Four loop baryon operator anomalous dimension

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Background

Baryons are established particles of nature corresponding to bound states of three quarks in the standard model

Their masses and widths are extracted from experiment with most if not all having a finite lifetime

In the case of the proton it has yet to be fully determined if it is stable or merely has an incredibly long lifetime

Some extensions of the standard model suggest that the proton may decay and if so this could indicate new physics although current experiment only provides a lower bound of around 10^{30} years

Another application where baryon operators become important relates to light cone distribution amplitudes which relate to hard exclusive processes with large momentum exchange

To improve the extraction of lattice estimates matching to the high energy behaviour of the matrix element is required as are the anomalous dimensions of moments of the baryonic operators

Anomalous dimensions for the basic 3-quark operator are available to three loops in the $\overline{\text{MS}}$ scheme [Lepage & Brodsky; Peskin; Pivovarov & Surguladze; Kränkl & Manashov; JAG]

In parallel the relevant matrix elements, where the 3-quark operators are inserted in a three quark Green's function, have to be evaluated perturbatively in the chiral limit and in a configuration which is the same as that used in lattice computations

The two main configurations are where the operator is inserted at zero and non-zero momentum [JAG; Kniehl & Veretin]

Recently the anomalous dimension of the first moment has been determined to two loops [Bali et al]

With other applications of baryon operator anomalous dimensions to BSM ideas, such as composite Higgs models based on $Sp(4)$ studied on the lattice there is now a need to extend results to four loops

Baryon operator

The basic baryon operators have the form

$$\mathcal{O}_1^{udu} = \epsilon^{IJK}\gamma^5 u^I \left(\left(u^J\right)^T C d^K\right) \quad , \quad \mathcal{O}_2^{udu} = \epsilon^{IJK} u^I \left(\left(u^J\right)^T C\gamma^5 d^K\right)$$

where C is the charge conjugation matrix and the gauge group has to be $SU(3)$ for the operators to be gauge invariant

More appropriate to carry out the renormalization in dimensional regularization for the general 3-quark operator [Kränkl, Manashov]

$$\mathcal{O}^{ijk}_{\alpha\beta\gamma} = \epsilon^{IJK}\psi^{iI}_\alpha \psi^{jJ}_\beta \psi^{kK}_\gamma$$

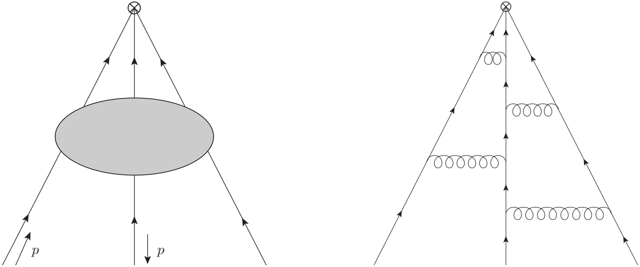
where i, j and k are flavour indices and the $SU(3)$ group generators are T^a_{IJ}

Dimensional regularization requires the extension of the γ-algebra to an infinite dimensional space with the totally antisymmetric matrices

$$\Gamma^{\mu_1\dots\mu_n}_{(n)} = \gamma^{[\mu_1}\dots\gamma^{\mu_n]}$$

for $0 \le n < \infty$, acting as the basis matrices

The operator anomalous dimension is deduced in the configuration on the left



where the momentum configuration means FORCER is applicable together with one of the 19061 four loop graphs on the right

Counting from the left there are 4, 8 and 4 γ matrices in the respective quark lines

To carry out the renormalization one has to have a strategy that balances the large amount of γ-algebra associated with open spinor strings and the retention of an integral representation that can be passed through FORCER

Four loop graphs are computed in the Feynman gauge with lower order ones calculated with an arbitrary gauge parameter

At four loops there are at most 8 quark propagators and in order to apply FORCER need to convert tensor integrals to scalars which is carried out by a projection

A general rank 8 Lorentz tensor integral with one external momentum p decomposes into 764 independent tensors built from $\eta_{\mu\nu}$ and p_μ

As the quarks are massless only even rank projectors are required but best to partition the decomposition into a basis of independent tensors built from

$$P_{\mu\nu}(p) ~=~ \eta_{\mu\nu} ~-~ \frac{p_\mu p_\nu}{p^2} ~~~,~~~ L_{\mu\nu}(p) ~=~ \frac{p_\mu p_\nu}{p^2}$$

The γ-matrices are stripped off each integral and uncontracted from any momentum before the γ-strings are converted to the $\Gamma_{(n)}^{\mu_1 \ldots \mu_n}$ basis

After the mapping of the Lorentz tensor integrals to scalars the $\Gamma_{(n)}$-matrices appear as triple tensor products

For the example of the earlier four loop graph the combination $\Gamma_{484} \equiv \Gamma_{(4)\,\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(8)}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8} \otimes \Gamma_{(4)\,\mu_5\mu_6\mu_7\mu_8}$

$$\mathbb{C}_{844} ~\equiv~ \Gamma_{844} ~+~ \Gamma_{484} ~+~ \Gamma_{448}$$

appears, among others, and symmetric combinations always arise

In particular the only general structures that are present are

$$\begin{aligned} \mathbb{C}_{qqq} &\equiv \Gamma_{qqq} ~~,~~ \mathbb{C}_{qqr} ~\equiv~ \Gamma_{qqr} + \Gamma_{qrq} + \Gamma_{rqq} ~=~ \mathbb{C}_{rqq} \\ \mathbb{C}_{qrs} &\equiv \Gamma_{qrs} + \Gamma_{qsr} + \Gamma_{rqs} + \Gamma_{rsq} + \Gamma_{srq} + \Gamma_{sqr} \end{aligned}$$

for distinct (even) q, r and s with the $\mathbb{C}_{pqr}$ satisfying a complicated algebra

Any $\mathbb{C}_{pqr}$ involving $\Gamma_{(6)}$ and $\Gamma_{(8)}$ corresponds to an evanescent operator which are not independent due to relations in d-dimensions

Several relations are

$$\begin{aligned}\mathbb{C}_{660} &= -12d(d-1)(2d-1)\mathbb{C}_{000} - 3(d-1)(7d-24)\mathbb{C}_{220} \\ &\quad - 6(2d-5)\mathbb{C}_{440} + 2(3d-4)\mathbb{C}_{220}^2 - \frac{1}{2}\mathbb{C}_{220}^3 + \frac{3}{2}\mathbb{C}_{220}\mathbb{C}_{440} + 3\mathbb{C}_{444} \\ \mathbb{C}_{642} &= 12d(d-1)(2d-7)\mathbb{C}_{000} + 9(d^2-9d+16)\mathbb{C}_{220} + 2(2d-5)\mathbb{C}_{440} \\ &\quad - 2(3d-10)\mathbb{C}_{220}^2 + \frac{1}{2}\mathbb{C}_{220}^3 - \frac{1}{2}\mathbb{C}_{220}\mathbb{C}_{440} - 3\mathbb{C}_{444}\end{aligned}$$

that reduce the anomalous dimension to purely four dimensional $\mathbb{C}_{pqr}$ which are $\mathbb{C}_{000}$, $\mathbb{C}_{220}$, $\mathbb{C}_{440}$ and $\mathbb{C}_{444}$ with

$$\Gamma_{444} = \Gamma_{(4)}^{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(4)\,\mu_1\mu_2}{}^{\mu_5\mu_6} \otimes \Gamma_{(4)\,\mu_3\mu_4\mu_5\mu_6}$$

The integration of the scalar integrals is carried out with FORCER

The gauge parameter cancels to three loops and the four loop Feynman gauge renormalization constant satisfies the lower loop prediction for the non-simple poles

Transition to four dimensions

The $\mathbb{C}_{pqr}$ involving non-evanescent $\Gamma_{(n)}$ matrices map to the following four dimensional γ-matrices

$$\begin{aligned}
\mathbb{C}_{000}\Big|_{d=4} &= I \otimes I \otimes I \\
\mathbb{C}_{220}\Big|_{d=4} &= [\sigma^{\mu\nu} \otimes \sigma_{\mu\nu} \otimes I \; + \; \sigma^{\mu\nu} \otimes I \otimes \sigma_{\mu\nu} \; + \; I \otimes \sigma^{\mu\nu} \otimes \sigma_{\mu\nu}] \\
\mathbb{C}_{440}\Big|_{d=4} &= 24 \left[\gamma^5 \otimes \gamma^5 \otimes I \; + \; \gamma^5 \otimes I \otimes \gamma^5 \; + \; I \otimes \gamma^5 \otimes \gamma^5 \right] \\
\mathbb{C}_{444}\Big|_{d=4} &= 0
\end{aligned}$$

Their eigenvalues have been computed for the four core baryons that arise in nucleon matrix elements [Kränkl, Manashov]

$$\mathcal{O}_{+}^{(\frac{1}{2},0)} = \epsilon^{IJK} \psi_L^I \left(\left(\psi_L^J \right)^T C \psi_L^K \right) \quad , \quad \mathcal{O}_{-}^{(\frac{1}{2},0)} = \epsilon^{IJK} \psi_R^I \left(\left(\psi_L^J \right)^T C \psi_L^K \right)$$

$$\mathcal{O}_{+}^{(\frac{3}{2},0)} = \epsilon^{IJK} \not{\Delta} \psi_L^I \not{\Delta} \psi_L^J \not{\Delta} \psi_L^K \quad , \quad \mathcal{O}_{-}^{(1,\frac{1}{2})} = \epsilon^{IJK} \not{\Delta} \psi_L^I \not{\Delta} \psi_L^J \not{\Delta} \psi_R^K$$

where $\Delta^2 = 0$, $\psi_R = \frac{1}{2}(1 + \gamma^5)\psi$ and $\psi_L = \frac{1}{2}(1 - \gamma^5)\psi$

The respective eigenvalues are deduced from the $\mathbb{C}_{pqr}$ eigenvalues for each $\mathbb{C}_{pqr}$ object via

Spin	Chirality	$\mathbb{C}_{000}$	$\mathbb{C}_{220}$	$\mathbb{C}_{440}$	$\mathbb{C}_{444}$
$(\frac{1}{2},0)$	$+$	1	12	72	0
$(\frac{1}{2},0)$	$-$	1	12	-24	0
$(\frac{3}{2},0)$	$+$	1	-12	72	0
$(1,\frac{1}{2})$	$-$	1	-4	-24	0

$$\begin{aligned}
\gamma_+^{(\frac{1}{2},0)}(a) &= 4a + \left[\frac{32}{9}N_f - \frac{140}{3}\right] a^2 \\
&+ \left[160N_f - 32\zeta_3 - \frac{10784}{9} - \frac{160}{3}\zeta_3 N_f - \frac{112}{27}N_f^2\right] a^3 \\
&+ \left[528\zeta_4 + 848\zeta_4 N_f - \frac{4928575}{162} - \frac{86600}{27}\zeta_5 - \frac{58972}{27}\zeta_3 N_f \right. \\
&\quad - \frac{29195}{243}N_f^2 - \frac{320}{81}N_f^3 - \frac{160}{3}\zeta_4 N_f^2 + \frac{64}{27}\zeta_3 N_f^3 + \frac{320}{27}\zeta_3 N_f^2 \\
&\quad \left. + \frac{18880}{9}\zeta_5 N_f + \frac{63670}{27}\zeta_3 + \frac{379232}{81}N_f\right] a^4 + O(a^5)
\end{aligned}$$

All four operator dimensions have been computed in the $\overline{\rm MS}$ scheme to four loop

Numerically for $N_f = 3$

$$
\begin{aligned}
\gamma_+^{(\frac{1}{2},0)}(a) &= 4.0000a - 36.0000a^2 - 986.3505a^3 - 16397.0035a^4 + O(a^5) \\
\gamma_-^{(\frac{1}{2},0)}(a) &= 4.0000a - 22.6667a^2 - 976.7735a^3 - 16302.4012a^4 + O(a^5) \\
\gamma_+^{(\frac{3}{2},0)}(a) &= -4.0000a - 96.0000a^2 - 1074.2363a^3 - 16594.0230a^4 + O(a^5) \\
\gamma_-^{(1,\frac{1}{2})}(a) &= -1.3333a - 66.2222a^2 - 1188.4358a^3 - 15092.0709a^4 + O(a^5)
\end{aligned}
$$

The critical exponents derived from the anomalous dimension in the conformal window, $9 < N_f \leq 16$, respect their respective unitarity bounds at successive loop orders

Banks-Zaks expansion

The Banks-Zaks expansion is a purely four dimensional perturbative expansion in the distance from the upper bound of the conformal window defined by the location of the Banks-Zaks fixed point $a^\star$

It is defined as the non-trivial zero of $\beta(a^\star) = 0$ closest to the origin where

$$\beta(a) \;=\; - \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a^2 - \left[\frac{34}{3} C_A^2 - 4 T_F C_F N_f - \frac{20}{3} T_F N_f C_A \right] a^3$$

Setting

$$\Delta_{\mathrm{BZ}} \;=\; \frac{11}{3} C_A \;-\; \frac{4}{3} T_F N_f$$

and replacing N_f with Δ_{BZ} then $a^\star$ is

$$a^\star \;=\; \frac{\Delta_{\mathrm{BZ}}}{3[11 C_F + 7 C_A]} \;+\; \frac{\left[924 C_F^2 + 1208 C_A C_F - 287 C_A^2\right] \Delta_{\mathrm{BZ}}^2}{216[11 C_F + 7 C_A]^3} \;+\; O(\Delta_{\mathrm{BZ}}^3)$$

Can examine the N_f dependence of the critical exponents of the baryon operators in the conformal window

For instance

$$\gamma_+^{(\frac{1}{2},0)}(a^\star) = \frac{4}{107}\Delta_{\mathrm{BZ}} + \frac{19379}{7350258}\Delta_{\mathrm{BZ}}^2 + \left[\frac{314021069}{1009837246104} - \frac{12224}{131079601}\zeta_3\right]\Delta_{\mathrm{BZ}}^3 + O(\Delta_{\mathrm{BZ}}^4)$$

or

$$\gamma_+^{(\frac{1}{2},0)}(a^\star) = 3.738318 \times 10^{-2}\,\Delta_{\mathrm{BZ}} + 2.636506 \times 10^{-3}\,\Delta_{\mathrm{BZ}}^2 + 1.988627 \times 10^{-4}\,\Delta_{\mathrm{BZ}}^3 + 7.358447 \times 10^{-5}\,\Delta_{\mathrm{BZ}}^4 + O(\Delta_{\mathrm{BZ}}^5)$$

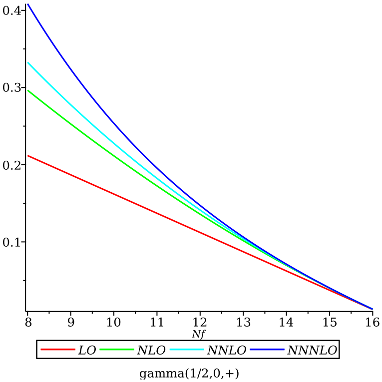
noting that the respective range of Δ_{BZ} in the conformal window, $9 \leq N_f \leq 16$, is $5 \geq \Delta_{\mathrm{BZ}} \geq \frac{1}{3}$

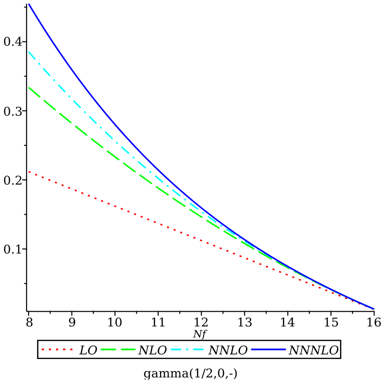
For example when $N_f = 9$

$$\gamma_+^{(\frac{1}{2},0)}(a^\star) = 0.186816 + 0.065663 + 0.024856 + 0.045990 + \ldots$$

where the convergence can be better gauged graphically

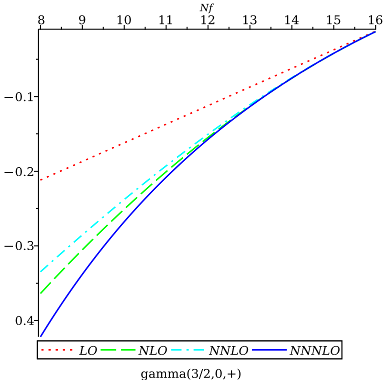
For spin-$\frac{1}{2}$ operators we have

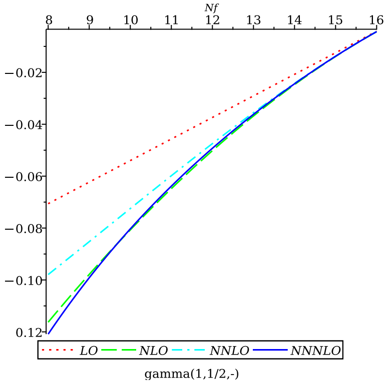




Perturbative reliability valid down to around $N_f = 12$

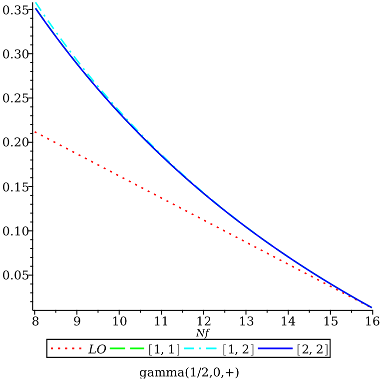
For other two operators we have

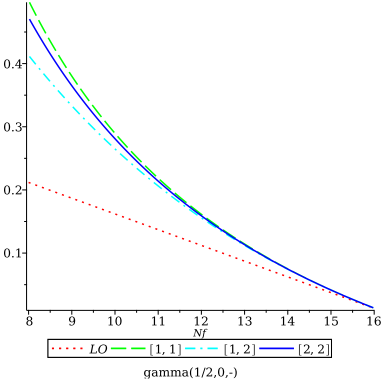




Again perturbative reliability valid down to around $N_f = 12$

Can refine the behaviour towards the lower end of the conformal window by using Padé approximants





Estimates for $N_f \leq 10$ may be reliable if the conformal window remains around $N_f = 8$

Padé approximants for other two operators are not as conclusive

Exponent	N_f	$P_{[1,1]}$	$P_{[1,2]}$	$P_{[2,2]}$
$\gamma_{+}^{(\frac{1}{2},0)}(a^{\star})$	8	0.352858	0.359503	0.352594
	12	0.142246	0.142809	0.142201
	16	0.012761	0.012761	0.012760
$\gamma_{-}^{(\frac{1}{2},0)}(a^{\star})$	8	0.499833	0.413023	0.472906
	12	0.161375	0.155779	0.159019
	16	0.012898	0.012894	0.012895
$\gamma_{+}^{(\frac{3}{2},0)}(a^{\star})$	8	−0.749623	−0.226711	−0.389838
	12	−0.180829	−0.139684	−0.155322
	16	−0.013010	−0.012980	−0.012982
$\gamma_{-}^{(1,\frac{1}{2})}(a^{\star})$	8	−0.199920	−0.068444	−0.110503
	12	−0.056850	−0.044097	−0.048697
	16	−0.004318	−0.004307	−0.004308

Quark mass dimension

The same Banks-Zaks expansion can be applied to the quark mass exponent, $\gamma_{\bar{\psi}\psi}(a^\star)$, as a way of determining where the lower end of the conformal window is and confining behaviour commences

Several lattice field theory groups have evaluated the $SU(3)$ β-function for even N_f as well as estimating $\gamma_{\bar{\psi}\psi}(a^\star)$

The general consensus from the lattice studies is that the conformal window exists down to at least and including $N_f = 12$ but it is not established if $N_f = 10$ is within the window as uncertainties on $a^\star$ need to be reduced

For $N_f = 8$ the evidence varies with a few groups convinced this value is inside the window

There have been various continuum approaches centred on renormalization group studies and resummation or improved convergence methods

One criterion used to define the lower end of the window is the solution of

$$\gamma_{\bar{\psi}\psi}(a^\star)\left[2 ~-~ \gamma_{\bar{\psi}\psi}(a^\star)\right] ~=~ 1$$

which is motivated by a condition that is connected to chiral symmetry breaking

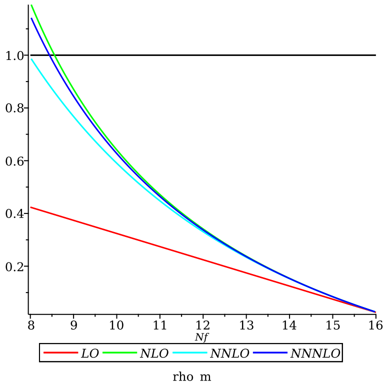
Given the Padé approximant improvement with the baryon operators exponent estimates with the Banks-Zaks expansion it is worthwhile repeating the exercise for the quark mass operator

As the five loop $\overline{\mathrm{MS}}$ QCD β-function ceases to have a Banks-Zaks fixed point below $N_f = 10$ we have used a different scheme to deduce $\gamma_{\bar{\psi}\psi}(a^\star)$ which is a renormalization group invariant

Instead we have used the five loop mini-MOM scheme β-function to find $a^\star$

The minimal MOM (mMOM) scheme is based on the ghost-gluon vertex and motivated by lattice considerations [von Smekal et al]

The scheme endeavours to preserve the relation of the Landau gauge coupling renormalization constant to the ghost and gluon renormalization constants for a general linear covariant gauge



Lower end of the conformal window can be estimated from solving for the value of N_f where the exponent is unity giving values of 8.56, 7.97 and 8.44

Next order thoughts

While the QCD β-function and quark mass dimension are available at five loops $\gamma_{\bar{\psi}\psi}(a^\star)$ has only been analysed to fourth order

The fifth order value for $\gamma_{\bar{\psi}\psi}(a^\star)$ requires the *six* loop β-function

This is because the one and two loop terms of $\beta(a)$ determine the leading order value of $a^\star$

Low loop computations of the β-function used straightforward approaches but some higher order calculations exploited properties of the underlying gauge theory such as Slavnov-Taylor identities and for example the background field gauge

For instance in that gauge the coupling renormalization constant is equivalent to the background gluon anomalous dimension

For any future next order renormalization a similar or improved strategy would be necessary in order to reduce the large bottleneck with integration by parts mostly driven by the second term of the gluon propagator

$$\langle A^a_\mu(p) A^b_\nu(-p) \rangle \;=\; -\,\frac{\delta^{ab}}{p^2}\left[\eta_{\mu\nu} \;-\; (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right]$$

Using the Feynman gauge would simplify matters but may necessitate a vertex renormalization

On the Slavnov-Taylor identity side in the Landau gauge the gluon-ghost vertex is finite

So the β-function is determined solely by a linear combination of the gluon and ghost anomalous dimensions but this would not avoid the bottleneck originating from the gluon propagator

Is it possible to have a setup where the β-function can be deduced from the gluon and ghost anomalous dimensions for a particular choice of the gauge parameter which preserves the benefit of $\alpha = 1$?

It transpires there is a (nonlinear) covariant gauge which has beneficial properties constructed by Curci and Ferrari

Curci-Ferrari gauge

The Curci-Ferrari gauge fixed Lagrangian is

$$
\begin{aligned}
L &= -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + i\bar{\psi}^i \not{D} \psi^i - \frac{1}{2\alpha} (\partial_\mu A^{a\,\mu})^2 + \bar{c}^a (\partial_\mu D^\mu c)^a \\
&\quad - \frac{g}{2} f^{abc} \bar{c}^a c^b\, \partial^\mu A^c_\mu + \frac{1}{8} \alpha g^2 f^{abcd} \bar{c}^a c^b \bar{c}^c c^d
\end{aligned}
$$

which differs only in the ghost sector from the canonical linear gauge but equates to the Landau gauge when $\alpha = 0$

Unlike the canonical linear gauge $\gamma_A(a,\alpha) + \gamma_\alpha(a,\alpha) \propto \alpha a$

Wschebor and Tissier showed that for all α

$$
\beta(a,\alpha) = a\left[\gamma_A(a,\alpha) + 2\gamma_c(a,\alpha) - 4(\gamma_A(a,\alpha) + \gamma_\alpha(a,\alpha))\right]
$$

which is the generalization of Taylor's theorem for calculating the β-function from the ghost-gluon vertex or

$$
\beta(a,\alpha) = a\left[2\gamma_c(a,\alpha) - 3\gamma_A(a,\alpha) - 4\gamma_\alpha(a,\alpha))\right]
$$

In other words the β-function can be deduced from renormalizing 2-point functions without evaluating a vertex function which substantially reduces the number of graphs to be computed

Moreover it also possesses the bottleneck simplification option of $\alpha = 1$

Using FORCER have checked the four loop gluon, ghost and gauge parameter anomalous dimensions reproduce the four loop $\overline{\text{MS}}$ QCD β-function for all α purely from renormalizing two 2-point functions

For example with the one loop anomalous dimensions

$$
\begin{aligned}
\gamma_A(a,\alpha) &= \left[\frac{4}{3} T_F N_f - \frac{13}{6} C_A + \frac{1}{2} C_A \alpha \right] a ~+~ O(a^2) \\
\gamma_\alpha(a,\alpha) &= \left[\frac{13}{6} C_A - \frac{4}{3} T_F N_f - \frac{1}{4} C_A \alpha \right] a ~+~ O(a^2) \\
\gamma_c(a,\alpha) &= \left[\frac{1}{4} C_A \alpha - \frac{3}{4} C_A \right] a ~+~ O(a^2)
\end{aligned}
$$

α cancels in the combination that produces $\beta(a,\alpha)$

Conclusions

In terms of the baryon operator renormalization for lattice distribution amplitude analyses there are several directions to head

The anomalous dimensions need to be extended to one or more moment and to three loops

Moreover the operator matrix element with non-zero momentum operator insertion needs to determined at three loops which is viable given the current loop technology

The resolution of the lower end of the conformal window using the Banks-Zaks resummation approach would benefit from the six loop QCD β-function

Whether using the Curci-Ferrari gauge is a viable strategy to achieve that is an open question that ought to be explored