Feynman-Fox integrals

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FIs are GHFs in the sense that their fundamental group of analytic continuations are generalizations of the fundamental (monodromy) group of the ordinary HGs.

Write multi-loop Feynman integrals in terms of Fox functions, respecting the original cut structure; compute the Fox functions numerically.

SYNOPSIS

In 1967 Regge suggested to consider FIs as a kind of GHFs

In 1973 Kershaw suggested that,by studying FIs as power series

we could derive the connection

one-loop box: sum of 192 dilogarithms collapses into one HF

hypergeometric A-systems of Gelfand, Kapranov, and Zelevinsky

FIs ➠ Fox functions

Each Feynman diagram is a multivalued analytic function of the relevant variables whose branching locus is in general an extremely complicated reducible algebraic variety; however, the set of singularities is very well defined by the Landau rules,

i.e. they are characterized by a branch cut structure determined by the Landau equations.

To give an example, in the most general one-loop triangle the physical Landau curve has six branches; when we consider the most general one-loop box we get 14 branches.

Furthermore we are interested in FI in the physical region, identified with the phase space for the corresponding process, i.e. the physical region of a given process is the set of all real initial and final energy-momenta variables subject to the mass-shell conditions and to energy-momentum conservation. Solutions that correspond to points outside the physical region are on the wrong sheet. Any process $\mathrm{n} \to \mathrm{m}$ is described by $3(\mathrm{m}+\mathrm{n})-10$ Mandelstam invariants and the physical region is dictated by the corresponding phase space.

Ingredients 

${}_qF_q$ ➠ Lauricella $F_D^{(N)}$ ➠ Meijer G ➠ Fox H. The two facets, EM ➠ MB;

Consider now the following Euler-Mellin integral:

$$I = \int_0^1 dx\, x^{b-1}\,(1-x)^{c-b-1}\,(1-zx)^{-a}\,.$$

If $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$ and $|\arg(1-z)| < \pi$

$$I = B(b,\, c-b)\; {}_2F_1(a, b; c; z)\,,$$

Note that with $z \to z - i\,\delta(\delta \to 0_+)$ the original integral can be interpreted as a Hadamard finite-part integral even if $z \in \mathbb{R}$ and $z > 1$. Next we would like to write I as a Mellin-Barnes integral, i.e.

$$I = \frac{\Gamma(c-b)}{\Gamma(a)} \int_L \frac{ds}{2\,i\,\pi}\,\Gamma(-s)\,\frac{\Gamma(a+s)\,\Gamma(b+s)}{\Gamma(c+s)}\,(-z)^s\,,$$

which requires $|\arg(-z)| < \pi$.

The Lauricella functions are defined by,

- EM

$$F_D^{(N)}(a;b_1,\ldots,b_N;c;z_1,\ldots,z_N)=\Gamma(c,a)\int_0^1 dx\, x^{a-1}\,(1-x)^{c-a-1}\prod_{n=1}^{N}(1-z_n x)^{-b_n}\,,$$

with $\mathrm{Re}\,c > \mathrm{Re}\,a > 0$ and $|\arg(1-z_i)| < \pi$.

- MB

$$F_D^{(N)}(a;b_1,\ldots,b_N;c;z_1,\ldots,z_N) = \frac{\Gamma(c)}{\Gamma(a)\prod_j\Gamma(b_j)}\left[\prod_{j=1}^{N}\int_{L_j}\frac{ds_j}{2i\pi}\right]\frac{\Gamma(a+\sum_j s_j)}{\Gamma(c+\sum_j s_j)} \times \prod_{j=1}^{N}\Gamma(b_j+s_j)\,\Gamma(-s_j)\,(-z_j)^{s_j}\,,$$

where L_j is a deformed imaginary axis curved so that only the poles of $\Gamma(-s_j)$ lie to the right of L_j.

$\mathcal{G}$

Definitions, $z = R\exp\{i\phi\}$ and $s = \sigma + it$

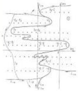
$$H^{m,n}_{p,q}(z) = \int_L \frac{ds}{2i\pi} \frac{\prod_{j=1}^{m}\Gamma(a_j + A_j s)\prod_{j=1}^{n}\Gamma(b_j - B_j s)}{\prod_{j=1}^{p}\Gamma(c_j + C_j s)\prod_{j=1}^{q}\Gamma(d_j - D_j s)} z^s,$$

exponential ← power-like

$$|\cdot| \sim \exp\{-\tfrac{1}{2}\alpha\pi|t| - \phi t\}\,|t|^{\beta\sigma+\lambda} R^{-\sigma}\rho^{\sigma} \qquad |t| \to \infty,$$

$$H^{m,n}_{p,q}(z) = \int_L \frac{ds}{2i\pi} \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)} z^{-s}.$$

$$|\cdot| \sim K_{\pm}\exp\{t\phi\}\left(\frac{e}{|\sigma|}\right)^{-\mu\sigma}\left(\frac{|z|}{\beta}\right)^{\mp\sigma}|\sigma|^{\delta} \qquad \sigma \to \pm\infty$$



There are three different paths of integration

- $L_{i\infty}$ — L runs from $-i\infty$ to $+i\infty$ separating the poles of the integrand.
- $L_{+\infty}$ — L is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma\left(b_j - s\right)$ once in the negative direction.
- $L_{-\infty}$ — L is a loop starting and ending at $-\infty$ and encircling all poles of $\Gamma\left(1 - a_j + s\right)$ once in the positive direction.

It is assumed that at least one of the three definitions makes sense. In cases where more than one make sense, they lead to the same result.

✵ With $L_{\pm\infty}$ compute (multidimensional) residues; with $L_{i\infty}$ (our choice) we compute $(s_j = \sigma_j + i t_j)$

$$\mathbf{H} = \left[\prod_{j=1}^{r} \int_{-\infty}^{+\infty} \frac{dt_j}{2\pi}\right] F\left(s_1 \ldots s_r\right) \prod_{j=1}^{r} z_j^{s_j},$$

Convergence, analytic structure

Convergence of H is controlled by five parametes, μ, β, ρ, δ and α (definitions in backup slides). The H function is an analytic function of z and exists in the following cases ($\mathrm{s} = \sigma + \mathrm{i}\,\mathrm{t}$):

① If $\mu > 0$ or $\mu = 0$ and $|\,\mathrm{z}\,| < \beta$ ➠ $\mathrm{L} = \mathrm{L}_{-\infty}$;

② If $\mu < 0$ or $\mu = 0$ and $|\,\mathrm{z}\,| > \beta$ ➠ $\mathrm{L} = \mathrm{L}_{+\infty}$;

③ If $\alpha > 0$ and $\arg(\mathrm{z}) < 1/2\,\alpha\,\pi$ or $\alpha = 0$ and $\beta\,\sigma + \mathrm{Re}\,\delta < -1$ ➠ $\mathrm{L} = \mathrm{L}_{\mathrm{i}\,\infty}$.

④ A frequent case is $\alpha = 2, \beta = 0$ but $\mathrm{z} \in \mathbb{R}^{-}$ which requires $\mathrm{Re}\,\delta < -1$.

First consequence, the MB splitting:

$$\mathrm{F} = (\mathrm{Q} + \mathrm{a} - \mathrm{i}\varepsilon)^{-\rho}\,,$$

where Q is a function of Feynman parameters and a is a positive parameter, with $\varepsilon \to 0_+$. We perform a MB splitting, i.e.

$$\mathrm{F} = \mathrm{a}^{-\rho} \int_{\mathrm{L}} \frac{\mathrm{ds}}{2\mathrm{i}\pi}\, \mathrm{B}(\mathrm{s},\, \rho - \mathrm{s}) \left(\frac{\mathrm{a}}{\mathrm{Q} - \mathrm{i}\varepsilon} \right)^{\mathrm{s}},$$

where $0 < \mathrm{Re}\, s < \rho$. The choice of L depends on $\mathrm{z} = \mathrm{a}/\mathrm{Q}$. Indeed F is proportional to a Meijer $\mathrm{G}^{1,1}_{1,1}$ function with parameters $\mathrm{a}_1 = 1$ and $\mathrm{b}_1 = \rho$.

If $|\,\mathrm{z}\,| < 1$ we select $\mathrm{L} = \mathrm{L}_{+\infty}$; if $|\,\mathrm{z}\,| > 1$ we select $\mathrm{L} = \mathrm{L}_{-\infty}$ and compute the residues of the poles.

☛ The parameters of the Meijer G function are such that the MB integral over $\mathrm{L}_{\mathrm{i}\infty}$ does not converge if $\mathrm{Q} \in \mathbb{R}^-$, despite the $-\mathrm{i}\varepsilon$ prescription. The main question will be how to use $\mathrm{L} = \mathrm{L}_{\mathrm{i}\infty}$.

Second consequence

$$I = {}_2F_1(a,b;c;z) = \frac{\Gamma(c-b)}{\Gamma(a)} \int_L \frac{ds}{2i\pi} \Gamma(-s) \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} (-z)^s,$$

☛ $|\arg(1-z)| < \pi$

☞ As long as $\mathrm{Re}(a+b-c) < 0$ the MB integral converges for $L = L_{i\infty}$ even if $z \in \mathbb{R}$ and $z > 0$.

☛ However, it is easily seen that the analytic continuation does not reproduce the cut structure of the original integral, namely we have a cut at $[0,\infty]$ instead of a cut at $[1,\infty]$.

Seen in terms of a Feynman integral this corresponds to the fact, with this procedure, we can describe the integral above its normal threshold but not below it.

The correct 

In order to understand the correct procedure we consider the following example:

$$I = \int_0^1 dx\, x^{-1/2}(1-zx)^{-1} = 2\,{}_2F_1\left(1, \frac{1}{2}; \frac{3}{2}; z\right),$$

with $z \in \mathbb{R}$. There are three cases:

$z < 0$ Here we immediately obtain

$$I = \int_{L_{i\infty}} \frac{ds}{2i\pi} \Gamma(-s) \frac{\Gamma(1+s)\,\Gamma(1/2+s)}{\Gamma(3/2+s)} (-z)^s \quad ✓$$

$0 < z < 1$ We use the following transformation of ${}_2F_1$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right),$$

where now the argument of ${}_2F_1$ is negative; we obtain

$$I = \sqrt{\pi}(1-z)^{-1} \int_{L_{i\infty}} \frac{ds}{2i\pi} \Gamma(-s) \frac{\Gamma^2(1+s)}{\Gamma(3/2+s)} \left(\frac{z}{1-z}\right)^s \quad ✓$$

$z > 1$ Here we use a contiguity relation to increase c (convergence requires $\mathrm{Re}(a+b-c) < 0$),

$$
\begin{aligned}
{}_2F_1(a,b;\mathbf{c};z) \quad = \quad & -\left[c(c+1)(z-1)\right]^{-1}\Big\{(c+1)\left[c-(2c-a-b+1)z\right] \\
\times \quad & {}_2F_1(a,b;\mathbf{c+1};z) \\
+ \quad & (c-a+1)(c-b+1)z\,{}_2F_1(a,b;\mathbf{c+2};z)\Big\},
\end{aligned}
$$

and obtain the following MB representation:

$$
\begin{aligned}
I \quad = \quad & \frac{5z-3}{2(z-1)}\int_{L_{i\infty}}\frac{ds}{2i\pi}\,\Gamma(-s)\,\frac{\Gamma(1+s)\,\Gamma(1/2+s)}{\Gamma(5/2+s)}\,(-z)^s \\
- \quad & 3\,\frac{z}{z-1}\int_{L_{i\infty}}\frac{ds}{2i\pi}\,\Gamma(-s)\,\frac{\Gamma(1+s)\,\Gamma(1/2+s)}{\Gamma(7/2+s)}\,(-z)^s \quad \checkmark
\end{aligned}
$$

where now the two MB integrals are convergent even for $z > 1$.

do we need MB reps?

$$\begin{aligned}
\mathrm{H} &= \int_0^1 \mathrm{d}x\,\mathrm{d}y\, x^{a-1}\,(1-x)^{c-a-1}\,(x^2-2\lambda\, x-y^2)^{-b} \\
&= \int_0^1 \mathrm{d}x\,\mathrm{d}y\, x^{a-1}\,(1-x)^{c-a-1}\,(x-x_-)^{-b}\,(x-x_+)^{-b}, \\
x_\pm &= \lambda \pm \sqrt{\lambda^2+y^2} = \lambda \pm \eta\,, \qquad \lambda = \lambda - i\,\delta\,, \quad \delta \to 0_+ \quad \Rightarrow F_D^{(2)}\,.
\end{aligned}$$

With $\lambda > 1$ we obtain $x_+ > 1$ and $x_- < 0$ We use the transformation

$$F_D^{(2)}(a;\mathbf{b};c;\mathbf{z}) = (1-z_1)^{c-a-b_1}\,(1-z_2)^{b_2}\,F_D^{(2)}(c-a;c-b_1-b_2\,,b_2;\zeta_1\,,\zeta_2)\,,$$

$$\zeta_1 = z_1 = 1/x_-\,, \quad z_2 = 1/x_+\,, \quad \zeta_2 = \frac{x_+ - x_-}{x_-\,(x_+-1)}\,.$$

Since $\zeta_j < 0$ we can use the MB representation obtaining

$$\begin{aligned}
\mathrm{H} &= \frac{\Gamma(a)}{\Gamma(c-2)} \int_0^1 \mathrm{d}y\,(\lambda-\eta)^{a-c}\,(\lambda-\eta-1)^{c-a-1} \\
&\times \Big[\prod_{j=1}^{2} \frac{\mathrm{d}s_j}{2i\pi}\,\frac{\Gamma(c-a+s_1+s_2)}{\Gamma(c+s_1+s_2)}\,\Gamma(-s_1)\,\Gamma(c-2+s_1)\,\Gamma(-s_2)\,\Gamma(1+s_2) \\
&\times (2\eta)^{s_2}\,(\eta-\lambda)^{-s_1-s_2}\,(\lambda+\eta-1)^{-1+s_2}\,, \quad \Rightarrow y = 2\lambda\, t/(t^2-1)\,.
\end{aligned}$$

Therefore the strategy is

① Use the Feynman parametrization and perform the first integral obtaining a (generalized) hypergeometric function (usually an $\mathrm{F}_{\mathrm{D}}^{(\mathrm{N})}$ Lauricella function),

② if needed transform it (Kummer transformation $z \to \frac{z}{z-1}$),

③ Use the MB representation of the result and compute the second integral,

④ repeat the procedure until the final result is obtained.

$\mathcal{G}$

Consider the following integral corresponding to a three-point Feynman function in arbitrary space-time dimensions $(\mathrm{d} = 4+\varepsilon)$, with a normal threshold at $\mathrm{s} = 4\,\mathrm{m}^2$ $(\lambda = \frac{\mathrm{m}^2}{\mathrm{s}} = 1/4)$

$$\begin{aligned} \mathrm{C} &= \pi^{\varepsilon/2}\,\Gamma(1-\varepsilon/2)\int_0^1 \mathrm{dx}\int_0^{\mathrm{x}} \mathrm{dy}\left[\mathrm{m}^2\,(1-\mathrm{x}) + (\mathrm{m}^2-\mathrm{s})\,\mathrm{y} + \mathrm{s\,x\,y}\right]^{\varepsilon/2-1} \\ &= \pi^{\varepsilon/2}\,\mathrm{m}^{\varepsilon-2}\,\frac{\Gamma^2(1-\varepsilon/2)\,\Gamma\,(1+\varepsilon/2)}{\Gamma\,(2+\varepsilon/2)}\,{}_3\mathrm{F}_2(1\,,1-\varepsilon/2,1+\varepsilon/2;\,\frac{3}{2}\,,2+\varepsilon/2;\,\frac{1}{4\,\lambda}), \end{aligned}$$

with $\lambda = \mathrm{m}^2/\mathrm{s}$. MB representation of ${}_3\mathrm{F}_2$ and Kummer transformations will not be discussed here; instead we study the case $\varepsilon = 0$

① $0 < s < m^2$, where $\lambda > 1$.

$$C = \frac{1}{s}\int_0^1 dx\, \frac{X}{\lambda - x}\, {}_2F_1(1,1;2;-X)\,, \qquad X = \frac{(\lambda - x)(1-x)}{\lambda\, x}\,.$$

Since $X > 0$ we can use a MB representation and perform the x integration obtaining a new hypergeometric function of argument $1/\lambda > 0$; therefore we use

$$ {}_2F_1\left(a, b; c; \lambda^{-1}\right) = \left(1 - \frac{1}{\lambda}\right)^{-a} {}_2F_1\left(a, c-b; c; \frac{1}{1-\lambda}\right)\,.$$

Using again the corresponding MB representation we get

$$C = \int_{L_1} \frac{ds_1}{2i\pi}\, \Gamma^2(-s_1)\, \Gamma^2(1+s_1) \left(1 - \frac{1}{\lambda}\right)^{s_1} {}_2F_1\left(-s_1, 2+s_1; 2; \frac{1}{1-\lambda}\right),$$

and we can use the standard MB representation for ${}_2F_1$ and no imaginary part will arise ✓

② $m^2 < s < 4m^2$ where $1/4 < \lambda < 1$. In this case it is more convenient to split the x integration introducing $C_<$ where $0 < x < \lambda$ and $C_>$ where $\lambda < x < 1$.

$$
\begin{aligned}
C_< &= \frac{1}{s}\int_{L_1}\frac{ds_1}{2i\pi}\frac{\Gamma^2(-s_1)\,\Gamma^3(1+s_1)}{\Gamma(2+s_1)}\,\lambda^{-1-s_1}(1-\lambda)^{1+s_1} \\
&\times\ {}_2F_1\left(-1-s_1, 1+s_1; 1; -\frac{\lambda}{1-\lambda}\right).
\end{aligned}
$$

$$
\begin{aligned}
C_> &= \frac{1}{s}\int_{L_1}\frac{ds_1}{2i\pi}\Gamma(-s_1)\frac{\Gamma^2(1+s_1)}{\Gamma(2+s_1)}\,J, \\
J &= \left(\frac{\lambda}{1-\lambda}\right)^{2s_1}\frac{\Gamma(1+s_1)\Gamma(2+s_1)}{\Gamma(3+2s_1)} \\
&\times\ F_D^{(2)}\left(1+s_1; 1+s_1, 1+s_1; 3+2s_1; \frac{1}{x_-}, \frac{1}{x_+}\right).
\end{aligned}
$$

Since $x_\pm = \frac{1}{2}(1-\lambda)^{-1}\left[1-2\lambda \pm i\sqrt{4\lambda-1}\right]$ are complex we can use the standard MB representation for the Lauricella function ✓

$\mathscr{G}$

Partial quadratization of Symanzik polynomials

A two loop diagram with K internal lines is described by the two Symanzik polynomials in K variables $\alpha_1, \dots, \alpha_K$. The diagram will have

- k_1 lines with momentum q_1,
- k_2 with momentum q_2 and
- k_{12} with momentum $q_1 - q_2$.

Partial quadratization is a change of variables defined as follows:

① to the k_1 lines we assign parameters $\alpha_1, \dots, \alpha_{k_1}$;

② to the k_{12} lines parameters $\alpha_{k_1+1}, \dots, \alpha_{k_1+k_{12}}$; to the k_2 lines parameters $\alpha_{k_1+k_{12}+1}, \dots, \alpha_{k_1+k_{12}+k_2}$.

Next we perform the following change of variables:

$$\alpha_1 = \rho_1\, x_1 \;, \;\ldots\;, \;\alpha_{k_1-1} = \rho_1\, x_{k_1-1} \;, \;\alpha_{k_1} = \rho_1 \left(1 - \sum_{j=1}^{k_1-1} x_j\right) \;.$$

For $k_{12} = 1$ we introduce

$$\alpha_{k_1+1} = \rho_3 \;, \;\alpha_{k_1+2} = \rho_2\, x_{k_1} \;, \;\ldots\;, \;\alpha_{k_1+k_2} = \rho_2\, x_{k_1+k_2-2} \;,$$

$$\alpha_{k_1+k_2+1} = \rho_2 \left(1 - \sum_{j=k_1}^{k_1+k_2-2} x_j\right) \;.$$

For $k_{12} = 2$ we introduce

$$\begin{aligned} \alpha_{k_1+1} &= \rho_3\, x_{k_1+k_2-1} \;, \;\alpha_{k_1+2} = \rho_3 \left(1 - x_{k_1+k_2-1}\right) \;, \\ \alpha_{k_1+3} &= \rho_2\, x_{k_1} \;, \;\ldots\;, \;\alpha_{k_1+k_2+1} = \rho_2\, x_{k_1+k_2-2} \;, \;\alpha_{k_1+k_2+2} = \rho_2 \left(1 - \sum_{j=k_1}^{k_1+k_2-2} x_j\right) \;, \end{aligned}$$

etc. As a result of the transformation we will have $\sum_j \rho_j = 1$;

➠ S_1 is a funtion of the ρ variables but not of the x variables; ➠ S_2 is a quadratic form in the x variables with coefficients that are ρ dependent.

A useful relation EM ➠ MB

$$F_D^{(N)}(a\,;\mathbf{b}\,;c\,;x\,,y_1\ldots y_{N-1}) = (1-x)^{-a}$$
$$\times \quad F_D^{(N)}\left(a\,;c-\sum_j b_j\,,b_2\ldots b_N\,;c\,;\frac{x}{x-1}\,,\frac{y_1-x}{1-x}\ldots\frac{y_{N-1}-x}{1-x}\right),$$

$$0<z_j<1\,,\quad j=1\ldots M\,,\quad z_j>1\,,\quad j=M+1\ldots N\,,\quad z_1=\max_{j=1\ldots M}\{z_j\}$$
$$\frac{z_1}{z_1-1}\,,\frac{z_2-z_1}{1-z_1}\ldots\frac{z_M-z_1}{1-z_1}\,,$$

where all variables are negative and

$$\frac{z_{M+1}-z_1}{1-z_1}\ldots\frac{z_N-z_1}{1-z_1}\,,$$

where all variables are greater than one.

Equal masses sunrise

After partial quadratization of the Symanzik polynomials we obtain

$$S = \int_0^1 d\rho\, dx\, \rho^3 (1-x)\,(a x^2 + b x + c)^{-1}$$

$$a = -b = \rho\,(m^2 - s\,\sigma)\,, \quad c = -m^2\,\sigma\,, \quad \sigma = 1 - \rho\,.$$

The integral can be rewritten as

$$S = \frac{1}{m^2} \int_0^1 d\rho\, dx\, \rho^3 (1-x) \left[a\,(x - \frac{1}{2})^2 + B\right]^{-1},$$

$$B = -\frac{1}{4}\lambda\,(\rho - \rho_-)\,(\rho - \rho_+)\,, \qquad \rho_\pm = \frac{1}{2\,\lambda}\left[\lambda + 3 \pm \sqrt{(\lambda - 1)\,(\lambda - 9)}\right]$$

It follows that $\lambda = 1$ corresponds to the pseudo-threshold while $\lambda = 9$ corresponds to the normal threshold. We have four different regions:

① $\lambda < 0$, where $\rho_0 > 1$ and $\rho_- > 1$, $\rho_+ < 0$.

② $0 < \lambda < 1$, where $\rho_0 < 0$ and $\rho_\pm > 1$.

③ $1 < \lambda < 9$, where $0 < \rho_0 < 1$ and $\rho_\pm$ are complex

④ $\lambda > 9$, where $0 < \rho_- < \rho_+ < 1$

In all cases we always start with ($\rho_0 = 1 - 1/\lambda$)

$$\mathrm{S} = -2 \int_0^1 \mathrm{d}\rho \, \rho^3 \left[(\rho - \rho_-)(\rho - \rho_+) \right]^{-1} {}_2\mathrm{F}_1\big(1, \frac{1}{2}; \frac{3}{2}; \frac{\rho\,(\rho - \rho_0)}{(\rho - \rho_-)(\rho - \rho_+)}\big) \,.$$

The strategy below the normal threshold will be as follows:

- for ${}_2\mathrm{F}_1(\dots; \mathrm{z} < 0)$ we use the standard MB representation.
- For ${}_2\mathrm{F}_1(\dots; \mathrm{z} > 0)$ and $0 < \mathrm{z} < 1$ we transform the HF before using the MB representation.

After that we perform the ρ integration, obtaining an $\mathrm{F}_\mathrm{D}^{(\mathrm{N})}$ Lauricella function. If needed we transform it (below the normal threshold) so that we always have to deal with $\mathrm{F}_\mathrm{D}^{(\mathrm{N})}$ with negative arguments; only at this point we use the corresponding MB representation ✓

One example only

$$
\begin{aligned}
S(0<\lambda<1) &= -\frac{1}{\sqrt{\pi}}\Big[\prod_{j=1}^{2}\int_{L_j}\frac{ds_j}{2\,i\,\pi}\Big]\frac{\Gamma^2(1/2+s_1)}{\Gamma(3/2+s_1)\,\Gamma(9/2+s_1)} \\
&\times\ \Gamma(-s_2)\,\Gamma(1/2-s_2)\,\Gamma(s_2-s_1)\,\Gamma(4+s_2+s_1) \\
&\times\ (1-\rho_0)^{s_1}\left(1-\frac{1}{\rho_-}\right)^{-1/2}\left(1-\frac{1}{\rho_+}\right)^{-1/2}\left(\frac{\lambda}{4}\right)^{1+s_1} \\
&\times\ F_D^{(2)}\left(\frac{1}{2}-s_2\,;\,\frac{1}{2},\frac{1}{2}\,;\,\frac{9}{2}+s_1\,;\,\frac{1}{1-\rho_-},\frac{1}{1-\rho_+}\right).
\end{aligned}
$$

General strategy:

Given an irreducible quadratic form in N variables always write it as

$$\mathrm{V} = (\mathrm{x}-\mathrm{X})^{\mathrm{t}}\,\mathrm{H}\,(\mathrm{x}-\mathrm{X}) + \mathrm{B}_{\mathrm{N}}\,,$$

since $\mathrm{B}_{\mathrm{N}} = 0$ induces a pinch (AT) con the integration contour at $\mathbf{x} = \mathbf{X}$ if

$$0 < \mathrm{X}_{\mathrm{N}} < \ldots < \mathrm{X}_1 < 1\,.$$

- for the vertex $\mathrm{C}_0 \sim \ln \mathrm{B}_2$,
- for the box $\mathrm{D}_0 \sim \mathrm{B}_3^{-1/2}$,
- for the pentagon $\mathrm{E}_0 \sim \mathrm{B}_4^{-1}$,
- no singularity for the hexagon F_0 in 4 dimensions.

Let L be the number of internal lines and ν the number of loops; define $\rho = 2\,\nu - 1/2\,(\mathrm{L}+1)$, the leading behavior of the diagram is given by

$$\mathrm{B}_{\mathrm{L}}^{\rho} \text{ for } \rho < 0\,, \quad \mathrm{B}_{\mathrm{L}}^{\mathrm{k}+1/2} \text{ for } \rho = \mathrm{k} + \frac{1}{2}\,, \quad \mathrm{B}_{\mathrm{L}}^{\mathrm{k}} \ln \mathrm{B}_{\mathrm{L}} \text{ for } \rho = \mathrm{k}\,, \quad \mathrm{k} \in \mathbb{Z}^*\,.$$

Therefore for $\mathrm{L} = 2\,(2\,\nu + n) - 1$ and $n \in \mathbb{Z}^+$ the AT is a pole of order n for the amplitude, e.g. a simple pole for the one-loop pentagon, for two-loop diagrams with 9 propagators etc. In all other cases it is a branch point.

Infrared poles

$$I = \int_0^1 dx\, x^{\varepsilon-1} (1-x)^{c-\varepsilon-1} (1-zx)^{-a} \,.$$

$$I = \Gamma(\varepsilon) \frac{\Gamma(c-\varepsilon)}{\Gamma(c)} \, {}_2F_1(a, \varepsilon; c; z) \,.$$

$$I = \frac{\Gamma(c-\varepsilon)}{\Gamma(a)} \int_{-i\infty}^{+i\infty} \Gamma(-s) \frac{\Gamma(\varepsilon+s)\,\Gamma(a+s)}{\Gamma(c+s)} (-z)^s \,.$$

- Solution: use contiguity

$$(c-b-1)\,{}_2F_1 = -\left[az - c + (b+1)(2-z)\right] {}_2F_1(b+1) - (b+1)(z-1)\,{}_2F_1(b+2) \,,$$

obtaining $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$

$$
\begin{aligned}
\mathrm{I}_1 &= \mathrm{H}_2\,(1+\varepsilon) \int_{-i\infty}^{+i\infty} \Gamma(-\mathrm{s}) \frac{\Gamma(\mathrm{a}+\mathrm{s})\Gamma(2+\varepsilon+\mathrm{s})}{\Gamma(\mathrm{c}+\mathrm{s})} (-\mathrm{z})^{\mathrm{s}} \\
&- \mathrm{H}_1\,(2-\mathrm{c}+2\mathrm{s}) \int_{-i\infty}^{+i\infty} \Gamma(-\mathrm{s}) \frac{\Gamma(\mathrm{a}+\mathrm{s})\Gamma(1+\varepsilon+\mathrm{s})}{\Gamma(\mathrm{c}+\mathrm{s})} (-\mathrm{z})^{\mathrm{s}} .
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{I}_2 &= \mathrm{H}_2\,(1+\varepsilon) \int_{-i\infty}^{+i\infty} \Gamma(-\mathrm{s}) \frac{\Gamma(\mathrm{a}+\mathrm{s})\Gamma(2+\varepsilon+\mathrm{s})}{\Gamma(\mathrm{c}+\mathrm{s})} (-\mathrm{z})^{\mathrm{s}+1} \\
&- \mathrm{H}_1\,(1-\mathrm{a}+\mathrm{s}) \int_{-i\infty}^{+i\infty} \Gamma(-\mathrm{s}) \frac{\Gamma(\mathrm{a}+\mathrm{s})\Gamma(1+\varepsilon+\mathrm{s})}{\Gamma(\mathrm{c}+\mathrm{s})} (-\mathrm{z})^{\mathrm{s}+1} .
\end{aligned}
$$

$$
\mathrm{H}_1 = \frac{\Gamma(\varepsilon)\,\Gamma(\mathrm{c}-\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(\mathrm{a})} \frac{1}{\mathrm{c}-1-\varepsilon}, \qquad \mathrm{H}_2 = \frac{\Gamma(\varepsilon)\,\Gamma(\mathrm{c}-\varepsilon)}{\Gamma(2+\varepsilon)\Gamma(\mathrm{a})} \frac{1}{\mathrm{c}-1-\varepsilon} .
$$

Numerical computation; we will limit the presentation to the univariate case

- The main point: the absolute value of the integrand in a Fox function is comparable with

$$\exp\{-\frac{1}{2}\alpha\pi\,|\,t\,|-\theta\,t\}\;|\,t\,|^{\beta\,\sigma+\lambda}\;R^{-\sigma}\rho^{\sigma},$$

where $z=R\,\exp\{i\,\theta\}$ and $s=\sigma+i\,t$; α,β and λ are parameters of H.

- Given a function $f(x)$ with $x\in\mathbb{R}$ we intoduce the Cardinal function

$$C(f,h)(x)=\sum_{k\in\mathbb{Z}}f(k,h)\,\mathrm{sinc}\left(\frac{x}{h}-k\right),\qquad \mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}.$$

- The Sinc approximation over the interval $[a,b]$ is defined by

$$f(x)\approx\sum_k f(x_k)\,\mathrm{sinc}\left(\frac{\phi(x)}{h}-k\right),$$

where ϕ is a one-to-one mapping of $[a,b]$ onto $\mathbb{R}$ and $x_k=\phi^{-1}(kh)$.

Sinc lattice: define the strip

$$\boxed{D_d = \{z \in \mathbb{C} : |\operatorname{Im} z| < d\}}\,.$$

❶ The interval is $\mathbb{R}$. If $z \in D_d$ and

$$\begin{aligned} \operatorname{Re} z &\leq 0\,, & |f(z)| &\leq c_- \exp\{-\alpha_- |z|\}\,,\\ \operatorname{Re} z &\geq 0\,, & |f(z)| &\leq c_+ \exp\{-\alpha_+ |z|\}\,,\\ \phi(z) &= z\,, & z_k &= kh \end{aligned}$$

❷ The interval is $\mathbb{R}$. If $z \in D_d$ with

$$\boxed{D_d = \left\{z \in \mathbb{C} : \left|\arg\left\{\sinh\left[z + (1+z^2)^{1/2}\right]\right\}\right| < d\right\}}\,,$$

$$\begin{aligned} \operatorname{Re} z &\leq 0\,, & |f(z| &\leq c_- |z|^{-\alpha_-}\,,\\ \operatorname{Re} z &\geq 0\,, & |f(z)| &\leq c_+ \exp\{-\alpha_+ |z|\}\,,\\ \phi(z) &= \ln\left\{\sinh\left[z + (1+z^2)^{1/2}\right]\right\}\,,\\ z_k &= \frac{1}{2}\left(u_k - u_k^{-1}\right)\,, & u_k &= \ln\left[\exp\{kh\} + (1+\exp\{2kh\})^{1/2}\right]\,. \end{aligned}$$

$\mathscr{G}$

❸ The interval is $\mathbb{R}$. If D_d is defined by

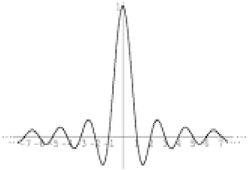
$$
\begin{aligned}
D_d &= \{z \in \mathbb{C} : |\arg\left[z + (1+z^2)^{1/2}\right]| < d\}, \\
\phi(z) &= \ln\left[z + (1+z^2)^{1/2}\right]
\end{aligned}
$$

If $z \in D_d$ and

$$
\begin{aligned}
\operatorname{Re} z &\le 0, \qquad |f(z)| \le c_- (1 + |z|)^{-\alpha_-}, \\
\operatorname{Re} z &\ge 0, \qquad |f(z)| \le c_+ (1 + |z|)^{-\alpha_+},
\end{aligned}
$$

then the Sinc points are defined by

$$
z_k = \sinh(kh), \qquad \frac{1}{\phi'(z_k)} = \cosh(kh).
$$



In all cases we introduce a positive integer N and define

$$M = \left[\frac{\alpha_+}{\alpha_-} N\right], \qquad h = \left(\frac{d}{\alpha_+ N}\right)^{1/2},$$

where $[x]$ is the integer part of x. Having defined all the auxiliary quantities we obtain

$$f(z) \approx \sum_{k=-M}^{N} f(z_k) \operatorname{sinc}\left(\frac{\phi(z)}{h} - k\right), \quad \int_a^b f(z) \approx h \sum_{k=-M}^{N} f(z_k) \left[\phi'(z_k)\right]^{-1}$$

In the computation of H we integrate over the real variable $\mathbf{t}$ but the analytic continuation, $\mathbf{t} \in \mathbb{C}$, is needed in order to determine the parameter $\mathbf{d}$ which defines the step size $\mathbf{h}$. The accuracy of the Sinc approximation on $\mathbb{R}$ is based on the fact that f is analytic and uniformly bounded on the strip D_d.

numerica

$$H = \Big[\prod_{j=1}^{2} \int_{L_j} \frac{ds_j}{2 i \pi}\Big] \Gamma(-s_1)\,\Gamma(-s_2)\, \frac{\Gamma(1+s_1+s_2)\,\Gamma(1+s_1)\,\Gamma(4+s_2)}{\Gamma(5+s_1+s_2)}\, z_1^{s_1}\, z_2^{s_2}\,,$$

$$z_1 = -\frac{1}{4}\,, \qquad z_2 = \frac{1}{3} + 0.01\,i\,.$$

	Re	Im
Korobov	8.16687758(4)	−0.0524562(2)
S_{10}	9.21579469	−0.152544264
S_{30}	8.26254429	−0.0540061617
S_{50}	8.19088582	−0.0525001759
S_{100}	8.16884067	−0.0524777335
S_{300}	8.16687874	−0.0524526209

$\mathscr{G}$

$$\mathrm{H} = \Big[\prod_{j=1}^{2} \int_{L_j} \frac{ds_j}{2i\pi}\Big] \frac{\Gamma\left(\frac{1}{2}+s_1+s_2\right)}{\Gamma\left(\frac{3}{2}+s_1+s_2\right)} \Gamma(-s_1)\,\Gamma(2+s_1)\,\Gamma(-s_2)\,\Gamma(1+s_2)\, z_1^{s_1} z_2^{s_2}\,.$$

Setting $z_1 = 0.15 + 0.01\,i$, $z_2 = 0.55 - 0.01\,i$ and $\sigma_1 = \sigma_2 = -0.1$ we can compare the exact result with the Korobov lattice and the Sinc lattice.

Exact		$62.6024046 - 0.183034716\,i$
Sinc	355152 calls	$62.6024046 - 0.183034712\,i$
Korobov	3071856 calls	$62.6024045(2) - 0.18303475(5)\,i$

$$\mathrm{H} = \Big[\prod_{j=1}^{4} \int_{L_j} \frac{ds_j}{2i\pi}\Big] \frac{\Gamma\left(\frac{1}{2}+\mathbf{s}\right)}{\Gamma\left(\frac{11}{2}+\mathbf{s}\right)} \prod_{j=1}^{4} \Gamma\left(-s_j\right) \Gamma\left(a_j+s_j\right) z_j^{s_j},$$

with $a_j = 1$, $\sigma_j = -0.1$ and

$$z_1 = -2.11, \quad z_2 = 0.22 + 0.1\,i, \quad z_3 = 0.33 + 0.1\,i, \quad z_4 = 0.44 + 0.1\,i,$$

obtaining

Lattice	Calls	Re	Im
Korobov	281437986	62.98(9)	−8.91(5)
Sinc	11298540	62.9799396	−8.91142516
	86972936	62.9370547	−8.91706193
	331085208	62.9329160	−8.91725259

We have studied the problem of writing the Mellin-Barnes representation (akas Fox functions) of Feynman integrals describing physical processes and taking into account their behavior below and above the thresholds characterizing the integrals.

Thank you for your attention


Backup Slides

Definition of H parameters

$$\mathrm{H}\left[\mathrm{z}\,;(\mathbf{a},\mathbf{A})\,;(\mathbf{b},\mathbf{B})\,;(\mathbf{c},\mathbf{C})\,;(\mathbf{d},\mathbf{D})\right]=\int_{\mathrm{L}}\frac{\mathrm{ds}}{2\,\mathrm{i}\,\pi}\,\frac{\prod_{j=1}^{m}\Gamma\left(a_j+A_j\,s\right)\prod_{j=1}^{n}\Gamma\left(b_j-B_j\,s\right)}{\prod_{j=1}^{p}\Gamma\left(c_j+C_j\,s\right)\prod_{j=1}^{q}\Gamma\left(d_j-D_j\,s\right)}\,z^{s}\,,$$

$$\overline{A}=\sum_{i=1}^{m}A_j \qquad \dots \qquad \overline{D}=\sum_{j=1}^{q}D_j\,,$$

$$\overline{a}=\mathrm{Re}\sum_{i=1}^{m}a_j \qquad \dots \qquad \overline{d}=\mathrm{Re}\sum_{j=1}^{q}d_j\,,$$

$$\begin{aligned}
\alpha &= \overline{A}+\overline{B}-\overline{C}-\overline{D}, \qquad \beta=\overline{A}-\overline{B}-\overline{C}+\overline{D},\\
\lambda &= \frac{1}{2}\left(p+q-m-n\right)+\overline{a}+\overline{b}-\overline{c}-\overline{d},\\
\rho &= \prod_{j=1}^{m}A_j^{A_j}\prod_{j=1}^{n}B_j^{-B_j}\prod_{j=1}^{p}C_j^{-C_j}\prod_{j=1}^{q}D_j^{D_j}\,.
\end{aligned}$$

Definition of H parameters

$$\mathrm{H}\left[z\,;(a_1,A_1)\ldots(a_p,A_p)\,;(b_1,B_1)\ldots(b_q,B_q)\right]=$$

$$\int_L \frac{ds}{2\,i\,\pi}\,\frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j\,s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_j\,s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j\,s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+A_j\,s\right)}\,z^{-s}\,.$$

$$\mu = \sum_{j=1}^{q} B_j-\sum_{j=1}^{p} A_j\,, \qquad \delta=\sum_{j=1}^{q} b_j-\sum_{j=1}^{p} a_j+\frac{1}{2}\,(p-q)\,,$$

$$\beta = \left[\prod_{j=1}^{p} A_j^{-A_j}\right]\left[\prod_{j=1}^{q} B_j^{B_j}\right], \qquad \alpha=\sum_{j=1}^{n} A_j-\sum_{j=n+1}^{p} A_j+\sum_{j=1}^{m} B_j-\sum_{j=m+1}^{q} B_j\,.$$

Multivariate H

given $\mathbf{s} = [s_1 \dots s_r]$, $\alpha = [\alpha_1 \dots \alpha_r]$, $\beta = [\beta_1 \dots \beta_r]$, $\arg(\mathbf{z}) = [\arg(z_1) \dots \arg(z_r)]$, we define

$$\mathbf{A} = \left(a_{j,k}\right)_{m \times r}, \qquad \mathbf{B} = \left(b_{j,k}\right)_{n \times r},$$

$$H\left[\mathbf{z}; (\alpha, \mathbf{A}); (\beta, \mathbf{B})\right] = \left[\prod_{j=1}^{r} \int_{L_j} \frac{ds_j}{2\,i\,\pi}\right] \Psi \prod_{j=1}^{r} (z_j)^{-s_j}, \quad \Psi = \frac{\prod_{j=1}^{m} \Gamma\left(\alpha_j + \sum_k a_{j,k}\, s_k\right)}{\prod_{j=1}^{n} \Gamma\left(\beta_j + \sum_k b_{j,k}\, s_k\right)},$$

where a and b are arbitrary real numbers. It is important to realize that the multiple integral may be overall divergent although the iterate integrals converge.

For an accurate Sinc approximation of a function f on a contour Γ, we require two conditions:

(a) analyticity of f in a domain D with $\Gamma \in \mathrm{D}$, and

(b) a set of Lipschitz conditions of f on Γ.

The infinite-point Sinc formula may be very accurate when the first condition is satisfied, even though the second condition is not. In this case, the use of Sinc approximation requires a large number of evaluation points in order to sum the series accurately.

If D is a n-dimensional hypercube and

$$\mathrm{V}_{\mathrm{N},n} = \sum_{i=0}^{\mathrm{N}} \mathrm{V}_{n}^{i}, \qquad \mathrm{V}_{\mathrm{n}}^{i} = \sum_{0 \le i_1 = \cdots i_{\mathrm{n}} \le i} \mathrm{a}^{i}_{\mathrm{i}_1 \dots i_{\mathrm{n}}} \, \mathrm{x}_1^{\mathrm{i}_1} \cdots \mathrm{x}_{\mathrm{n}}^{\mathrm{i}_{\mathrm{n}}},$$

where the $\mathrm{V}_{\mathrm{n}}^{\mathrm{i}}$ are homogeneous polynomials and $\mathrm{V}_{\mathrm{N},mrn}$ is a generic polynomial in the ring of polynomials of degree N, it is convenient to determine the $(\mathrm{N}-1)^{\mathrm{n}}$ n-tuples $\mathrm{X}_1^{\mathrm{i}} \dots \mathrm{X}_{\mathrm{n}}^{\mathrm{i}}$ such that

$$\mathrm{V}_{\mathrm{N},n}\left(\mathrm{x}_1 - \mathrm{X}_1^{\mathrm{i}} \dots \mathrm{x}_{\mathrm{n}} - \mathrm{X}_{\mathrm{n}}^{\mathrm{i}}\right) = \Delta + \sum_{i=2}^{\mathrm{N}} \mathrm{V}_{\mathrm{n}}^{\mathrm{i}} \left(\mathrm{x}_1 - \mathrm{X}_1^{\mathrm{i}} \dots \mathrm{x}_{\mathrm{n}} - \mathrm{X}_{\mathrm{n}}^{\mathrm{i}}\right), \quad \mathrm{i} = 1 \dots (\mathrm{N}-1)^{\mathrm{n}},$$

so that the solutions of $\Delta(w_1 \dots w_{\mathrm{k}}) = 0$, are the potential (leading) pinch singularities if $\mathrm{X}_{\mathrm{j}}^{\mathrm{i}} \in \mathbb{R},\ 0 < \mathrm{X}_{\mathrm{j}}^{\mathrm{i}} < 1\ \forall \mathrm{j}$. For $\mathrm{V}_{\mathrm{n}}^{2} = \dots = \mathrm{V}_{\mathrm{n}}^{\mathrm{k}} = 0$ the singular point will have multiplicity $\mathrm{k}+1$.