributing to the polynomial F for the square diagram



Dimension of MB integrals depends on factorizations of F and U!

$$\mathbf{U} = \mathbf{x_1 + x_2 + x_3 + x_4}$$! *1-loop* $\longrightarrow 1$

$$\mathbf{F} = \mathbf{t \cdot x_1 x_3 + s \cdot x_2 x_4}$$

Cuts of internal lines such that:

- U: (i) every vertex is still connected to every other vertex by a sequence of uncut lines; (ii) no further cuts without violating (i)
- F: (iii) divide the graph into two disjoint parts such that within each part (i) and (ii) are obeyed and such that at least one external momentum line is connected to each part;

Cheng–Wu Theorem

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod\limits_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The Cheng–Wu theorem states that the same formula holds with the delta function

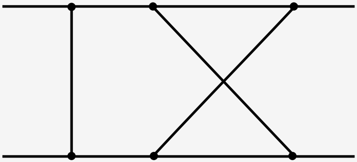
$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1, \ldots, L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.
One can prove this theorem in a simple way starting from the alpha representation using

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i=1}^{N}\alpha_i\right) \Leftrightarrow 1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i\in\Omega}\alpha_i\right)$$

and change variables from α_i to $\alpha_i = \lambda x_i$ as shown above.

Non–Planar DoubleBox

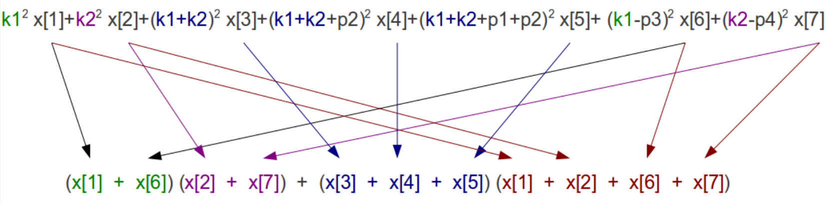


$$B_7^{NP} = \iint d^d k_1 d^d k_2 \frac{1}{[(k_1+k_2+p_1+p_2)^2]^{n_1}[(k_1+k_2+p_2)^2]^{n_2}[(k_1+k_2)^2]^{n_3}}$$
$$\frac{1}{[(k_1-p_3)^2]^{n_4}[(k_1)^2]^{n_5}[(k_2-p_4)^2]^{n_6}[(k_2)^2]^{n_7}}$$

$$U(x) = x[1]x[2] + x[1]x[4] + x[2]x[4] + x[1]x[5] + x[2]x[5] + x[2]x[6] + x[4]x[6] + x[5]x[6] + x[1]x[7] + x[4]x[7] + x[5]x[7] + x[6]x[7] \quad \textit{11d}$$

$$F(x) = -s\ x[1]x[2]x[5] - s\ x[1]x[3]x[5] - s\ x[2]x[3]x[5] - u\ x[2]x[4]x[6] - s\ x[3]x[5]x[6] - t\ x[1]x[4]x[7] - s\ x[3]x[5]x[7] - s\ x[3]x[6]x[7] \quad \textit{7d}$$

In this case F, U polynomials are the following



Factorization scheme

$$U(x) = (x[1]+x[6])(x[2]+x[7]) + (x[3]+x[4]+x[5])(x[1]+x[2]+x[6]+x[7])$$

$$F(x) = -t\ x[1]x[4]x[7] - u\ x[2]x[4]x[6] - s\ x[1]x[2]x[5] \\ - s\ x[3]x[6]x[7] - s\ x[3]x[5](x[1]+x[2]+x[6]+x[7])$$

Now we can apply the Cheng-Wu theorem and integrations will look as follows

$$B_7^{NP} = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - d\right)}{\Gamma(n_1)\dots\Gamma(n_N)} \int\limits_0^\infty dx_3 dx_4 dx_5 \int\limits_0^1 dx_1 dx_2 dx_6 dx_7 \delta(1-(x_1+x_2+x_6+x_7)) \\ \frac{((x_1+x_6)(x_2+x_7)+x_3+x_4+x_5)^{N_\nu - \frac{3d}{2}}}{(-t\ x_1 x_4 x_7 - u\ x_2 x_4 x_6 - s\ x_1 x_2 x_5 - s\ x_3 x_6 x_7 - s\ x_3 x_5)^{N_\nu - d}}$$

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\dots\Gamma(n_N)} \int\limits_{-i\infty}^{i\infty} dz_1 \dots dz_4 \int dx_1 \dots dx_7\, (-s)^{-N_\nu+d-z_2-z_3}(-t)^{z_2}(-u)^{z_3} \\ \times \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(N_\nu - d + z_1 + z_2 + z_3 + z_4) \\ \times x_1^{-N_\nu+d-z_1-z_2-z_3} x_2^{z_2+z_3} x_3^{-N_\nu+d-z_2-z_3-z_4} x_4^{z_1+z_3} x_5^{z_2+z_4} x_6^{z_1+z_2} x_7^{z_3+z_4} \\ \times (x_3+x_4+x_5+(x_1+x_6)(x_2+x_7))^{N_\nu - \frac{3d}{2}}$$

Integration over Cheng–Wu variables

$$\int_0^\infty dx\ x^{N_1}(x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)}$$

4-dim result:

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_7)} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_4 (-s)^{4-2\epsilon-N_\nu-z_{23}} (-t)^{z_3} (-u)^{z_2}$$
$$\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(2-\epsilon-n_{45})\Gamma(2-\epsilon-n_{67})}{\Gamma(4-2\epsilon-n_{4567})\Gamma(n_{45}+z_{1234})\Gamma(n_{67}+z_{1234})\Gamma(6-3\epsilon-N_\nu)}$$
$$\Gamma(n_2+z_{23})\Gamma(n_4+z_{24})\Gamma(n_5+z_{13})\Gamma(n_6+z_{34})\Gamma(n_7+z_{12})\Gamma^3(-2+\epsilon+n_{4567}+z_{1234})$$
$$\Gamma(4-2\epsilon-n_{124567}-z_{123})\Gamma(4-2\epsilon-n_{234567}-z_{234})\Gamma(-4+2\epsilon+N_\nu+z_{1234})$$

---

with notations $z_{i\ldots j\ldots k} = z_i + \ldots + z_j + \ldots + z_k$
and $n_{i\ldots j\ldots k} = n_i + \ldots + n_j + \ldots + n_k$

In general: $\Gamma[\Lambda_i] = \Gamma[\sum_l \alpha_{ij} z_j + \beta_i]$, *massless cases:* $\alpha_{ij} = \pm 1$

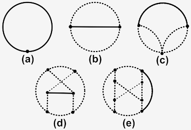
AMFlow method, $\eta = \infty \longrightarrow \eta = 0^+$ analytic continuation (auxiliary mass flow)

2. A set of Jan 27 2022 papers by Zhi-Feng Liu, Yan-Qin Ma and Xiao Liu:
https://inspirehep.net/literature/2020677, https://inspirehep.net/literature/2020676, https://inspirehep.net/literature/2020880 and 1711.09572 https://inspirehep.net/literature/1639025.

$$\begin{aligned}
\widetilde{I}_{\vec{\nu}}(\eta) &= \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{D/2}} \right) \frac{\widetilde{\mathcal{D}}_{K+1}^{-\nu_{K+1}} \cdots \widetilde{\mathcal{D}}_N^{-\nu_N}}{\widetilde{\mathcal{D}}_1^{\nu_1} \cdots \widetilde{\mathcal{D}}_K^{\nu_K}}. \\
\widetilde{\mathcal{D}}_1 &= \ell_1^2 - m^2 + \mathrm{i}\eta \\
I_{\vec{\nu}} &= \lim_{\eta \to 0^+} \widetilde{I}_{\vec{\nu}}(\eta) \\
\mathrm{i}\frac{\partial}{\partial \eta} \overrightarrow{\widetilde{J}}(\eta) &= A(\eta) \widetilde{J}(\eta)
\end{aligned}$$

Key point: boundary conditions at $\eta \to \infty$ are single mass scale bubble integrals, solved iteratively.

MIs with high accuracy by AMFlow, results



$$I_{\vec{\nu}} = \int \left(\prod_{i=1}^{L} \frac{{}^{D}\ell_i}{\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}}, \quad \mathcal{D}_1 = \ell_1^2 - m^2 + 0^+$$

$$\widehat{I}_{\vec{\nu}'}(\ell_1^2) = \int \left(\prod_{i=2}^{L} \frac{{}^{D}\ell_i}{\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2} \cdots \mathcal{D}_K^{\nu_K}}, \quad I_{\vec{\nu}} = \{\Gamma[\ldots]\} \widehat{I}_{\vec{\nu}'}(-m^2)$$

$I_{\vec{\nu}}$ ↑ **L-loop**; $\widehat{I}_{\vec{\nu}'}$ ↑ **(L-1)-loop**

$$\begin{aligned} I[(e)] = & -2.073855510286740\epsilon^{-2} - 7.812755312590133\epsilon^{-1} \\ & - 17.25882864945875 + 717.6808845492140\epsilon \\ & + 8190.876448160049\epsilon^2 + 78840.29598046500\epsilon^3 \\ & + 566649.1116484678\epsilon^4 + 3901713.802716081\epsilon^5 \\ & + 23702384.71086095\epsilon^6 + 14214293.68205112\epsilon^7, \end{aligned}$$

10 orders in ϵ, 16-digit precision.

Such an exact boundary point can be transported by `DiffExp` *to any physical point*.

Methods of brackets

1. Exponential function expansion

$$\exp(-xA) = \sum_n \frac{(-1)^n}{\Gamma(n+1)} x^n A^n .$$

2. Integration symbol and its equivalent bracket

   The structure $\int x^{a_1+a_2+\ldots+a_n-1}\, dx$ is replaced by its respective bracket representation

$$\int x^{a_1+a_2+\ldots+a_n-1} dx = \langle a_1 + a_2 + \ldots + a_n \rangle .$$

3. Polynomials expansion

$$(A_1 + \ldots + A_r)^{\pm\mu} = \sum_{n_1} \ldots \sum_{n_r} \phi_{n_1} \ldots \phi_{n_r}\ (A_1)^{n_1} \ldots (A_r)^{n_r} \times \frac{\langle \mp\mu + n_1 + \ldots + n_r \rangle}{\Gamma(\mp\mu)} .$$

4. Finding the solution

   For the case of a generic series of brackets J

$$J = \sum_{n_1} \ldots \sum_{n_r} \phi_{n_1} \ldots \phi_{n_r}\ F(n_1, \ldots, n_r)$$

$$\times\ \langle \mathbf{a_{11} n_1} + \ldots + \mathbf{a_{1r} n_r} + \mathbf{c_1} \rangle \ldots \langle \mathbf{a_{r1} n_1} + \ldots + \mathbf{a_{rr} n_r} + \mathbf{c_r} \rangle ,$$

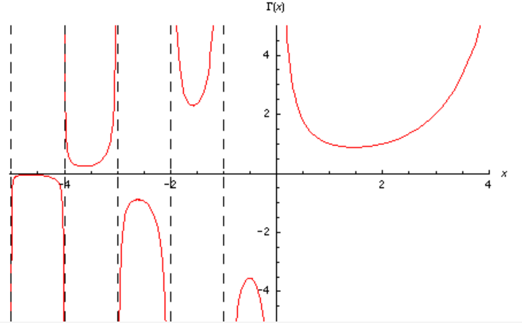
   the solution is obtained using the general formula (Ramanujan's Master Theorem)

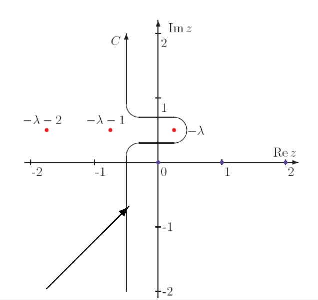
$$J = \frac{1}{|\det(\mathbf{A})|} \Gamma(-\mathbf{n_1^*}) \ldots \Gamma(-\mathbf{n_r^*})\, F(\mathbf{n_1^*}, \ldots, \mathbf{n_r^*})$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\int dz \Gamma[z+\lambda]$$

SINGULARITIES REGULAR





Contours: shifts, deformations

Types of contour deformations

$$V(s) = \frac{e^{\epsilon\gamma_E}}{i\pi^{d/2}} \int \frac{d^dk}{[(k+p_1)^2 - m^2][k^2][(k-p_2)^2 - m^2]} = \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots,$$

$$V_{-1}(s) = -\frac{1}{2s} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dz}{2\pi i} \underbrace{(-s)^{-z}}_{\textbf{Problem I}} \overbrace{\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)}}^{\textbf{Problem II}}$$

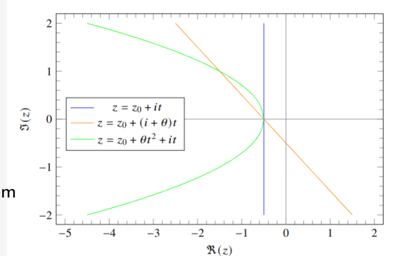
$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{2\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}},$$

$$z = \Re[z] + i\, y, \quad y \in (-\infty, +\infty),$$

$z(t) = x_0 + \theta t + it$

$\int_{-\infty}^{+\infty} (\theta + i)\ dt\ I[z(t)]$

high accuracy, no problem

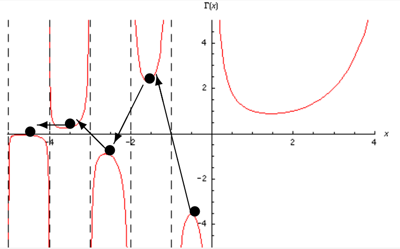


$$\int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + n + Im[z_k], \ldots) \qquad I_{orig}$$

$$= \quad Residue[\int dz_1 \ldots \not{dz_k} \ldots I]_{Re[z_k]+n} \qquad I_{Res}$$

$$+ \quad \int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + (n+1) + Im[z_k], \ldots) \qquad I_{new}$$

1. Residues **lower** dimensionality of original MB integrals.
2. Integral after passing a pole (proper shifts) **can be made smaller**.



Numerical integration approaches

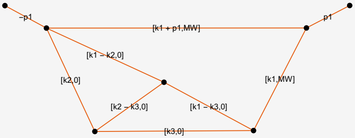
- integration over contours parallel to imaginary axis
  - requires combination of different types of transformation to finite integration region $(-\infty, +\infty) \to [0, 1]$

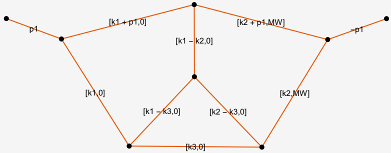
$$t_i \to \ln\left(\frac{x_i}{1 - x_i}\right), \quad t_i \to \tan\left(\pi(x_i - \frac{1}{2})\right)$$

  - low numerical stability
  - can be improved by new integration methods/libraries
- contours deformation (restoring of the exponential damping factor)

```
In[1]: INT = -((-s)^(-z1) Gamma[-z1]^3 Gamma[1+z1])/(2*s*Gamma[-2*z1])/(2*Pi*I);
In[2]: NIntegrate[ D[-1/2+theta*t+I*t, t]*INT
       /. s->1 /. z1-> -1/2 + theta t + I t /. theta-> -1, {t,-Infinity,Infinity},
       Method -> DoubleExponential]
```

  - steepest descent method - $z_i = z_{i0} + f_i(t_1, ..., t_n) + it_i$ (JG, Jeliński, Kosower '17), only one-dimensional cases
  - rotation of integration contours - $z_i = z_{i0} + (i + \theta)t_i$ (Freitas '10) Works well for certain integrals, but is not general The core of the MB integral (gammas) becomes non-smooth





1-dim

$-18.779406962 - 6.390785027i$

4-dim

$-22.5213 + 4.74442i \pm (0.001 + 0.001i)$

$$I = -\frac{1}{(-s)^{1+3\epsilon}} \int_{-i\infty}^{+i\infty} \prod_{i=1}^{4} dz_i \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-\epsilon - z_1)\Gamma(-z_1)\Gamma(1+2\epsilon+z_1)}{\Gamma(1-2\epsilon)\Gamma(1-3\epsilon-z_1)}$$
$$\times \frac{\Gamma(-2\epsilon - z_{12})\Gamma(1-\epsilon+z_2)\Gamma(1+z_{12})\Gamma(1+\epsilon+z_{12})\Gamma(1+3\epsilon+z_3)\Gamma(1-\epsilon-z_4)}{\Gamma(1-z_2)\Gamma(2+\epsilon+z_{12})}$$
$$\times \frac{\Gamma(-\epsilon - z_2)\Gamma(-z_2)\Gamma(1+z_3-z_4)\Gamma(-z_4)\Gamma(-z_3+z_4)\Gamma(-3\epsilon-z_3+z_4)}{\Gamma(1-4\epsilon-z_3)\Gamma(2+2\epsilon+z_3-z_4)}.$$

$$I = \frac{3}{s} \int_{-i\infty-\frac{17}{28}}^{+i\infty-\frac{17}{28}} dz_3 \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-1-z_3)\Gamma(-z_3)(\Gamma(1-z_3)\Gamma(-z_3) - \Gamma(-2z_3))\Gamma(1+z_3)\psi^{(2)}(z_1)}{\Gamma(1+z_3)\Gamma(-2z_3)}.$$

Numerical integration of MB integrals

In the most general form MB integral can be represented as follows:

$$I = \frac{1}{(2\pi i)^r} \int_{-i\infty+z_{10}}^{+i\infty+z_{10}} \cdots \int_{-i\infty+z_{r0}}^{+i\infty+z_{r0}} \prod_i^r dz_i \; f_S(Z) \, \frac{\prod_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod_{k=1}^{N_d} \Gamma(\Lambda_k)} f_\psi(Z)$$

$f_S(Z)$ depends on: Z – some subset of integration variables
S – kinematic parameters and masses

Λ_i : linear combinations of z_i, e.g., $\Lambda_i = \sum_l \alpha_{il} z_l + \gamma_i$

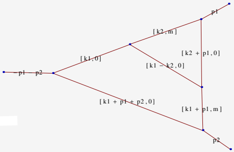
An example:

$$I^{0h0w}_{5,\epsilon^{-2}} = \frac{1}{2s} \frac{1}{2\pi i} \int_{-i\infty-\frac{1}{2}}^{+i\infty-\frac{1}{2}} dz \left(\frac{M_Z^2}{-s} \right)^z \frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)}$$

Progress for critical cases (quasi-Monte Carlo).

https://www.actaphys.uj.edu.pl/R/50/11/1993/pdf

With QMC, we can approach MB integrals with dim > 5.



$$I = \frac{1}{(2\pi i)^3}\frac{1}{s^2}\int_{-i\infty}^{i\infty} dz_1 \int_{-i\infty}^{i\infty} dz_2 \int_{-i\infty}^{i\infty} dz_3 \left(\frac{m^2}{-s}\right)^{z_1} \frac{\Gamma(-1-z_1)\ldots\Gamma(-z_1-z_2+z_3)}{\Gamma(-z_1)\Gamma(1-z_2)\Gamma(1-z_1+z_3)}.$$

Overlaped integrals

Numerical results for I with $s = m^2 = 1$.

`Analytical`	$-1.\mathbf{199526183135}$	$+5.\mathbf{567365907880}i$	
`MB`	$-1.\mathbf{199526183}168$	$+5.\mathbf{567365907}904i$	Cuhre, 10^7, 10^{-8}
`MB`	$-1.\mathbf{20}4597845834$	$+5.\mathbf{567}518701898i$	Vegas, 10^7, 10^{-3}
`MB`	$-1.\mathbf{19951}6455248$	$+5.\mathbf{5673}76681167i$	QMC, 10^6, 10^{-5}
`MB`	$-1.\mathbf{19952}7580305$	$+5.\mathbf{56736}7345229i$	QMC, 10^7, 10^{-6}

In a nutshell:

- This is a stationary phase method leading to optimal steepest decent contours.
- Contours found using Lefschetz thimbles (exact contours) or Pade approximations.

Lefschetz thimbles (LT) is a fascinating subject, crossing many issues (far way to real multidimensional MB applications):

- Behaviour of LT in presence of poles, singularities and branch cuts, behaviour in the complex infinity, Stokes phenomenon, relation to relative homology of a punctured Riemann sphere etc.
- Other applications of LT: analytical continuation of 3d Chern-Simons theory, QCD with chemical potential, resurgence theory, counting master integrals, repulsive Hubbard model,...

Some basic technicalities of Lefschetz thimbles

- Deform the original integration contour $\mathcal{C}_0$ to a Lefschetz thimble $\mathcal{J}(z_*)$, $f = \mathrm{Re}\, f + i\,\mathrm{Im}\, f$, $f \equiv -lnF$, F - original MB integrand

$$\int_{\mathcal{C}_0} dz\, e^{-f} = \overbrace{e^{-i\,\mathrm{Im}\, f|_{\mathcal{J}(z_*)}}}^{\textbf{Overal factor}} \int\limits_{\mathcal{J}(z_*)} \overbrace{dz\, e^{-\mathrm{Re}\, f}}^{\textbf{Damping factor}} + 2\pi i \sum_{\mathcal{C}_0 \to \mathcal{J}(z_*)} \overbrace{\mathrm{Res}\, e^{-f}}^{\textbf{Remnants}}$$

- Definition: Lefschetz thimble $\mathcal{J}(z_*)$ is a union of curves $t \to z(t) \in \mathbb{C}^n$ which satisfy

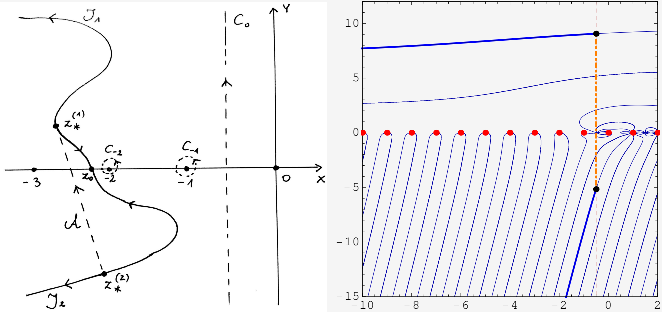
$$\overbrace{\frac{d\mathbf{z^i(t)}}{dt}}^{\textbf{Contour}} \overset{!}{=} -\overbrace{\left(\frac{\partial \mathbf{f(z)}}{\partial z^i}\right)^*}^{\textbf{function}}, \quad z(+\infty) = z_*,$$

where z_* is a saddle point of a meromorphic function f.
- Properties of $\mathcal{J}(z_*)$: $\mathrm{Im}\, f$ = const. while $\mathrm{Re}\, f$ is monotonically decreasing when $t \to +\infty$ and goes to $+\infty$ when $t \to -\infty$.
- $\mathrm{Im}\, f$ generates Hamiltonian flow on $\mathbb{R}^{2n}$, e.g. for $n = 1$:

$$\frac{dx(t)}{dt} = \frac{\partial \mathrm{Im}\, f}{\partial y}, \quad \frac{dy(t)}{dt} = -\frac{\partial \mathrm{Im}\, f}{\partial x}.$$

JG, Tomasz Jeliński, David Kosower, https://doi.org/10.1103/PhysRevD.95.076016

Exact Lefschetz contour



Shortcut (simplifies integration)

<u>Basic literature:</u>

D. Harlow, J. Maltz, and E. Witten, "Analytic Continuation of Liouville Theory",

L. Nicolaescu, "An Invitation to Morse Theory"

Y. Tanizaki and T. Koike, "Real-time Feynman path integral with Picard-Lefschetz theory and its applications to quantum tunneling"

Other directions (1)

K.H. Phan and T. Riemann, Phys. Lett. B791 (2019) 257 (The general d-dependence of 1-loop Feynman integrals) + numerics,

(a) ${}_2F_1$ Gauss hypergeometric functions are needed for self-energies;
(b) F_1 Appell functions are needed for vertices;
(c) F_S Lauricella-Saran functions are needed for boxes.

New approach to Mellin–Barnes integrals for massive one-loop Feynman integrals, Johann Usovitsch, Tord Riemann Tera-Z report, section E.6., arXiv:1809.01830, doi:10.23731/CYRM-2019-003

`MBOneLoop` package.

$$J_n = (-1)^n \Gamma(n-d/2) \int_0^1 \prod_{i=1}^{n} dx_i \delta\left(1-\sum_{j=1}^{n} x_j\right) \frac{1}{F_n(x)^{n-d/2}}$$

F-function rewritten with $\delta(1-\sum x_i)$ which makes the n-fold x-integration to be an integral over an $(n-1)$-simplex.

$$J_n(d,\{q_i,m_i^2\}) = \frac{-1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s)\Gamma(\frac{d-n+1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} \left(\frac{1}{R_n}\right)^s \times \sum_{k=1}^{n} \left(\frac{1}{R_n}\frac{\partial r_n}{\partial m_k^2}\right) \mathbf{k}^- J_n(d+2s;\{q_i,m_i^2\}).$$

- *Recursion formula which gives the minimal integration dimension for 1-loop Mellin-Barnes integrals compared to following the U and F polynomial aproach (e.g. 9dim box → 3-dim). We would like to see such recursion formulas at multi-loop level*

Other directions (2)

Summations, asymptotics, hypergeometric functions

- J. Davies, G. Mishima, M. Steinhauser, D. Wellmann, JHEP 03 (2018) 048;

$$\int_C \frac{dz}{2\pi i} \frac{\Gamma[a_1 - z, a_2 - z, a_3 + z, a_4 + z, a_5 + z]}{\Gamma(-a_6 + z)}$$
$$= \frac{\Gamma[a_{13}, a_{23}, a_{14}, a_{24}, a_{15}, a_{25}]}{\Gamma[a_{1235}, a_{1245}, -a_{56}]} \, {}_3F_2\left(\begin{matrix} a_{15}, a_{25}, a_{123456} \\ a_{1235}, a_{1245} \end{matrix} ; 1 \right) ,$$