SCATTERING AMPLITUDES ON ELLIPTIC GEOMETRIES

(AND BEYOND)

Loopsummit 2, Cadenabbia (Italy)

July 24th 2025

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BASED ON

MAINLY:

Concept of **pure UT integrals** in **elliptic case** [Brödel, Duhr, Dulat, Penante, Tancredi, arXiv:1809.10698]

Integrand analysis and ***canonical bases*** in elliptic case *(and beyond)* [Görges, Nega, Tancredi, Wagner arXiv:2305.14090]

Generalization to **Calabi-Yau geometries** [Duhr, Maggio, Nega, Sauer, Tancredi, Wagner arXiv:2503.20655]

Applications to **elliptic amplitudes** and **correlators**
[Duhr, Gasparotto, Nega, Tancredi, Weinzierl arXiv:2408.05154]
[Forner, Nega, Tancredi arXiv:2411.19042]
[Marzucca, McLeod, Nega arXiv:2501.14435]
[Becchetti, Coro, Nega, Tancredi, Wagner arXiv:2502.00118]
[more applications coming soon!]

NOTE ALSO:

Applications by other groups [Becchetti, Dlapa, Zoia arXiv:2503.03603]

Generalizations to higher genus [Duhr, Porkert, Stawinski arXiv:2412.02300]

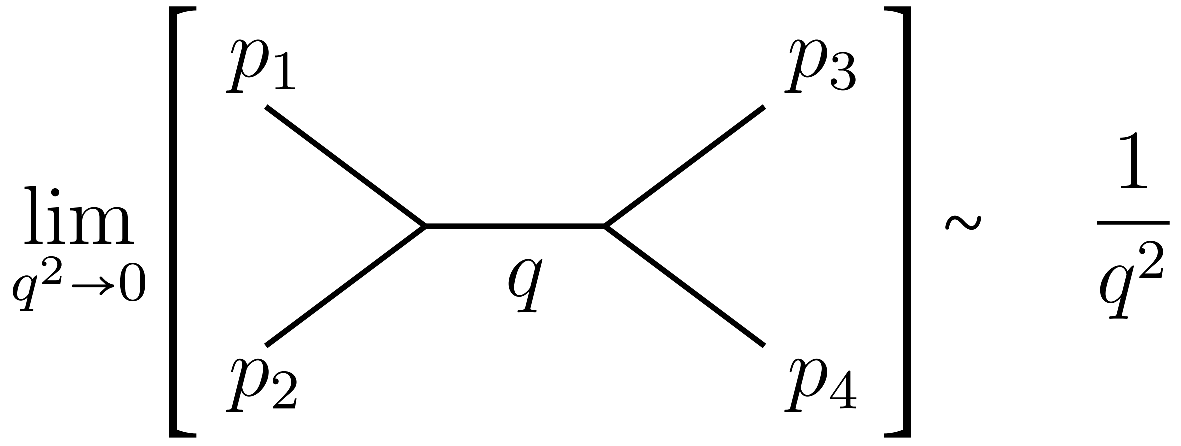
Applications to CY in Gravitational Waves [Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch ’24]

+ a lot of parallel work by *Adams, Frellesvig, Morales, Pögel, Wang, Weinzierl, Wilhelm,…* [See previous talk by S. Weinzierl]

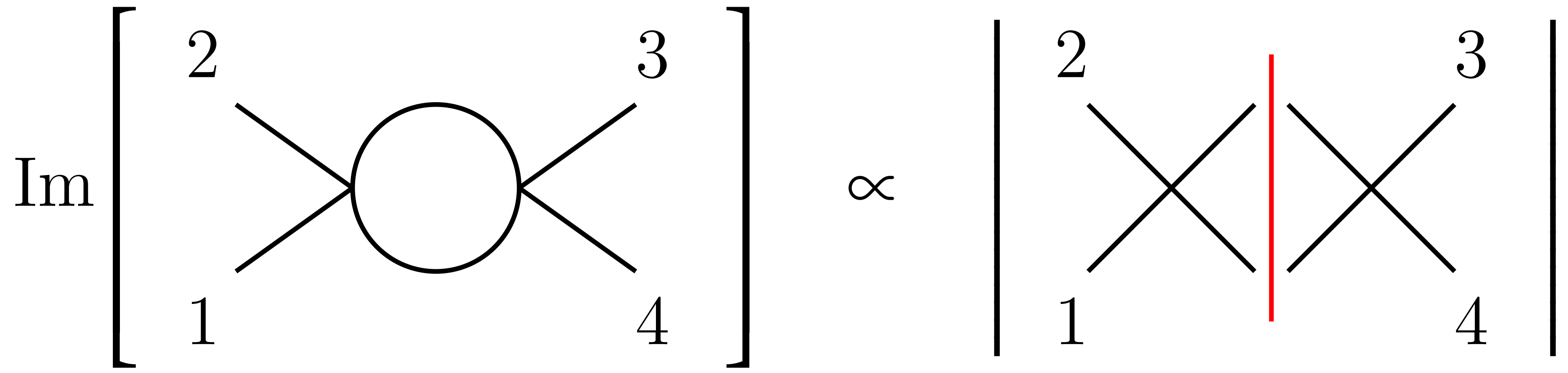
SCATTERING AMPLITUDES: POLES AND CUTS

$\mathcal{A}$

Amplitudes have **poles** where *single-particle states* go on-shell



They develop **branch-cuts** *(logarithmic and algebraic!)* when *multi-particle states* go on-shell



POLYLOGARITHMIC SCATTERING AMPLITUDES

In the well understood case of **polylogarithmic amplitudes**, there is a clear "separation"

$$\mathscr{A} \longrightarrow \sum_i R_i(s_{ij}) \int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1$$

Rational functions:

Information from poles

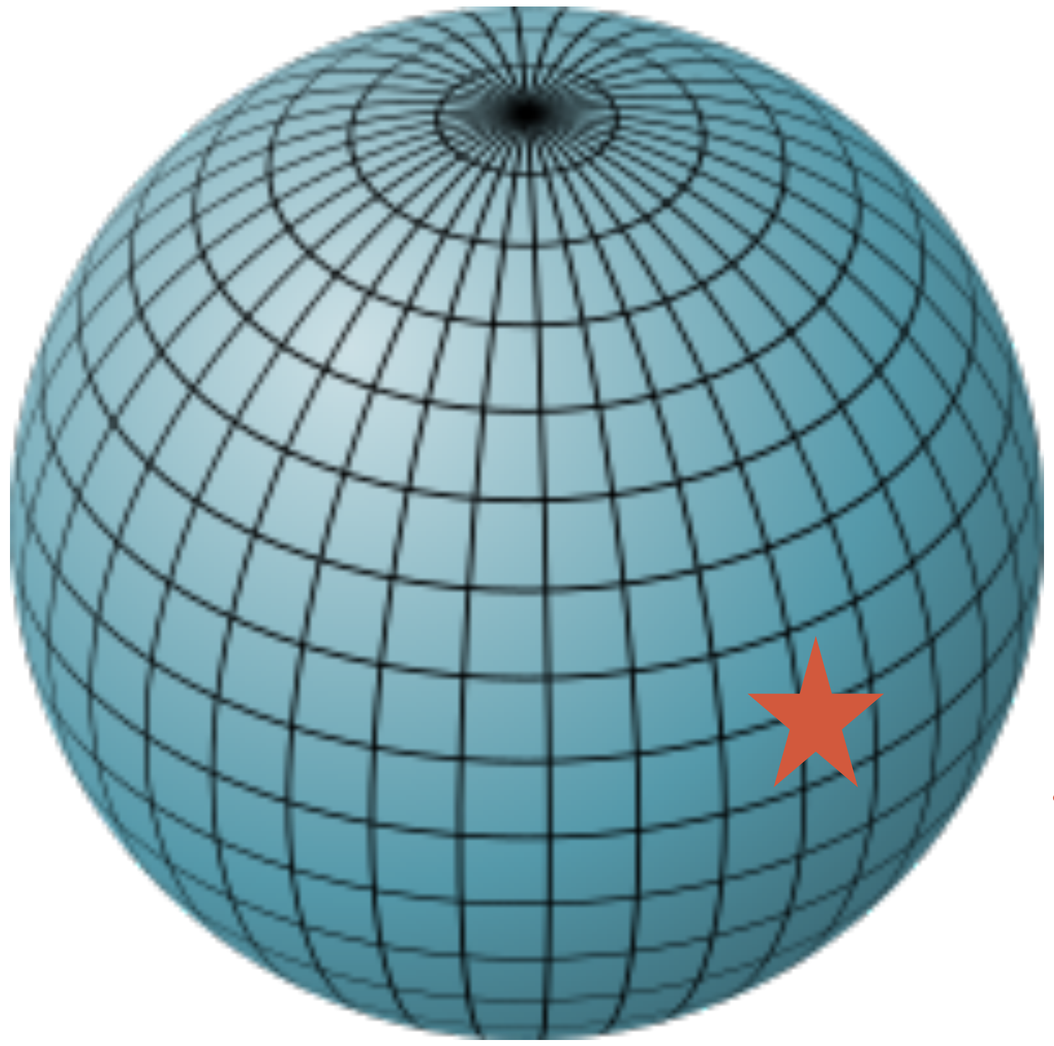
special functions (logarithms and more):

irreducible "trascendental" information from "Feynman Integrals"

Becomes clear once we choose the **right "integrals"**

How do we generalize this to "special functions" on **more complicated geometries**?

DIFFERENTIAL FORMS ON ELLIPTIC GEOMETRIES



$x = a$

entire space of functions spanned by single poles

$$\log(1 - x/a) = \int_0^x \frac{dt}{t-a}$$

Global statement

Multiple polylogarithms have log-singularities everywhere

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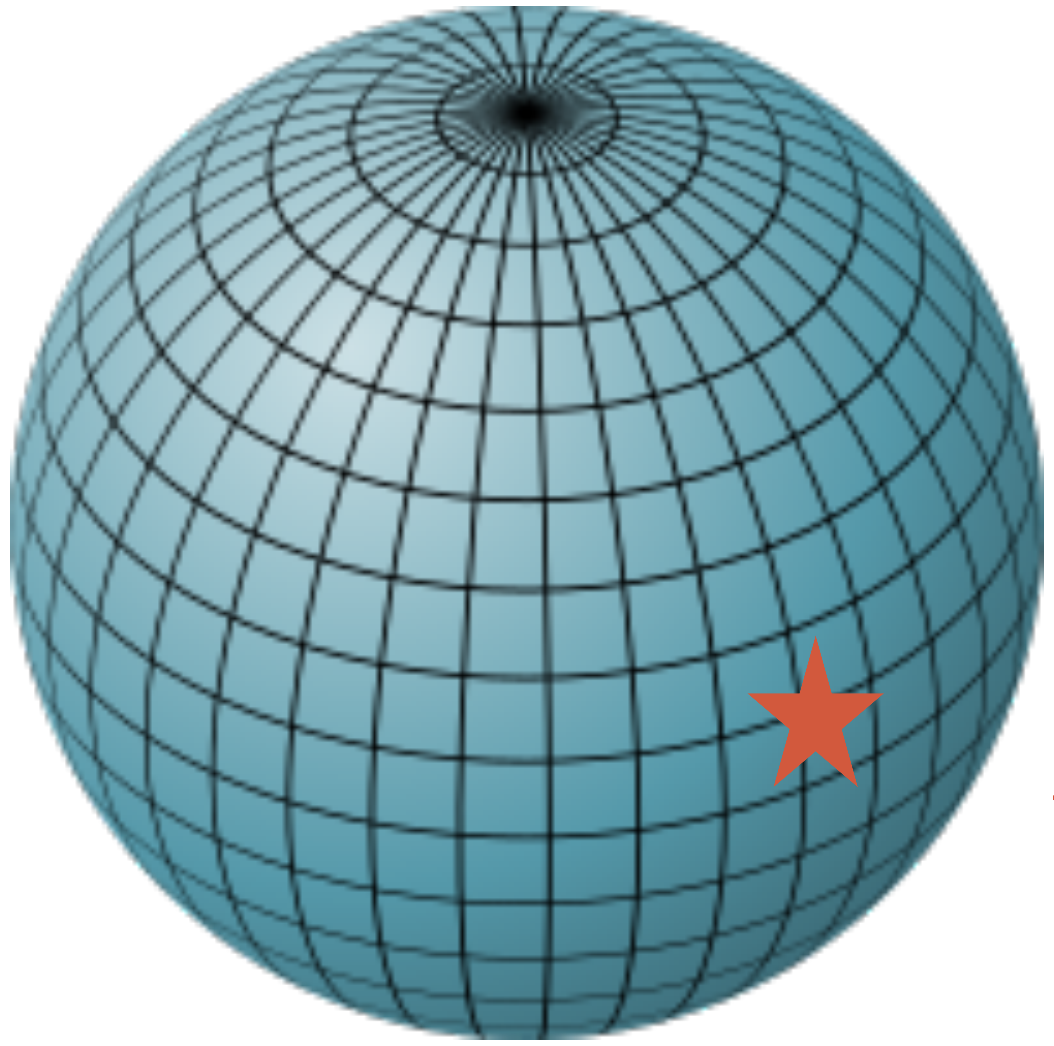

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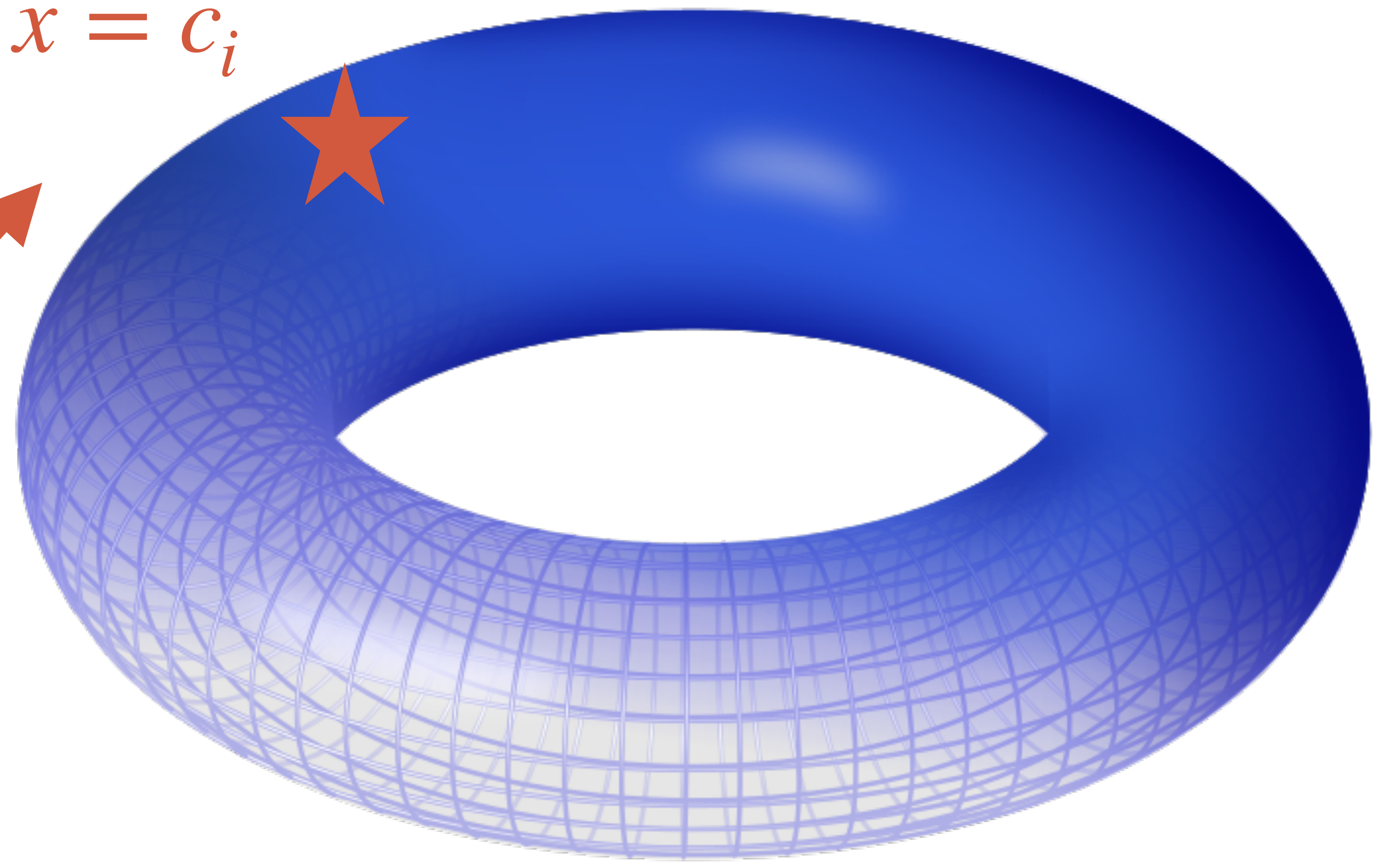
$x = c_i$



genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

Third kind

single poles $\quad g \sim \int \frac{dx}{(x-c_i)y}$

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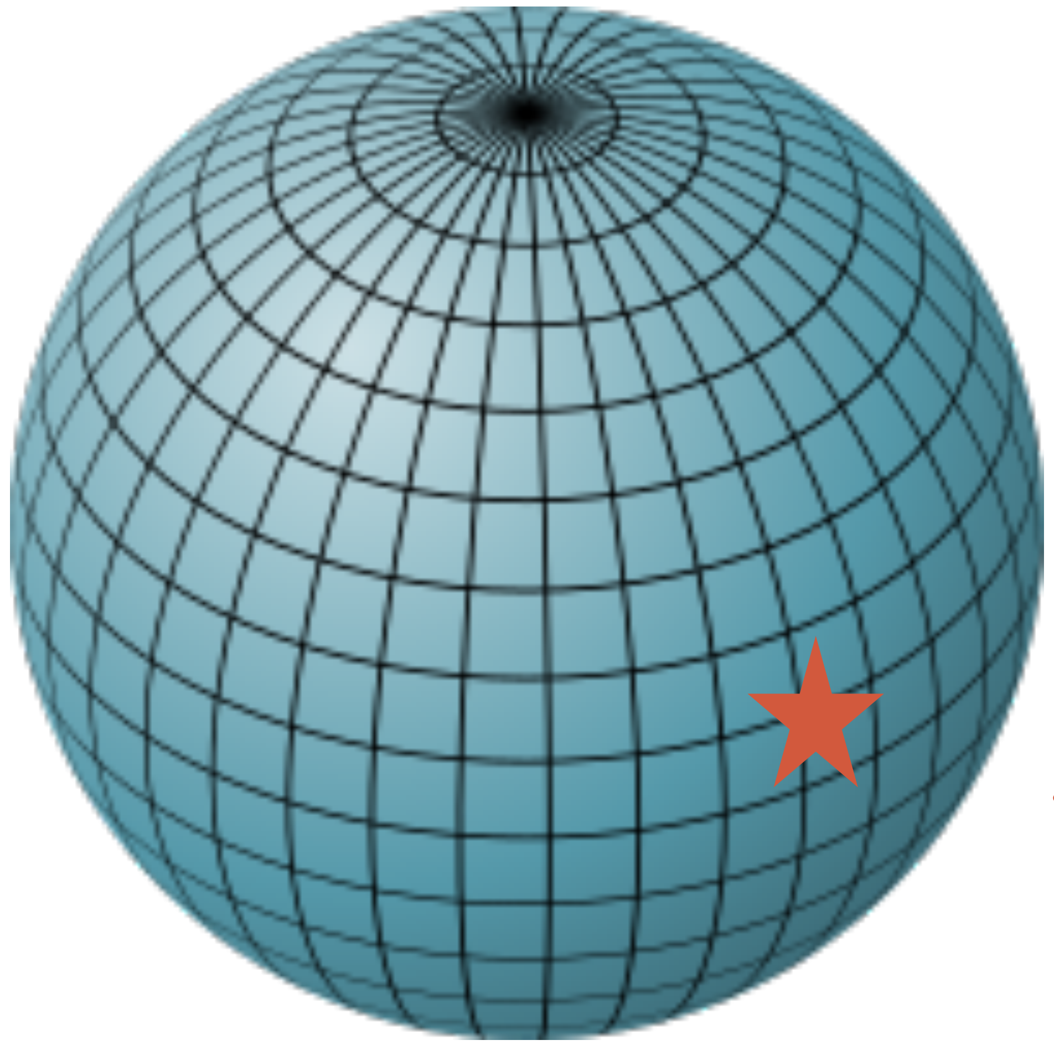

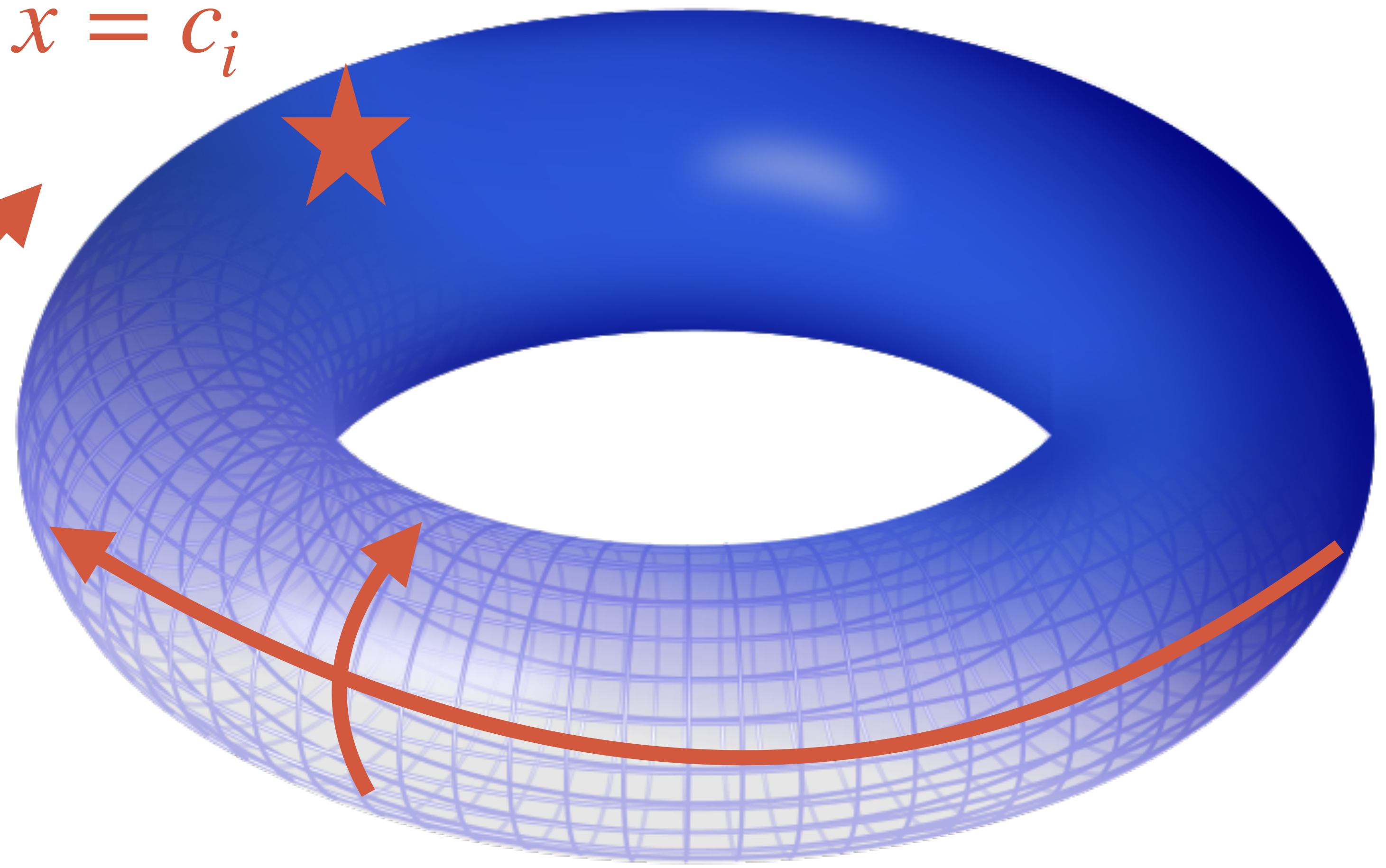

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genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

First kind

No poles $\quad \omega \sim \int \frac{dx}{y}$

Second kind

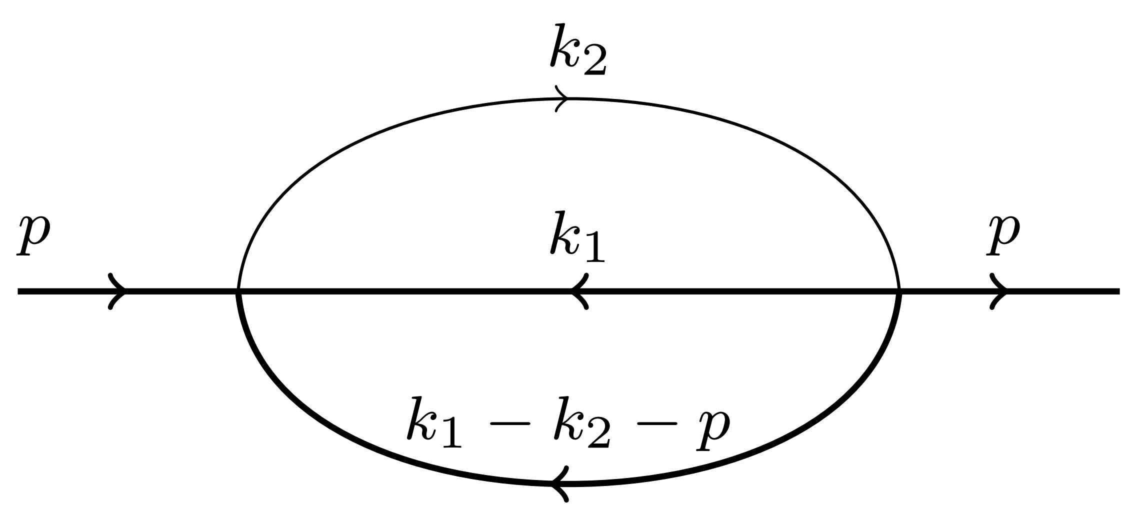
double poles $\quad \eta \sim \int \frac{dx\, x}{y}$

Third kind

single poles $\quad g \sim \int \frac{dx}{(x-c_i)y}$

FROM INTEGRANDS TO INTEGRALS TO SPECIAL FUNCTIONS

Let us consider a *(in)famous* Feynman graph: **the two-loop sunrise**



Consider the case with 2 different masses m, M

When $M \to 0$ polylogs

While $M \neq 0$ is elliptic

$$I_{\nu_1,\ldots,\nu_5}(\underline{z};d) = \int \left(\prod_{j=1}^{2} \frac{\mathrm{d}^d k_j}{i\pi^{d/2}} \right) \frac{(k_1 \cdot p)^{-\nu_4}(k_2 \cdot p)^{-\nu_5}}{(k_1^2 - m^2)^{\nu_1}(k_2^2 - M^2)^{\nu_2}((k_1 - k_2 - p)^2 - m^2)^{\nu_3}}$$

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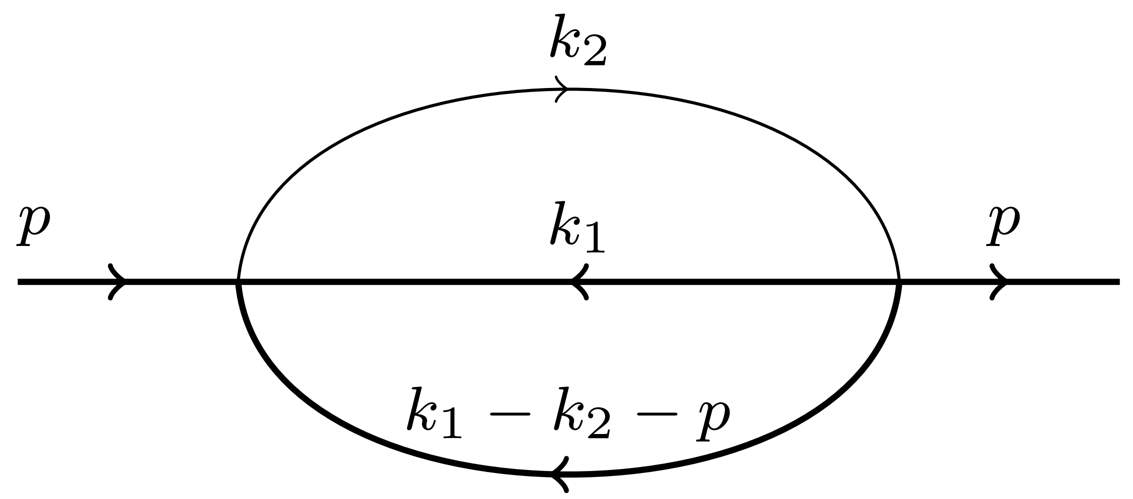

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Following the idea of **local canonical integrals** [Arkani-Hamed et al '10] [Henn '13]

Analyse its "integrand" to choose ***"good" integrals to represent scattering amplitudes*** (role of diff forms seen before!)

THE INTEGRAND IN D=2

Use "some" parametric representation for the integrand of sunrise, with numerator in last scalar prod $z_5^{\nu_5}$

I choose **Baikov**, but choose your favourite

$$I_{1,1,1,0,\nu_5} = (s)^{(2-D)/2} \int_\gamma \frac{dz_1 \ldots dz_5}{z_1 z_2 z_3} \frac{z_5^{-\nu_5}}{[B(z_j, m^2, M^2, s)]^{(4-D)/2}}$$

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Fix integer number of dimensions: we choose $D = 2$ *(more later about $D = 2 - 2\epsilon$)*

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The integrand has a bunch of singularities:

e.g. @ $z_j = 0$, $j = 1{,}2{,}3$ and many others when $B = 0$

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For reason of space, let us ***focus on a subset of them***, the ones that correspond to $z_1 = z_2 = z_3 = 0$

This is the so-called ***maximal cut*** of the graph: ***subset of its analytic structure***

THE INTEGRAND IN D=2

On the max cut the integral becomes

$$I_{1,1,1,0,\nu_5}\Big|_{z_1=z_2=z_3=0} = \int dz_5\, z_5^{-\nu_5} \int \frac{dz_4}{(z_4 - A^+(z_5))(z_4 - A^-(z_5))}$$

with $A^\pm(z_5) = \frac{1}{2}\left(s + z_5 \pm \frac{\sqrt{\Delta}}{s + M^2 + 2z_5}\right)$

and $\Delta = (2z_5 + s + M^2)(M^2 s - z_5^2)(4m^2 - M^2 - s - 2z_5)$

There are 2 single poles in z_4, with same residue *(up to a sign)* $\frac{1}{\sqrt{\Delta}}$ $\rightarrow$ **Global Residue Theorem**

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$$I_{1,1,1,0,\nu_5}\Big|_{z_1=z_2=z_3=0} \sim \int \frac{dz_5\, z_5^{-\nu_5}}{\sqrt{(2z_5 + s + M^2)(M^2 s - z_5^2)(4m^2 - M^2 - s - 2z_5)}} \int \frac{d \log[f(z_4, z_5, m^2, M^2, s)]}{dz_4} dz_4$$

POLYLOG CASE: THE INTEGRAND IN D=2

We are not done: more structure from residue in z_5. **Separate two cases**

First case: $M^2 = 0$

$$I_{1,1,1,0,\nu_5}\Big|_{z_1=z_2=z_3=0} \longrightarrow \int \frac{dz_5\, z_5^{-\nu_5}}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \int \frac{d\log[f(z_4,z_5,m^2,s)]}{dz_4} dz_4$$

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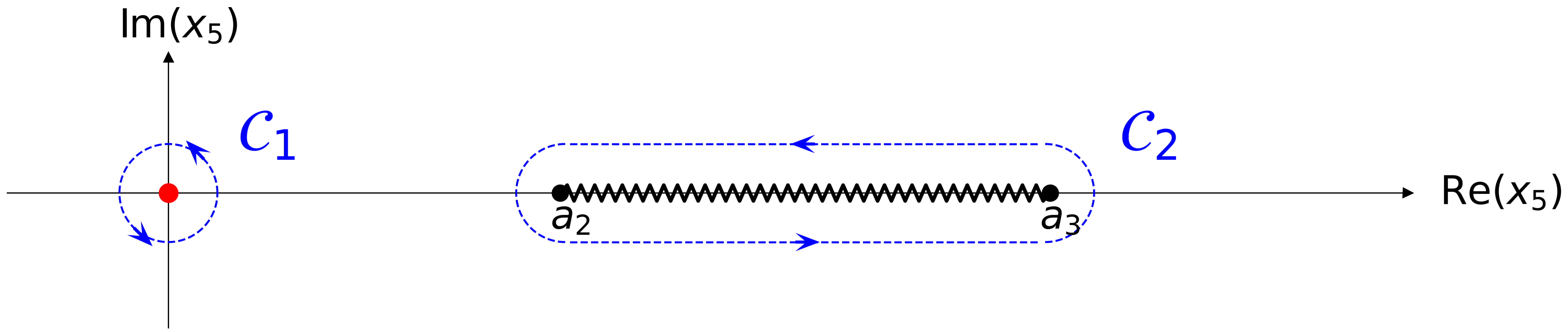

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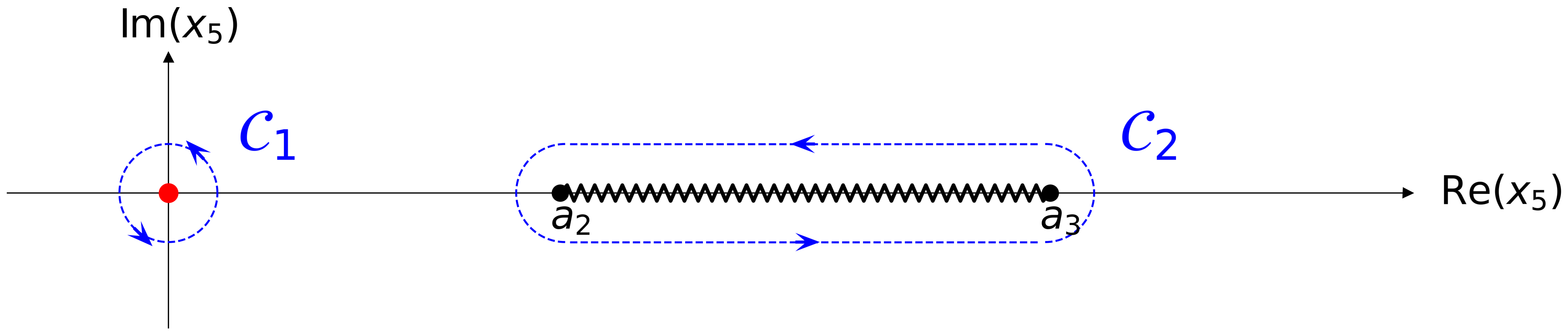
Focus on integrand in z_5 and $\nu_5 = 0$

GRT: not independent!

$$\int_{C_1} \frac{dz_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{C_2} \frac{dz_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \frac{1}{\sqrt{s(s-4m^2)}}$$



$\rightarrow I_{1,1,1,0,0}$ produces 1 independent logarithmic "master integral", with residue $\neq 1$

take $\sqrt{s(s-4m^2)}I_{1,1,1,0,0}$ as normalized integral

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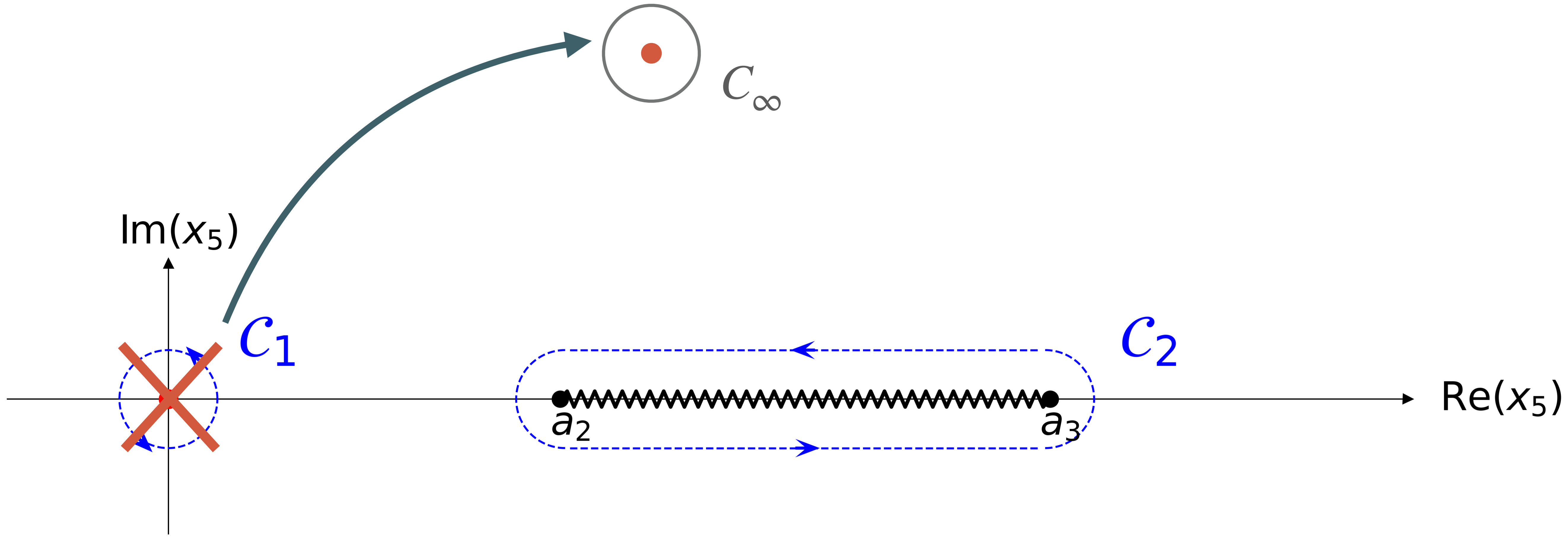
What happens for other values of ν_5? $\nu_5 = -1$ removes pole at zero and produces a *new simple pole at infinity*



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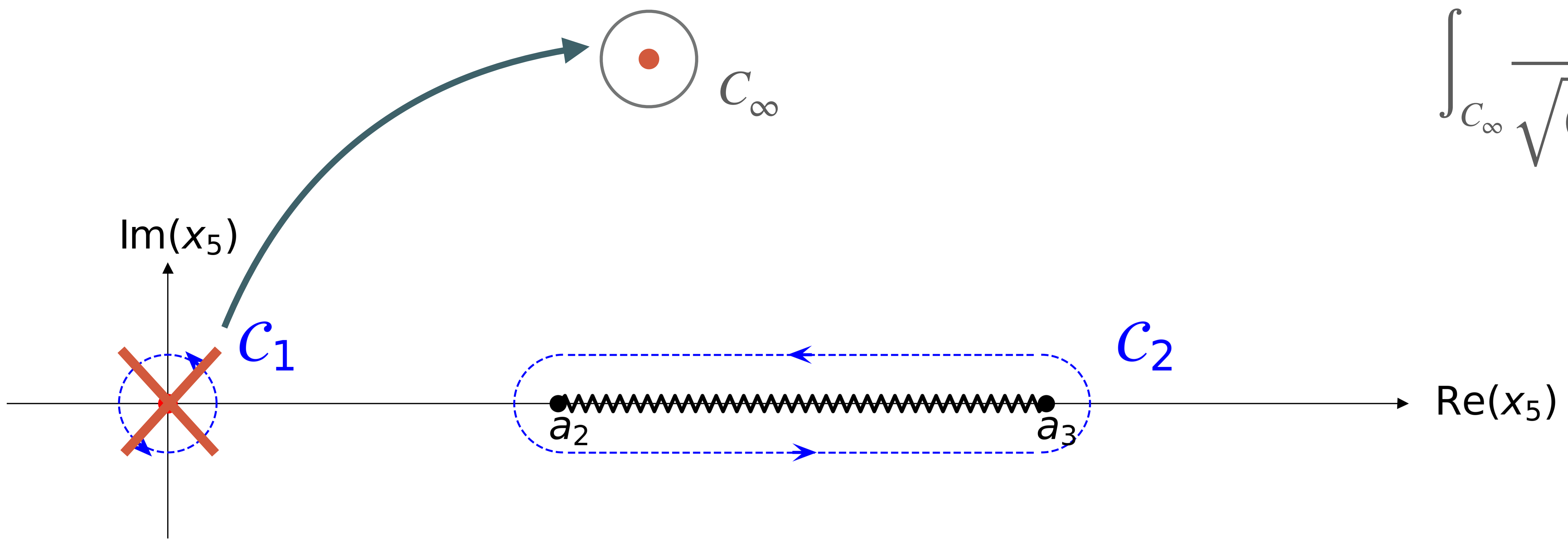

$$\int_{C_\infty} \frac{dz_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{C_2} \frac{dz_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto 1$$

$\rightarrow I_{1,1,1,0,-1}$ produces 1 independent logarithmic "master integral", with residue $= 1$

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What happens for other values of ν_5? As soon as $\nu_5 < -1$ we produce ***higher poles: not independent!***

$$\int_{C_\infty} \frac{dz_5\, z_5^n}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}}, \qquad n \geq 1 \quad \rightarrow \quad \text{no residue, algebraic, nothing new}$$

→ just looking at the integrand we know that there are **2 master integrals**, both "**logarithmic**"…

POLYLOG CASE: A GOOD BASIS IN D=2

First case: $M^2 = 0$

Moreover, the **2 master integrals** are **in dlog form in D=2** *(analysis can be easily extended beyond max cut)*

$$J_1 = \sqrt{s(s-4m^2)}I_{1,1,1,0,0} \propto \int \frac{d\log g_1(z_5, s, m^2)}{dz_5} dz_5 \int \frac{d\log f(z_4, z_5, s, m^2)}{dz_4} dz_4$$

$$J_2 = I_{1,1,1,0,-1} \propto \int \frac{d\log g_2(z_5, s, m^2)}{dz_5} dz_5 \int \frac{d\log f(z_4, z_5, s, m^2)}{dz_4} dz_4$$

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Removed algebraic "residue" gives "pure" integrals that contribute only to "irreducible" transcendental part

$$\mathscr{A} \longrightarrow \sum_i R_i(s_{ij}) \boxed{\int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1}$$

POLYLOG CASE: A GOOD BASIS IN D=2-2ϵ

First case: $M^2 = 0$

What if we deform $D = 2 - 2\epsilon$? It's ***easy to restore full ϵ dependence*** noticing that we would only get

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But new object just adds more "logs" once it is expanded close to $\epsilon = 0$

$$\Big(G(z_4, z_5, m^2, M^2, s)\Big)^{k\epsilon} \sim 1 + k\,\epsilon \log G + \mathcal{O}(\epsilon^2)$$

POLYLOG CASE: DIFFERENTIAL EQUATIONS IN D=2-2ϵ

These integrals fulfil **canonical diff-equations** [Kotikov '10] [Henn '13]

$$d\vec{I} = \epsilon \left[\quad \epsilon\text{-indep} \quad \right] \vec{I}\,, \qquad \rightarrow \qquad \left[\quad \epsilon\text{-indep} \quad \right] = \sum_i B_i \, \mathrm{d}\log f_i$$

Solution as **path-ordered exponential:** *naturally polylogs if* f_i *are rational functions!*

$$\vec{I} = \mathbb{P}\exp\left[\epsilon \sum_i B_i \int_\gamma \mathrm{d}\log f_i\right] \vec{I}_0$$

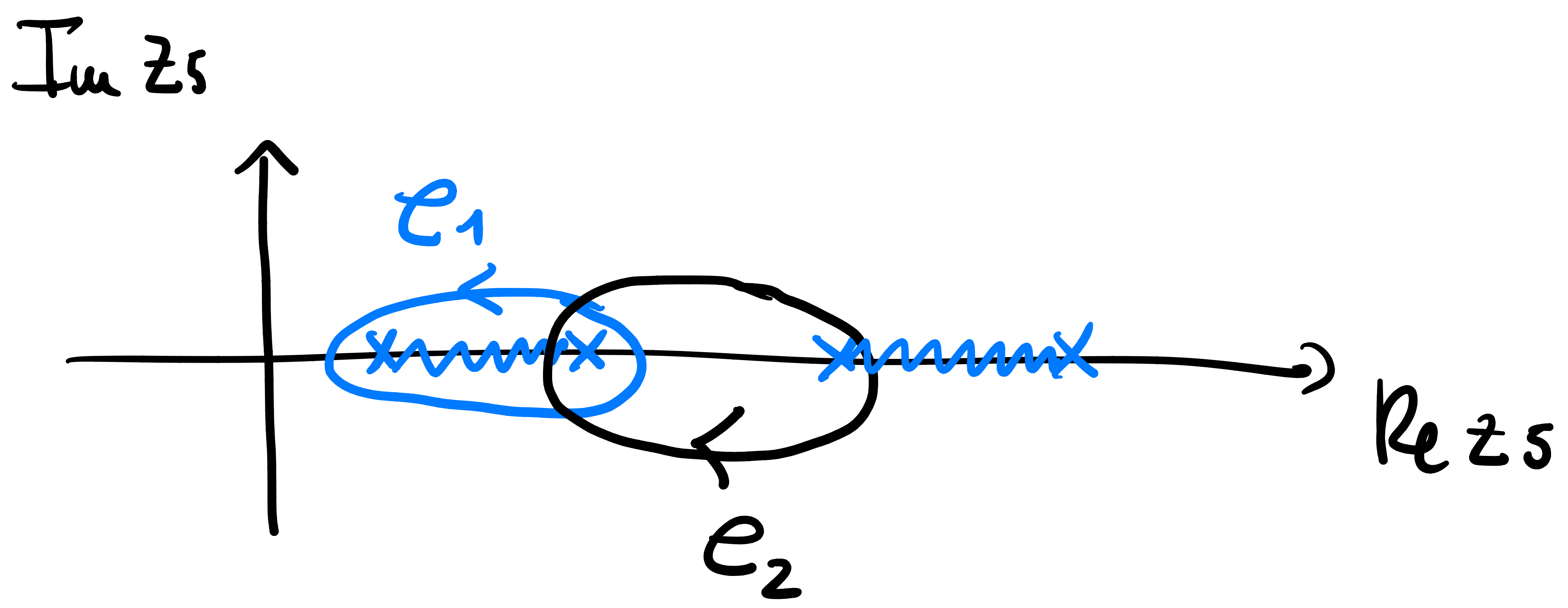
Integrals have (at most) **logarithmic singularities** close to each *regular singular point*

ELLIPTIC CASE: THE INTEGRAND IN D=2

Second case: $M^2 \neq 0$

$$I_{1,1,1,0,\nu_5}\Big|_{z_1=z_2=z_3=0} \longrightarrow \int_\gamma \frac{dz_5\, z_5^{-\nu_5}}{\sqrt{(2z_5+s+M^2)(M^2 s - z_5^2)(4m^2 - M^2 - s - 2z_5)}} \int \frac{d\log[f(z_4, z_5, m^2, s)]}{dz_4} dz_4$$

Polynomial of degree 4 in square-root → for $\nu_5 = 0$, there is **NO POLE** but ***two independent contours*** among 4 roots:



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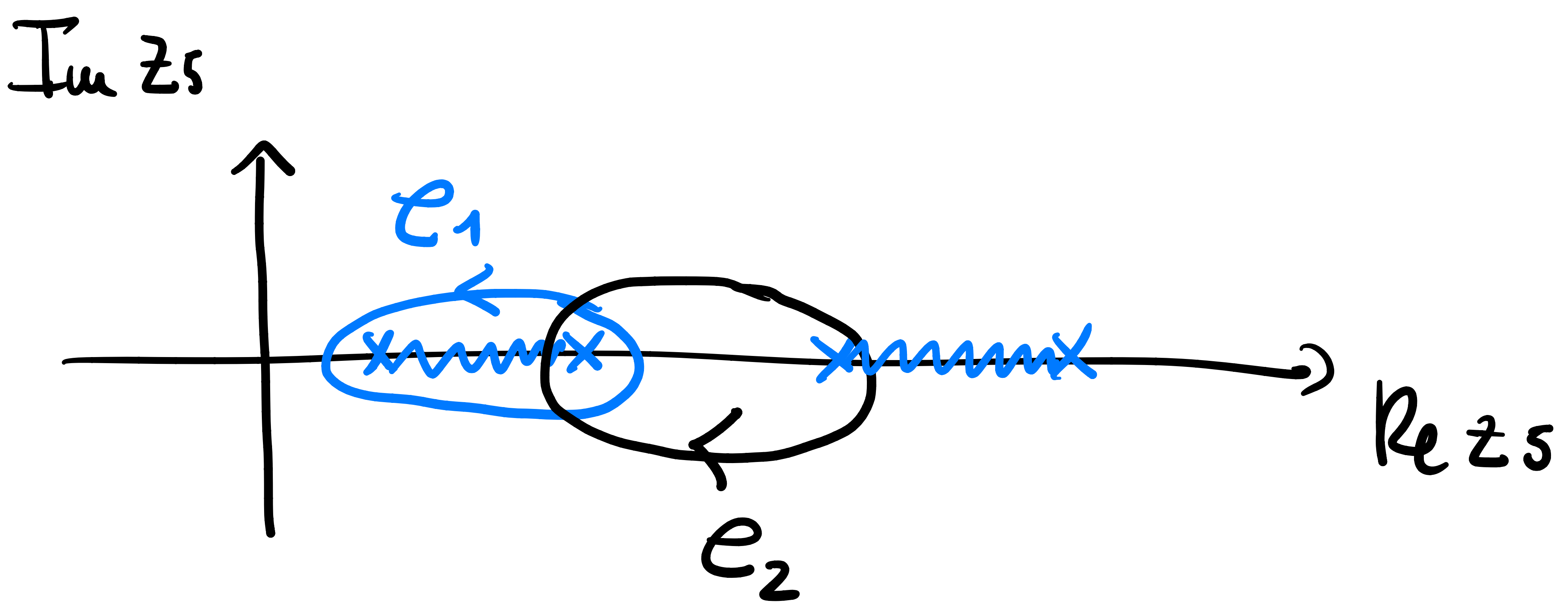

$$\int_{C_1} \frac{dz_5}{\sqrt{P_4(z_5)}} \propto \omega_0 \underset{s\to 0}{\simeq} 1 + \sum_{n=1}^{\infty} c_n s^n$$

$$\int_{C_2} \frac{dz_5}{\sqrt{P_4(z_5)}} \propto \omega_1 \underset{s\to 0}{\simeq} \omega_0 \log s + \sum_{n=1}^{\infty} d_n s^n$$

True locally !!!

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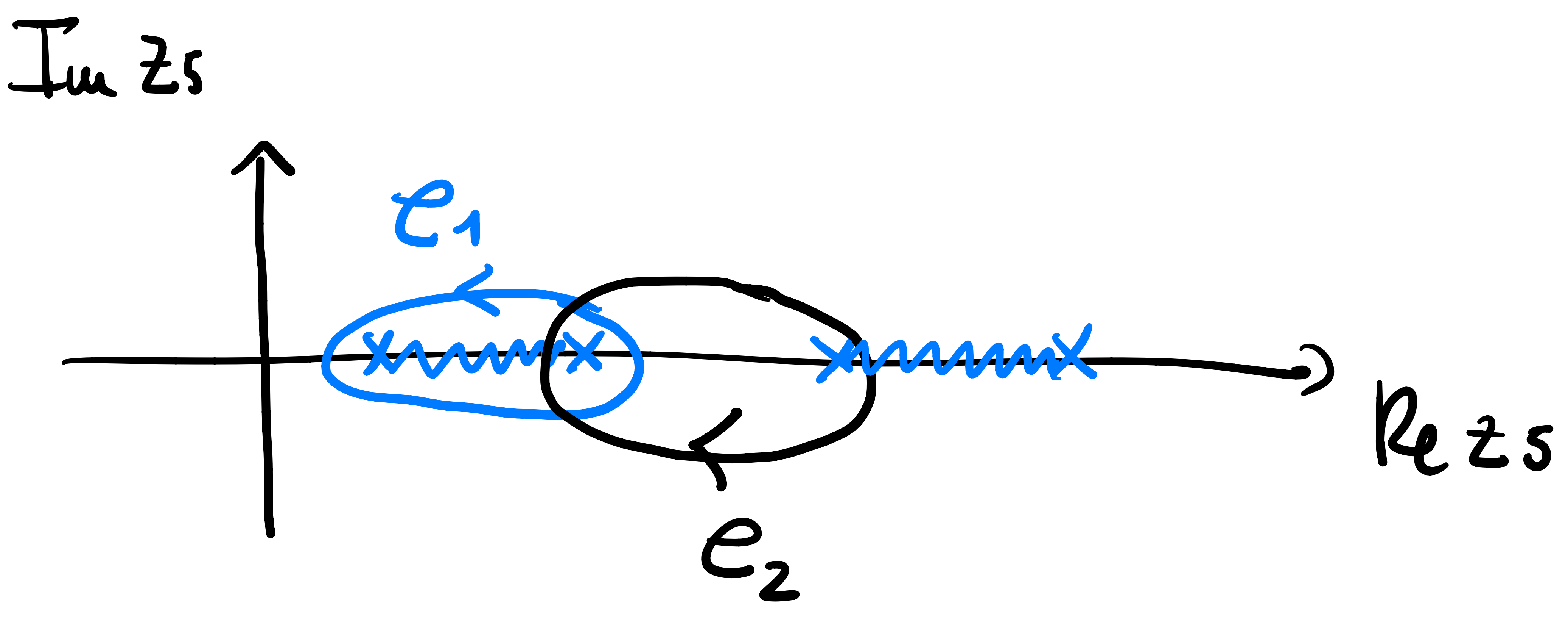
Polynomial of degree 4 in square-root → for $\nu_5 = 0$, there is **NO POLE** but ***two independent contours*** among 4 roots:



$$\int_{C_1} \frac{dz_5}{\sqrt{P_4(z_5)}} \propto \omega_0 \sim 1 + \sum_{n=1}^{\infty} c_n s^n$$ *1st kind integral*

<u>*locally*</u>, ***holomorphic solution*** ω_0 generalization of algebraic prefactor (no trascendental weight)

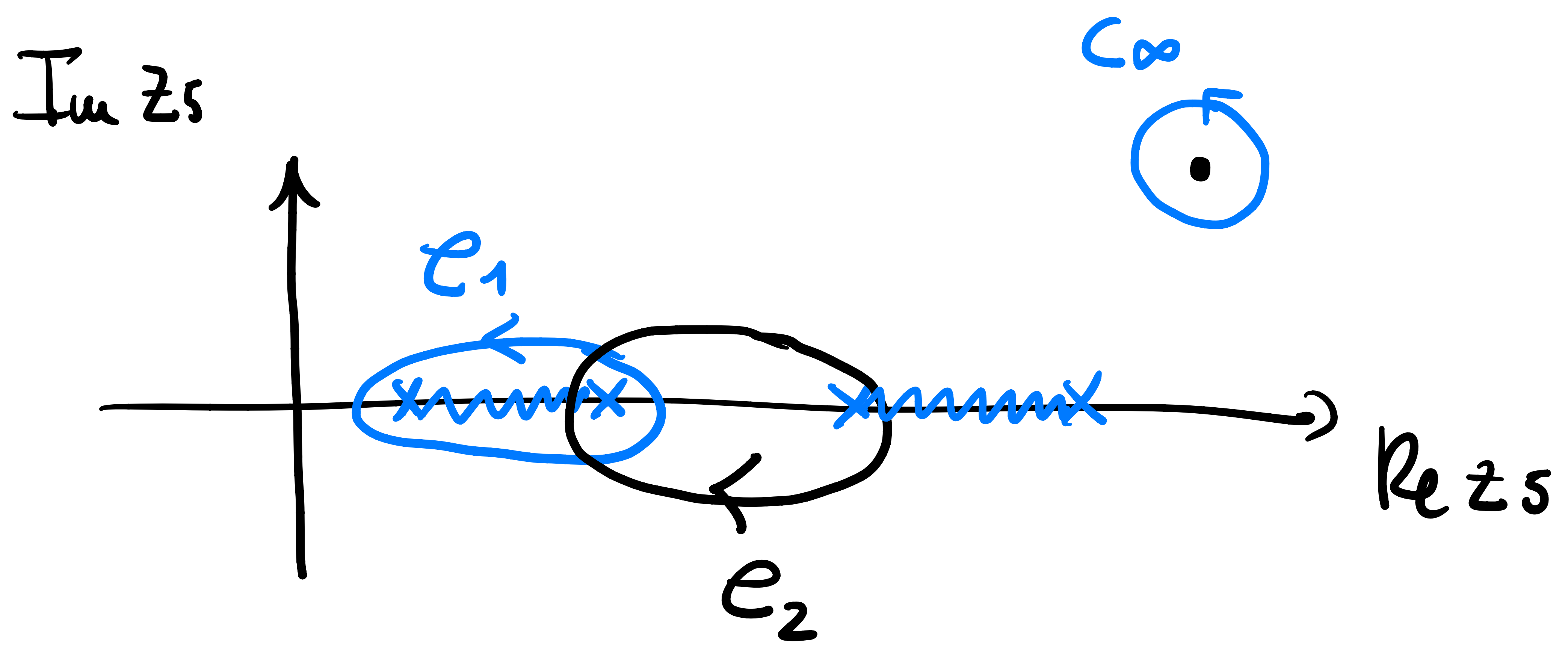
$\frac{1}{\omega_0} I_{1,1,1,0,0}$ generalization of integral with *unit leading singularities beyond logarithmic case*

ELLIPTIC CASE: THE INTEGRAND IN D=2

Second case: $M^2 \neq 0$

$$I_{1,1,1,0,\nu_5}\Big|_{z_1=z_2=z_3=0} \longrightarrow \int_\gamma \frac{dz_5\, z_5^{-\nu_5}}{\sqrt{(2z_5+s+M^2)(M^2 s - z_5^2)(4m^2-M^2-s-2z_5)}} \int \frac{d\log[f(z_4,z_5,m^2,s)]}{dz_4}dz_4$$

What happens increasing ν_5? For $\nu_5 = -1$ there is a ***single pole at infinity,*** now there are three contours



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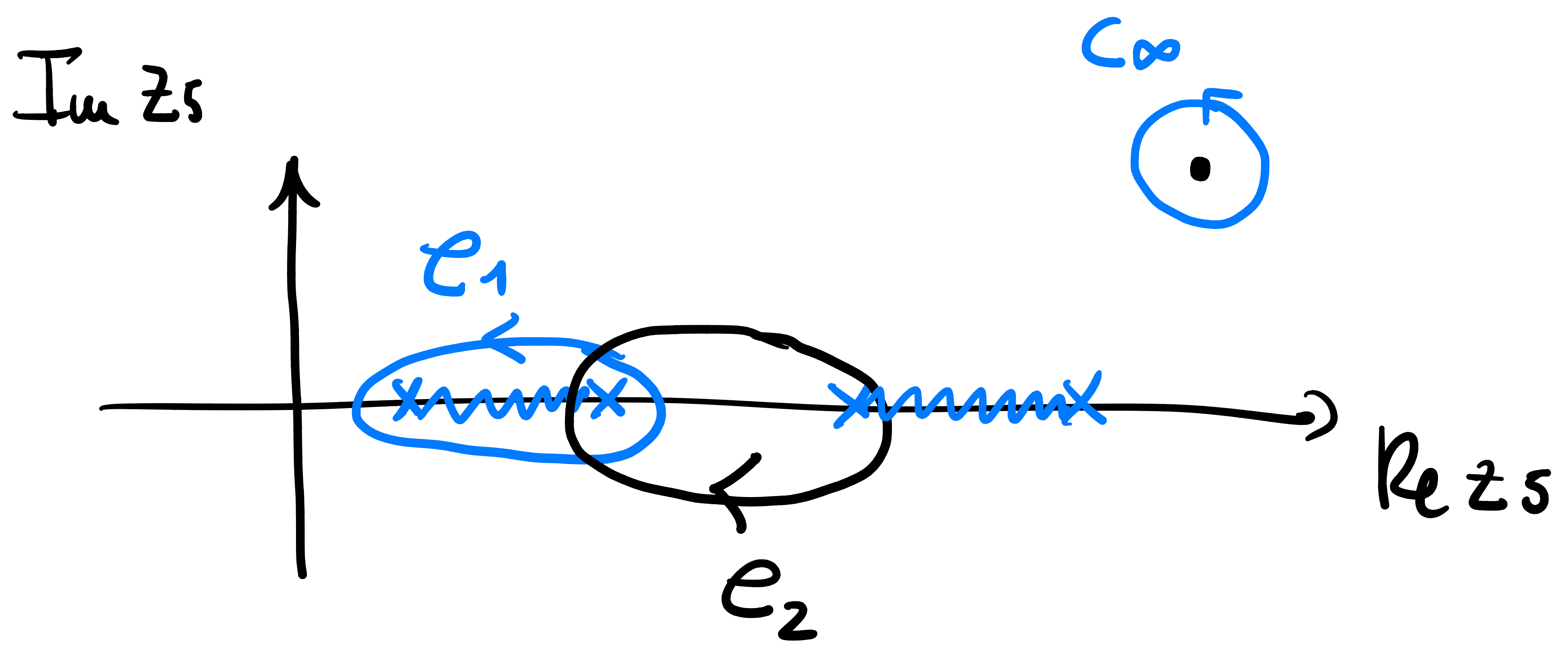

$$\int_{C_1} \frac{dz_5\, z_5}{\sqrt{P_4(z_5)}} \propto \Pi_0 \qquad \int_{C_2} \frac{dz_5\, z_5}{\sqrt{P_4(z_5)}} \propto \Pi_1$$ *3rd kind integrals*

$$\int_{C_\infty} \frac{dz_5\, z_5}{\sqrt{P_4(z_5)}} \propto \mathrm{Res}_\infty \left[\frac{z_5}{\sqrt{P_4(z_5)}}\right] \propto 1$$

Extra residue: it **decouples** from the others

$I_{1,1,1,0,-1}$ is a second *good integral, already normalized!*

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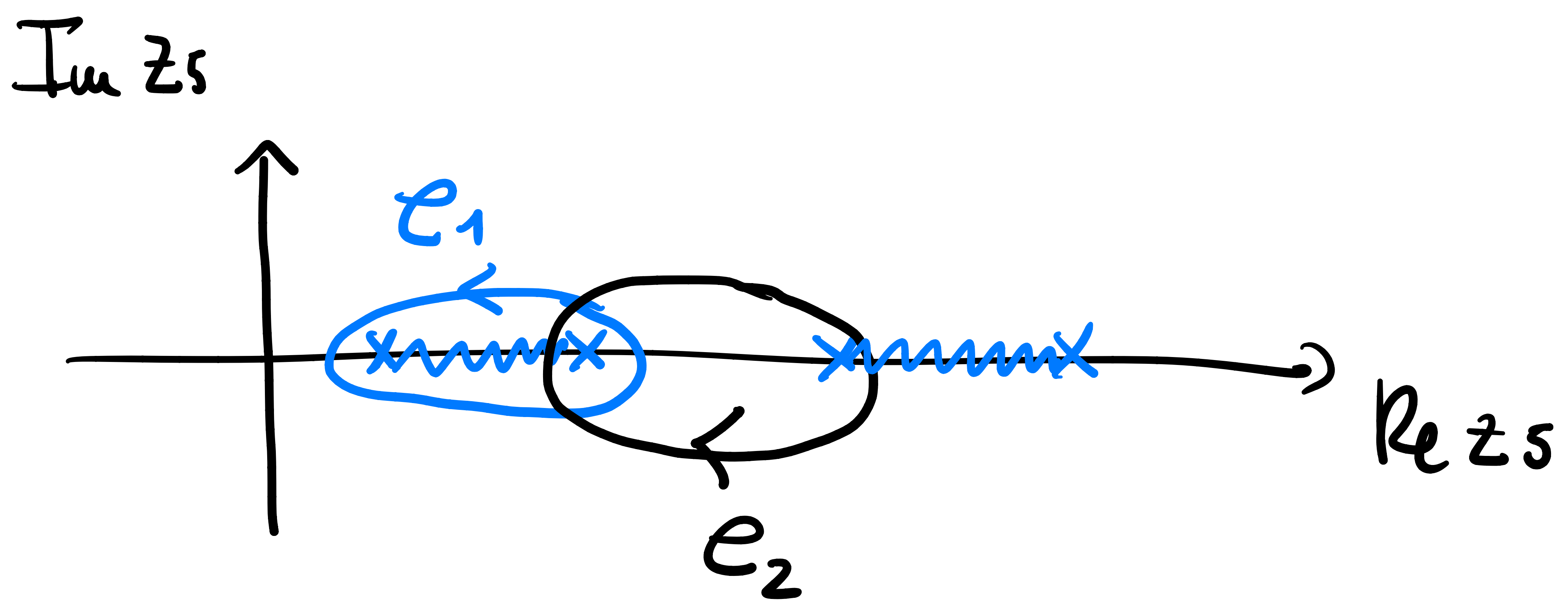
Can we increase ν_5 more? Contrary to polylog case $\nu_5 = -2$ is independent! ***Double pole at infinity (no residue)***



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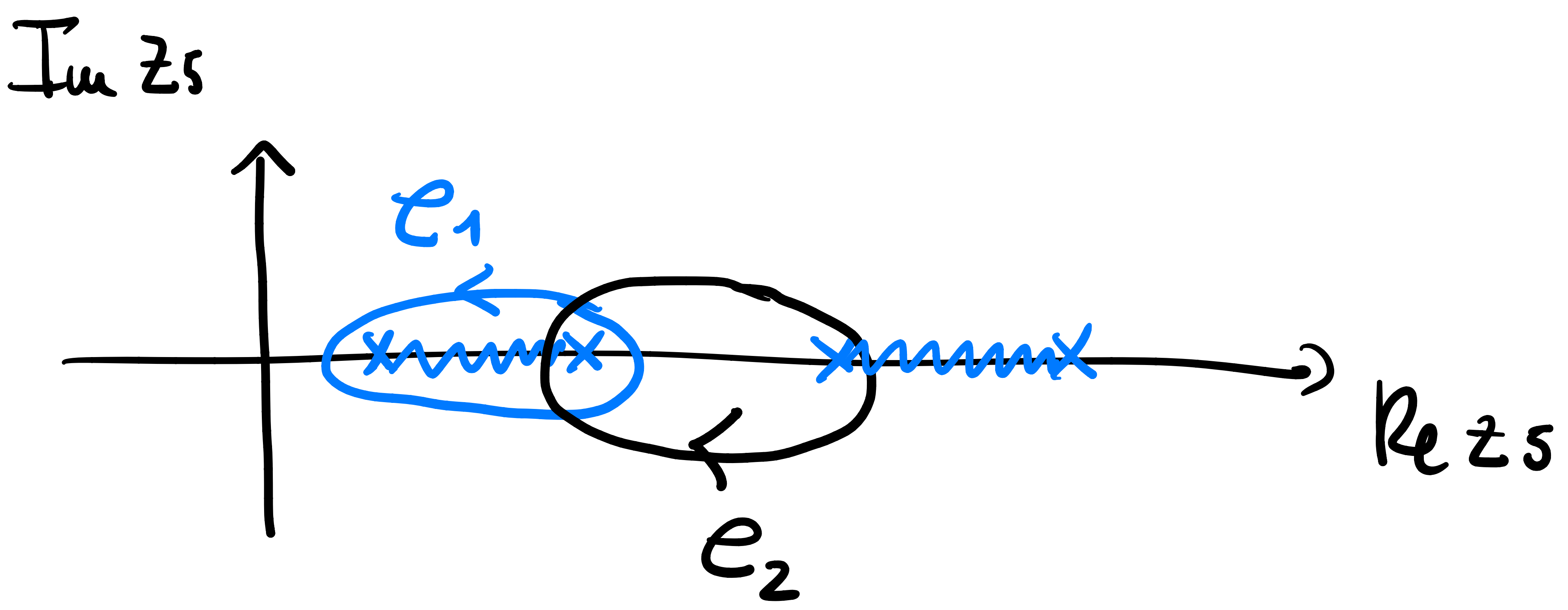

Subtlety: $\displaystyle \int_{C_1} \frac{dz_5\, z_5^2}{\sqrt{P_4(z_5)}} \to (z_5 \to 1/x)$

$$\to \int dx \Bigg[\underbrace{\frac{1}{2x^2}}_{\text{double pole}} - \underbrace{\frac{s + M^2 - 2m^2}{4x}}_{\text{contamination single pole}} + \mathcal{O}(x^0) \Bigg]$$

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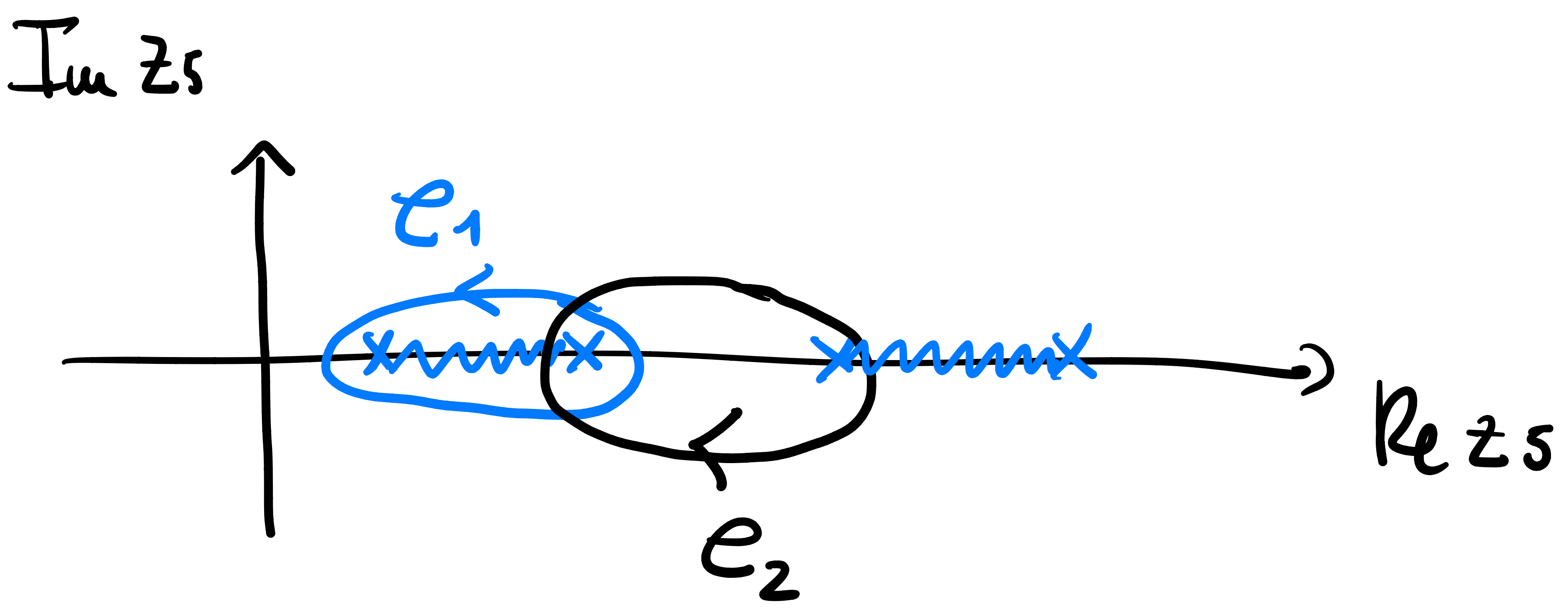

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Candidate with ***pure double pole*** in $D=2$

$$I_{1,1,1,0,-2} + \left[\frac{s+M^2-2m^2}{2}\right] I_{1,1,1,0,-1} + C_0\, I_{1,1,1,0,0}$$

ELLIPTIC CASE: A GOOD BASIS IN D=2

$$J_1 = \frac{1}{\omega_0} I_{1,1,1,0,0} = \frac{1}{\omega_0} \int \frac{dz_5}{\sqrt{P_4(z_5)}} \int \frac{d \log f(z_4, z_5, s, m^2)}{dz_4} dz_4$$

$$J_2 = I_{1,1,1,0,-2} + \left[\frac{s + M^2 - 2m^2}{2} \right] I_{1,1,1,0,-1} + C_0 \, I_{1,1,1,0,0} = \int \frac{dz_5}{\sqrt{P_4(z_5)}} \left(z_5^2 + \frac{s_1}{2} z_5 + C_0 \right) \int \frac{d \log f(z_4, z_5, s, m^2)}{dz_4} dz_4$$

$$J_3 = I_{1,1,1,0,-1} = \int \frac{dz_5 \, z_5}{\sqrt{P_4(z_5)}} \int \frac{d \log f(z_4, z_5, s, m^2)}{dz_4} dz_4$$

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?

Unfortunately, J_2 is **not right yet** to generalize the decomposition.

Double pole would generate ***extra "poles" in the special functions!***

Not just logarithmic singularities

$$\mathscr{A} \longrightarrow \sum_i R_i(s_{ij}) \boxed{\int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1}$$

ELLIPTIC CASE: A GOOD BASIS IN D=2-2ϵ

Serious problem:

second integral ***cannot easily be lifted to*** $D = 2 - 2\epsilon$ and give rise to a **"real canonical basis"**

$$J_1 = \int \frac{dz_5}{\sqrt{P_4(z_5)}} \int \frac{d \log f(z_4, z_5, s, m^2)}{dz_4} dz_4 \Big(G(z_4, z_5, m^2, M^2, s) \Big)^{k\epsilon}$$

GOOD as for polylogs

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BAD!?!?

Double pole requires "integration by parts", OK strictly in $D = 2$, "bad" in $D = 2 - 2\epsilon$

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INTERMEZZO: USING DERIVATIVES FOR POLYLOGS

Imagine we have found a perfectly "canonical" integral. It's expression will be

$$I = c^{(0)} + \epsilon \sum_k c_k^{(1)} \mathcal{I}_k^{(w=1)} + \epsilon^2 \sum_k c_k^{(2)} \mathcal{I}_k^{(w=2)} + \mathcal{O}(\epsilon^3)$$

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Consider now its derivative

$$\partial I \;\propto\; \epsilon \sum_k R_k\, c_k^{(1)} + \epsilon^2 \sum_k c_k^{(2)} \sum_j R_j\, \mathcal{I}_{k,j}^{(w=1)} + \mathcal{O}(\epsilon^3)$$

It generates a new ***uniform weight integral with lower weight***, ***not pure due to*** R_k

Not perfect, but **after an ϵ-rescaling**, can be transformed into a canonical integral by an **ϵ-independent rotation**

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For **polylogs** *we can live without* (but they can still be useful, see the **INITIAL** algorithm)

[Dlapa, Henn, Yan ’20]

ELLIPTIC CASE: DO WE NEED DERIVATIVES?

$$\cancel{J_2 = \int \frac{dz_5}{\sqrt{P_4(z_5)}} \left(z_5^2 + \frac{s_1}{2} z_5 + C_0 \right) \int \frac{d \log f(z_4, z_5, s, m^2)}{dz_4} dz_4 \Big(G(z_4, z_5, m^2, M^2, s) \Big)^{k\epsilon}}$$

→ trade by **derivative of first one**, with **full ϵ-dependence!**

$$J_2 \propto \partial \left[\int \frac{dz_5}{\sqrt{P_4(z_5)}} \int \frac{d \log f(z_4, z_5, s, m^2)}{dz_4} dz_4 \Big(G(z_4, z_5, m^2, M^2, s) \Big)^{k\epsilon} \right]$$

[part of Ansatz procedure by S. Weinzierl et al!]

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[part of Ansatz procedure by S. Weinzierl et al!]

At exactly $D = 2$ ***no difference*** with previous choice *(derivative completes cohomology without generating single poles)*

BUT derivative guarantees that ***when we turn on*** ϵ we can reach a <u>“generalized” canonical basis</u> by **an ϵ-independent rotation** *modulo overall rescaling due to weight drop*

IMPORTANT: up to this point, this is the <u>*only difference*</u> in our proposal versus [Chaubey, Sotnikov arXiv 2504.20897]

ELLIPTIC CASE: A CANONICAL BASIS IN D=2-2ϵ

Second integral ***still has double poles*** *(think about polylog integral before removing LS)*

$$J_2 \propto \partial \left[\int \frac{dz_5}{\sqrt{P_4(z_5)}} \int \frac{d\log f(z_4, z_5, s, m^2)}{dz_4} dz_4 \Big(G(z_4, z_5, m^2, M^2, s) \Big)^{k\epsilon} \right]$$

If we want to remove them, we must perform a "rotation" $\rightarrow$ ***defined locally close to a singular point***

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In this basis, third integral ***decouples*** and differential equations are:

$$\partial_z \begin{pmatrix} \mathrm{Cut}(\widetilde{I}_1) \\ \mathrm{Cut}(\widetilde{I}_2) \\ \mathrm{Cut}(\widetilde{I}_3) \end{pmatrix} = \left[A(\underline{z}) + \epsilon\, B(\underline{z}) + \epsilon^2\, C(\underline{z}) \right] \begin{pmatrix} \mathrm{Cut}(\widetilde{I}_1) \\ \mathrm{Cut}(\widetilde{I}_2) \\ \mathrm{Cut}(\widetilde{I}_3) \end{pmatrix} \qquad A(\underline{z}) = \begin{pmatrix} 0 & 1 & 0 \\ a_{21}(\underline{z}) & a_{22}(\underline{z}) & 0 \\ a_{31}(\underline{z}) & a_{32}(\underline{z}) & 0 \end{pmatrix}$$

$$\longrightarrow \qquad \partial_z \begin{pmatrix} \mathrm{Cut}(\widetilde{I}_1) \\ \mathrm{Cut}(\widetilde{I}_2) \end{pmatrix} = \left[\widehat{A}(\underline{z}) + \mathcal{O}(\epsilon) \right] \begin{pmatrix} \mathrm{Cut}(\widetilde{I}_1) \\ \mathrm{Cut}(\widetilde{I}_2) \end{pmatrix}, \quad \text{with} \quad \widehat{A}(\underline{z}) = \begin{pmatrix} 0 & 1 \\ a_{21}(\underline{z}) & a_{22}(\underline{z}) \end{pmatrix}$$

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Focus on 2×2 system at $D = 2$

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$$W = \begin{pmatrix} \varpi_0 & \varpi_1 \\ \partial_z \varpi_0 & \partial_z \varpi_1 \end{pmatrix}, \qquad \text{with} \qquad \partial_z W = \begin{pmatrix} 0 & 1 \\ a_{21}(\underline{z}) & a_{22}(\underline{z}) \end{pmatrix} W$$

Period matrix $=$ homogeneous solution

mixes **transcendental weight** ***(not UT)***

not pure

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Period matrix = homogeneous solution

mixes **transcendental weight** ***(not UT)***

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$$\left.\begin{aligned} \varpi_0(z) &= 1 + \sum_{j=1}^{\infty} c_j z^j \\ \varpi_1(z) &= \varpi_0(z) \log(z) + \sum_{j=1}^{\infty} d_j z^j \end{aligned}\right\}$$

$$\tau(z) = \frac{\varpi_1(z)}{\varpi_0(z)} = \log(z) + \mathcal{O}(z)$$

Transcendental weight 1
close to MUM point

$$\Delta = \det W$$

Algebraic function: weight 0

ELLIPTIC CASE: A GOOD BASIS IN D=2

A possible solution: split period matrix into ***semi-simple and unipotent part***

[Brödel, Duhr, Dulat, Penante, Tancredi, arXiv:1809.10698]

$$\underbrace{\begin{pmatrix} \varpi_0 & \varpi_1 \\ \partial_z \varpi_0 & \partial_z \varpi_1 \end{pmatrix}}_{W} = \underbrace{\begin{pmatrix} \varpi_0 & 0 \\ \partial_z \varpi_0 & \frac{\Delta}{\varpi_0} \end{pmatrix}}_{W^{\mathrm{ss}}} \underbrace{\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}}_{W^{\mathrm{u}}}$$

(splitting becomes unique requiring special form of W^u)

W^{ss} is algebraic: generalizes LS, must be rotated away

W^u is transcendental, weight 1 as a logarithm

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W^{ss} is algebraic: generalizes LS, must be rotated away

W^u is transcendental, weight 1 as a logarithm

$$\partial_z W^{\mathrm{u}} = \begin{pmatrix} 0 & \partial_z \tau \\ 0 & 0 \end{pmatrix} W^{\mathrm{u}} = \begin{pmatrix} 0 & \frac{\Delta}{\varpi_0^2} \\ 0 & 0 \end{pmatrix} W^{\mathrm{u}}$$

Unipotent part fulfils <u>*generalized dlog-equation*</u>

dlog not properly multiplied by ϵ

$$\mathrm{d}\tau \xrightarrow{M \to 0} \mathrm{dlog}\left(\frac{s - \sqrt{s(s-4m^2)}}{s + \sqrt{s(s-4m^2)}}\right) \Bigg\}$$

Second integral has "weight drop", ***rescale by*** $\frac{1}{\epsilon}$

Proved that splitting produces same result as ***Ansatz procedure*** by S. Weinzierl et al

[Duhr, Maggio, Nega, Sauer, Tancredi, Wagner arXiv:2503.20655]

A CANONICAL BASIS?!

After ***splitting*** and some ***minor clean up***, integrals fulfil ***"generalized" canonical differential equations:***

Analytic structure manifest in terms of a set of ***independent differential forms*** with ***at most single poles***

$$\mathrm{d}\vec{J} = \epsilon \left(\sum_i G_i \, \omega_i \right) \vec{J} \qquad \longleftrightarrow \qquad f_i(x)\mathrm{d}x = \omega_i$$

NOTE:

Differential equations degenerate to standard ***dlog canonical equations*** close to ***singular points of elliptic curve***
$\rightarrow$ for this to happen without ***non-trivial ϵ-rotation***, ***it is crucial to have chosen derivative as second-kind form!***

Some differential forms which can be associated to **form of second kind** **drop from Amplitudes at $\mathcal{O}(\epsilon^0)$**

AN ELLIPTIC AMPLITUDE: TOP CORRECTIONS TO $pp \to \gamma\gamma$

$q\bar{q} \to \gamma\gamma$ and $gg \to \gamma\gamma$ mediated by a top quark ***perfect laboratory***:

- realistic amplitude of elliptic type (studied only numerically before us)

[Becchetti, Bonciani, Cieri Coro, Ripani '23]

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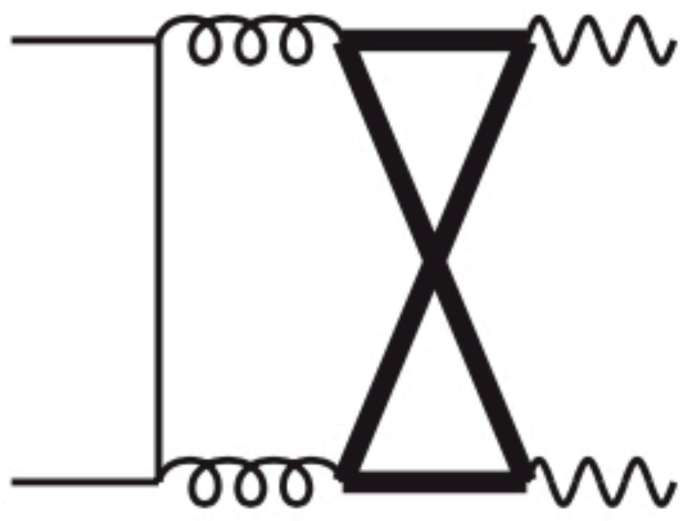

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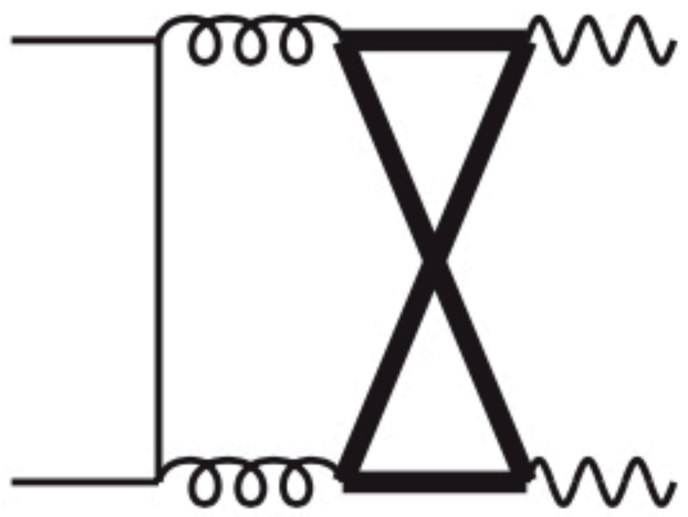

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- ***not too complex*** algebraically but still ***rich in physical and mathematical features in NPL graphs***

[Becchetti, Coro, Nega, LT, Wagner '25]

See also [Ahmed, Chakraborty, Chaubey, Kaur, Maggio '24, '25]

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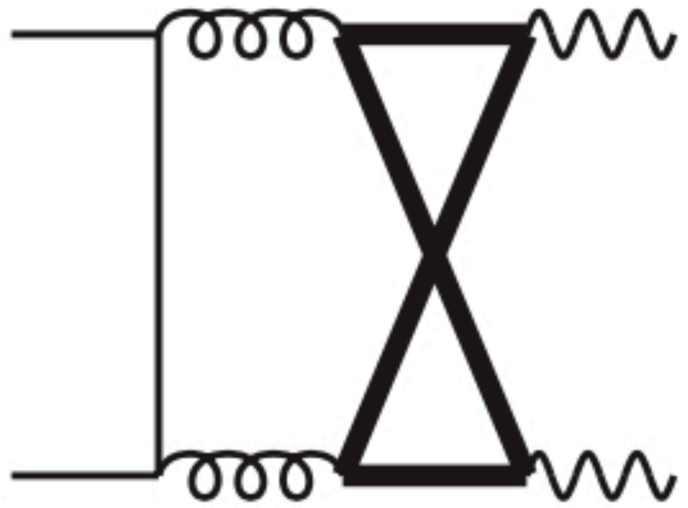

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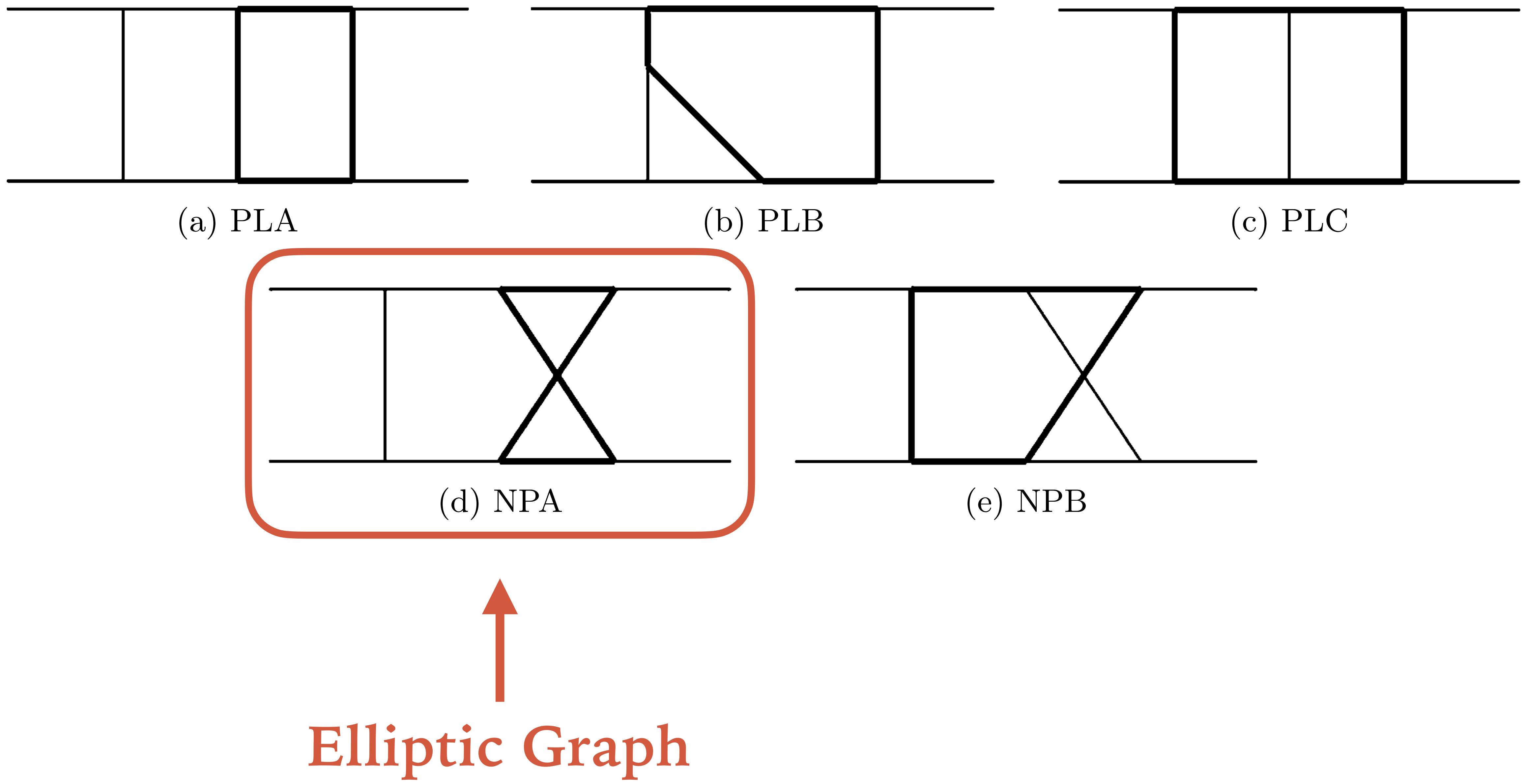
[Becchetti, Bonciani, Cieri Coro, Ripani '23]

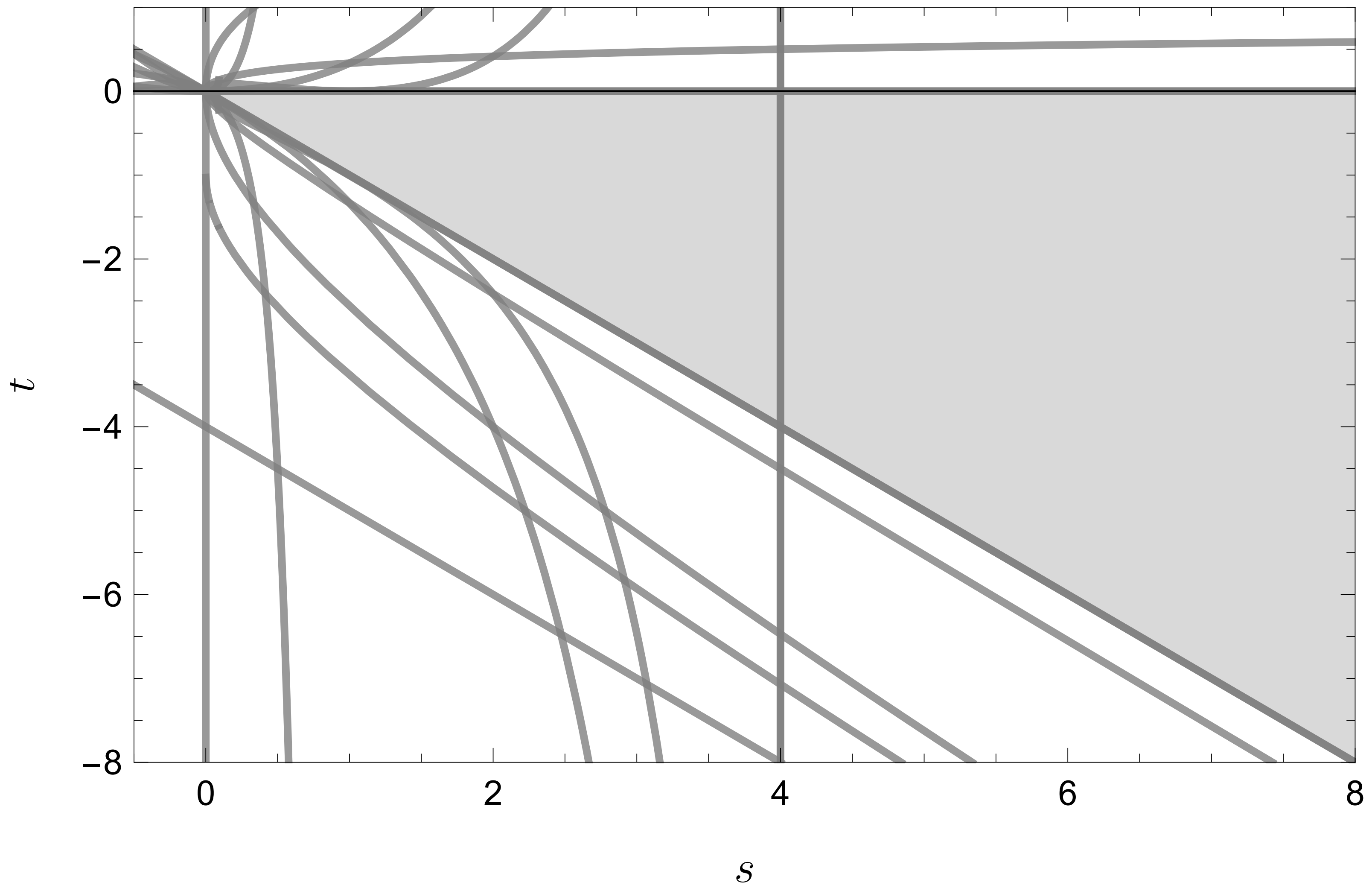
[Maltoni, Mandal, Zhao '18]

- ***not too complex*** algebraically but still ***rich in physical and mathematical features in NPL graphs***

[Becchetti, Coro, Nega, LT, Wagner '25]

See also [Ahmed, Chakraborty, Chaubey, Kaur, Maggio '24, '25]





THE ELLIPTIC DOUBLE BOX: SELECTING A GOOD BASIS

Construction of basis of master integrals mapped to “right” differential forms on the elliptic curve

Total of 4 master integrals on the maximal cut

[Gorges, Nega, LT, Wagner ’23]
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$$\begin{aligned}&\mathrm{MaxCut}\left[(m^2-t)\mathcal{I}_{\mathrm{NPA}}(1,1,1,1,0,1,1,1,0)-\mathcal{I}_{\mathrm{NPA}}(1,1,1,1,0,1,1,1,-1)\right]\\&\quad\propto\frac{1}{s}\int\mathrm{d}z_5\mathrm{d}z_9\frac{(m^2-t-z_9)}{P_{2,3}(z_5,z_9)}=\frac{1}{s}\int\frac{\mathrm{d}z_9}{\sqrt{P_4(z_9)}}\int\mathrm{d}\log\left(\frac{1+f(z_5,z_9)}{1-f(z_5,z_9)}\right)\end{aligned}$$

Form of first kind

+ its derivatives for second kind

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single poles:

two forms of 3rd kind

NUMERICAL RESULTS: TOP CORRECTIONS TO $pp \to \gamma\gamma$

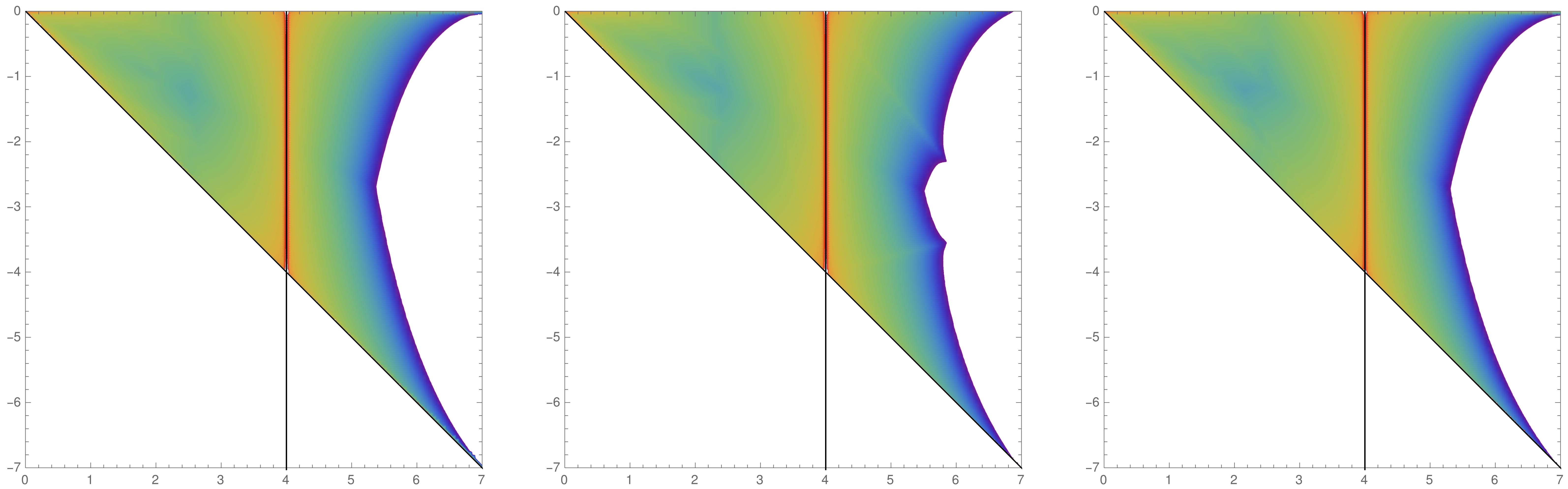
From analytic representation, we can obtain <u>*few*</u> *fast converging series expansions for numerical evaluation:*

With only 2 series, reliable numerical evaluation across large portion of phase space ***due to cancellation of unphysical singularities in full amplitude***

$$A_{gg}^{++++} = \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} f_{++++}(x,y)\,,$$

$$A_{gg}^{--++} = \frac{\langle 12\rangle[34]}{[12]\langle 34\rangle} f_{--++}(x,y)\,,$$

$$A_{gg}^{-+-+} = \frac{\langle 13\rangle[24]}{[13]\langle 24\rangle} f_{-+-+}(x,y)\,,$$



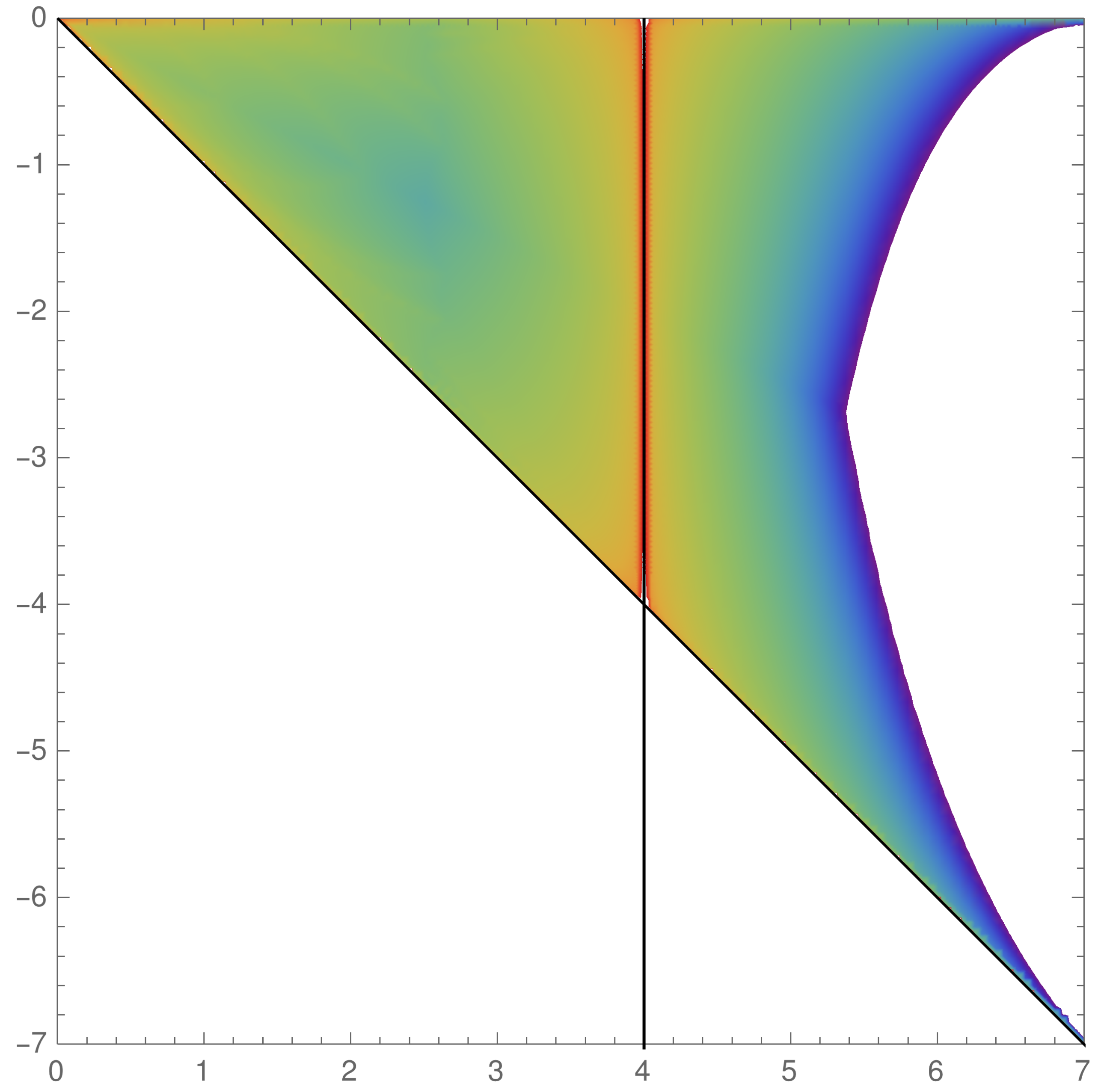
(a) $f^{(2,b)}_{++++}$ (b) $f^{(2,b)}_{--++}$ (c) $f^{(2,b)}_{-+-+}$

[Becchetti, Coro, Nega, LT, Wagner ’25]

NUMERICAL RESULTS: TOP CORRECTIONS TO $pp \to \gamma\gamma$

Preliminary: extend the convergence using "Bernoulli-like" variables in 2 dimensions

[Becchetti, Coro, Nega, LT, Wagner *to appear soon*]



(a) $f^{(2,b)}_{++++}$

roughly

$$x \propto \log\left(1 + \frac{s - 4m^2}{4m^2}\right)$$

push $s = 0$ singularity to infinity

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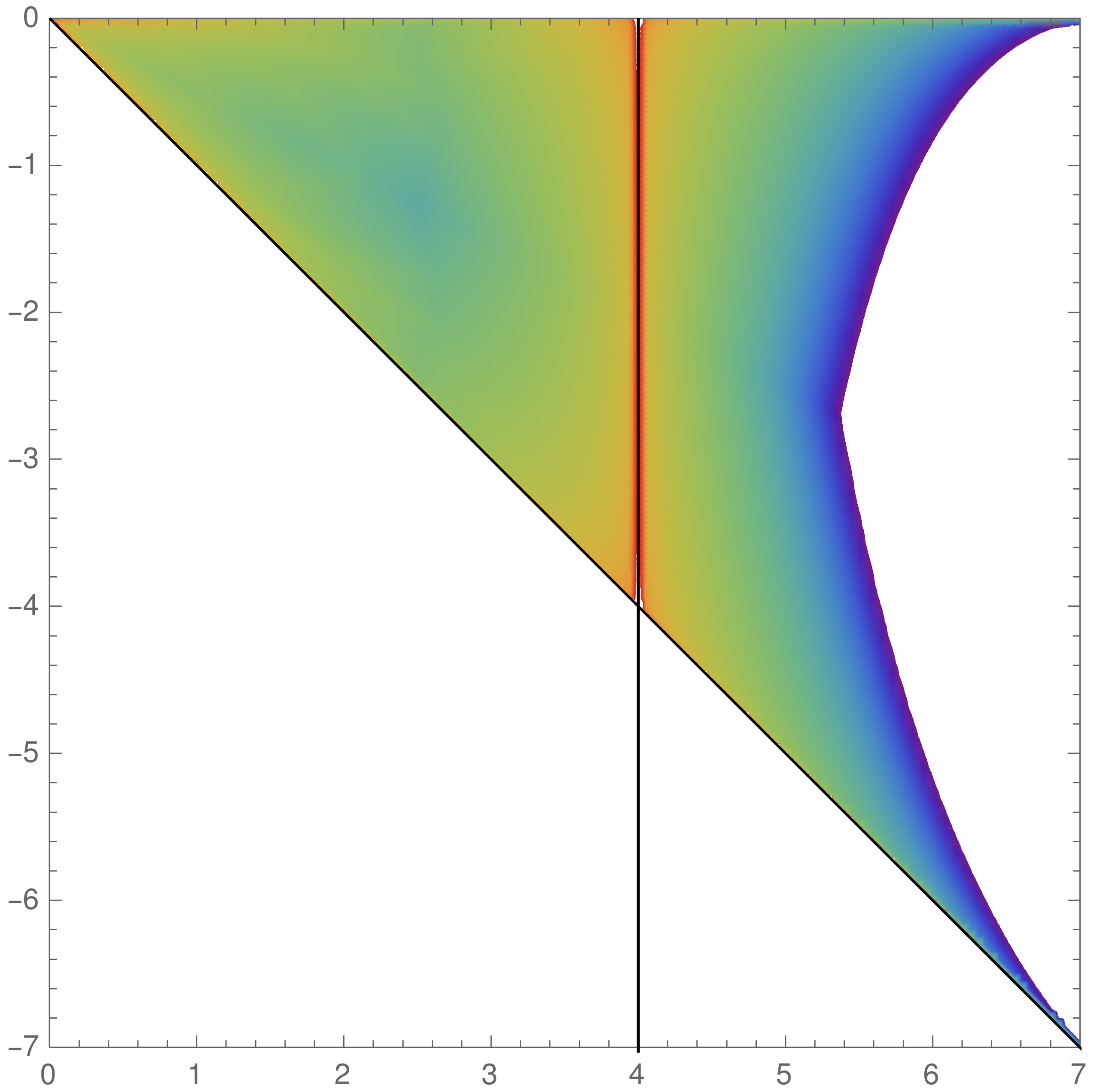

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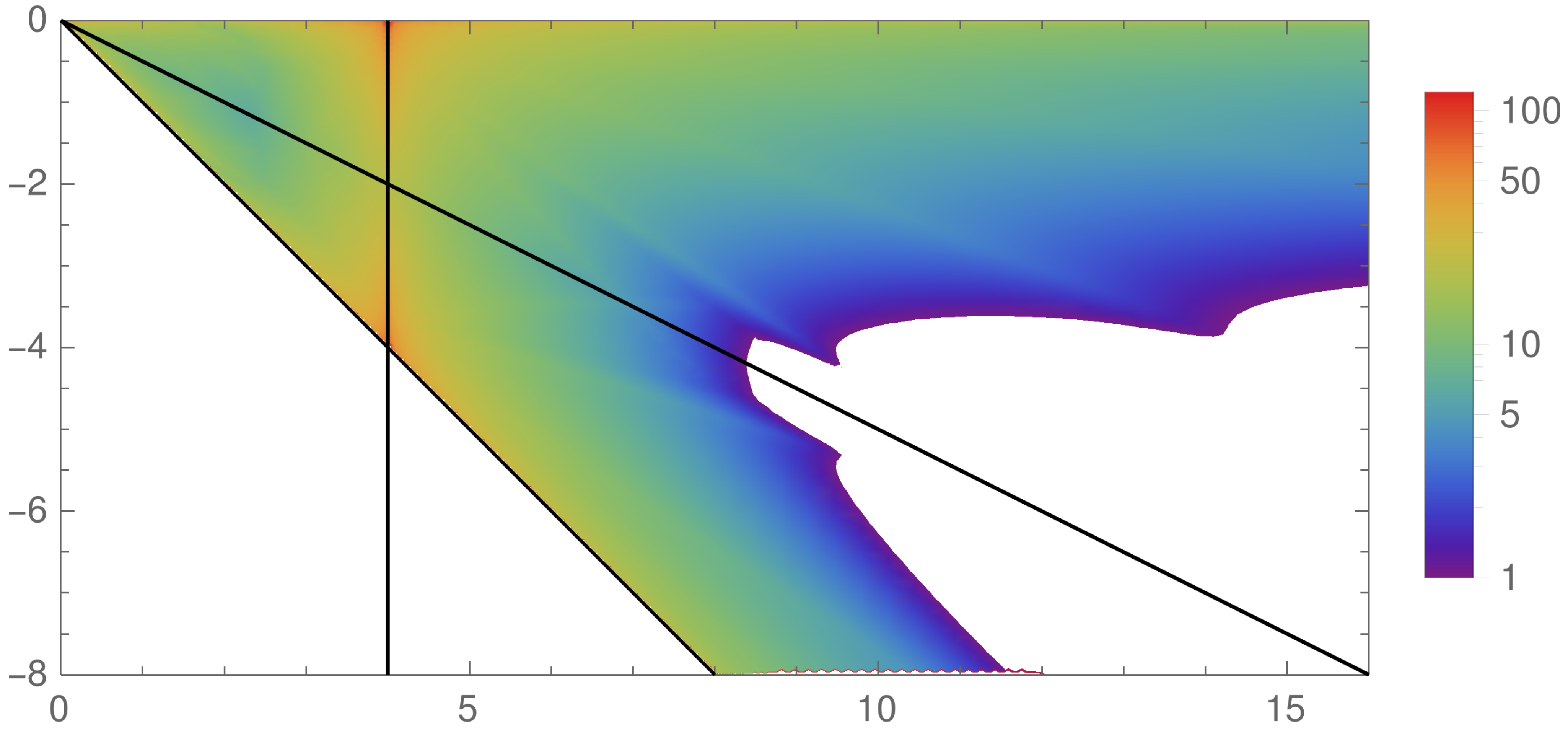
Preliminary!!!!!!



push convergence to very high values of s

small mass expansion nicely complements the rest

CONCLUSIONS

- Elliptic amplitudes are **fundamental building blocks in QFT**, for precision collider physics and beyond
- Controlling them "analytically" requires understanding relations among integrals, analytic continuation, and being able to evaluate them numerically *(i.e. doing series expansions, see Matthias' talk)*
- Choosing **good integrals** to make analytic structure manifest has been **fundamental to solve many polylogarithmic problems**
- I described today a ***path towards the generalization of those ideas to elliptic amplitudes and beyond***
- Thanks to these developments, ***first "fully analytic" results obtained for elliptic amplitudes and more!***

CUTTING-EDGE PROBLEMS ADDRESSED

Description	References	Geometry
Equal-mass banana graphs	[41], this paper	CY 2-, 3- and 4-folds
Single scale triangle graphs	[41]	Elliptic curve
3-loop corrections to the electron and photon self-energies in QED	[58, 59]	Sunrise elliptic curve, banana K3 surface
3- and 4-loop ice cone integrals	[41], this paper	Two copies of sunrise elliptic curve and banana K3 surface
Deformed CY operators	this paper	CY 2-, 3- and 4-folds
Equal-mass banana graphs with one massless propagator	unpublished	CY 1-, 2-, 3-folds
Gravitational scattering at 5PM-1SF	[60, 61, 162]	Sym. square of Legendre curve, CY 3-fold AESZ 3
Gravitational scattering at 5PM-2SF	this paper	Apéry family of K3 surfaces, CY 3-fold

Generic three-mass sunset	[41]	Elliptic curve
2-loop 3-point integrals for $gg \to H$	[170]	Two-mass sunrise elliptic curve
2-parameter triangle graph	[41]	Elliptic curve
2-loop 4-point integrals for Bhabha and Møller scattering	unpublished	Elliptic curve
2-loop 4-point integrals for diphoton	[56]	Elliptic curve
2-loop 4-point acnode integral (diagonal box)	unpublished	Elliptic curve
3-parameter double box	[41]	Elliptic curve
2-loop 5-point integrals for $t\bar{t}$+jet	[57]	Elliptic curve
3-loop two-mass banana graph	unpublished	K3 surface
4-loop two-mass banana graph	unpublished	CY 3-fold
Maximal cut of a non-planar double box	[62]	Hyperelliptic curve of genus 2

THANK YOU!

BACK-UP SLIDES

RECAP: CONSTRUCTION OF CANONICAL BASES BEYOND POLYLOGS

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[Görges, Nega, LT, Wagner '23] [Duhr, Maggio, Nega, Sauer, LT, Wagner '25]

1) For each geometry, identify the master integrals corresponding to the **form of the first kind**

This is a ***differential form without poles*** (holomorphic) — In ***elliptic case*** $\rightarrow \int_C \frac{dx}{y}$ $\qquad y = \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}$

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   These are ***differential forms with higher poles*** — In ***elliptic case***, just one with a double pole $\rightarrow \int_C dx\, \frac{x^2}{y} \sim \partial \int_C \frac{dx}{y}$

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3) Identify all master integrals corresponding to the forms of the third kind

Special: like "dlogs", punctures, ***differential forms with single poles*** — In *elliptic case* → $\left[\int_C dx\,\frac{x}{y},\quad \int_C dx\,\frac{1}{(x-c)y}\right]$

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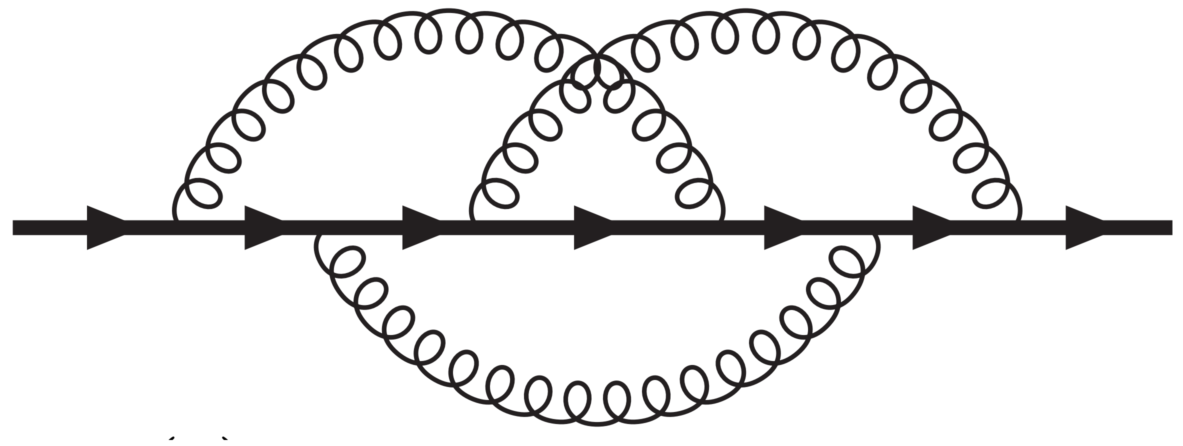
4) ***Locally*** close to a singular point: rotate away the ***semi-simple part*** + ***clean up*** for a **full ϵ-factorization**

A TWO-POINT CORRELATOR: THE THREE-LOOP QED SELF-ENERGY

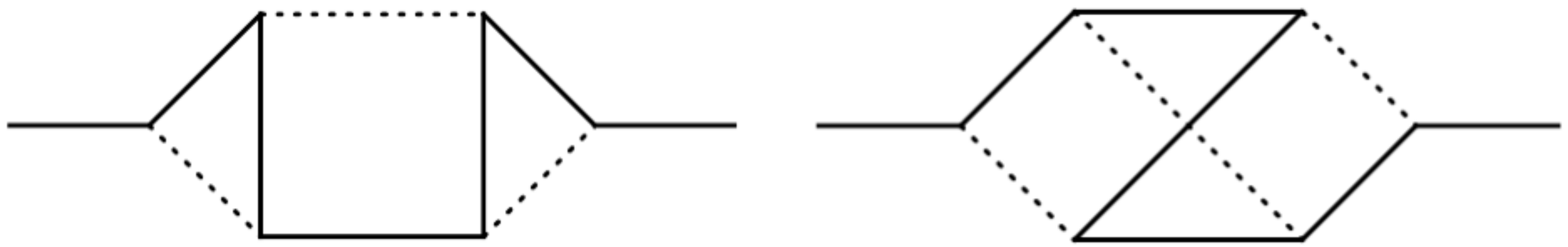
[Duhr, Gasparotto, Nega, Tancredi, Weinzierl ’24]



$$\hat{p}\ \Sigma_V(p^2, m^2) + m\ \Sigma_S(p^2, m^2)$$

Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

2 “top graphs”



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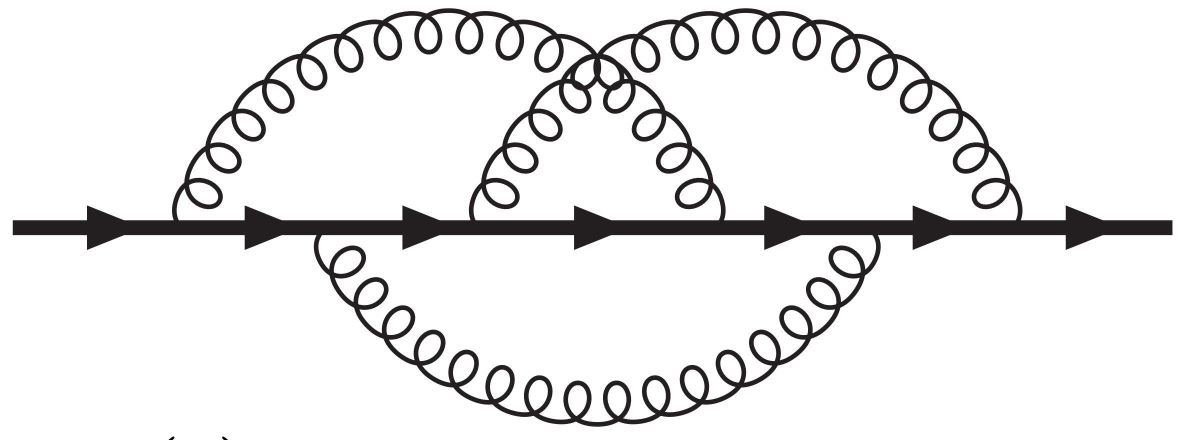

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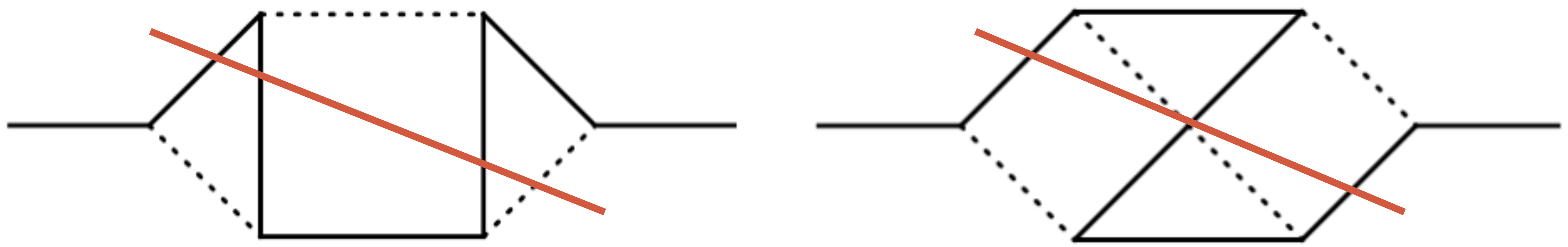
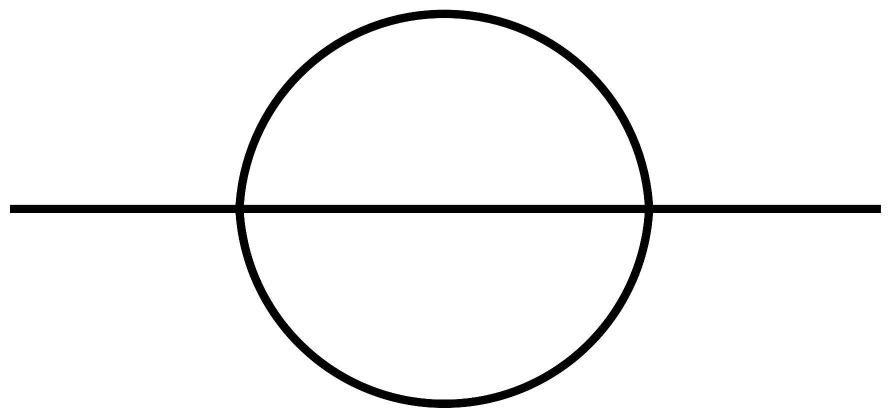


mix of elliptic and polylogarithmic sectors

same elliptic curve as 2loop sunrise graph



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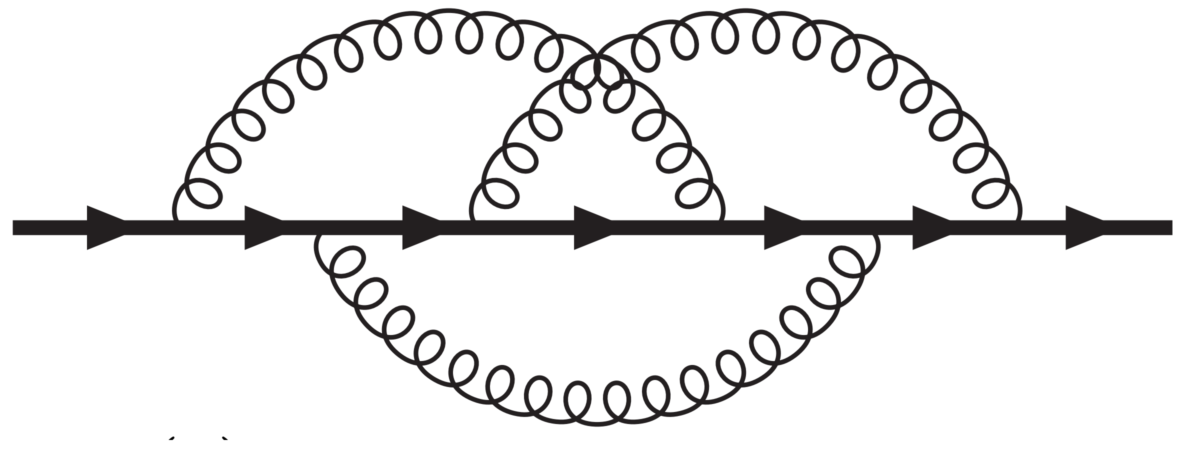

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Following prescription described before:

$$\mathrm{d}\vec{J} = \epsilon\left(\sum_i G_i\,\omega_i\right)\vec{J} \longleftrightarrow f_i(x)\mathrm{d}x = \omega_i$$

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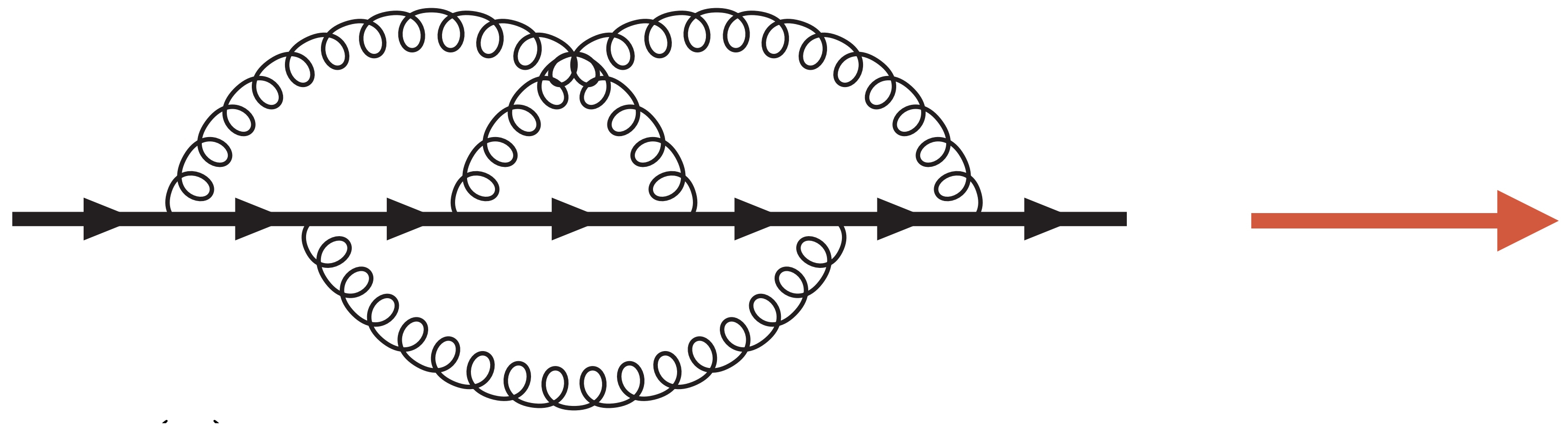

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Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

7 (independent) elliptic differential forms: full analytic control over **iterated integrals** over these forms

$$f_i \in \left\{ \frac{1}{x(x-1)(x-9)\varpi_0(x)^2}, \varpi_0(x), \frac{\varpi_0(x)}{x-1}, \frac{(x-3)\varpi_0(x)}{\sqrt{(1-x)(9-x)}}, \frac{(x+3)^4\varpi_0(x)^2}{x(x-1)(x-9)}, \frac{(x+3)(x-1)\varpi_0(x)^2}{x(x-9)}, \frac{\varpi_0(x)^2}{(x-1)(x-9)} \right\} \quad \text{for } i = 10, \ldots, 16\,,$$

$\varpi_0(x)$ is the first elliptic period

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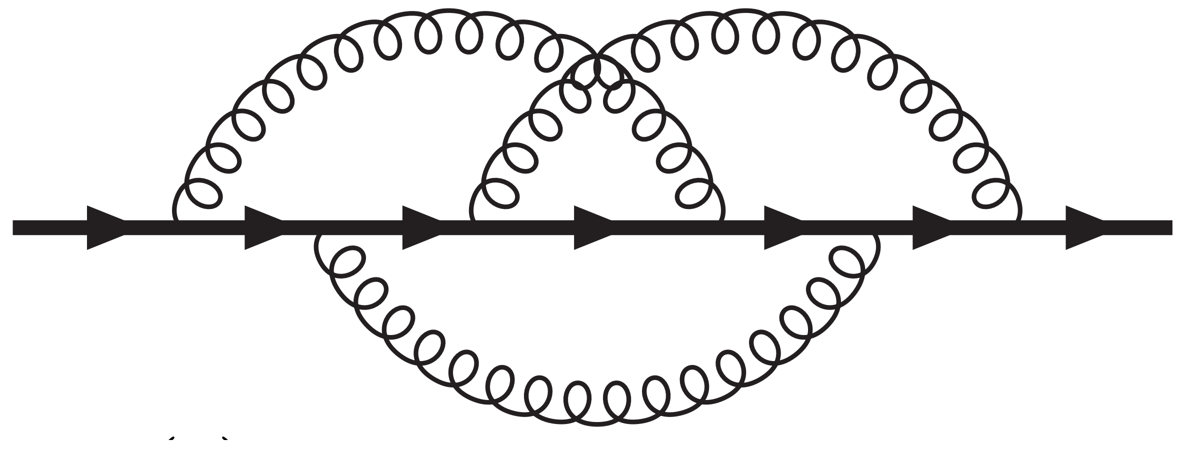
[Duhr, Gasparotto, Nega, Tancredi, Weinzierl ’24]



$$\longrightarrow \quad \hat{p}\,\Sigma_V(p^2,m^2) + m\,\Sigma_S(p^2,m^2)$$

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$$\left. \cancel{\frac{(x+3)(x-1)\varpi_0(x)^2}{x(x-9)}},\ \cancel{\frac{\varpi_0(x)^2}{(x-1)(x-9)}} \right\}$$

3 of the kernels drop in the physical amplitude: they are related to forms of the second kind with “double poles” → a hint for bootstrap program?