

Gradient-flowed Operator Product Expansion without Infrared Renormalons

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MB, H. Takaura, work in progress



External current correlation functions

$$\Pi(Q) = i \int d^4x e^{iqx} \langle J(x) J(0) \rangle$$

- Electromagnetic or electroweak currents $\bar{\psi} \Gamma \psi$
- $\rangle = |\Omega\rangle$, or $|\bar{B}\rangle$ or $|\text{nucleon}\rangle$ (DIS)
- $Q \gg \Lambda_{\text{QCD}}$ large momentum or heavy quark mass $\rightarrow$ perturbative expansion

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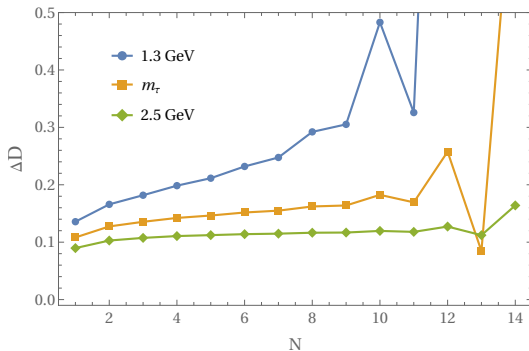
Adler function (flavour non-singlet vector current, vacuum, $n_f = 3$ massless flavours)

$$\begin{aligned} D(Q^2) &= Q^2 \frac{d\Pi}{dQ^2} = \frac{N_c}{12\pi^2} \left[1 + \sum_{n=1}^{\infty} d_n a_Q^n \right] \quad (a_Q = \alpha_s(Q)/\pi) \\ &= \frac{N_c}{12\pi^2} \left[1 + a_Q + 1.64a_Q^2 + 6.37a_Q^3 + 49.08a_Q^4 + \dots \right] \end{aligned}$$

[5-loop d_4 : Baikov, Chetyrkin, Kühn, 2008]

Adler function, perturbative approximation

$$\Delta D(Q^2) \equiv \sum_{n=1}^N d_n a_Q^n$$



Asymptotic behaviour in dimensional regularization

Factorial growth, zero radius of convergence.

Several components of factorial divergence of form

$$c_n \stackrel{n \gg 1}{\sim} \alpha_s^{n+1} K (a\beta_0)^n n! n^b \left(1 + \frac{s_1}{n} + O(1/n^2)\right)$$

Borel transform

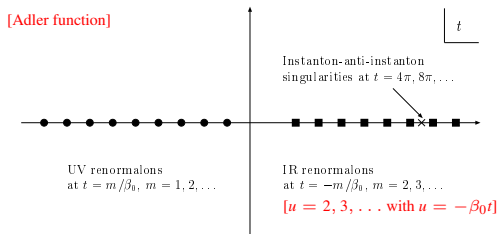
$$F = \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} \implies B[F](t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \implies F(\alpha) = \int_0^{\infty} dt e^{-t/\alpha} B[F](t)$$

$$c_n = K a^n \Gamma(n+1+b) \iff B[F](t) = \frac{K \Gamma(1+b)}{(1-at)^{1+b}}.$$

Minimal term at $n \approx 1/(|a\beta_0|\alpha_s(Q))$ of size

$$\Delta \approx e^{-1/(|a\beta_0|\alpha_s(Q))} \approx \left(\frac{\Lambda^2}{Q^2}\right)^{1/a} \approx \text{size of ambiguity}$$

Renormalon (and instanton) singularities



- **UV renormalons** – from large loop momentum
Sign-alternating, singularity structure related to higher-dim operators in the cut-off QCD Lagrangian [Parisi, 1977; MB, Kivel, Braun 1997]
- **IR renormalons** – from small loop momentum
Fixed-sign, singularity structure related to higher-dim operators in the OPE [Gross, Neveu, 1974; Lautrup, 1977; 't Hooft, 1977; David, 1984; Mueller, 1985; Zakharov, 1992; MB, 1993]
- **Instanton-anti-instanton** – number of diagrams
[Bogomolny, Fadeev, 1977; Balitsky, 1991]

IR renormalons and the operator product expansion (OPE)

Large momentum/small coupling expansion of the Adler function:

$$\begin{aligned}
 D(Q^2) &= Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \sum_{k=0} \underbrace{c_k(\alpha_s(Q))}_{\text{perturbative series}} \times \frac{1}{(Q^2)^k} \times \underbrace{\langle \mathcal{O}_k \rangle}_{\substack{\text{condensates } (k > 0) \\ \text{"power corrections"}}} \\
 &= \sum_{k=0} \underbrace{\left[\frac{\langle \tilde{\mathcal{O}}_k \rangle}{\Lambda_{\text{QCD}}^{2k}} \right]}_{\text{pure number}} \times \left[e^{-\frac{1}{(-\beta_0)\alpha_s(Q)}} \right]^k (-\beta_0\alpha_s(Q))^{k\beta_1/\beta_0^2 - \gamma_{0,k}/\beta_0} \times \sum_{n=0} c_k^{(n)} \alpha_s(Q)^n
 \end{aligned}$$

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 \end{aligned}$$

Transseries structure: Series $c_k^{(n)}$ divergent and not Borel-summable (IR renormalons) but the entire double expansion is unambiguous.

- Condensates must be ambiguous in dim reg and their values related to the summation prescription (or vice versa!) [Can be checked in the 2d O(N) non-linear sigma model, David, 1984, 1986; MB, 1998, MB, Braun, Kivel, 1998; Marino et al., 2024, 2025]
- But in QCD cannot do nonperturbative computations in dim reg
- Ambiguities related to **UV power divergences** of local operators

Asymptotic behaviour and cut-off / Wilsonian OPE

Main obstruction from IR renormalons

Cut-off factorization [advocated by Novikov, Shifman, Vainshtein, Zakharov (1984, 1985)]

$$D(Q^2) = \sum_{k=0} \left[\frac{\langle \mathcal{O}_k \rangle(\Lambda_f)}{Q^{2k}} \right] \times \sum_{n=0} c_k^{(n)}(\Lambda_f, Q) \alpha_s(Q)^n$$

- Restricts loop momenta to $k > \Lambda_f$, where $\Lambda_f \gg \Lambda_{\text{QCD}}$ is a hard factorization scale.
- Perturbative series ($c_k^{(n)}(\Lambda_f, Q)$) don't have IR renormalon factorial divergence.
- Condensates unambiguous but contain terms proportional to Λ_f^{2k} from power UV divergences.

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Has never been used in practice: Calculation with hard cut-offs too difficult beyond one loop. No gauge invariant practical implementation of a hard cut-off, but a gauge-invariant regulator is crucial.

Here: continuum version (“Wilson flow” on lattice)

Define “flowed” gluon field $B_\mu(t, x)$ by

$$\partial_t B_\mu = \tilde{D}_\nu \tilde{G}_{\nu\mu} + \xi_0 \tilde{D}_\mu \partial_\nu B_\nu, \quad B_\mu|_{t=0} = A_\mu$$

t = flow “time”, $\tilde{G}_{\mu\nu}$, $\tilde{D}_\mu$ usual definitions but with B_μ .

Interpretation: Smeared gluon field over distance $\sqrt{8t}$. LO solution

$$B_\mu(t, x) = \int d^d y K(t, x - y) A_\mu(y) \quad K(t, z) = \frac{e^{-z^2/(4t)}}{(4\pi t)^{d/2}}$$

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Key idea: $1/\sqrt{8t}$ provides a gauge-invariant, Wilsonian cut-off. Formulate OPE in terms of gradient-flow regularized local operators at finite flow time $\Lambda_{\text{QCD}} \ll 1/\sqrt{8t} \ll Q$

- Well-defined condensates and short-distance coefficients: Divergent series from IR renormalon disappear
- Can be done non-perturbatively in the continuum and on the lattice ($a \ll \sqrt{8t} \ll L$). Decouples continuum limit from power divergences and factorization

$$D(Q) = C_0(\alpha_s, Q/\mu) + C_{GG}(\alpha_s, Q/\mu) \frac{1}{Q^4} \langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu) + O(1/Q^6)$$

[C_0 known to $\mathcal{O}(\alpha_s^4)$, see above, C_{GG} to $\mathcal{O}(\alpha_s^2)$ (Harlander 1998)]

Any definition / subtraction of the divergent perturbative series implies a renormalization scheme for the quartic power-divergences of the operator $\langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu)$.

Action density

$$E(t) = \frac{g^2}{4} G_{\mu\nu}^A(t) G^{A\mu\nu}(t)$$

Its expectation value, $\langle E(t) \rangle$, can be regarded as a gauge-invariant non-perturbative definition of the gluon condensate with cut-off $\Lambda_{UV} \propto 1/\sqrt{t}$, which can replace the ill-defined $\langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu)$ in the $\overline{\text{MS}}$ -OPE.

Relatively simple quantity for lattice simulations, has become a new standard for setting the scale on the lattice.

OPE of the action density / subtracted Adler function

Small flow-time expansion $t \ll 1/\Lambda_{\text{QCD}}^2$ [Lüscher, 1006.4518; Harlander, Neumann 1606.03756 (NNLO)]

$$\frac{1}{\pi^2} \langle E(t) \rangle = \frac{\tilde{C}_0(t)}{t^2} + \tilde{C}_{GG}(t) \langle \frac{\alpha_s}{\pi} GG \rangle + \mathcal{O}(t \times \text{dim-6})$$

$\tilde{C}_0$ known to NLO [Lüscher, 1006.4518] and NNLO [Harlander, Neumann 1606.03756], $\tilde{C}_{GG}(t)$ to NNLO [Harlander, Kluth, Lange, 1808.09837]

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Eliminate $\langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu)$ in the standard OPE for the Adler function

$$D(Q) = \underbrace{\left[C_0(Q) - \frac{1}{t^2 Q^4} \frac{C_{GG}(Q)}{\tilde{C}_{GG}(t)} \times \tilde{C}_0(t) \right]}_{u = 2 \text{ renormalon cancels}} + \underbrace{\frac{1}{Q^4} \frac{C_{GG}(Q)}{\tilde{C}_{GG}(t)} \frac{1}{\pi^2} \langle E(t) \rangle}_{\text{non-perturbatively defined}} + \mathcal{O}(1/Q^6)$$

Only fixed-order calculations needed to obtain a better-behaved expansion.

Incorporate the knowledge of asymptotic behaviour into an Ansatz for the **Adler function** that reproduces known $c_{n,1}$ to $n = 4$ and $c_{5,1} = 283$.

$$B[D](u) = B[D_1^{\text{UV}}](u) + B[D_2^{\text{IR}}](u) + B[D_3^{\text{IR}}](u) + d_0^{\text{PO}} + d_1^{\text{PO}} u$$

- Fit Stokes constants K_p for $u = -1, 2, 3$ to $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and adjust $d_{0,1}^{\text{PO}}$ to reproduce $c_{1,1}$ and $c_{2,1}$.
- Pole ansatz works well already at $n = 2$ (d_1^{PO} small). Apparently the series is very regular.

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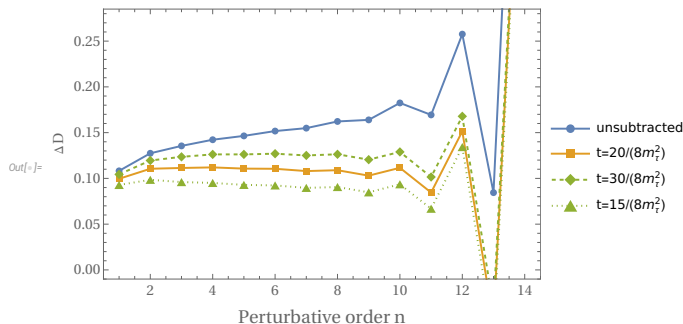
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The **gradient flow action density** has no UV renormalons. Ansatz (in practice for the entire subtraction term including $C_{GG}(Q)/\tilde{C}_{GG}(t)$):

$$B[E](u) = B[E_2^{\text{IR}}](u) + B[E_3^{\text{IR}}](u) + e_0^{\text{PO}} + e_1^{\text{PO}} u$$

Three unknowns (one fixed by $u = 2$ cancellation). Matches the available three exactly known coefficients of $\tilde{C}_0$

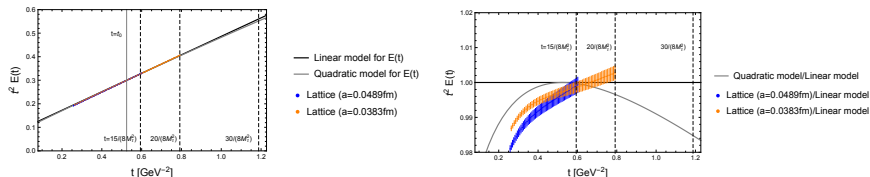
Adler function series with gradient-flow subtraction



Perturbative growth stops around second to third order, until UV renormalon divergence sets in.

Non-perturbative gradient-flowed gluon condensate

Lattice data from 1411.3982, 1712.04884 (CLS), reweighting according to Kuberski, 2306.02385, kindly provided by [Saez Gonzalvo](#)



Almost linear dependence of $t^2 E(t)$ on t in the t -range of interest. Parameterize as

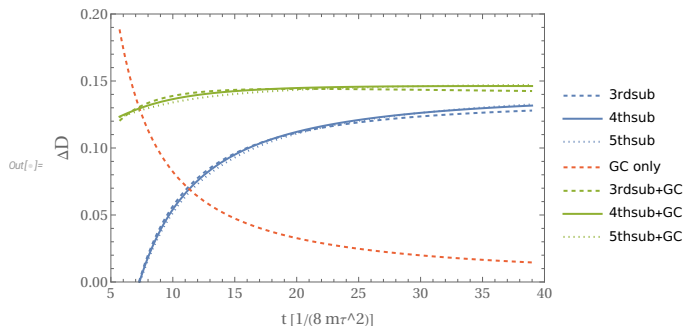
$$t^2 E(t)|_{\text{lin+quad}} = 0.3 + \frac{0.3}{w_0^2} (t - t_0) + x(t - t_0)^2$$

$[t_0, w_0, x]$ defined from

$$t^2 E(t)|_{t=t_0} = 0.3, \quad t \frac{d}{dt} (t^2 E(t)) \Big|_{t=w_0^2} = 0.3, \quad t^2 E(t)|_{t=t_2} = 0.2$$

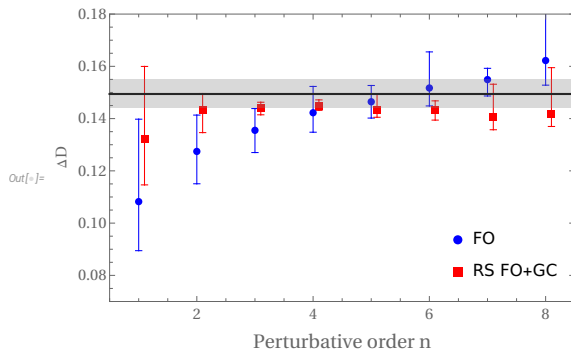
and given by the FLAG averages $\sqrt{t_0} = 0.14292 \text{ fm}$, $w_0 = 0.17256 \text{ fm}$ and x via the “ t_2 ” scale, obtained as $\sqrt{t_0/t_2} = 1.3867(18)$ in Larsen et al., 2502.08061]

Adler function series with gradient-flow subtraction + gluon condensate



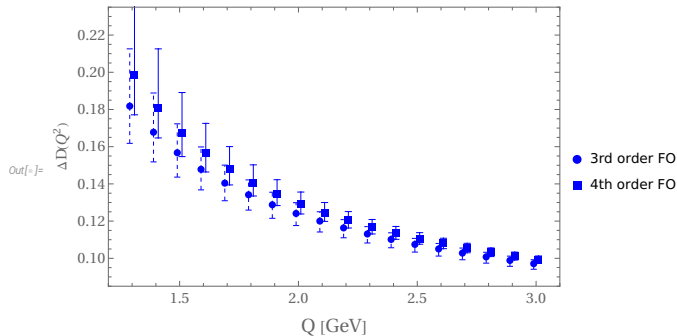
Independence of t in the $8m_\tau^2$ window 15 – 30, little dependence on perturbative truncation order.

Adler function, perturbative vs. non-perturbative (I)

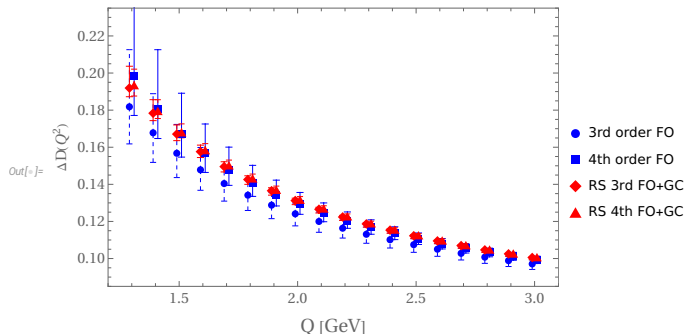


Best approximation at 3rd or 4th order. Increasing uncertainty at higher orders due to UV renormalon (removable).

Adler function, perturbative vs. non-perturbative (II) - Q dependence



Adler function, perturbative vs. non-perturbative (II) - Q dependence

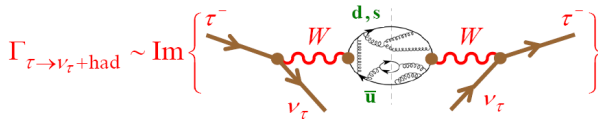


Extends the range where perturbation theory is reliable to smaller Q , larger coupling. Applications to QCD sum rules, α_s and strange quark mass determinations

The τ hadronic width

$$R_\tau \equiv \frac{\Gamma[\tau^- \rightarrow \text{hadrons } \nu_\tau(\gamma)]}{\Gamma[\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau(\gamma)]} = \frac{1 - \mathcal{B}_e - \mathcal{B}_\mu}{\mathcal{B}_e} = R_{\tau,V} + R_{\tau,A} + R_{\tau,S} = 3.6381 \pm 0.0075$$

[HFLAV, 2206.07501]

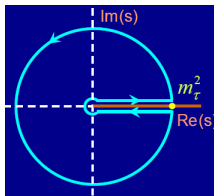


Focus on non-strange final states

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \text{Im } \Pi^{(1)}(s) + \text{Im } \Pi^{(0)}(s) \right]$$

$$\Pi_{\mu\nu}^{V/A}(p) \equiv i \int dx e^{ipx} \langle \Omega | T \{ J_\mu^{V/A}(x) J_\nu^{V/A}(0)^\dagger \} | \Omega \rangle = (p_\mu p_\nu - g_{\mu\nu} p^2) \Pi^{V/A,(1)} + p_\mu p_\nu \Pi^{V/A,(0)}$$

$$R_\tau = -i\pi \oint_{|x|=1} \frac{dx}{x} (1-x)^3 \left[3(1+x) D(M_\tau^2 x) + 4D^{(0)}(M_\tau^2 x) \right]$$

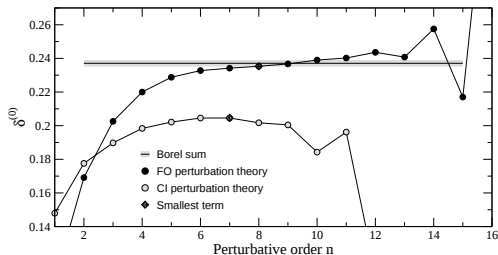


FOPT $\delta_{\text{FO}}^{(0)} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^n k c_{n,k} J_{k-1} \quad J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^l(-x)$

CIPT $\delta_{\text{CI}}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(M_\tau^2) \quad J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-M_\tau^2 x)$

[Le Diberder, Pich, 1993] - Sums π^2 terms

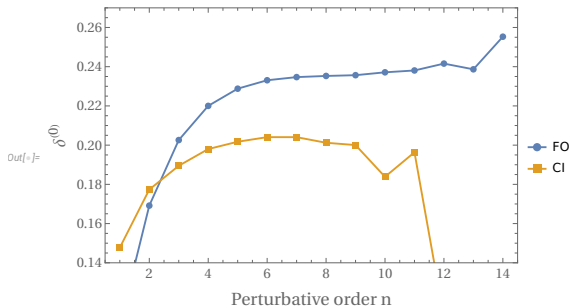
The problem [MB, Jamin, 2008]



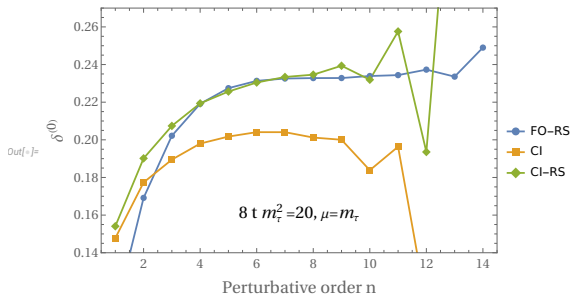
$$(\alpha_s(M_\tau^2) = 0.34)$$

- FO/CI difference *increases* by adding more orders. Systematic problem.
- FO converges to Borel sum
- CI converges more quickly than FO at low orders, but to the wrong value
- α_s from CI too large.
- Problem arises from renormalon pole related to the gluon condensate [MB, Boito, Jamin, 2012]
CIPT is inconsistent with OPE [Regner, Hoang, 2021; Gracia, Hoang, Mateu, 2022]

Hadronic τ decay with gradient-flow subtraction [MB, Takaura, 2309.10853]



Hadronic τ decay with gradient-flow subtraction [MB, Takaura, 2309.10853]



- CI and FO approach now similar values.
(How well, depends on choice of t .)
- CI converges more quickly than FO at low orders, now to the correct value.

Conclusion

- The gradient-flowed OPE realises a gauge-invariant Wilsonian cut-off scheme. It eliminates IR renormalons from short-distance coefficients and allows to consistently add ambiguity-free non-perturbative corrections.
- The gradient flow subtraction scheme works in low orders without explicit knowledge of asymptotic behaviour at a low subtraction scale.
- The gradient flow separates the continuum limit $a \rightarrow 0$ on the lattice from the cut-off scale $1/\sqrt{t}$ defining the renormalon subtraction. The flowed action density / gluon condensate can be computed on the lattice (t does not have to be very small).
- Extends the range where perturbation theory is reliable to smaller Q , larger coupling. Applications to QCD sum rules, α_s and strange quark mass determinations
- FOPT/CIPT resolved by gluon-condensate renormalon subtraction. FOPT α_s value from τ decay is the correct one.

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The proposal is general and applies to HQET matrix elements, DIS twist-four moments etc. in the same way

Backup slides

Condensate ambiguities at order $\mathcal{O}(1/N)$

Define condensates with a momentum cut-off $\mu_f \gg m$ [Novikov, Shifman, Vainshtein, Zakharov (1984)]

$$\langle \alpha^2 \rangle(\mu_f, m) = \text{diagram} = \int_{p^2 < \mu^2} \frac{d^2 p}{(2\pi)^2} D_\alpha(p)$$

$$= m^4 [F(\ln A) + F(-\ln A) - 2\gamma_E]$$

$$A = \left(\sqrt{1 + \frac{\mu_f^2}{4m^2}} + \sqrt{\frac{\mu_f^2}{4m^2}} \right)^4, \quad \ln A \xrightarrow{\mu_f \gg m} \frac{2}{g(\mu_f)} \gg 1$$

$F(x) = \text{Ei}(-x) - \ln x$ has an essential singularity at $x = 0$ but no discontinuity. Asymptotic expansion for large x has a Stokes discontinuity at negative arguments

$$F(-x) = e^x \left[\sum_{n=0}^{\infty} \frac{n!}{x^{n+1}} - e^{-x} (\ln x \mp i\pi) \right]$$

$$\langle \alpha^2 \rangle = \mu^4 \sum_{n=0}^{\infty} \left(\frac{g}{2} \right)^{n+1} n! + 2g \mu^2 m^2 + m^4 \left[-2 \ln \frac{2}{g} \pm i\pi - 2\gamma_E - 4g + \frac{g^2}{2} \right] + \mathcal{O}\left(\frac{m^2}{\mu^2}\right)$$

Condensate ambiguities at order $\mathcal{O}(1/N)$

Define condensates in dimensional regularization [David (1982, 1984, 1986)]

Computation of the ambiguity [MB (1998)]

$$\langle \alpha^2 \rangle(\mu, m) = \mu^\epsilon \int \frac{d^d p}{(2\pi)^d} D_\alpha(p, d) = \frac{m^2/(4\pi)}{\Gamma(1 - \epsilon/2)} \left(\frac{m^2}{4\pi\mu^2} \right)^{-\epsilon} \int_0^\infty d \left(\frac{p^2}{m^2} \right) \left(\frac{p^2}{m^2} \right)^{-\epsilon} D_\alpha(p, d)$$

Can set $d = 2$ in $D_\alpha(p, d)$. Define $u = p^2/m^2$ and use

$$D_\alpha(p) = 4\pi m^2 u \sum_{k,l} \frac{c_{kl}}{u^k} \frac{1}{\ln^l u}, \quad \frac{1}{\ln^l u} = \int_0^\infty dv v^{l-1} u^v.$$

Then (keeping only singular terms in v -integral)

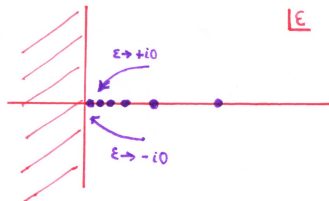
$$\langle \alpha^2 \rangle(\mu, m) \sim \frac{m^4}{\Gamma(1 - \epsilon/2)} \left(\frac{m^2}{4\pi\mu^2} \right)^{-\epsilon} \sum_{k,l} c_{kl} \int_0^\infty dv \frac{v^{l-1}}{-2 + \epsilon + k + v}$$

$$\left[\lim_{\epsilon \rightarrow +i0} - \lim_{\epsilon \rightarrow -i0} \right] \langle \alpha^2 \rangle(\mu, m) = 2\pi i m^4 \sum_{k=0}^1 \sum_{l=1} c_{kl} (2-k)^{l-1} = 2\pi i m^4$$

Condensate ambiguities at order $\mathcal{O}(1/N)$

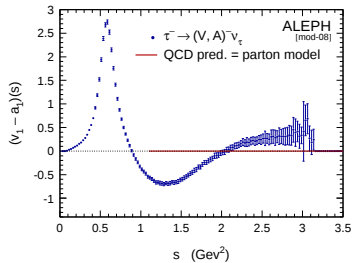
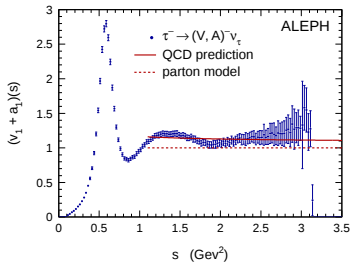
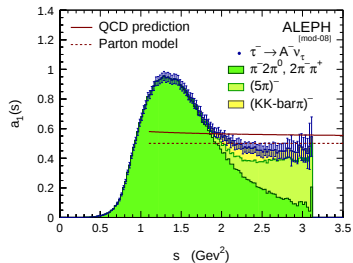
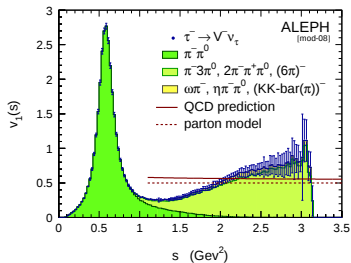
The ambiguity arises from power divergences. They do not lead to $1/\epsilon$ poles but to accumulating poles near $\epsilon = 0$.

For a condensate of dimension D , the poles of the v -integral are located at $\epsilon = 2k/l$ ($k < d$, l positive integer).

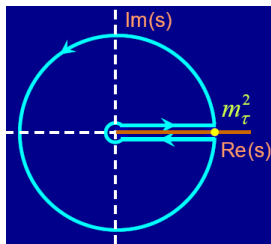


No ambiguity at finite ϵ . No renormalon divergence either. Limits $\epsilon \rightarrow 0$ and $n \rightarrow \infty$ don't commute.

ALEPH spectral functions



The τ hadronic width in QCD



- Analyticity
- Condensate expansion
- Slightly Euclidean $[(1-x)^3 \text{ suppression}]$
- $D^{(1+0)}(s) \equiv -s \frac{d}{ds} [\Pi^{(1+0)}(s)]$ (Adler fn)

$$\begin{aligned}
 R_\tau &= 6\pi i \oint_{|s|=M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \Pi^{(1)}(s) + \Pi^{(0)}(s) \right] \\
 &= -i\pi \oint_{|x|=1} \frac{dx}{x} (1-x)^3 \left[3(1+x) D^{(1+0)}(M_\tau^2 x) + 4D^{(0)}(M_\tau^2 x) \right] \\
 &= N_c S_{\text{EW}} |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta'_{\text{EW}} + \sum_{D \geq 2} \frac{C_D(s, \mu) \langle O_D(\mu) \rangle}{(-s)^{D/2}} \right]
 \end{aligned}$$

[Braaten, Narison, Pich, 1992]

Condensate expansion

- **D=2** m_q^2
 $\longrightarrow (3.1 \pm 8.6) \cdot 10^{-5}$

- **D=4** $m_q^4, m_q \langle \bar{q}q \rangle, \langle \frac{\alpha_s}{\pi} GG \rangle$
 $\longrightarrow (6.3 \pm 3.3) \cdot 10^{-4}$

Suppression of the D=4 contribution due to the kinematic weight function
 $(1-x)^3(1+2x) = 1 - 2x + 2x^3 - x^4$

- **D=6** $\langle \bar{q}q\bar{q}q \rangle, \langle \alpha_s G^3 \rangle$
 $\longrightarrow (-4.8 \pm 2.9) \cdot 10^{-3}$ – dominant

Factor 10 cancellation between V and A. This explains $R_{\tau, V-A} \approx 0.08$ ($\rightarrow$ Fig.)

- **S+P** $D^{(0)}(s)$ contribution dominated by the calculable pion pole contribution
 $\longrightarrow (-2.64 \pm 0.05) \cdot 10^{-3}$

Non-perturbative terms very small [3.5% of perturbative contribution!] due to V+A cancellation and kinematic suppression

$$\delta_{\text{PC}} = (-6.8 \pm 3.5) \cdot 10^{-3}$$

Nevertheless, the gluon condensate will play an important role in the following.