Loop Summit 2: New perturbative results and methods and precision physics: Quantum Field Theory and Collider Physics

Cadenabbia, July 2025

QCD scattering in the Regge limit

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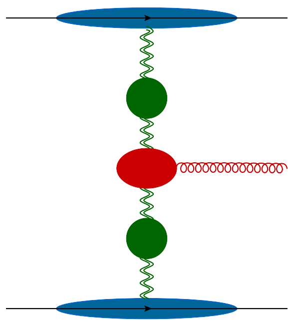
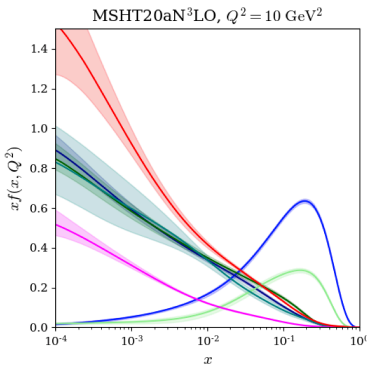


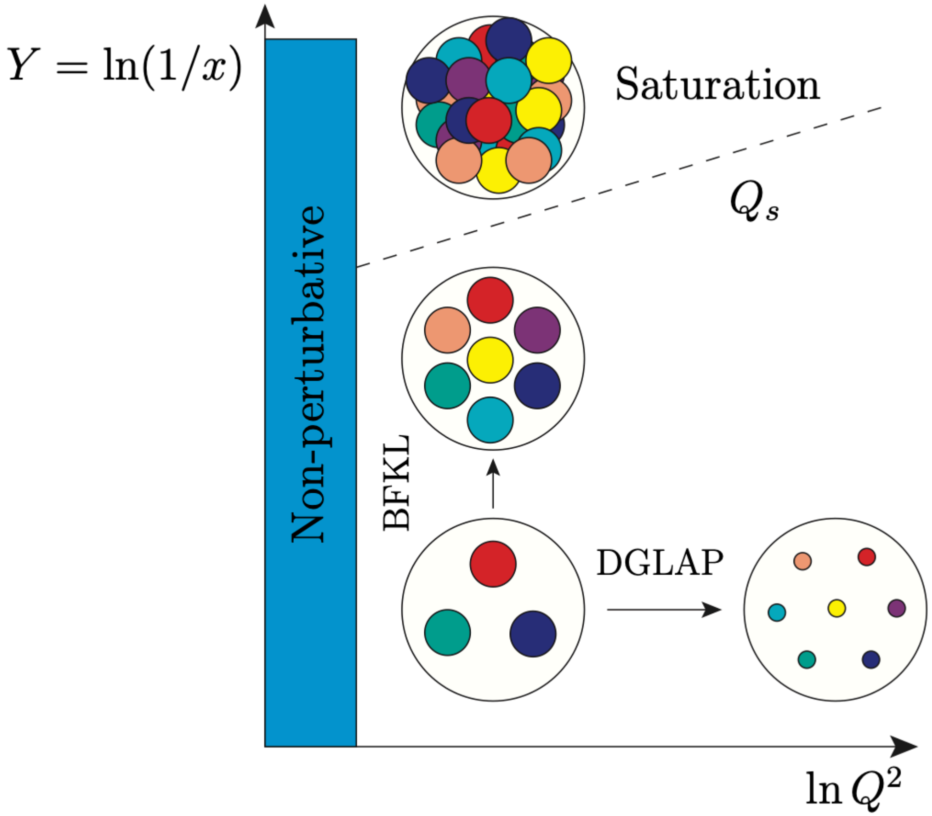
parton density evolution in Q^2 and in rapidity

- DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) evolution resums logarithms of Q^2/μ_F^2
- BFKL (Balitsky-Fadin-Kuraev-Lipatov) evolution resums energy logarithms (= rapidity Y)

  The high-gluon-density saturation regime requires a generalisation: Balitsky-JIMWLK (Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner) non-linear evolution in rapidity

Parton densities increase at small Bjorken x, and have large uncertainties there.
[plot from MSHT collaboration, 2207.04739]

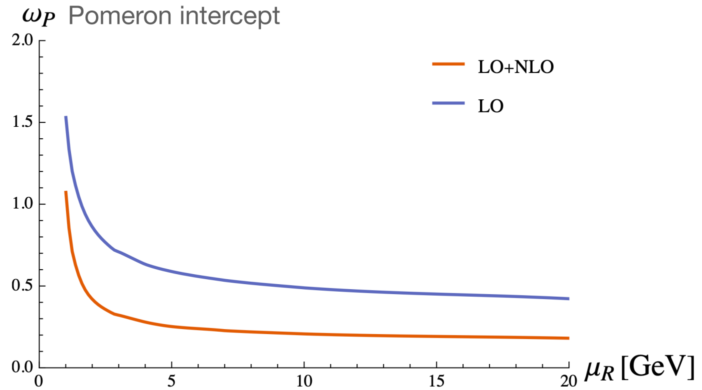




While DGLAP evolution is using approximate N^{3}LL, BFKL remains at NLL…

QCD scattering in the Regge limit

Motivation



[Polizzi, Fucilla, Papa 2505.18074]

- NLL BFKL displays poor convergence*
- NNLL is a qualitatively different problem to NLL:
  BFKL relies of gluon Reggization.
  But at NNLL the Reggeization assumption breaks down, requiring to generalise the evolution equation.
- Understanding multi-Reggeon interactions in amplitudes paves the way forward!
- It is a key step towards an effective theory for the Regge limit

*J. Blümlein and A. Vogt (1997), Blümlein, W.L. van Neerven, V. Ravindran, A. Vogt, hep-ph/9806368; D.A. Ross, Phys. Lett. B 431 (1998) 161; G.P. Salam, JHEP 07, 019 (1998).

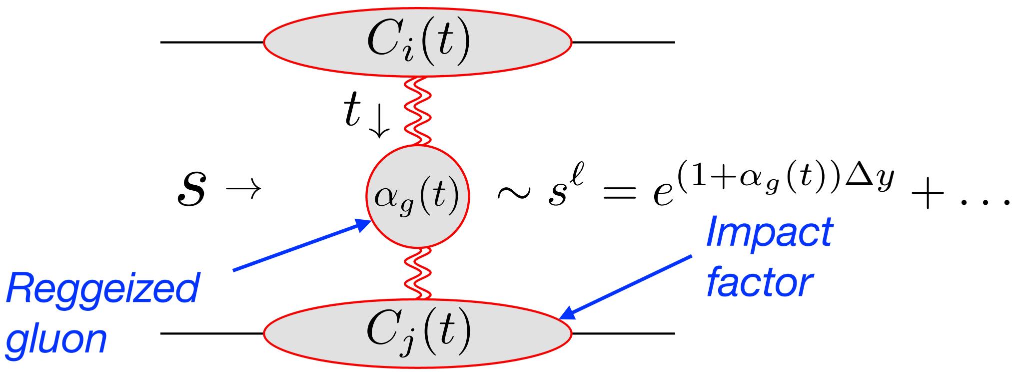
The high-energy limit of 2 → 2 gauge-theory amplitudes

- **Reggeization** (Regge-pole):
  $s \gg -t$

$$\frac{s}{t} \longrightarrow \frac{s}{t}\left(\frac{s}{-t}\right)^{\alpha_g(t)}$$

← *Gluon Regge Trajectory*

- Regge-pole factorization:



- Regge-pole factorization amounts to a **relation** between gg→gg, qg→qg, qq→qq

- This holds for the **real part** of the amplitude (its signature odd part) through NLL.
  Beyond that factorization is violated by **non-planar** corrections associated with **multi-Reggeon** exchange forming **Regge cuts.** These effects are now under control at NNLL.

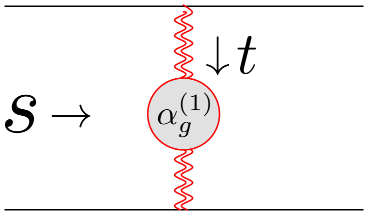
Signature-odd amplitudes: Regge-pole factorisation and its breaking

Regge factorization and violation in the signature odd amplitude:

$$\mathcal{M}^{(-)}_{ij\to ij} = C_i(t)\, e^{\alpha_g(t)\, C_A\, L}\, C_j(t) \mathcal{M}^{\text{tree}}_{ij\to ij} + \text{MR}$$

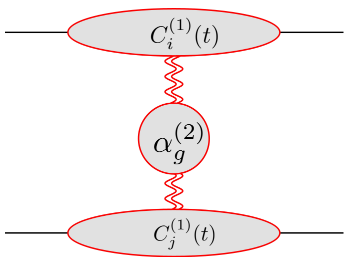
Colour **octet** exchange in the t channel: single Reggeon

✓ LL $\alpha_s^n \log^n\left(\frac{s}{-t}\right)$



one-loop Regge trajectory

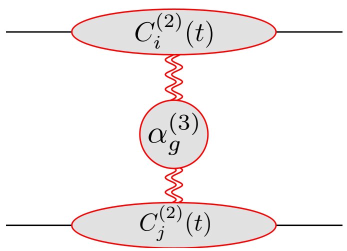
✓ NLL $\alpha_s^n \log^{n-1}\left(\frac{s}{-t}\right)$



two-loop Regge trajectory

one-loop impact factors

✗ NNLL $\alpha_s^n \log^{n-2}\left(\frac{s}{-t}\right)$



three-loop Regge trajectory

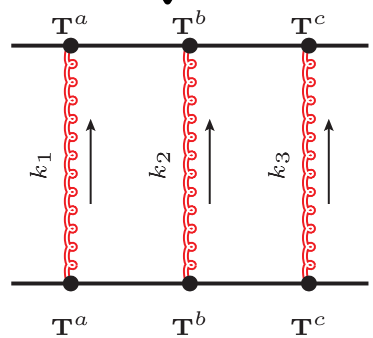
two-loop impact factors

Regge factorisation breaking (starting at 2 loops) can be inferred from comparing gg→gg, qg→qg, qq→qq amplitudes

[Del Duca, Glover '01]

[Del Duca, Falcioni, Magnea, Vernazza '14]

But until recently unknown how to account for it



Reggeons from the shockwave formalism

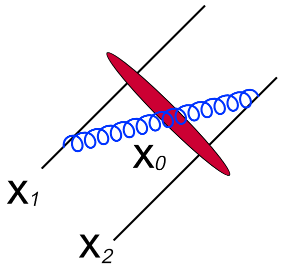
- Each parton in the projectile → an infinite lightlike Wilson line going through a shockwave

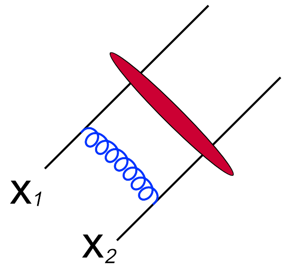
Balitsky-JIMWLK

$$-\frac{d}{d\eta}\left[U(\mathbf{x}_1)\ldots U(\mathbf{x}_n)\right] = H\left[U(\mathbf{x}_1)\ldots U(\mathbf{x}_n)\right]$$

$$H = \frac{\alpha_s}{2\pi^2}\int d\mathbf{x}_i d\mathbf{x}_j d\mathbf{x}_0 \frac{\mathbf{x}_{0i}\cdot\mathbf{x}_{0j}}{\mathbf{x}_{0i}^2\mathbf{x}_{0j}^2}\left(T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\mathrm{adj}}(\mathbf{x}_0)(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R})\right)$$

$$T^a_{i,L} \equiv T^a U(\mathbf{x}_i)\frac{\delta}{\delta U(\mathbf{x}_i)}, \qquad T^a_{i,R} \equiv U(\mathbf{x}_i)T^a\frac{\delta}{\delta U(\mathbf{x}_i)}$$





- Complete separation between the lightcone directions and the transverse plane

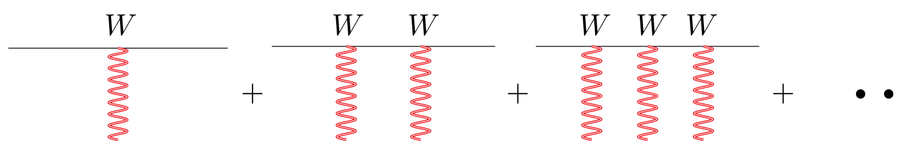
- In the perturbative regime $U(\mathbf{x}) \simeq 1$ it is natural to expand in terms of W Simon Caron-Huot (2013)

$$U(\mathbf{x}) = \mathcal{P}\exp\left\{ig_s\int_{-\infty}^{\infty} dx^+ A^a_+(x^+, x^- = 0; \mathbf{x})T^a\right\} = e^{ig_s\, T^a W^a(\mathbf{x})}$$

W sources a Reggeon. It has odd signature.

- Scattered particles are expanded in states of a definite number of Reggeons

$$|\psi_i\rangle \equiv \frac{Z_i^{-1}}{2p_1^+}a_i(p_4)a_i^\dagger(p_1)|0\rangle \sim g_s\,|W\rangle + g_s^2\,|WW\rangle + g_s^3\,|WWW\rangle + \ldots =$$



Linearlising the Balitsky-JIMWLK rapidity evolution

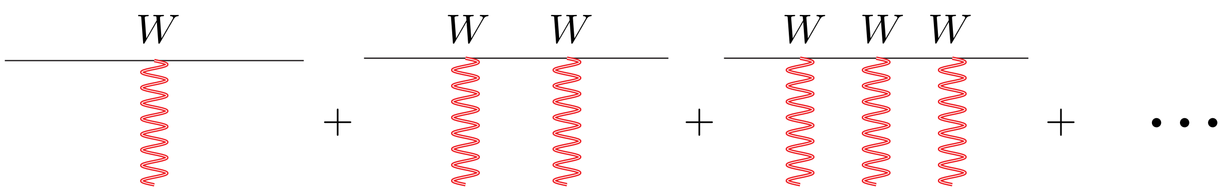
Amplitudes are governed by rapidity evolution between the target and projectile:

$$-\frac{d}{d\eta}\left|\psi_i\right\rangle = H\left|\psi_i\right\rangle$$

$$\boxed{\frac{i(Z_iZ_j)^{-1}}{2s}\mathcal{M}_{ij\to ij} = \langle\psi_j|e^{-HL}|\psi_i\rangle}$$

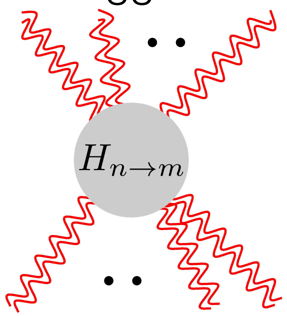
Each action of the Hamiltonian generates an extra power of the high-energy log L

Projectile $\left|\psi_i\right\rangle =$



n Reggeons

n Reggeons to *m* $\sum_{n,m}$

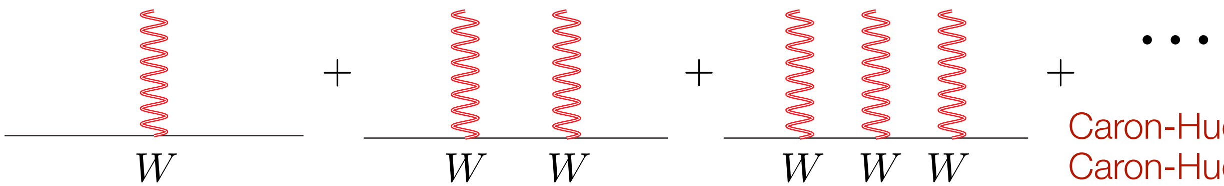


m Reggeons

$$H\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \cdots \\ 0 & H_{2\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & \cdots \\ 0 & g_s^2 & 0 & \cdots \\ g_s^4 & 0 & g_s^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

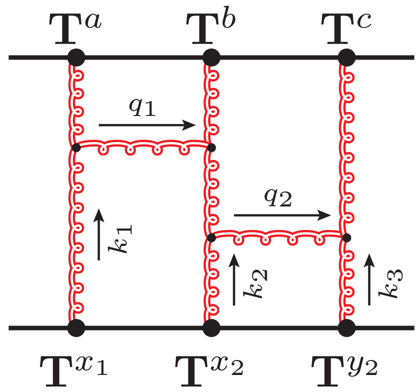
Target $\left\langle\psi_j\right| =$



Caron-Huot (2013)
Caron-Huot, EG, Vernazza, 1701.05241

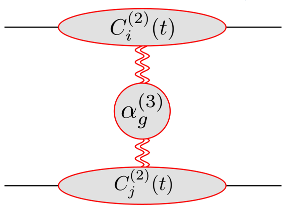
Signature-odd 2 → 2 amplitudes: understanding the NNLL tower

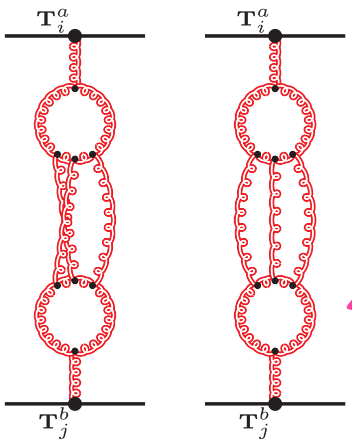
- Using **non-linear** rapidity evolution, the NNLL tower is determined to all orders in terms of **one** and **three** Reggeon exchanges

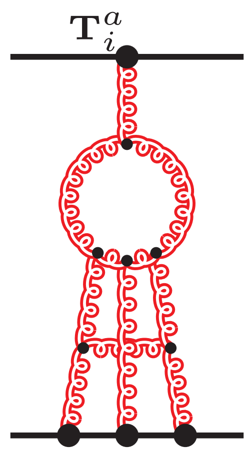


- Expanding in $X \equiv \frac{\alpha_s}{\pi} r_\Gamma L$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ \langle j_1|i_1\rangle^{\mathrm{NNLO}} + r_\Gamma^2\pi^2 \sum_{k=0}^{\infty} \frac{(-X)^k}{k!} \Bigg[\langle j_3|\hat{H}^k_{3\to3}|i_3\rangle + \Theta(k\geq 1)\Big[\langle j_1|\hat{H}_{3\to1}\hat{H}^{k-1}_{3\to3}|i_3\rangle + \langle j_3|\hat{H}^{k-1}_{3\to3}\hat{H}_{1\to3}|i_1\rangle\Big] + \Theta(k\geq 2)\langle j_1|\hat{H}_{3\to1}\hat{H}^{k-2}_{3\to3}\hat{H}_{1\to3}|i_1\rangle \Bigg]^{\mathrm{LO}} \Bigg\}$$







Caron-Huot, EG, Vernazza
JHEP 06 (2017) 016
Falcioni, EG, Milloy, Vernazza
Phys. Rev. D 103 (2021) L111501

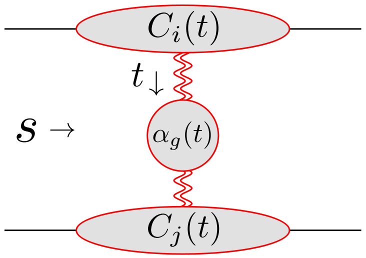
- All diagrams computed to four loops; only non-planar corrections violate Regge-pole factorization.

Signature odd amplitude at NNLL: Regge pole and cut properties

All-order structure through NNLL for any gauge theory, any representation:

$$\mathcal{M}^{(-)}_{ij\to ij} = Z_i(t)\,\bar{D}_i(t)\,Z_j(t)\,\bar{D}_j(t)\left[\left(\frac{-s}{-t}\right)^{C_A\alpha_g(t)} + \left(\frac{-u}{-t}\right)^{C_A\alpha_g(t)}\right]\mathcal{M}^{\text{tree}}_{ij\to ij} + \sum_{n=2}^{\infty} a^n L^{n-2}\mathcal{M}^{(\pm,n,n-2)\,\text{cut}}$$

Regge pole-factorized



✓ single Reggeon; colour octet
✓ dominant in planar limit

✓ Trajectory and impact factors at NNLL are **fixed****
by matching to (qq, gg, qg) scattering amplitudes*

Regge cut: breaks factorization

✓ multiple Reggeons; various colour reps.
✓ suppressed in planar limit
✓ proportional to $(i\pi)^2$

✓ no dependence on the matter content:
the same for any gauge theory!

✓ Sensitive to soft singularities beyond the dipole formula.

*qq amplitude: Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi, JHEP 10 (2021) 206

Determination of the two-loop impact factors and **three-loop gluon Regge trajectory:
Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi, Phys.Rev.Lett. 128 (2022) 21, 21
Falcioni, EG, Maher, Milloy, Vernazza, Phys.Rev.Lett. 128 (2022) 13, 13; *JHEP* 03 (2022) 053.

Regge-pole factorisation for multi-leg amplitudes in MRK

Multi-Regge Kinematics (MRK)

4-momentum $p = (p^+, p^-; \mathbf{p})$

target $p_1 = (0, p_1^-; \mathbf{0})$

projectile $p_2 = (p_2^+, 0; \mathbf{0})$

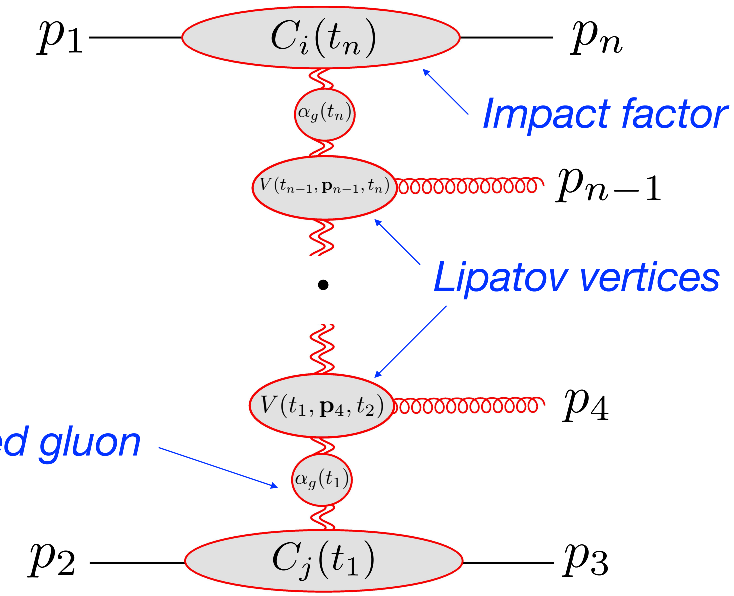
strong hierarchy of light-cone components $p_3^+ \gg p_4^+ \gg ... \gg p_n^+$

$p_3^- \ll p_4^- \ll ... \ll p_n^-$

no ordering of transverse components $|\mathbf{p}_3| \sim |\mathbf{p}_4| \sim ... \sim |\mathbf{p}_n|$

Regge (pole) factorization holds in MRK for the dispersive (real part) of the amplitudes through NLL; established using unitarity [Fadin et al. 2006]

Regge (pole) factorization in MRK



Planar limit:

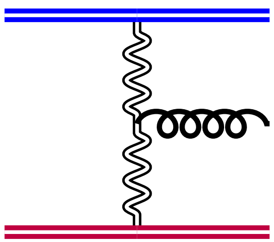
- Four- and five-point planar amplitudes have only Regge poles. Essential for the BDS ansatz in SYM.
- Six and higher-point planar amplitudes have also Regge cuts in some special kinematic regions [Bartels, Lipatov, Sabio Vera (2008)]. All multiplicity planar results are available [Del Duca *et al.* (2019)]

Extracting the Lipatov vertex from one-loop amplitudes

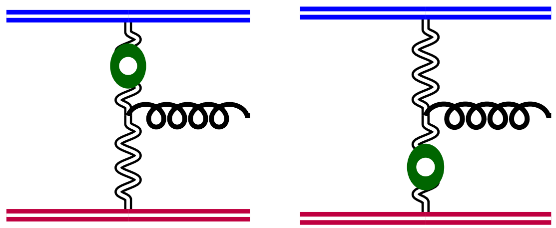
1-loop vertex extracted/computed in Fadin & Lipatov (1993); Del Duca and Schmidt (1998); Del Duca, Duhr, Glover (2009); Fadin, Fucilla, Papa (2023), Buccioni, Caola, Devoto, Gambuti — 2411.14050 [hep-ph] (Figures below), Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza — 2412.20578 [hep-ph]

The full $2 \to 3$ amplitude compared to the expected factorised Regge pole + multi-Regge interactions:

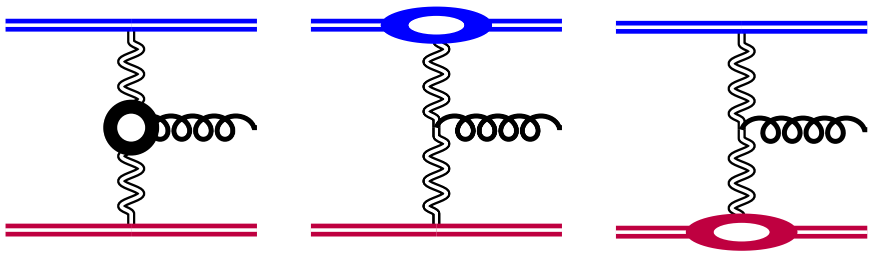
Tree



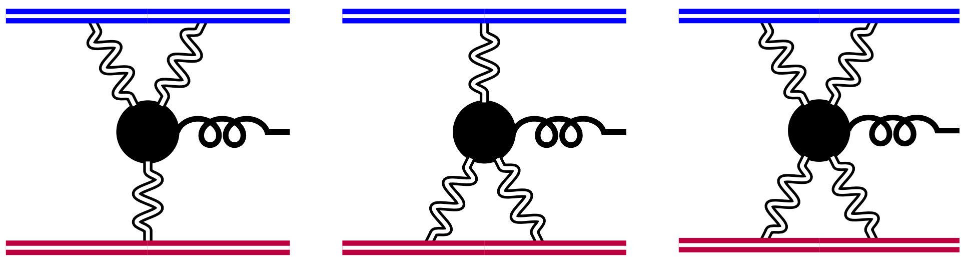
One-loop: $(-,-)$ signature
Leading Logarithms



One-loop
$(-,-)$ signature
No Logarithms ("Next to Leading")



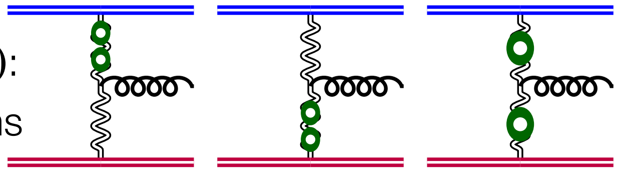
One-loop
$(-,+)$, $(+-)$, $(+,+)$ signature
No Logarithms
No [8,8] colour component



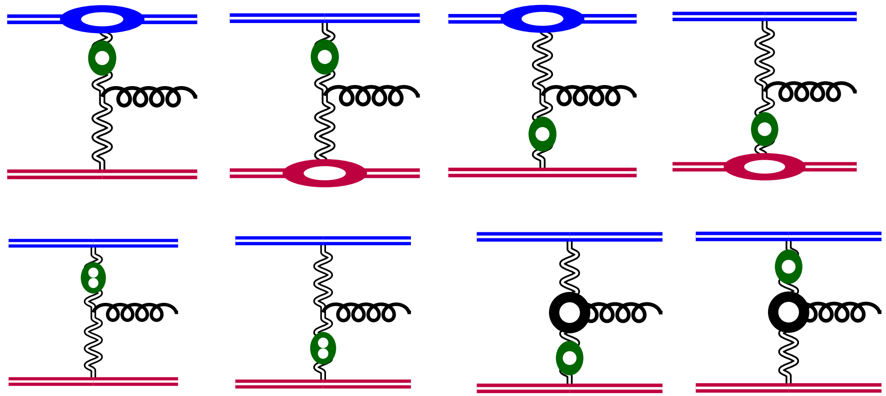
Extracting the Lipatov vertex from 2 → 3 (− , −) signature amplitudes

Two-loop amplitudes are available since last year: G. De Laurentis, H. Ita , M. Klinkert, V. Sotnikov 2311.10086, 2311.18752,
B. Agarwal, F. Buccioni, F. Devoto, G. Gambuti, A. von Manteuffel, L. Tancredi, 2311.09870

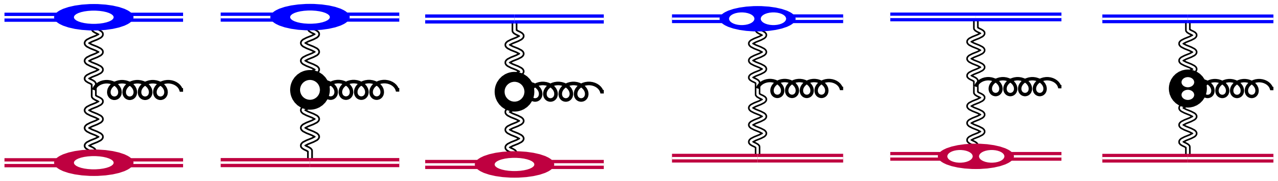
Two-loop (− , −): Leading Logarithms

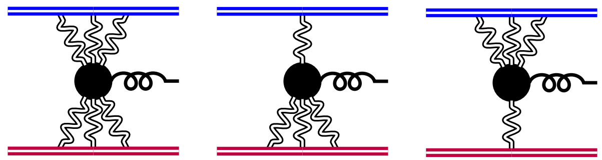


Two-loop (− , −): NLL



Two-loop (− , −): Next-to-next-to Leading Logarithms





Multi-Reggeon [8,8] contributions

Robust check: we obtained the (same) expression for the 2-loop QCD Lipatov Vertex from all three partonic channels!

Abreu, Falcioni, EG, Milloy and Vernazza — *PoS* LL2024 (2024) 085

Buccioni, Caola, Devoto, Gambuti 2411.14050; Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza: 2412.20578

Conclusions

(1) The shock-wave formalism and rapidity evolution equations

facilitate efficient computation of multiple-Reggeon interactions in the (multi) Regge limit:

NLL for signature even $2 \to 2$ amplitudes (14 loops)

NNLL for signature odd $2 \to 2$ amplitudes (so far to four loops)

NNLL for signature odd-odd $2 \to 3$ amplitudes (so far to two loops)

(2) **Regge-pole factorization violations in $2 \to 2$ and $2 \to 3$ amplitudes** - Regge cut contributions - **are non-planar**

(3) Based on (1), (2) and 3-loop 4-point calculations **we now know all Regge-pole parameters to NNLO.**

(4) Based on (1), (2) and (3) and recent 2-loop 5-point calculations **we determined the 2-loop Lipatov vertex in QCD.**

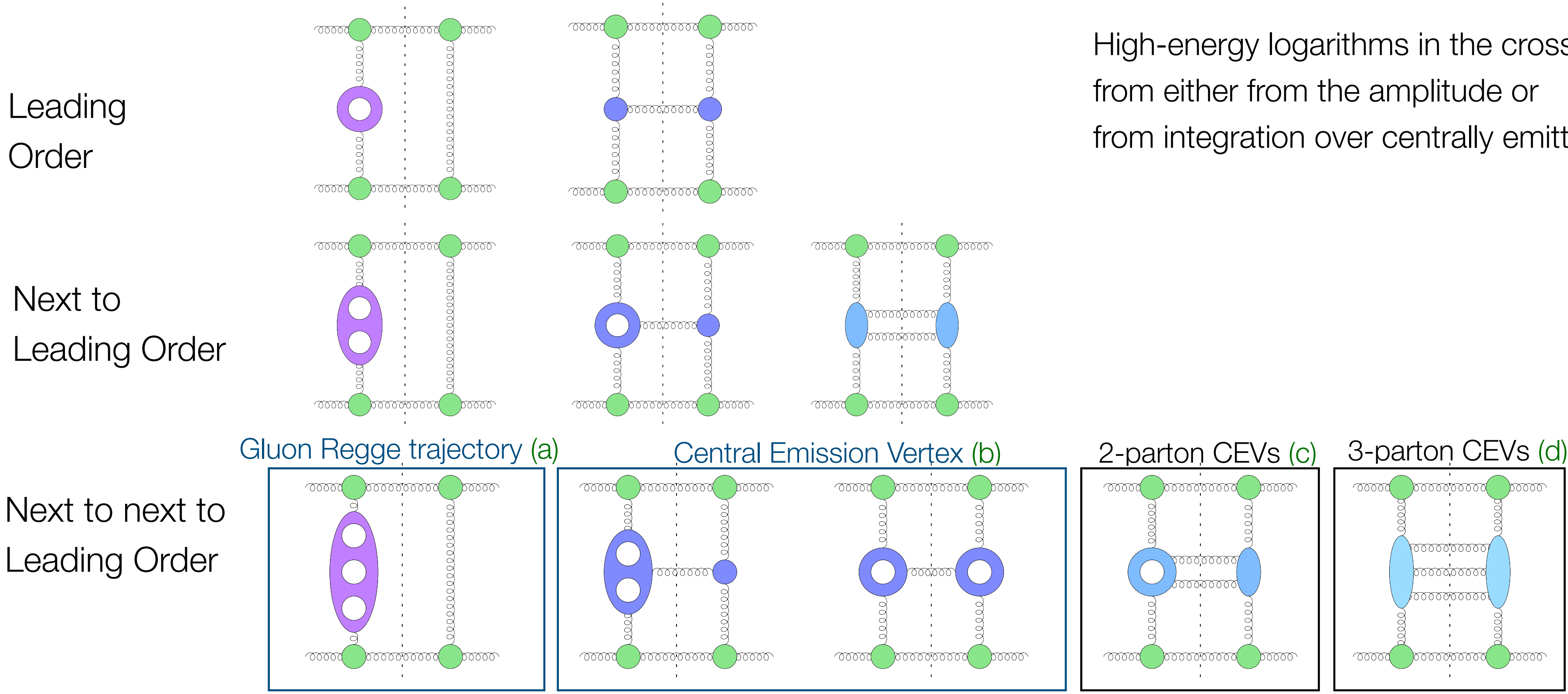
The study of amplitudes confirms Caron-Huot's proposition that W sources a Reggeon: important step towards an effective theory for the Regge limit to any log accuracy!

At the NNLL level we disentangled Regge pole and cut contributions in $2 \to 2$ and $2 \to 3$ QCD amplitudes

Progress towards determining all ingredients for NNLL rapidity evolution!

Contributions to BFKL kernel at increasing logarithmic accuracy

High-energy logarithms in the cross section arise
from either from the amplitude or
from integration over centrally emitted partons.



(a) 3-loop Regge trajectory Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi, Phys.Rev.Lett. 128 (2022) 21, 21,
Falcioni, EG, Maher, Milloy, Vernazza, Phys.Rev.Lett. 128 (2022) 13, 13; *JHEP* 03 (2022) 053;

(b) 2-loop Lipatov Vertex Buccioni, Caola, Devoto, Gambuti *JHEP* 03 (2025) 129,
Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza: *JHEP* 04 (2025) 161;

(c) 1-loop 2-gluon CEV **in sYM:** Byrne, Del Duca, Dixon, EG, Smillie *JHEP* 08 (2022) 271;
and **in QCD** with Byrne, De Laurentis, Del Duca, EG, Smillie (on-going work);

(d) All tree-level 3- (and 4-) parton CEVs and PEVs in QCD: Byrne, Del Duca, EG, Mo, Smillie 2506.10644.