Planar Six-Point Feynman Integrals for Four-Dimensional Gauge Theories

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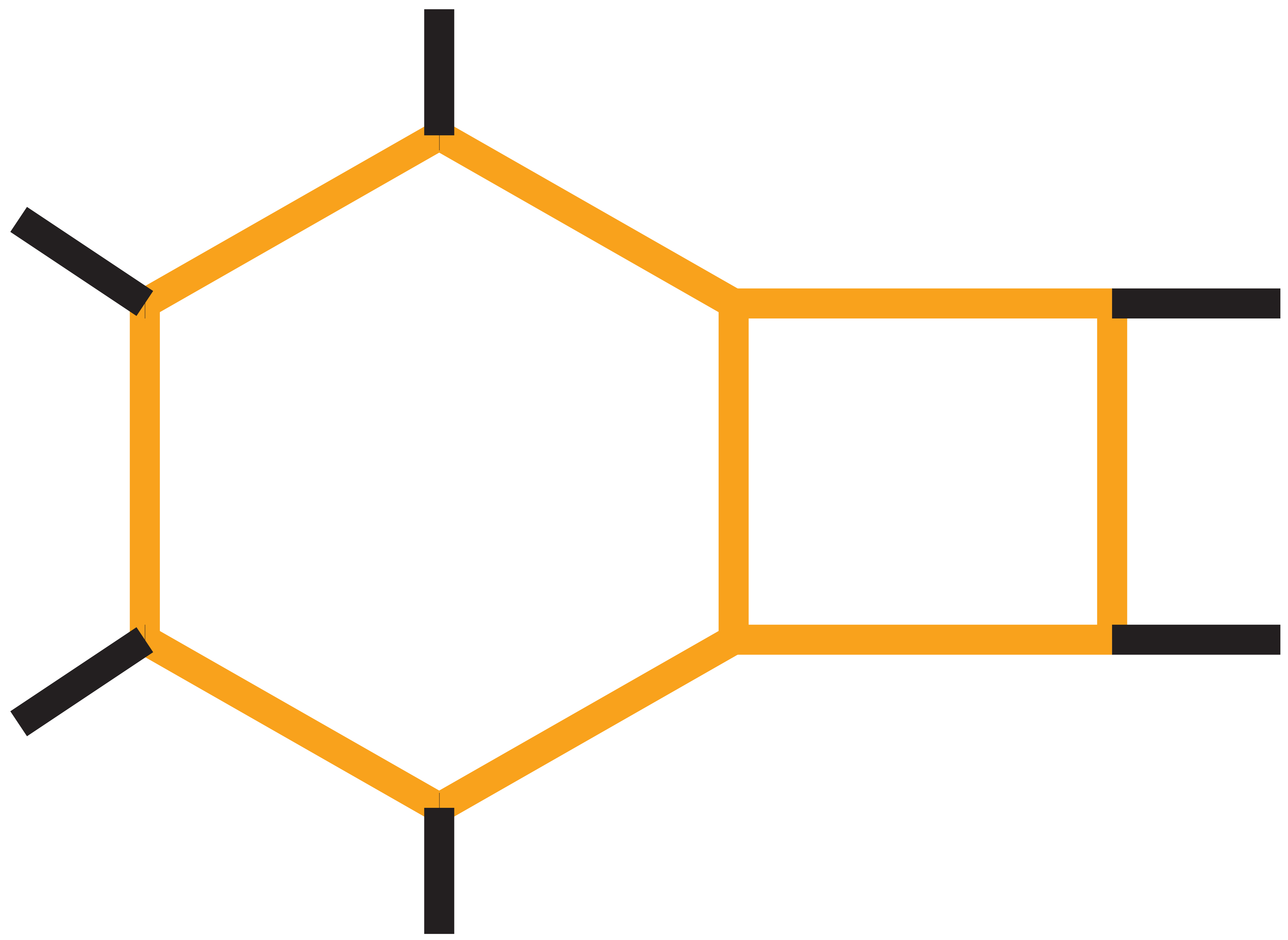
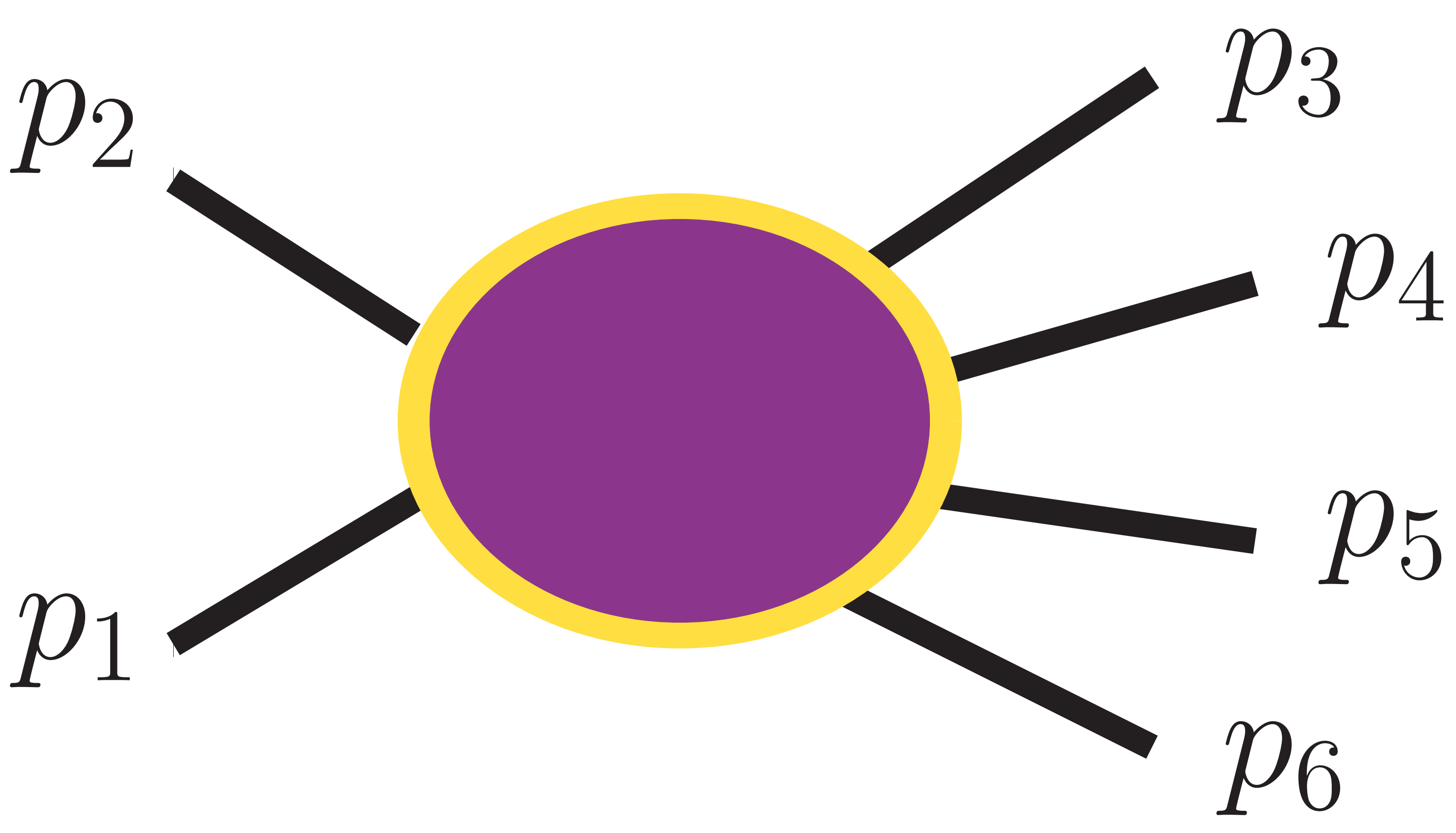
Outline

- Setup
- Kinematics
- One-Loop Warm-Up
- Two-Loop Planar Integrals
- Summary and Discussion



Setup

- **Scattering of six massless particles**
  - The p_i are four-dimensional momenta
  - Momentum conservation $\Rightarrow$ 5 independent momenta
  - Four-dimensionality $\Rightarrow$ **4 independent momenta**



- **Regulate singularities in Dim Reg**
  - Loop momenta are in $D = 4 - 2\epsilon$ dimensions

- **Consider theories with renormalisable power counting** (e.g., QCD, but no gravity)

How to include these conditions? What are the simplifications?

- Similar work, with the same conclusions

[J. Henn, T. Peraro, Y. Xu, and Y. Zhang, 2021]

[J. M. Henn, A. Matijašić, J. Miczajka, T. Peraro, Y. Xu, and Y. Zhang, 2024 and 2025]

Kinematics

- **Variables**
  - With $s_{ij} = (p_i + p_j)^2$ and $s_{ijk} = (p_i + p_j + p_k)^2$, need $\vec{s} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\}$
  - Implement four-dimensional constraint: $\Delta_6 = G(p_1, p_2, p_3, p_4, p_5) = 0$
- **Gram determinants**
  - Two-line Gram $G\begin{pmatrix} u_1 \ldots u_n \\ v_1 \ldots v_n \end{pmatrix} = \det(2u_i \cdot v_j)$, and one-line Gram $G(u_1 \ldots u_n) = \det(2u_i \cdot u_j)$
  - **Important property**: vanish if any of the u_i (or v_i) are linearly dependent
  - Δ_6: degree 5 polynomial in $\vec{s}$, quadratic in each variable

$$-2\,s12\,s123\,s16^2\,s34 + 2\,s123^2\,s16^2\,s34 + 2\,s12\,s123\,s16\,s23\,s34 + 2\,s12\,s123\,s16\,s234\,s34 - 2\,s123^2\,s16\,s234\,s34 -$$
$$2\,s12\,s123\,s23\,s234\,s34 + 2\,s123^2\,s16\,s34^2 + 2\,s123\,s16^2\,s34^2 - 2\,s123\,s16\,s23\,s34^2 - 2\,s12\,s123\,s16\,s23\,s345 +$$
$$2\,s12\,s123\,s16\,s234\,s345 - 2\,s123^2\,s16\,s234\,s345 + 2\,s12\,s123\,s23\,s234\,s345 - 2\,s12\,s123\,s234^2\,s345 + 2\,s123^2\,s234^2\,s345 -$$
$$2\,s123^2\,s16\,s34\,s345 + 2\,s123\,s16\,s23\,s34\,s345 - 2\,s123^2\,s234\,s34\,s345 - 4\,s123\,s16\,s234\,s34\,s345 + 2\,s123\,s23\,s234\,s34\,s345 +$$
$$2\,s123^2\,s234\,s345^2 - 2\,s123\,s23\,s234\,s345^2 + 2\,s123\,s234^2\,s345^2 + 2\,s12^2\,s16\,s23\,s45 - 2\,s12^2\,s16\,s234\,s45 +$$
$$2\,s12\,s123\,s16\,s234\,s45 - 2\,s12^2\,s23\,s234\,s45 + 2\,s12^2\,s234^2\,s45 - 2\,s12\,s123\,s234^2\,s45 + 2\,s12\,s123\,s16\,s34\,s45 -$$
$$4\,s12\,s16\,s23\,s34\,s45 + 2\,s12\,s123\,s234\,s34\,s45 + 2\,s12\,s16\,s234\,s34\,s45 + 2\,s123\,s16\,s234\,s34\,s45 + 2\,s12\,s23\,s234\,s34\,s45 -$$
$$2\,s123\,s16\,s34^2\,s45 + 2\,s16\,s23\,s34^2\,s45 - 4\,s12\,s123\,s234\,s345\,s45 + 2\,s12\,s23\,s234\,s345\,s45 - 2\,s12\,s234^2\,s345\,s45 -$$
$$2\,s123\,s234^2\,s345\,s45 + 2\,s123\,s234\,s34\,s345\,s45 - 2\,s23\,s234\,s34\,s345\,s45 + 2\,s12^2\,s234\,s45^2 + 2\,s12\,s234^2\,s45^2 -$$
$$2\,s12\,s234\,s34\,s45^2 + 2\,s12\,s16^2\,s34\,s56 - 2\,s123\,s16^2\,s34\,s56 - 4\,s12\,s16\,s23\,s34\,s56 + 2\,s123\,s16\,s23\,s34\,s56 + 2\,s12\,s23^2\,s34\,s56 +$$
$$2\,s12\,s16\,s23\,s345\,s56 + 2\,s123\,s16\,s23\,s345\,s56 - 2\,s12\,s23^2\,s345\,s56 - 2\,s12\,s16\,s234\,s345\,s56 + 2\,s123\,s16\,s234\,s345\,s56 +$$
$$2\,s12\,s23\,s234\,s345\,s56 - 4\,s123\,s23\,s234\,s345\,s56 + 2\,s123\,s16\,s34\,s345\,s56 + 2\,s123\,s23\,s34\,s345\,s56 + 2\,s16\,s23\,s34\,s345\,s56 -$$
$$2\,s23^2\,s34\,s345\,s56 - 2\,s123\,s23\,s345^2\,s56 + 2\,s23^2\,s345^2\,s56 - 2\,s123\,s234\,s345^2\,s56 - 2\,s23\,s234\,s345^2\,s56 -$$
$$4\,s12\,s16\,s23\,s45\,s56 + 2\,s12\,s16\,s234\,s45\,s56 - 2\,s123\,s16\,s234\,s45\,s56 + 2\,s12\,s23\,s234\,s45\,s56 - 4\,s12\,s16\,s34\,s45\,s56 +$$
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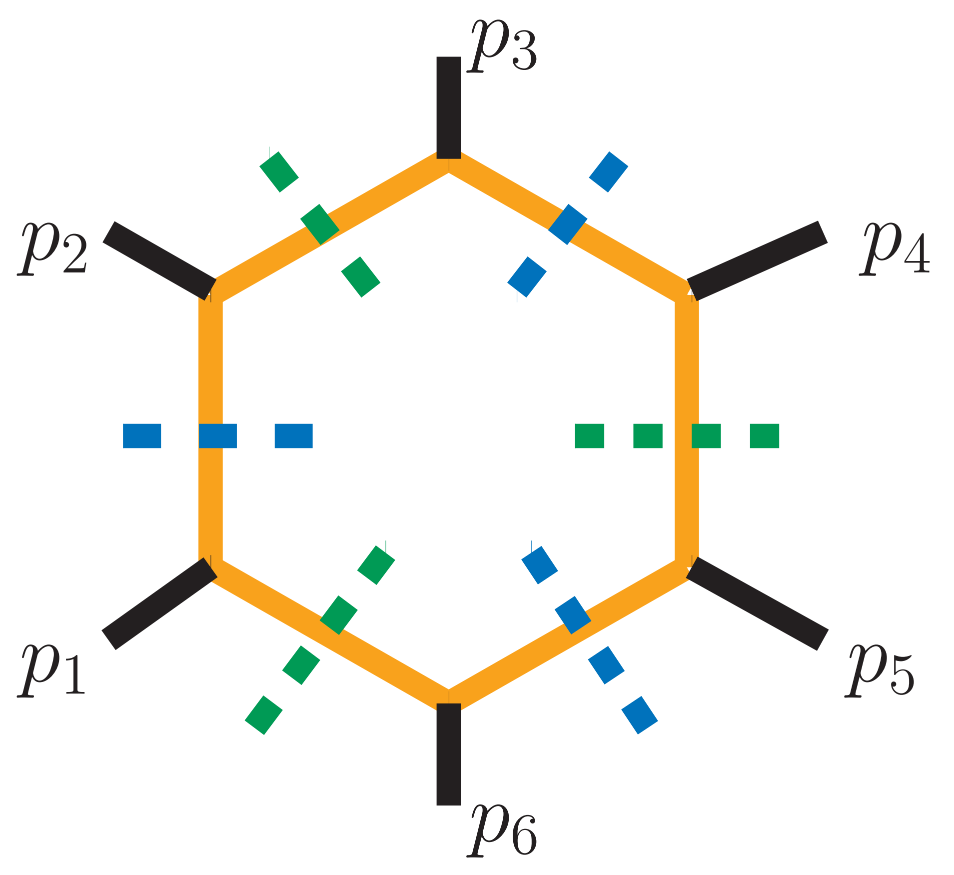
$$2\,s123\,s234\,s345\,s45\,s56 + 2\,s23\,s234\,s345\,s45\,s56 - 2\,s123\,s34\,s345\,s45\,s56 + 2\,s23\,s34\,s345\,s45\,s56 - 2\,s12\,s234\,s45^2\,s56 +$$
$$2\,s12\,s34\,s45^2\,s56 - 2\,s16\,s23\,s345\,s56^2 + 2\,s23^2\,s345\,s56^2 + 2\,s23\,s345^2\,s56^2 + 2\,s16\,s23\,s45\,s56^2 - 2\,s23\,s345\,s45\,s56^2$$

Kinematics — square roots



- **3-point-like square roots**

  $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$

  - Planar: $G(p_1 + p_2, p_3 + p_4) = -\lambda(s_{12}, s_{34}, s_{56})$, $G(p_2 + p_3, p_4 + p_5) = -\lambda(s_{23}, s_{45}, s_{16})$
  - And also a \`non-planar' version: $G(p_1 + p_4, p_2 + p_3) = -\lambda(s_{14}, s_{23}, s_{56})$

from 5pt one mass, [S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow, M. Zeng, 2020]

- **5-point-like square root(s)**
  - Levi-Civita contraction of 4 vectors: $\mathrm{tr}_5 = 4i\varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$; $\mathrm{tr}_5^2 = G(1234)$
  - Take canonical choice: $\sqrt{\Delta_5} \equiv \sqrt{G(1234)}$
  - All 5-point like roots polynomially related: $\sqrt{\Delta_5}\sqrt{\Delta^{(5)}_{i_1i_2i_3i_4}} = G\begin{pmatrix} i_1i_2i_3i_4 \\ 1234 \end{pmatrix} \mod \Delta_6 = 0$

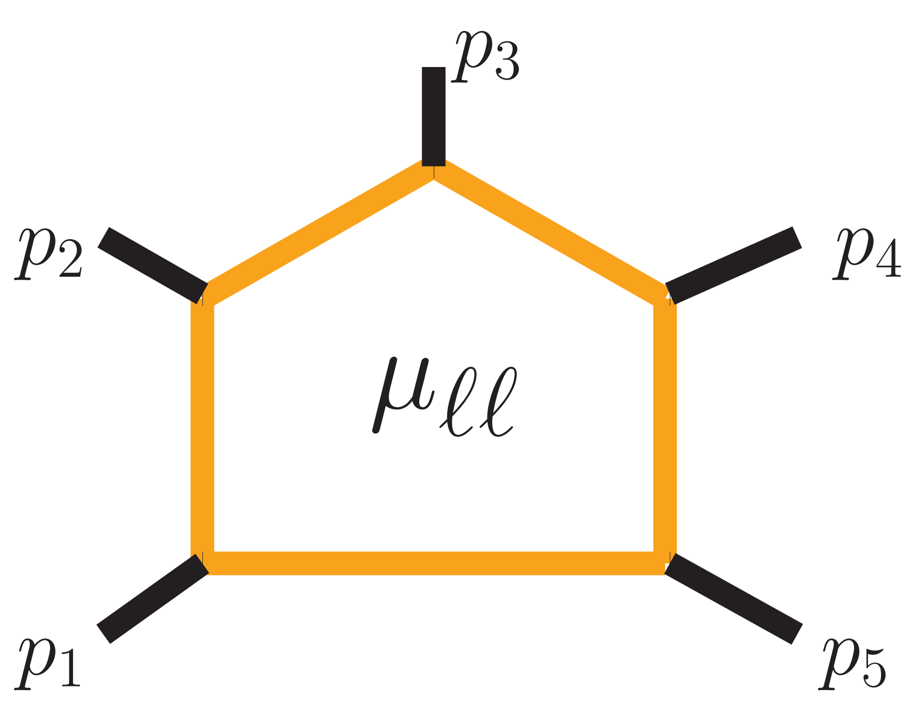
Kinematics — 4d condition

- **Solution 1: solve $\Delta_6 = 0$**
  - choose variable to eliminate ⇒ breaks symmetry of the problem 👎
  - quadratic equation introduces extra square root ⇒ nested square roots 👎

- **Solution 2: twistor variables** [A. Hodges, 2009]
  - four-dimensional variables, rationalise $\sqrt{\Delta_5}$ 👍
  - distort phase-space, makes it harder to understand singularities 👎

- **Solution 3: work modulo $\Delta_6 = 0$**
  - keeps full symmetry of the problem and doesn't distort phase-space 👍
  - expressions not unique, valid modulo $\Delta_6 = 0$. Unique when evaluated on 4d phase-space point 👍

One-loop Warm-up

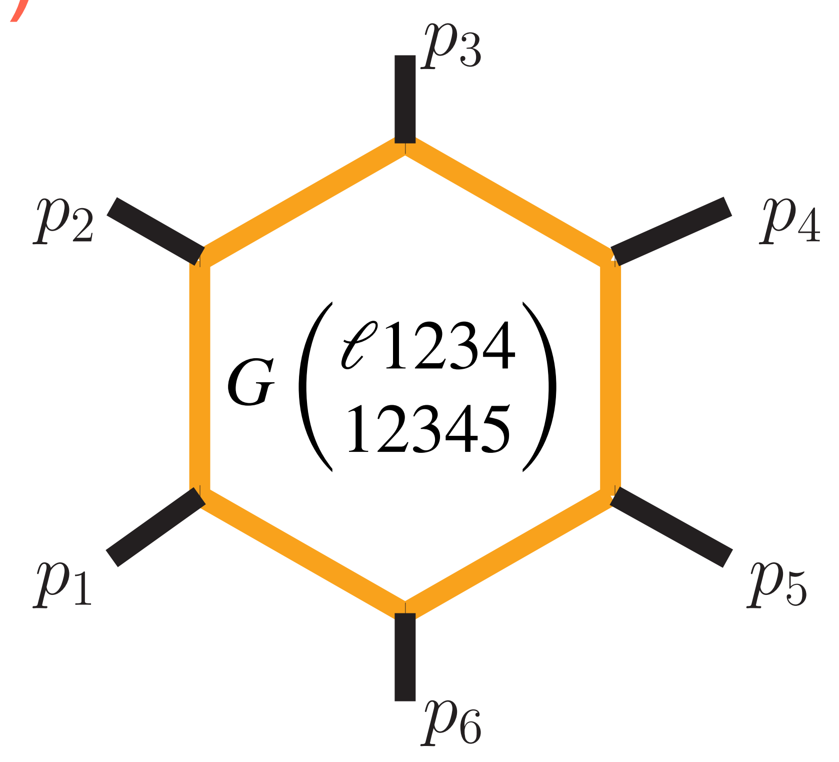


$\mu_{\ell\ell} = G(\ell 1234)$

see e.g. [Z. Bern, L. J. Dixon, and D. A. Kosower, 1993]

- "At one loop we only need bubbles, triangles and boxes"
  - basis dependent and only up to terms that vanish when $\epsilon \to 0$
  - beyond ϵ^0 pentagons contribute
  - ~ 6-dimensional pentagon, evanescent integral

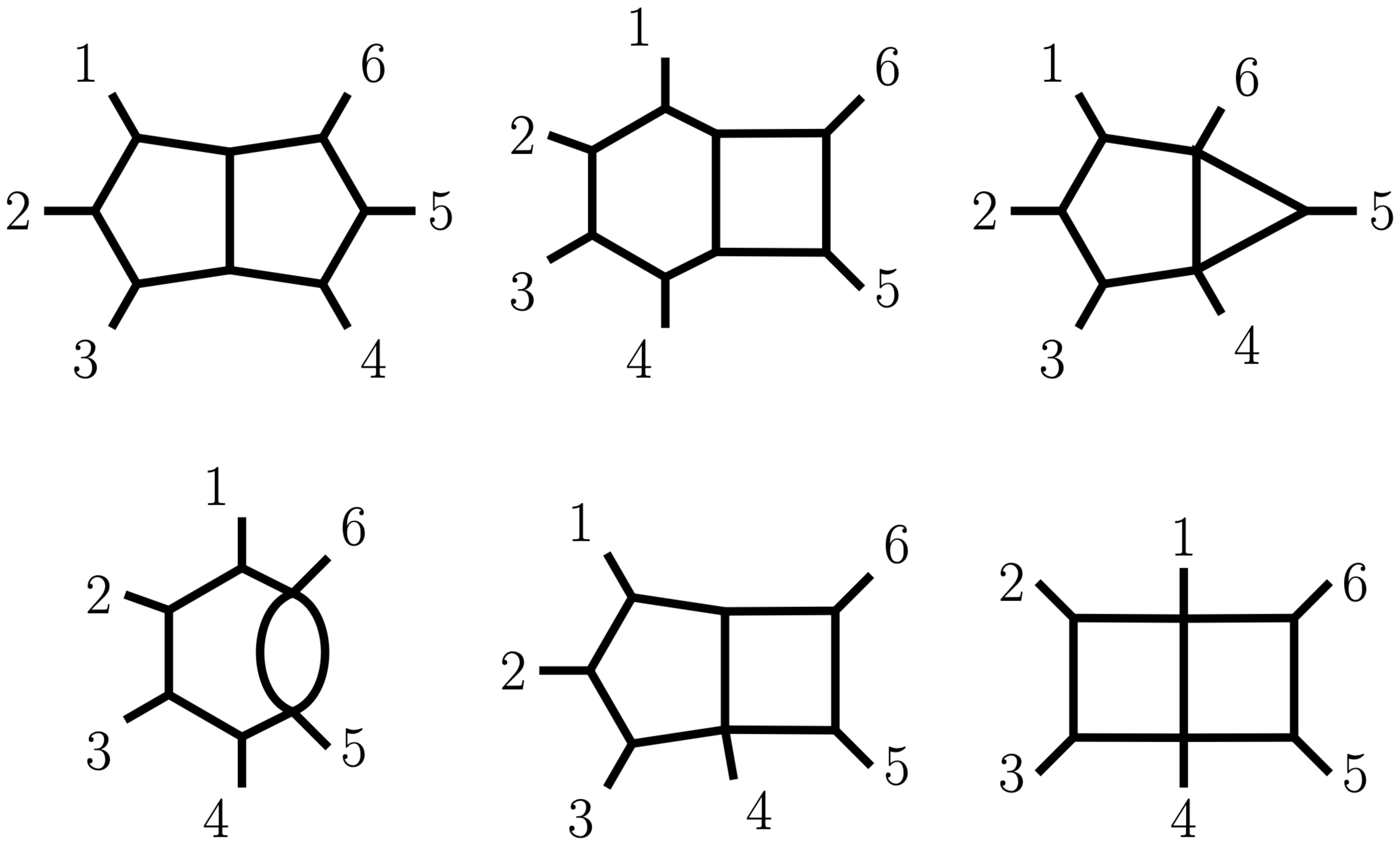
- For four-dimensional external legs, we never need hexagons (or bigger diagrams)
  - one-line Gram version: $G(\ell 12345) = p(\vec{s}) + \sum \rho_i \, p_i(\vec{s}) + \sum \rho_i \rho_j \, p_{ij}(\vec{s}) = 0$
  - two-line Gram version: $G\begin{pmatrix} \ell 1234 \\ 12345 \end{pmatrix} = q(\vec{s}) + \sum \rho_i \, q_i(\vec{s}) = 0$
  - trivial to write hexagon as linear combination of 6 pentagons
  - NB: not an IBP relation!



see e.g. [J. Gluza, K. Kajda, and D. A. Kosower, 2010]

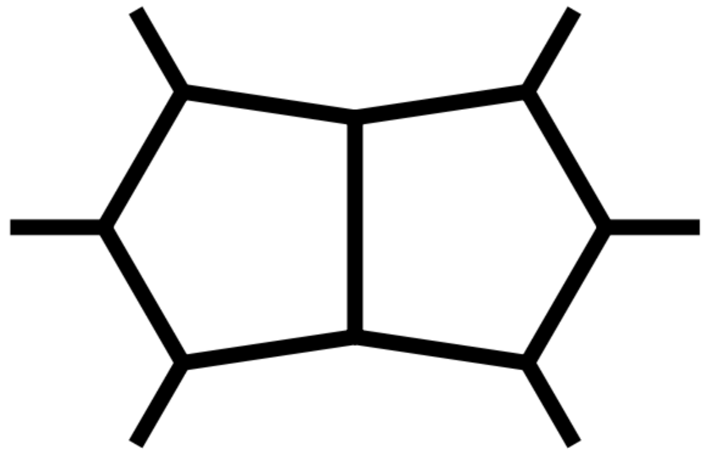
Two-Loop Planar Integrals

- Six topologies, can be embedded in the first two (double pentagon and hexabox)

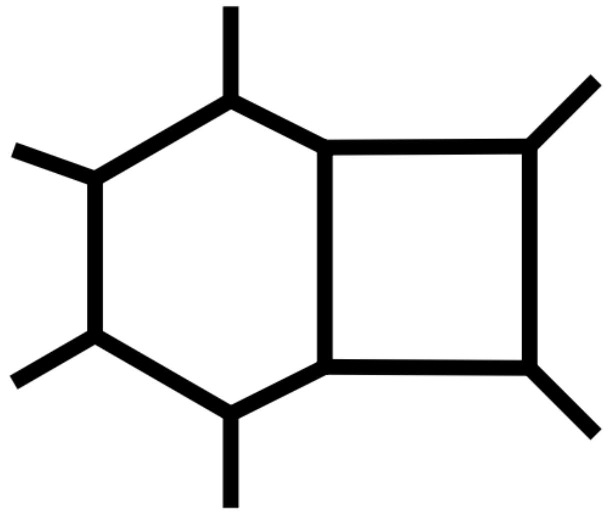


Two-Loop Integrals — counting masters

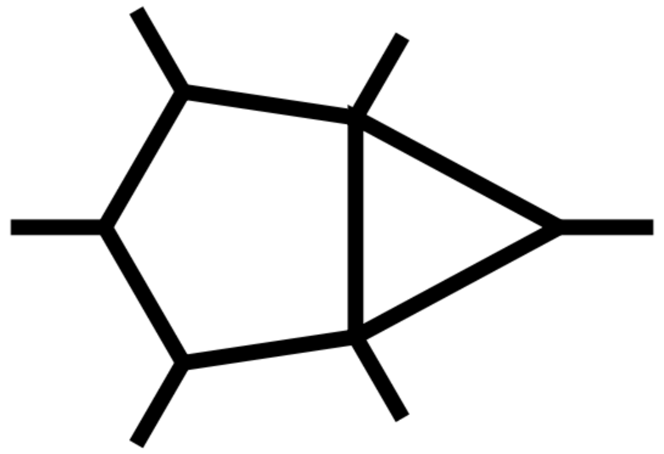
- Drop in number of masters from four-dimensional external legs



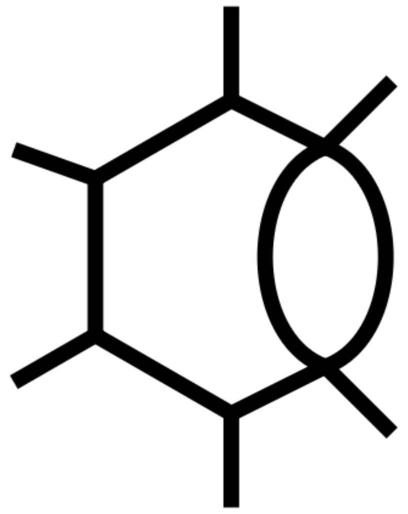
$8 \to 5$



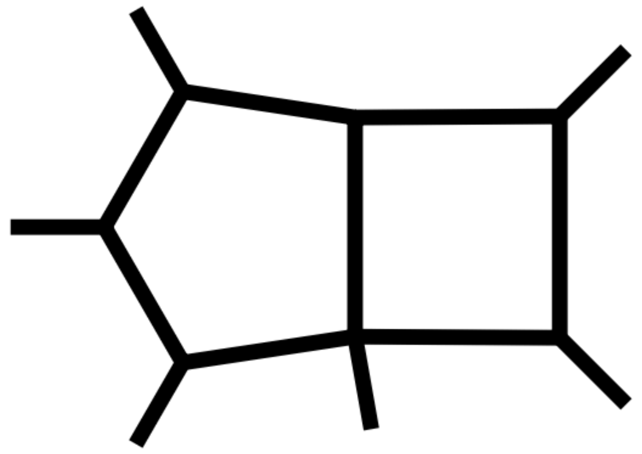
$3 \to 1$



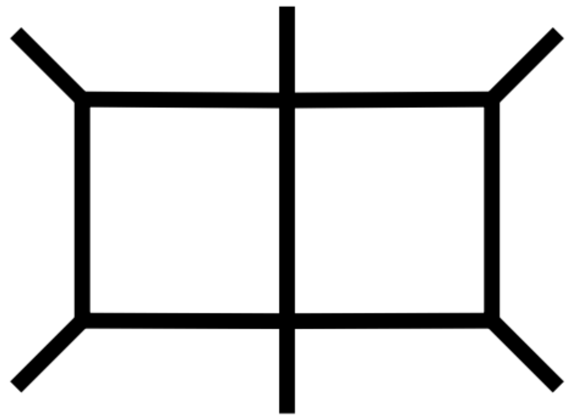
$2 \to 1$



$2 \to 1$



$4 \to 3$

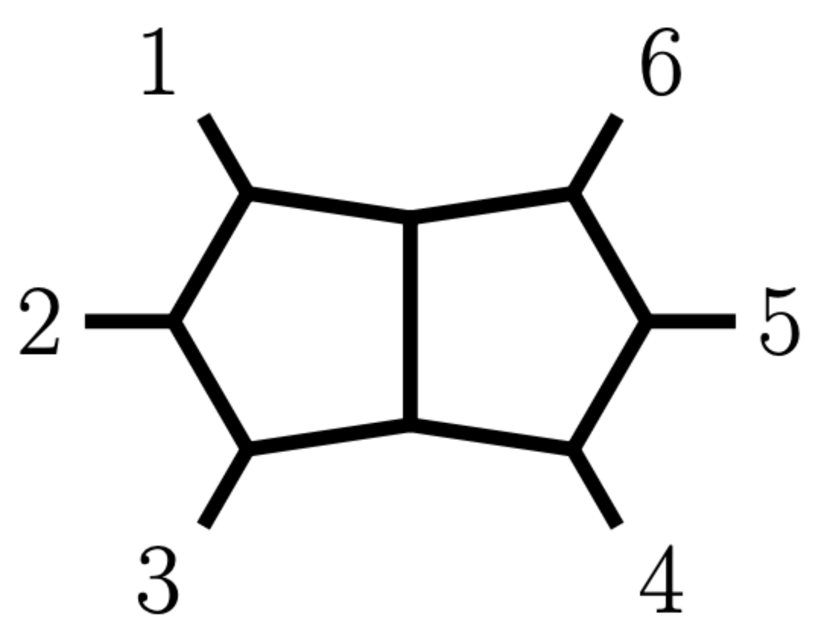


$9 \to 7$

IBP relations + $\left(G\begin{pmatrix} \ell_1 1234 \\ 12345 \end{pmatrix} = G\begin{pmatrix} \ell_2 1234 \\ 12345 \end{pmatrix} = 0 \right)$

Two-Loop Integrals — evanescent integrals

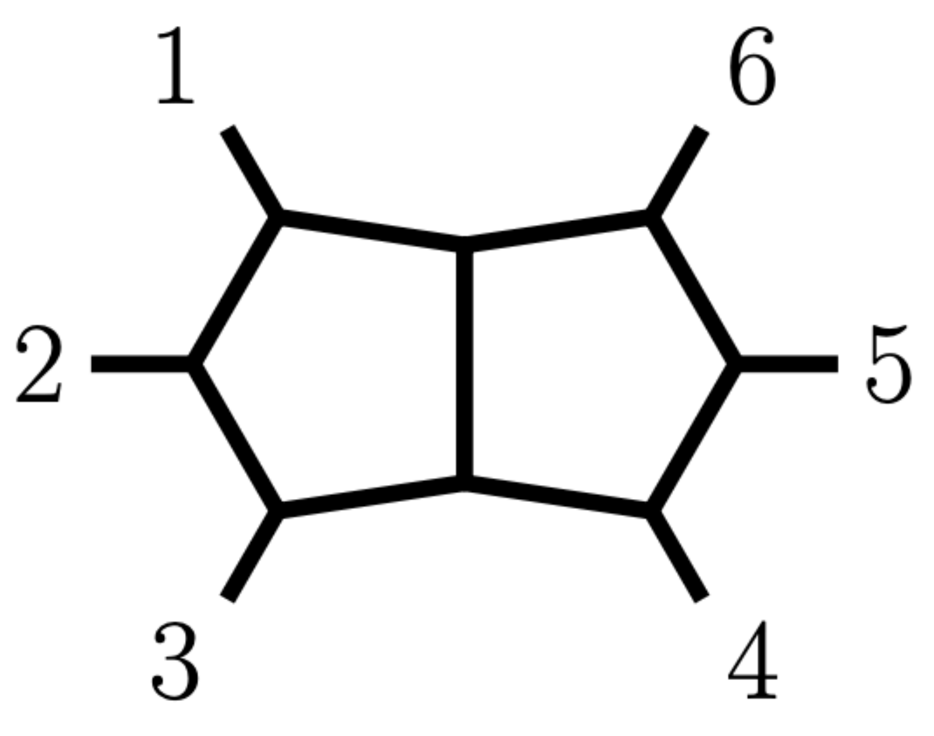
- How many integrals can we make $\mathcal{O}(\epsilon)$? Can we **remove a complete topology?**



see also [J. Gluza, K. Kajda, and D. A. Kosower, 2010]
[G. Gambuti, D. A. Kosower, P. P. Novichkov, and L. Tancredi, 2023]

- Integrand is polynomial in components of ℓ_1 and ℓ_2
- There are 11 independent integration variables (that cannot be trivially integrated over)
- Renormalisable power counting: 5 powers of ℓ_1, 5 powers of ℓ_2, 8 powers of $\ell_1 \cup \ell_2 \Rightarrow$ **UV finite!**
- Transverse variables: $\alpha_1 = \varepsilon_{\mu\nu\rho\sigma}\ell_1^\mu p_1^\nu p_2^\rho p_3^\sigma$; $\alpha_2 = \varepsilon_{\mu\nu\rho\sigma}\ell_2^\mu p_4^\nu p_5^\rho p_6^\sigma$
- First set of 11 independent variables: $\{\rho_1, \ldots, \rho_9, \alpha_1, \alpha_2\}$
- Integrand is polynomial in ρ_i and α_i ; monomials with ρ_i belong to lower topology $\Rightarrow$ polynomial in α_i only
- Count independent monomials in α_1 and α_2 that satisfy power counting $\Rightarrow$ **33-dimensional space**

Two-Loop Integrals — evanescent integrals



- Can we **fill the 33-dimensional space with evanescent integrals?**
  - new set of variables:

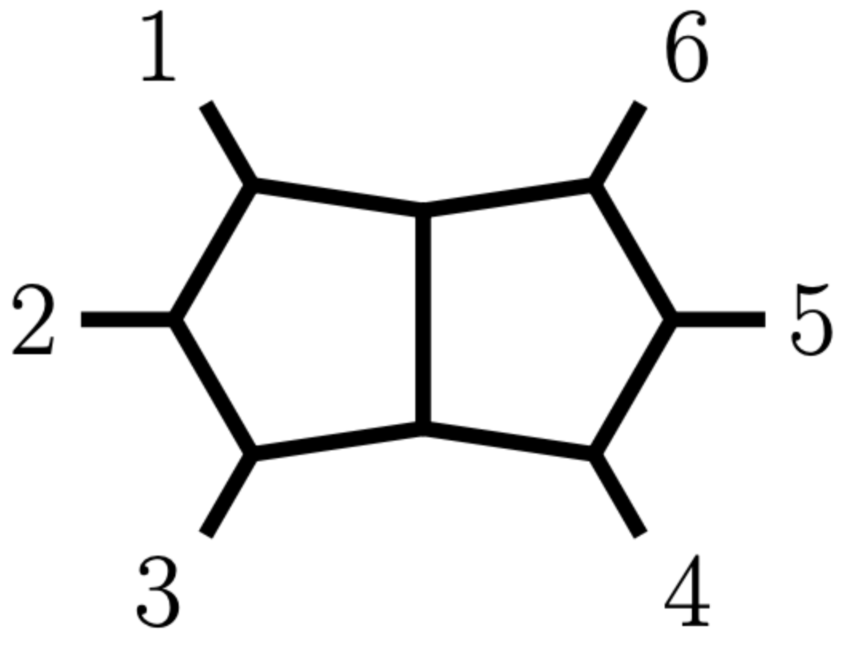
$$\alpha_1 = \varepsilon_{\mu\nu\rho\sigma}\ell_1^\mu p_1^\nu p_2^\rho p_3^\sigma \qquad \alpha_2 = \varepsilon_{\mu\nu\rho\sigma}\ell_2^\mu p_4^\nu p_5^\rho p_6^\sigma \qquad \mu_{ij} = \ell_i^\epsilon \cdot \ell_j^\epsilon = G\begin{pmatrix}\ell_i 1234\\ \ell_j 1234\end{pmatrix}$$

  - soft regions of ℓ_1: $\ell_1 - p_1 \sim \lambda, \quad \ell_1 - p_1 - p_2 \sim \lambda$
  - collinear regions of ℓ_1: $\ell_1 // p_1, \quad \ell_1 - p_1 // p_2, \quad \ell_1 - p_1 - p_2 // p_3$

  } $\alpha_1, \mu_{11}, \mu_{12}$ vanish in these limits!

  - consider monomials $m_\beta = \alpha_1^{\beta_1}\alpha_2^{\beta_2}\mu_{11}^{\beta_3}\mu_{22}^{\beta_4}\mu_{12}^{\beta_5}, \quad$ for $\beta_i \geq 0$
  - **finite integrals**: $1 \leq \beta_1 + 2\beta_3 + \beta_5 \leq 5\,, \quad 1 \leq \beta_2 + 2\beta_4 + \beta_5 \leq 5\,, \quad \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 \leq 8$
  - **evanescent integrals**: $\beta_3 + \beta_4 + \beta_5 \geq 1$
  - there are 165 monomials leading to evanescent integrals, but not independent on maximal cut

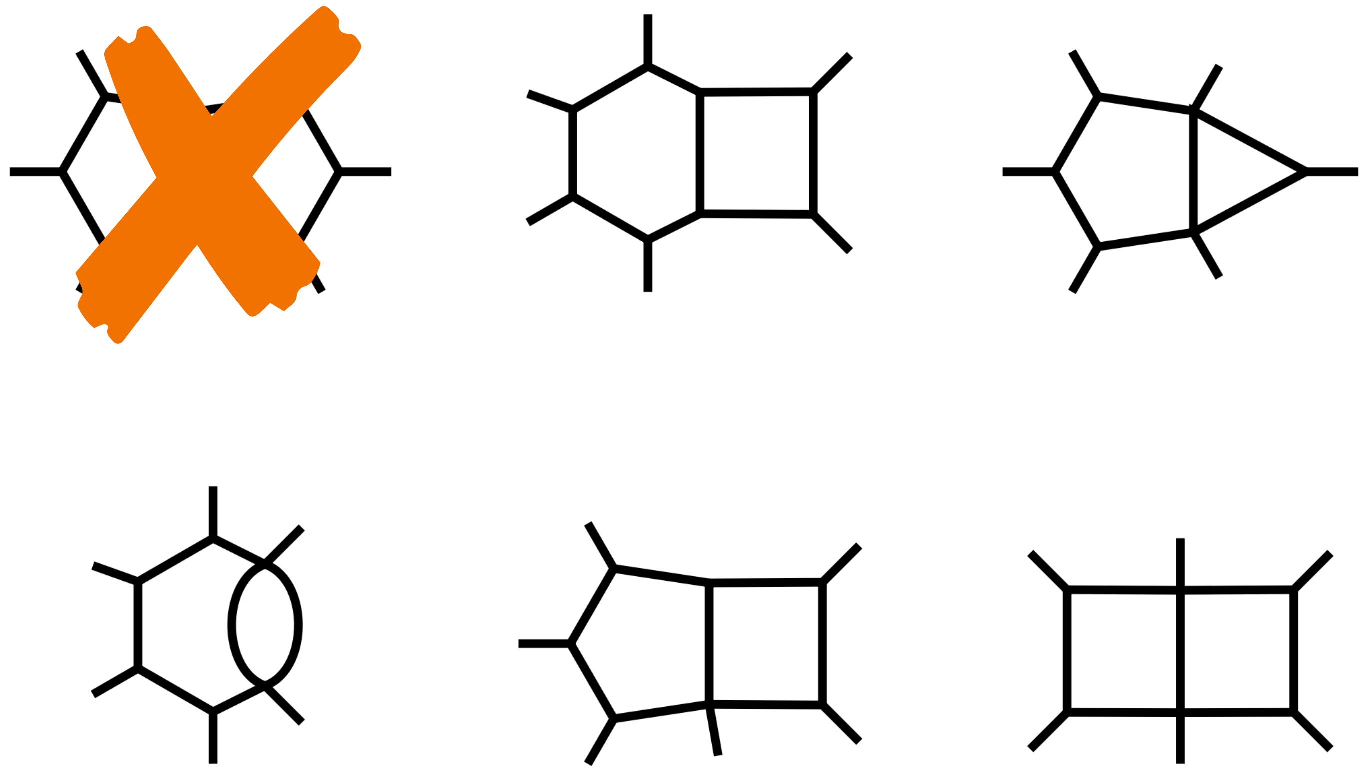
Two-Loop Integrals — evanescent integrals



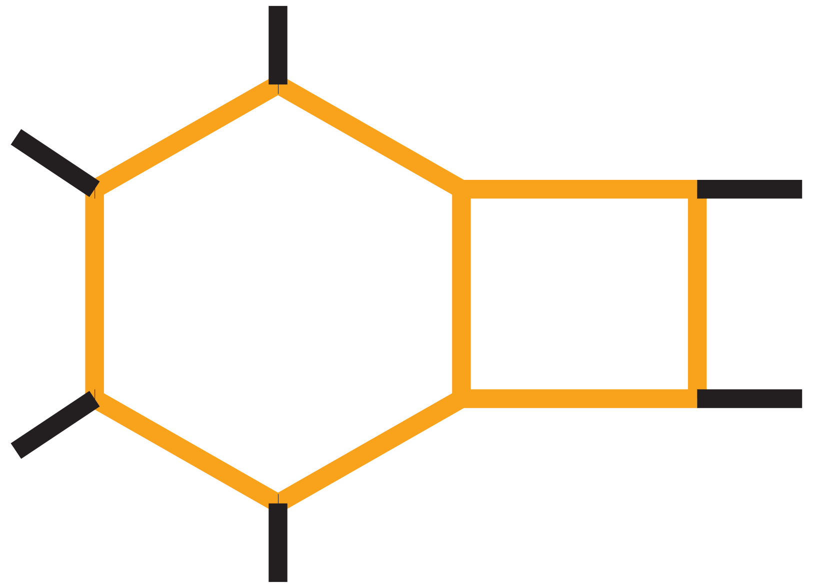
- Can we **fill the 33-dimensional space with evanescent integrals?**
  - Evaluate evanescent monomials on on-shell configurations of ℓ_1 and ℓ_2
  - There are **33 independent monomials**!

$$\{\mu_{12}, \alpha_2\mu_{12}, \alpha_2\mu_{11}, \alpha_1\mu_{12}, \alpha_1\mu_{22}, \mu_{12}^2, \mu_{12}\mu_{22}, \mu_{11}\mu_{12}, \mu_{11}\mu_{22}, \alpha_2^2\mu_{11}, \alpha_1\alpha_2\mu_{22}, \alpha_1\alpha_2\mu_{11}, \alpha_1^2\mu_{22}, \alpha_2\mu_{12}^2, \alpha_2\mu_{12}\mu_{22}, \alpha_2\mu_{11}\mu_{12}, \alpha_2\mu_{11}^2, \alpha_1\mu_{22}^2, \alpha_1\mu_{11}\mu_{12}, \mu_{12}^3,$$
$$\mu_{12}^2\mu_{22}, \mu_{12}\mu_{22}^2, \mu_{11}\mu_{12}^2, \mu_{11}^2\mu_{12}, \alpha_1\alpha_2\mu_{22}^2, \alpha_1\alpha_2\mu_{11}^2, \alpha_2\mu_{12}^3, \alpha_2\mu_{12}^2\mu_{22}, \alpha_2\mu_{11}\mu_{12}^2, \alpha_2\mu_{11}^2\mu_{12}, \mu_{12}^4, \mu_{12}^3\mu_{22}, \mu_{11}\mu_{12}^3\}$$

  - For renormalisable power counting, we can **make the whole double pentagon topology $\mathcal{O}(\epsilon)$!**

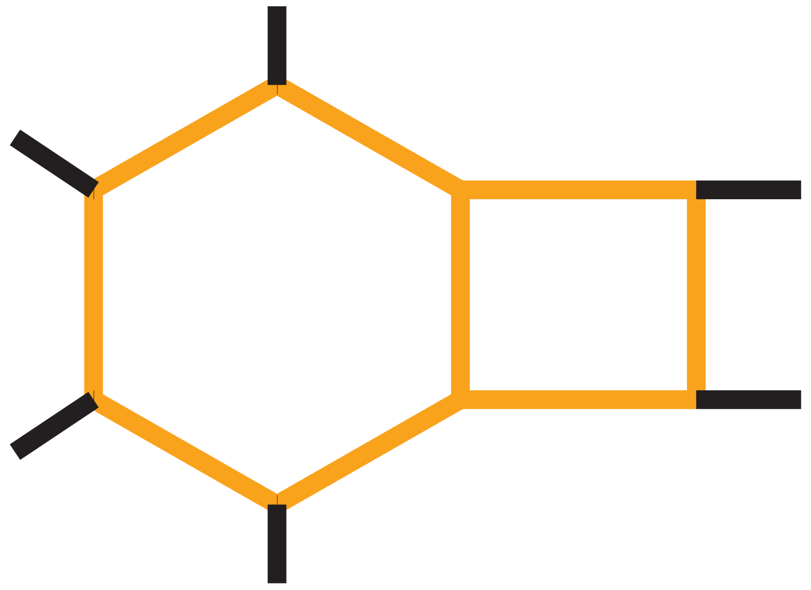


$\Rightarrow$



NB: the single master in the hexabox topology can also be made $\mathcal{O}(\epsilon)$, but it is very easy to compute so we keep it

Two-Loop Integrals — Hexabox Topology



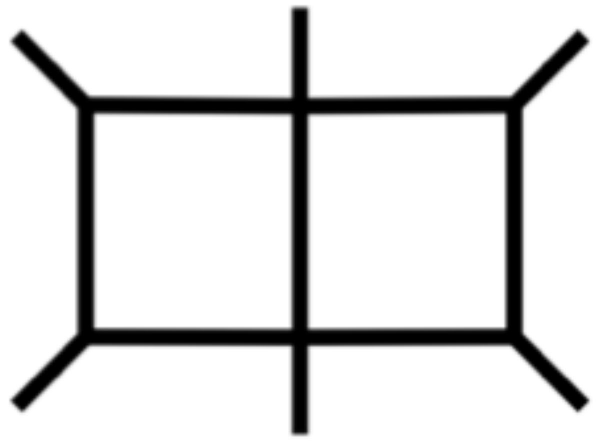
- **Approach**: DEs in canonical basis, powered by finite field tech

  [A.V. Kotikov, 1991] [E. Remiddi, 1997] [T. Gehrmann and E. Remiddi, 1999]
  [J. Henn, 2013]
  [A. von Manteuffel and R. M. Schabinger, 2014] [T. Peraro, 2016]

- **Difficulty 1:** four-dimensional differential operators ⇒ build them numerically

- **Difficulty 2:** fast enough IBPs ⇒ KIRA with finite fields

  [J. Klappert, F. Lange, P. Maierhöfer, and J. Usovitsch, 2021]



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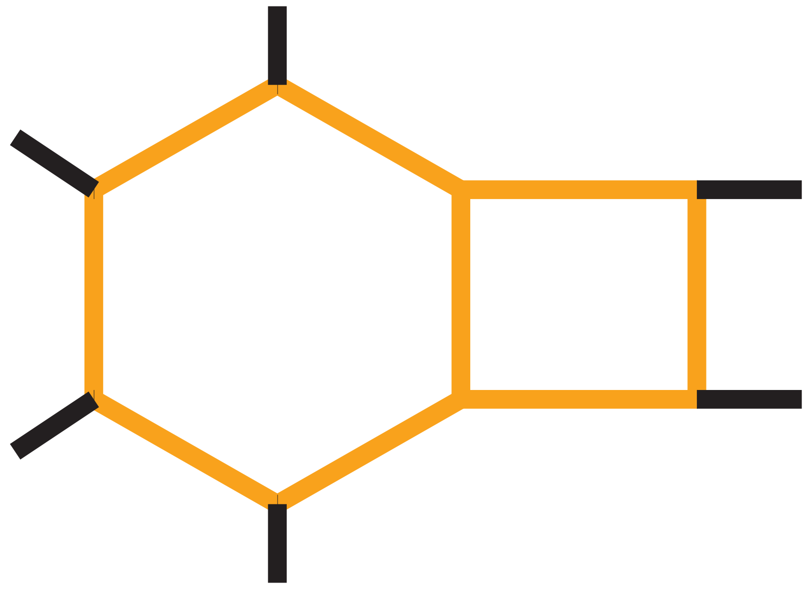
- **Difficulty 3:** pure basis for double-box
  - Start from on-shell pure basis — similar to [J. Henn, T. Peraro, Y. Xu, and Y. Zhang, 2021]
  - Go off-shell and rotate with Magnus expansion — [M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, and L. Tancredi, 2014]
  - Functional reconstruction techniques for rotation matrix, many non-trivial technical improvements to account for $\Delta_6 = 0$ condition — ideas from [G. De Laurentis and B. Page, 2022]

Two-Loop Integrals — Hexabox Topology

$$\mathrm{d}I = \epsilon\, \mathbf{M}\, I\,, \qquad \mathbf{M} = M_\alpha\, \mathrm{d}\log W_\alpha$$



- **202 master integrals**, in 126 different sectors
  - 185 are five-point one-mass topologies. All this for 17 new integrals!
- **128 letters**, only 11 new ones (associated with six-point topologies)
  - 9 appear in the differential equation on the maximal cut, 2 appear in the off-shell penta-triangle
  - 2 \`off-shell letters' are odd under $\sqrt{\Delta_5}$, 9 \`on-shell' letters are even
- Closure of the alphabet under dihedral symmetry: **245 letters**
- From here we know how to proceed to get basis of functions, evaluate integrals, …

see Simon's talk from Monday

Two-Loop Integrals — Numerical Checks

- Two four-dimensional rational points (start from twistor variables)

[J. Henn, T. Peraro, Y. Xu, and Y. Zhang, 2021]

$$P_1 = \{x_1 = -1, x_2 = -24, x_3 = 9, x_4 = 54, x_5 = 38, x_6 = -97, x_7 = 3, x_8 = 95\}$$
$$= \{s_{12} = -1, s_{23} = -38, s_{34} = -\frac{1039}{6}, s_{45} = -2712776, s_{56} = -50409, s_{16} = -1662120, s_{123} = -95, s_{234} = -19926, s_{345} = -2752175\}$$

$$P_2 = \{x_1 = -1, x_2 = -74, x_3 = 7, x_4 = 53, x_5 = 34, x_6 = -68, x_7 = -91, x_8 = 76\}$$
$$= \{s_{12} = -1, s_{23} = -34, s_{34} = -\frac{827628}{37}, s_{45} = -7995952, s_{56} = -147756, s_{16} = -4804450, s_{123} = -76, s_{234} = -56882, s_{345} = -6669452\}.$$

- Compute with $\texttt{AMFlow}$ at P_1 and P_2
- Evolve from P_1 to P_2 with $\texttt{DiffExp}$

} **Everything agrees ⇒ DE is correct!**

[X. Liu and Y.-Q. Ma, 2023]

[M. Hidding, 2020]

  - **NB**: $\texttt{DiffExp}$ evolution in twistor space to ensure we stay in four-dimensional kinematics

Summary and discussion

- Strong impact of four-dimensional kinematics beyond five-point integrals
  - Reduced number of masters
  - Decouple topologies at $\mathcal{O}(\epsilon)$

- Possible to work with Mandelstam variables modulo four-dimensional condition
  - Keeps symmetry of problem
  - More transparent interpretation of singularities of differential equation
  - Non-trivial generalisation of functional reconstruction technology

- Not a lot of new information in the genuine six-point integrals…
  - 17 new integrals, 11 new letters

Summary and discussion

- Solving the four-dimensional DE
  - Are twistors really the only way?
- Beyond six points
  - Do we see even more simplification? see e.g. [P. Bargiela, T. Yang, 2025]
- What to do with the integrals?
  - one-loop six-gluon amplitudes are already very very complicated…

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Thank you