The geometry of Feynman integrals

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1. Improving integration-by-parts
2. A systematic algorithm for ε-factorised differential equations

Section 1

Introduction

The Laporta algorithm

- Integration-by-parts identities provide **linear relations** among Feynman integrals.
- One defines an **order relation** between Feynman integrals and eliminates the more complicated ones in favour of the simpler ones.

Order relations

Typical order relations: Lexicographical order of tuples

$$\textbf{ISP-basis}: \quad \left(N_{\mathrm{prop}}, N_{\mathrm{id}}, r_{\mathrm{dot}}, s_{\mathrm{ISP}}, \dots \right)$$
$$\textbf{dot-basis}: \quad \left(N_{\mathrm{prop}}, N_{\mathrm{id}}, s_{\mathrm{ISP}}, r_{\mathrm{dot}}, \dots \right)$$

where

$$N_{\mathrm{prop}} = \sum_{j=1}^{n_{\mathrm{int}}} \Theta\left(\nu_j > 0\right), \qquad N_{\mathrm{id}} = \sum_{j=1}^{n_{\mathrm{int}}} 2^{j-1} \Theta\left(\nu_j > 0\right),$$
$$r_{\mathrm{dot}} = \sum_{j=1}^{n_{\mathrm{int}}} \nu_j \Theta\left(\nu_j > 0\right), \qquad s_{\mathrm{ISP}} = \sum_{j=1}^{n_{\mathrm{int}}} \left|\nu_j\right| \Theta\left(\nu_j < 0\right).$$

Comments

- The number of master integrals is independent of the order relation.
- The set of master integrals depends on the order relation, and so do the reduction coefficients.
- The reduction coefficients are rational functions of ε and x.
  We would like to avoid in the denominator irreducible polynomials, which depend on ε and x.
  Such polynomials lead to an expression swell.
- We may exploit the **freedom of choosing an order relation**.

Differential equations

1. Using integration-by-parts obtain a differential equation of the form (always possible)

$$dI = A(\varepsilon, x)\, I.$$

(Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99)

2. **Bottle neck**: Find a transformation $I = RK$ such that

$$dK = \varepsilon A(x)\, K$$

(Henn '13)

3. Solve the latter differential equation with appropriate boundary conditions in terms of iterated integrals (always possible).

(Chen '77)

Sectors and block-triangular structure

Order the set of master integrals $I = (I_1, \dots, I_{N_F})$ such that I_1 is the simplest integral and I_{N_F} the most complicated integral.

The integration-by-parts identities and the matrix A have a lower block-triangular structure:

$$\begin{pmatrix} D_1 & 0 & 0 & 0 \\ N_{21} & \multicolumn{2}{c}{D_2} & 0 \\ & & & 0 \\ N_{31} & \multicolumn{2}{c}{N_{32}} & D_3 \end{pmatrix}$$

Diagonal blocks:	D_1, D_2, D_3
Non-diagonal blocks:	N_{21}, N_{31}, N_{32}

Diagonal blocks

- The challenging part are the diagonal blocks (i.e. the maximal cut).
- The size of the diagonal blocks can be sizeable, e.g. $\mathcal{O}(10)$.
- Once the correct masters are known on the maximal cut, it is rather straightforward to extent these masters beyond the maximal cut.
- In the following we denote by V the **vector space of Feynman integrals on the maximal cut**.

Transformation to an ε-factorised form

A two-step procedure:

$$I = R_1 J = R_1 R_2 K$$

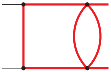
1. Construct an intermediate basis $J = R_1^{-1} I$, such that the differential equation for J is compatible with a filtration, in particular it is in Laurent polynomial form:

$$dJ = \sum_{k=k_{\min}}^{1} \varepsilon^k A^{(k)}(x) J,$$

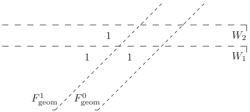
2. Construct a matrix R_2, which leads to a basis $K = R_2^{-1} J$, such that the differential equation for K is in ε-factorised form.

Motivating example

- An elliptic sector, contributing to $pp \to t\bar{t}$, with three master integrals:



- We may **decompose** the three-dimensional vector-space into



- We are interested in methods, which work for **any geometry**.

Section 2

Constructing the basis J

The Baikov representation

We study the integrands of Feynman integrals on the maximal cut in a (loop-by-loop) **Baikov representation**.

Baikov polynomials $p_i(z)$ defined by

$$\int_{\mathcal{C}_{\mathrm{maxcut}}} \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod\limits_{j=1}^{n_{\mathrm{edges}}} \sigma_j} \sim \int d^n z \prod_{i \in I_{\mathrm{all}}} \left[p_i\left(z\right)\right]^{\alpha_i}.$$

The exponents α_i are **always** of the form

$$\alpha_i = \frac{1}{2}\left(a_i + b_i \varepsilon\right), \quad \text{with} \quad a_i, b_i \in \mathbb{Z}.$$

Define I_{odd} as the set of indices for which a_i is odd and I_{even} as the set of indices for which a_i is even.

Integrands of Feynman integrals

- Recall: V denotes vector space of Feynman **integrals** on the maximal cut mod linear relations.
- We denote the vector space spanned by the **integrands** by Ω_ω and the vector space mod linear relations by H_ω.
- There is an injective map

$$\iota \quad : \quad V \rightarrowtail H_\omega.$$

- In general, this map is not surjective due to
  - **Symmetries**

$$z_1 dz_1 \wedge dz_2 \neq z_2 dz_1 \wedge dz_2 \quad \text{but} \quad \int\limits_{[0,1]^2} z_1 dz_1 \wedge dz_2 = \int\limits_{[0,1]^2} z_2 dz_1 \wedge dz_2$$

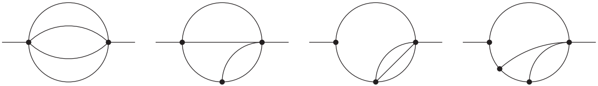
  - **Super-sectors**

Super-sectors

- It may happen that

$$p_i(z) = z_r.$$

- In this case, Ω_ω will also contain the integrands of the sector where the exponent of this inverse propagator is positive. If this sector has additional master integrals, they will also appear in H_ω.



- Explains "magic relations".

The minimal twist

- Instead of working in the affine chart $z = (z_1, \dots, z_n)$, go to projective space $\mathbb{CP}^n$ with homogeneous coordinates $[z_0 : z_1 : \dots : z_n]$.
- Define the **minimal twist** U and its $(\varepsilon = 0)$-part U_0 by

$$
\begin{aligned}
U(z_0, z_1, \dots, z_n) &= \prod_{i \in I^0_{\mathrm{odd}}} P_i^{-\frac{1}{2} + \frac{1}{2} b_i \varepsilon} \prod_{j \in I^0_{\mathrm{even}}} P_j^{\frac{1}{2} b_j \varepsilon}, \\
U_0(z_0, z_1, \dots, z_n) &= \prod_{i \in I^0_{\mathrm{odd}}} P_i^{-\frac{1}{2}}.
\end{aligned}
$$

The integrand

$$\Psi_{\mu_0\dots\mu_{N_D}}[Q] = \underbrace{\mathbf{C}_{\text{Baikov}}\mathbf{C}_{\text{abs}}\mathbf{C}_{\text{rel}}\mathbf{C}_{\text{clutch}}}_{z-\text{independent prefactors}} \underbrace{\mathbf{U}(\mathbf{z})}_{\text{minimal twist}} \underbrace{\frac{\mathbf{Q}}{\prod\limits_{\mathbf{i}\in\mathbf{I}^0_{\text{all}}} \mathbf{P}_{\mathbf{i}}^{\mu_{\mathbf{i}}}}}_{\text{rational function}} \eta,$$

where η is the standard n-form on $\mathbb{CP}^n$:

$$\eta = \sum_{j=0}^{n} (-1)^j z_j \, dz_0 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n,$$

and C_{Baikov} and C_{abs} are independent of Q and $\mu_0 \dots \mu_{N_D}$,

$$C_{\text{rel}} = \prod_{i\in I^0_{\text{odd}}} \left(-\frac{1}{2}+\frac{1}{2}b_i\varepsilon\right)_{\mu_i} \prod_{i\in I^0_{\text{even}}} \left(\frac{1}{2}b_i\varepsilon\right)_{\mu_i}, \qquad (a)_n = \frac{\Gamma(a+1)}{\Gamma(a+1-n)},$$

$$C_{\text{clutch}} = \varepsilon^{-|\mu|}, \qquad |\mu| = \sum_{i\in I^0_{\text{all}}} \mu_i.$$

Linear relations

1. Integration-by-parts identities:

$$0 = \frac{1}{\varepsilon} \Psi_{\mu_0 \dots \mu_i \dots \mu_{N_D}} \left[\partial_{z_j} Q_+ \right] + \sum_{i \in I_{\mathrm{all}}^0} \Psi_{\mu_0 \dots (\mu_i + 1) \dots \mu_{N_D}} \left[Q_+ \cdot \left(\partial_{z_j} P_i \right) \right]$$

2. Distribution identities:

$$\Psi_{\mu_0 \dots \mu_{N_D}} \left[Q_1 + Q_2 \right] = \Psi_{\mu_0 \dots \mu_{N_D}} \left[Q_1 \right] + \Psi_{\mu_0 \dots \mu_{N_D}} \left[Q_2 \right]$$

3. Cancellation identities:

$$\Psi_{\mu_0 \dots (\mu_j + 1) \dots \mu_{N_D}} \left[P_j \cdot Q \right] = \frac{1}{\varepsilon} \frac{\mathbf{C}_{\mathrm{rel}}^{(\mathrm{j})}}{\mathbf{C}_{\mathrm{rel}}} \Psi_{\mu_0 \dots \mu_j \dots \mu_{N_D}} \left[Q \right]$$

Remarks

- We compute a basis in twisted cohomology with the Laporta algorithm (and not with intersection numbers).
- We may reduce the subsystem formed by the integration-by-parts identities and the distribution identities by setting $\varepsilon = 1$, and recover the ε-dependence in the end from the $|\mu|$-grading.
- This is **only spoiled** by the ratio $C_{\mathrm{rel}}^{(j)} / C_{\mathrm{rel}}$ in the cancellation identities:

$$\frac{C_{\mathrm{rel}}^{(j)}}{C_{\mathrm{rel}}} = \frac{1}{2} a_j - \mu_j + \frac{b_j}{2} \varepsilon.$$

Algebraic geometry

- We are interested in a method, which is independent of any specific geometry (elliptic curve, higher-genus curve, Calabi-Yau, ...)
- In algebraic geometry we may look at
  - **poles**
  - **residues**

The pole order

Define $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$ by replacing U with U_0 in the definition of $\Psi_{\mu_0 \dots \mu_{N_D}}[Q]$.

The **pole order** o of $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$ is defined as follows:

- The pole order is the maximum of pole orders at individual points.
- For $\alpha > 0$, the pole order of $z^{-\alpha} dz$ at $z = 0$ is $\lfloor \alpha \rfloor$, where $\lfloor x \rfloor$ denotes the floor function, e.g. the pole order of $z^{-\frac{3}{2}} dz$ at $z = 0$ is 1.
- For normal-crossing singularities, the pole order is additive, i.e. the pole order of $\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2^2}$ at $(z_1, z_2) = (0, 0)$ is 3.
- For non-normal-crossing singularities, we first need to perform a blow-up.

The number of residues

Let r be the largest number such that the r-fold residue of $\Psi^{0}_{\mu_0 \dots \mu_{N_D}}[Q]$ is non-zero.

Example in the affine chart $z_0 = 1$:

$$r = 1 : \qquad \frac{dz_1}{z_1}$$

$$r = 0 : \qquad \frac{dz_1}{\sqrt{z_1 \left(z_1 - 1\right)\left(z_1 - 2\right)\left(z_1 - 3\right)}}$$

Filtrations

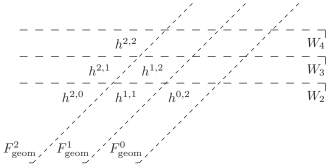
With the help of r, o and $|\mu|$ we define three **filtrations** of Ω_ω:

$$\begin{array}{lll}\Psi_{\mu_0\dots\mu_{N_D}}[Q] \in W_w\Omega_\omega & \text{if} & n+r \le w \\ \Psi_{\mu_0\dots\mu_{N_D}}[Q] \in F^p_{\mathrm{geom}}\Omega_\omega & \text{if} & n+r-o \ge p \\ \Psi_{\mu_0\dots\mu_{N_D}}[Q] \in F^{p'}_{\mathrm{comb}}\Omega_\omega & \text{if} & n-|\mu| \ge p'\end{array}$$

The $W_\bullet$-filtration and the $F^\bullet_{\mathrm{geom}}$-filtration define the decomposition

$$\Omega^{p,q}_{\mathrm{geom}} = \mathrm{Gr}^p_{F_{\mathrm{geom}}} \mathrm{Gr}^W_{p+q} \Omega_\omega$$

and similar for $H^{p,q}_{\mathrm{geom}}$ and $V^{p,q}$.



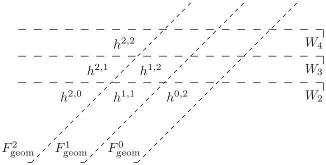
Order relation for the Laporta algorithm

Definition (order relation)

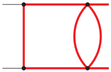
$$(a, w, o, |\mu|, \dots)$$

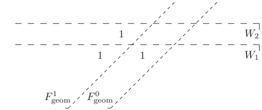
where

$$a = \begin{cases} -r & \text{if } \Psi \text{ is the pre-image of a master integrand of a sub-problem} \\ & \text{localised on } P_i = 0 \text{ with } i \in I^0_{\mathrm{even}} \\ 0 & \text{otherwise} \end{cases}$$

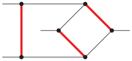


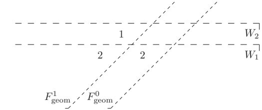
Examples



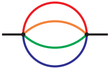


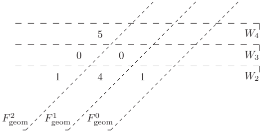
1
W_2
1 1
W_1
F^1_{geom} F^0_{geom}





1
W_2
2 2
W_1
F^1_{geom} F^0_{geom}





5
W_4
0 0
W_3
1 4 1
W_2
F^2_{geom} F^1_{geom} F^0_{geom}

The differential equation

Let $\Psi = (\Psi_1, \dots, \Psi_{N_F})$ be the basis of master integrands obtained from this algorithm and

$$d\Psi = A(\varepsilon, x)\Psi.$$

We observe that the **differential equation is compatible with the $F^\bullet_{\mathrm{comb}}$-filtration**: If $\Psi_i \in \mathrm{Gr}^{n-|\mu|_i}_{F_{\mathrm{comb}}} \Omega_\omega$ and $\Psi_j \in \mathrm{Gr}^{n-|\mu|_j}_{F_{\mathrm{comb}}} \Omega_\omega$, then

$$A_{ij}(\varepsilon, x) = \sum_{k=-(|\mu|_i-|\mu|_j)}^{1} \varepsilon^k A_{ij}^{(k)}(x).$$

The compatibility condition implies Griffiths transversality.

Section 3

Constructing the basis K

The transformation R_2

Rewrite:

$$A = \sum_{k=-n}^{1} \varepsilon^k A^{(k)}(x) = \sum_{k=-n}^{1} B^{(k)}(\varepsilon, x)$$

Example $n = 2$:

$$B^{(1)} = \left(\begin{array}{c|c|c} \varepsilon B_{11}^{(1)} & \varepsilon B_{12}^{(1)} & 0 \\ \hline \varepsilon B_{21}^{(1)} & \varepsilon B_{22}^{(1)} & \varepsilon B_{23}^{(1)} \\ \hline \varepsilon B_{31}^{(1)} & \varepsilon B_{32}^{(1)} & \varepsilon B_{33}^{(1)} \end{array} \right),$$

$$B^{(0)} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline B_{31}^{(0)} & 0 & 0 \end{array} \right), \quad B^{(-1)} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline B_{21}^{(-1)} & 0 & 0 \\ \hline \frac{1}{\varepsilon} B_{31}^{(-1)} & B_{32}^{(-1)} & 0 \end{array} \right), \quad B^{(-2)} = \left(\begin{array}{c|c|c} B_{11}^{(-2)} & 0 & 0 \\ \hline \frac{1}{\varepsilon} B_{21}^{(-2)} & B_{22}^{(-2)} & 0 \\ \hline \frac{1}{\varepsilon^2} B_{31}^{(-2)} & \frac{1}{\varepsilon} B_{32}^{(-2)} & B_{33}^{(-2)} \end{array} \right).$$

The transformation R_2

- We look for a transformation $J = R_2 K$.
- Set

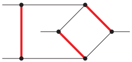
$$R_2 = R_2^{(-n)} R_2^{(-n+1)} \dots R_2^{(-1)} R_2^{(0)}$$

- $R_2^{(k)}$ removes the terms of $B^{(k)}$.
- The entries of $R_2^{(k)}$ are determined by **ε-independent** first-order differential equations (a differential ideal).
- In general, the entries of $R_2^{(k)}$ are transcendental functions (periods, ...).
- The procedure **does not depend on properties of a special point** (for example, properties of a point of maximal unipotent monodromy).

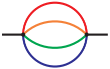
Examples

With this algorithm we were able to compute state-of-the-art integrals (including sub-topologies):

- A **genus-two** non-planar double box contributing to Møller scattering and Drell-Yan.



- The unequal-mass three-loop banana integral (involving a **K3 surface**)



Section 4

Summary and conclusions

Details on the conjecture

Conjecture: The order relation $(a, w, o, |\mu|, \dots)$ leads to a $F^\bullet_{\mathrm{comb}}$-compatible differential equation.

Equivalent to: In deriving the differential equation the cancellation identities

$$\Psi_{\mu_0 \dots (\mu_j+1) \dots \mu_{N_D}} [P_j \cdot Q] = \frac{1}{\varepsilon} \frac{C^{(j)}_{\mathrm{rel}}}{C_{\mathrm{rel}}} \Psi_{\mu_0 \dots \mu_j \dots \mu_{N_D}} [Q]$$

are always used with pivot elements from the left-hand side.

Conclusions

- **A well-chosen order relation in integration-by-parts reductions improves the efficiency.**
  - Sub-system can be reduced by setting $\varepsilon = 1$.
  - Full system: work with Laurent polynomials.
- **A systematic algorithm for ε-factorised differential equations.**
  - Step 1: Obtain the intermediate basis J directly from the order relation.
  - Step 2: Rotation to the basis K introduces transcendental functions.
- **Input from mathematics**:
  - Twisted cohomology
  - Ideas from Hodge theory