Four loop anomalous dimensions in QCD

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Work at four loops:

- *Non-Singlet Splitting Functions at Four Loops in QCD – The Fermionic Contributions –*
  B. Kniehl, S. M., V. Velizhanin, and A. Vogt arXiv:2505.09381
- *Four-loop splitting functions in QCD – The gluon-to-gluon case –*
  G. Falcioni, F. Herzog, S. M., A. Pelloni and A. Vogt arXiv:2410.08089
- *Constraints for twist-two alien operators in QCD*
  G. Falcioni, F. Herzog, S. M., and S. Van Thurenhout arXiv:2409.02870
- *Four-loop splitting functions in QCD – The quark-to-gluon case –*
  G. Falcioni, F. Herzog, S. M., A. Pelloni and A. Vogt arXiv:2404.09701
- *Additional moments and x-space approximations of four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt arXiv:2310.05744
- *The double fermionic contribution to the four-loop quark-to-gluon splitting function*
  G. Falcioni, F. Herzog, S. M., J. Vermaseren and A. Vogt arXiv:2310.01245
- *Four-loop splitting functions in QCD – The gluon-to-quark case –*
  G. Falcioni, F. Herzog, S. M., and A. Vogt arXiv:2307.04158
- *Four-loop splitting functions in QCD – The quark-quark case –*
  F. Herzog, G. Falcioni, S. M., and A. Vogt arXiv:2302.07593

Work at four loops (cont'd):

- *Low moments of the four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2111.15561`
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`

Past:

- Many papers of MVV and friends . . . `2001 - ...`

Parton evolution

- Evolution equations for parton distributions
  - non-singlet valence PDFs $q_{\rm ns}^{\rm v} = \sum_f (q_f - \bar{q}_f)$
  - flavor asymmetries $q_{{\rm ns},ff'}^{\pm} = (q_f \pm \bar{q}_f) - (q_{f'} \pm \bar{q}_{f'})$

$$\frac{d}{d\ln\mu^2}\, q_{\rm ns}^{\pm,{\rm v}} \quad = \quad P_{\rm ns}^{\pm,{\rm v}} \otimes q_{\rm ns}^{\pm,{\rm v}}$$

  - quark-flavor singlet PDFs $q_{\rm s} = \sum_f (q_f + \bar{q}_f)$ and gluon PDF g
  - 2x2 matrix equation

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \quad = \quad \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$$

- Splitting functions P up to N^{3}LO (work in progress)

$$P_{\rm ij} \quad = \quad \underbrace{\alpha_s\, P_{\rm ij}^{(0)} + \alpha_s^2\, P_{\rm ij}^{(1)} + \alpha_s^3\, P_{\rm ij}^{(2)}}_{\text{NNLO: standard approximation}} + \alpha_s^4\, P_{\rm ij}^{(3)} + \ldots$$

- Anomalous dimensions (Mellin transform)

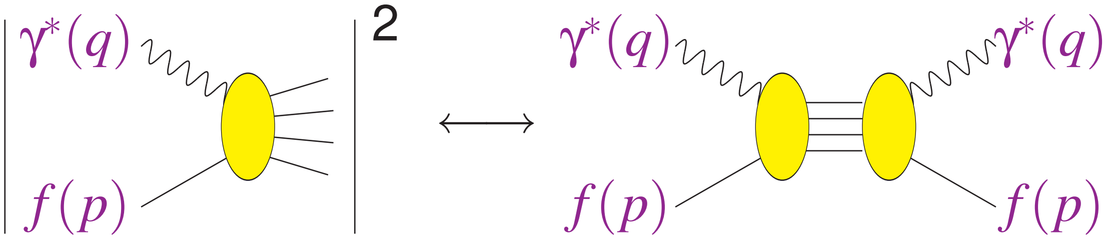
$$\gamma_{\rm ij}(N) = -\int_0^1 dx\, x^N\, P_{\rm ij}(x) = \alpha_s\, \gamma_{\rm ij}^{(0)} + \alpha_s^2\, \gamma_{\rm ij}^{(1)} + \alpha_s^3\, \gamma_{\rm ij}^{(2)} + \alpha_s^4\, \gamma_{\rm ij}^{(3)} + \ldots$$

Operator product expansion (I)

- Direct computation of physical observable
  - structures functions in deep-inelastic scattering (DIS)

Optical theorem

- Total cross section related to imaginary part of Compton amplitude
  - Bjorken variable $x = Q^2/(2p \cdot q)$ and momentum transfer $Q^2 = -q^2$



- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu} = \mathrm{i} \int d^4z \, \mathrm{e}^{\mathrm{i}q\cdot z} \langle \mathrm{P} | \, T \left(j_\mu^\dagger(z) j_\nu(0) \right) | \mathrm{P} \rangle$

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x,Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x,Q^2)$$

- OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, x fixed
  Wilson ‘72; Christ, Hasslacher, Mueller ‘72

Operator product expansion (II)

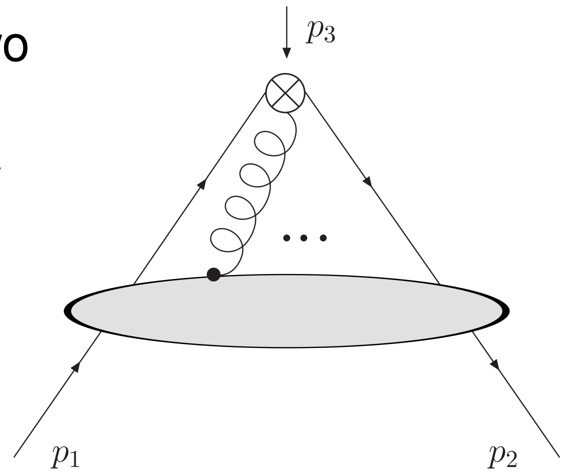
- OPE for parton states gives coefficient functions in Mellin space $C_{a,i}^N$

$$T_{\mu\nu,\mathrm{k}} \;=\; \sum_{N,\mathrm{j}} \left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}\, C_{L,\mathrm{j}}^N \left(\frac{Q^2}{\mu^2}, \alpha_s\right) + d_{\mu\nu}\, C_{2,\mathrm{j}}^N \left(\frac{Q^2}{\mu^2}, \alpha_s\right) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p\cdot q} C_{3,\mathrm{j}}^N \left(\frac{Q^2}{\mu^2}, \alpha_s\right) \right] A_{\mathrm{jk}}^N \left(\mu^2\right) \;+\; \text{higher twists}$$

- Operator matrix elements $A_{\mathrm{ij}}^N = \langle j | O_{\mathrm{i}}^N | j \rangle$ in parton state
- Anomalous dimensions $\gamma_{\mathrm{ij}}(N)$ from collinear singularities of Compton amplitude $T_{\mu\nu}$ after mass factorization
  - established computational approach through four loops
    one loop Buras '80; two loops Kazakov, Kotikov '90; S.M., Vermaseren '99; three loops S.M., Vermaseren, Vogt '04; four loops Davies, Vogt, Ruijl, Ueda, Vermaseren '17; S.M., Ruijl, Ueda, Vermaseren, Vogt to appear
- Versatile calculation method
  - photon-DIS $\longrightarrow \gamma_{\mathrm{qq}}\,, \gamma_{\mathrm{qg}}$
  - Higgs (scalar)-DIS $\longrightarrow \gamma_{\mathrm{gq}}\,, \gamma_{\mathrm{gg}}$
  - graviton-DIS $\longrightarrow \Delta\gamma_{\mathrm{ij}}$ (polarized quantities) S.M., Vermaseren, Vogt '14

Operator matrix elements

- Scalar singlet operators of spin-N and twist two from contraction with light-like vector Δ_μ
  - quarks ψ, field strenghth $F^{\mu;a} = \Delta_\nu F^{\mu\nu;a}$
  - N covariant derivatives $D = \Delta_\mu D^\mu$



$$\mathcal{O}_{\rm q}^{(N)} = \frac{1}{2}\text{Tr}\left[\overline{\psi}\not{\Delta}\, D^{N-1}\psi\right]$$

$$\mathcal{O}_{\rm g}^{(N)} = \frac{1}{2}\text{Tr}\left[F_\nu\, D^{N-2} F^\nu\right]$$

- Direct computation of OMEs $A_{\rm ij}^N = \langle j|O_{\rm i}^N|j\rangle$ in parton state
  - anomalous dimensions $\gamma_{\rm ij}(N)$ from renormalization of operators
- Physical operators mix under renormalization with alien operators
  Dixon, Taylor '74; Kluberg-Stern, Zuber '75; Joglekar, Lee '76

Workflow

- Zero-momentum transfer through operator gives 2-point functions
- Feynman diagrams generation with **Qgraf** Nogueira '91
- Four-loop IBP reduction with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and **TForm** Tentyukov, Vermaseren '07

Quark pure-singlet splitting function $P_{\rm qq} = P^{+}_{\rm ns} + P_{\rm ps}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

Moments of pure-singlet splitting function

- Moments $N = 2, \ldots 20$ for pure-singlet anomalous dimension $\gamma_{\rm ps}^{(3)}(N)$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N\!=\!2) &= -691.5937093\, n_f + 84.77398149\, n_f^2 + 4.466956849\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N\!=\!4) &= -109.3302335\, n_f + 8.776885259\, n_f^2 + 0.306077137\, n_f^3 \,, \\
&\ldots \\
\gamma_{\rm ps}^{(3)}(N\!=\!20) &= -0.442681568\, n_f + 0.805745333\, n_f^2 - 0.020918264\, n_f^3 \,.
\end{aligned}
$$

- Results $N \leq 8$ agree with inclusive DIS S.M., Ruijl, Ueda, Vermaseren, Vogt '21 (also for $N = 10$ and $N = 12$)
- Quartic color terms $d_R^{abcd} d_R^{abcd}$ agree with S.M., Ruijl, Ueda, Vermaseren, Vogt '18
- Large-n_f parts agree with all-N results Davies, Vogt, Ruijl, Ueda, Vermaseren '17;
- ζ_4 terms in $\gamma_{\rm ps}^{(3)}(N)$ agree with Davies, Vogt '17 based on no-π^2 theorem Jamin, Miravitllas '18; Baikov, Chetyrkin '18
- Checked by n_f^2 terms at all-N Gehrmann, von Manteuffel, Sotnikov, Yang '23

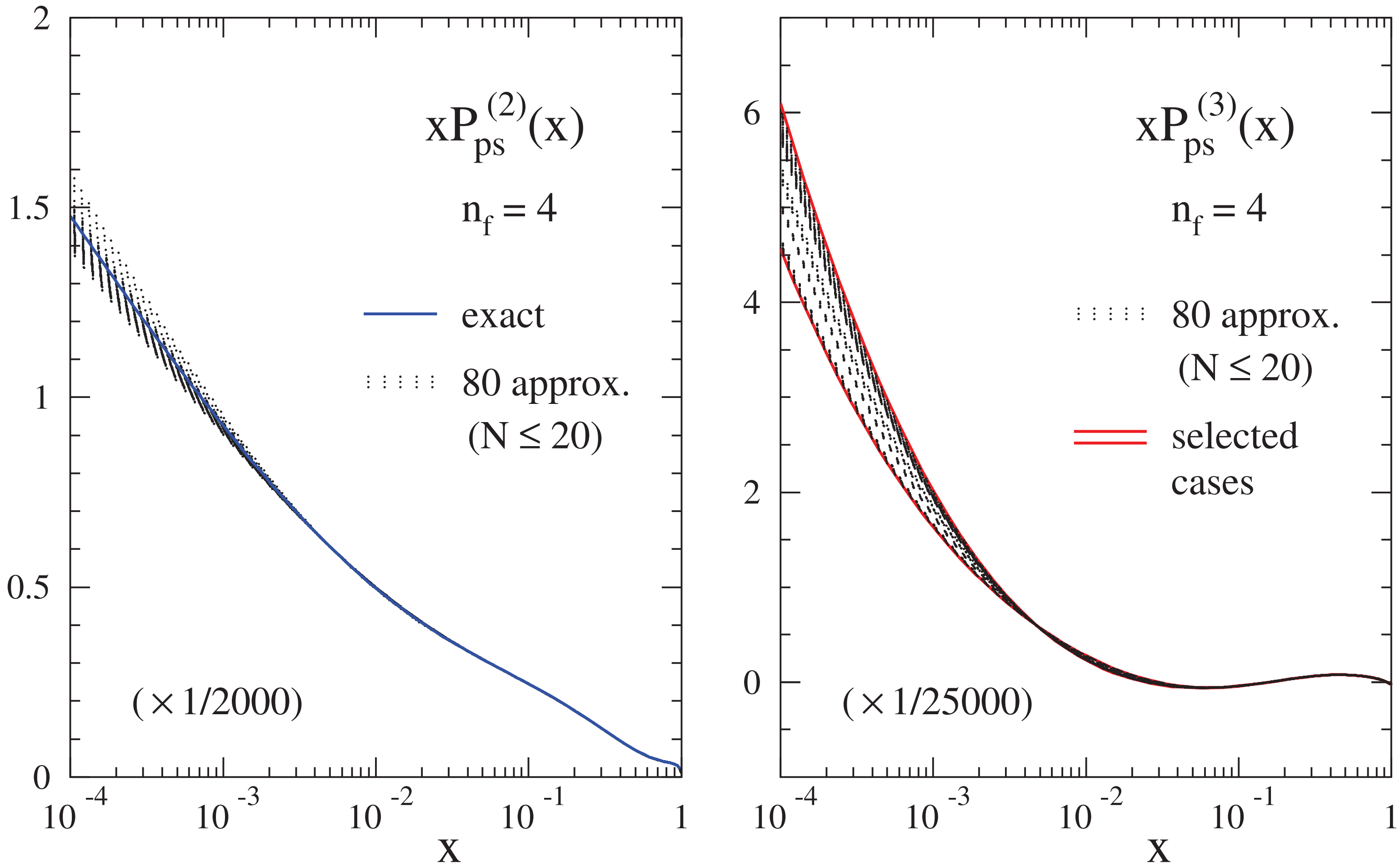
Outlook

- Higher moments $N = 22, \ldots$ to be published

Approximations in x-space

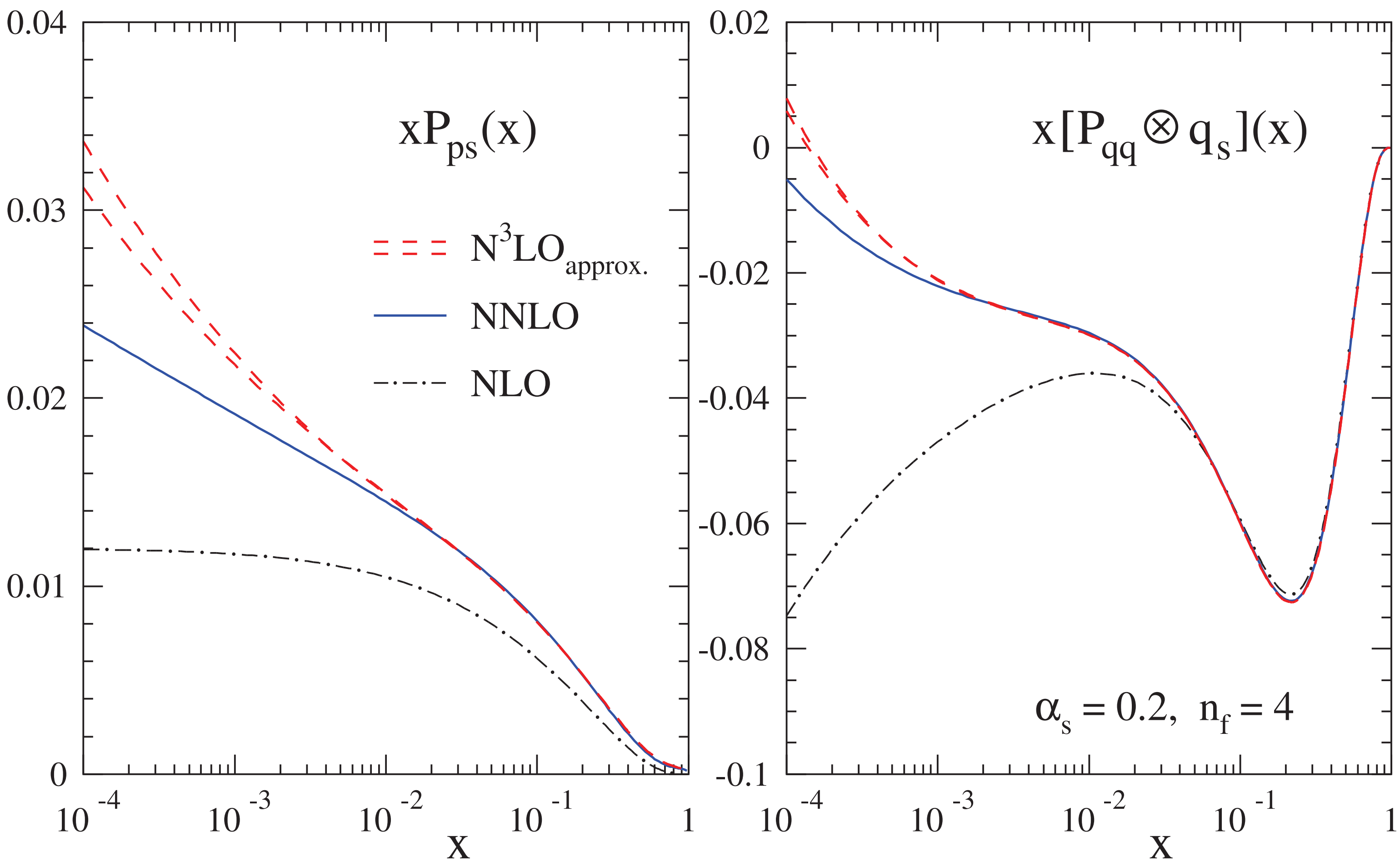
- Large- and small-x information about four-loop splitting function $P_{\rm ps}^{(3)}(x)$
  - leading logarithm $(\ln^2 x)/x$ Catani, Hautmann '94
  - sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$ Davies, Kom, S.M., Vogt '22
  - leading large-x terms $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$ with $k = 4, 3$ known Soar, S.M., Vermaseren, Vogt '09
- Approximation of four-loop splitting function $P_{\rm ps}^{(3)}(x)$ with suitable ansatz
  - unknown leading small-x terms: $(\ln x)/x$, $1/x$
  - unknown sub-dominant logarithms: $\ln^k x$ with $k = 3, 2, 1$
  - two remaining large-x terms $(1-x)\ln^k(1-x)$ with $k = 2, 1$
  - different two-parameter polynomials together one function (dilogarithm $\mathrm{Li}_2(x)$ or $\ln^k(1-x)$ with $k = 2, 1$, suppressed as $x \to 1$)
- Approximations for phenomenology with fixed $n_f = 3, 4, 5$
  - easy-to-use
  - no correlations between different n_f dependenct terms accounted for

Pure-singlet splitting function



- Approximations to pure-singlet splitting function $P_{\rm ps}^{(n)}(x)$ at $n_f = 4$ with 80 trial functions
  - left: three-loops ($n = 2$) with comparison to known result
  - right: four-loops ($n = 3$) with remaining uncertainty

Pure-singlet splitting function



- Left: results for $P_{\rm ps}(x)$ up to N^{3}LO; $\alpha_s \ = \ 0.2$ fixed, $n_f \ = 4$
- Right: contribution to evolution kernel $d \ln q_{\rm s}/d \ln \mu_f^2$ up to N^{3}LO for typical quark-singlet shape

$$xq_{\,\rm s}(x, \mu_0^2) \ = \ 0.6\, x^{\,-0.3}(1-x)^{3.5}\,(1 + 5.0\, x^{\,0.8})$$

Scale stability of evolution (I)

- PDF evolution
  - splitting functions enter PDF evolution via convolution

$$\frac{d}{d\ln\mu^2}\, f_{\rm i}(x) \quad = \quad \sum_{\rm j} \int_x^1 \frac{dz}{z}\, P_{\rm ij}(z)\, f_{\rm j}\left(\frac{x}{z}\right)$$

- Interplay between $P(z \sim x \to 0)$ and $f(\frac{x}{z} \to 1)$
  - $P(\mathbf{z \sim x \to 0})$ has largest uncertainty
  - $f(\mathbf{\frac{x}{z}} \to \mathbf{1})$ is suppressed
- Model singlet PDFs

$$\begin{aligned} x\mathrm{q}_{\,\rm s}(x,\mu_0^2) \quad &= \quad 0.6\, x^{\,-0.3}(\mathbf{1-x})^{\mathbf{3.5}}\,\left(1 + 5.0\, x^{\,0.8}\right) \\ x\mathrm{g}(x,\mu_0^2) \quad &= \quad 1.6\, x^{\,-0.3}(\mathbf{1-x})^{\mathbf{4.5}}\,\left(1 - 0.6\, x^{\,0.3}\right) \end{aligned}$$

- Residual small-x uncertainty in four-loop splitting functions at $x \sim \mathcal{O}(10^{-4})$ affects PDFs only at $x \sim \mathcal{O}(10^{-5})$
  - edge of LHC parton kinematics (low scales, forward region)
  - $x \sim 10^{-5}$ corresponds to $y \sim 4$ and $Q \sim 10$ GeV

All-N results

Analytic reconstruction (I)

- Sufficiently many Mellin moments allow for reconstruction of analytic all-N expressions through solution of Diophantine equations

  Lenstra, Lenstra, Lovász '82

- Harmonic sums define basis in space of functions for $\gamma_{\rm ij}(N)$

$$S_{\pm m_1,\dots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\dots,m_k}(i)$$

  - at weight w there are $2 \cdot 3^{w-1}$ harmonic sums
  - l-loop $\gamma_{\rm ij}^{(l-1)}(N)$ contains harmonic sums up to weight $2l-1$
    $\longrightarrow$ numbers grow quickly: $2, 18, 162, 1458$ sums for $l = 1, 2, 3, 4$

- Some applications in QCD
  - three-loop non-singlet transversity $\gamma_{\rm tr}^{(2)}$ Velizhanin'12
  - three-loop polarized $\Delta\gamma_{\rm ij}^{(2)}$ S.M., Vermaseren, Vogt '14
  - four-loop non-singlet $\gamma_{\rm ns}^{(3)\pm}$ (large-n_c) S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - four-loop non-singlet DIS $C_{\rm ns}^{(4)}$ (large-n_f)

    Basdew-Sharma, Pelloni, Herzog, Vogt '22
  - ...

Analytic reconstruction (II)

Conformal symmetry and integrability

- Gribov-Lipatov reciprocity relation (RR)
  - diagonal splitting functions $P_{\rm ii}^{(0)}(x)$ invariant under mapping $x \to \frac{1}{x}$
- RR realized for universal $\gamma_{\rm u}(N)$ in $N = 4$ SYM theory
  - uniform transcendentality sums with $w = 2l - 1$ only at l-loops
- RR in N-space for QCD implies $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(\alpha_s)\right)$
- RR constraints for $\gamma_{\rm u}$ reduce number to 2^{w-1} sums at weight w for $\gamma_{\rm u}$
  - $2^{w+1} - 1$ objects with denominators $1/(N+1)$ added (255 at $w = 7$)

Example

- Large-n_c limit of $\gamma_{\rm ns}^{(3)\pm}$ only needs harmonic sums with positive index
  - weight w RR sums given by Fibonacci number $F(w)$
  - total number of unknowns (including powers $1/(N+1)$) amount to $F(w+4) - 2$ (87 at $w = 7$)
- Additional 46 constraints from large-x/small-x ($N \to \infty/N \to 0$) limit
- Solution becomes feasible with 18 Mellin moments for $\gamma_{\rm ns}^{(3)\pm}$

Analytic reconstruction (III)

- Mellin moments suffice to determine all-N result for parts of $\gamma_{\rm ps}^{(3)}(N)$
  - harmonic sums and Riemann ζ_n terms up to total weight $w = 7$
- Terms proportional to ζ_5 are particularly simple
  - N-dependent terms respect RR
  - RR implies invariance under mapping $N \to -N - 1$
- Combinations of denominators $\eta = \dfrac{1}{N} - \dfrac{1}{N+1}$ and $\nu = \dfrac{1}{N-1} - \dfrac{1}{N+2}$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N)\Big|_{\zeta_5} \;=\;& 160\, n_f\, C_F^3 \left(9\,\eta + 6\,\eta^2 - 4\,\nu\right) + 80/3\, n_f\, C_A\, C_F^2 \left(-\,9\,\eta - 6\,\eta^2 + 4\,\nu\right) \\
& + 40/9\, n_f\, C_A^2\, C_F \left(-\,1 - 214\,\eta - 144\,\eta^2 + 104\,\nu\right) \\
& + 320/3\, n_f\, \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(-\,1 + 56\,\eta + 36\,\eta^2 - 16\,\nu\right)
\end{aligned}
$$

- Inverse Mellin transformation generates additional terms with ζ_n
  - ζ_n in N-space $\neq$ ζ_n in x-space

Non-singlet splitting functions $P_{\rm ns}^{\pm,\rm v}$

Small-x behavior (I)

The small x-limit: $x \to 0$

- Structure of non-singlet splitting functions $P_{\rm ns}^{\pm}$ at small x
  - double-logarithmic contributions with very large coefficients
  - resummation for $P_{\rm ns}^{+}$ to leading logarithmic (LL) accuracy in Mellin-N space — Kirschner, Lipatov '83

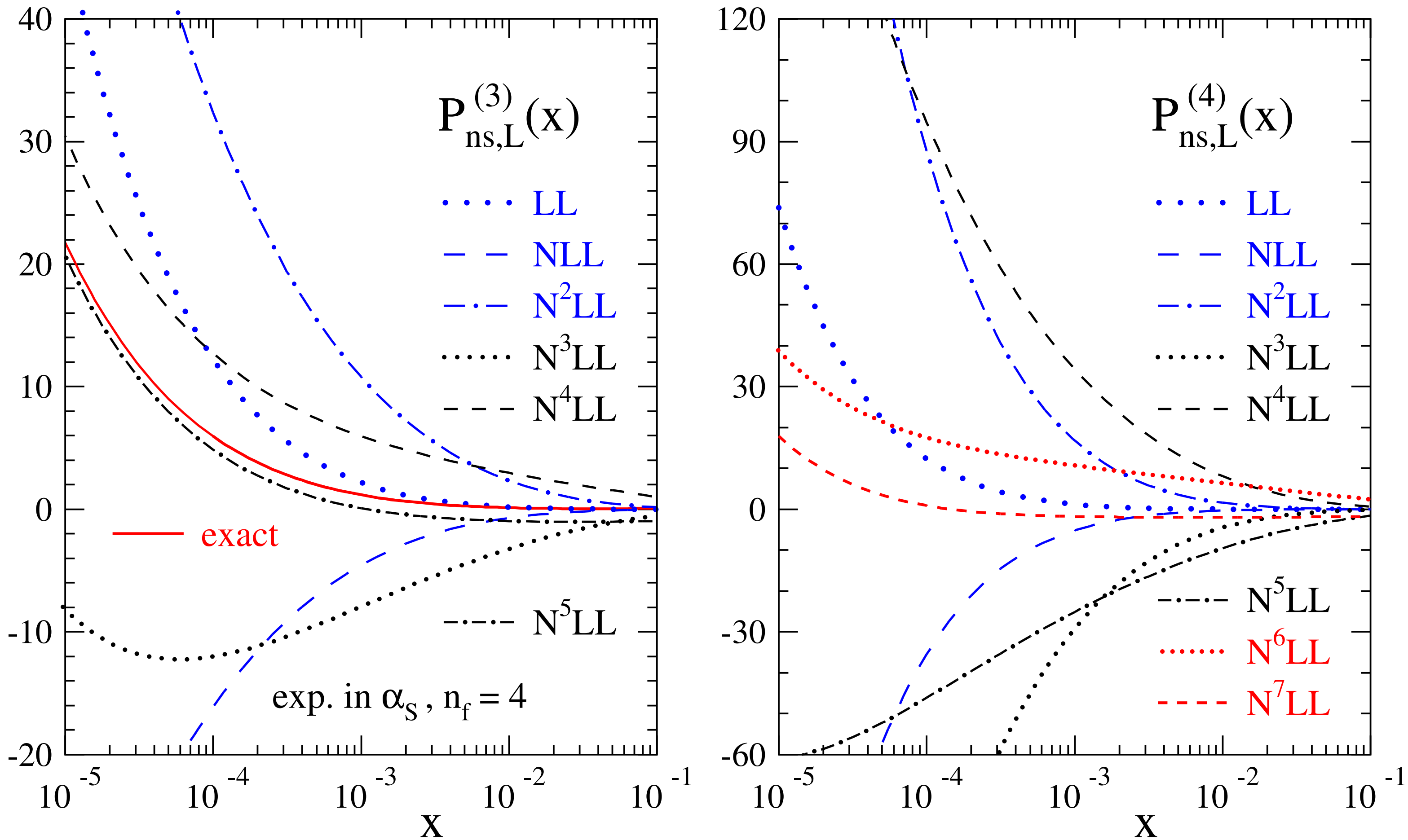
$$\gamma^{+}_{\rm ns,LL}(N,\alpha_s) \;=\; -\frac{N}{2}\left\{1-\left(1-\frac{2\alpha_s C_F}{\pi N^2}\right)^{1/2}\right\}$$

- Large-n_c limit with intriguing structure — Velizhanin '14

$$\gamma^{+}_{\rm ns}(N,\alpha_s)\,\Big(N+\gamma^{+}_{\rm ns}(N,\alpha_s)-\beta(\alpha_s)/\alpha_s\Big) \;=\; O(1)$$

  - Laurent expansion about $N=0$
  - product of anomalous dimension for operator and its shadow is finite
  - $\gamma^{+}_{\rm ns}(N,\alpha_s)$ is not analytic function at intersection of two (or more) Regge trajectories
  - interesting relations to twist-two operators in $O(N)\,\phi^4$ theory and Gross-Neveu models — Manashov, S.M., Shumilov '25

Small-x behavior (II)

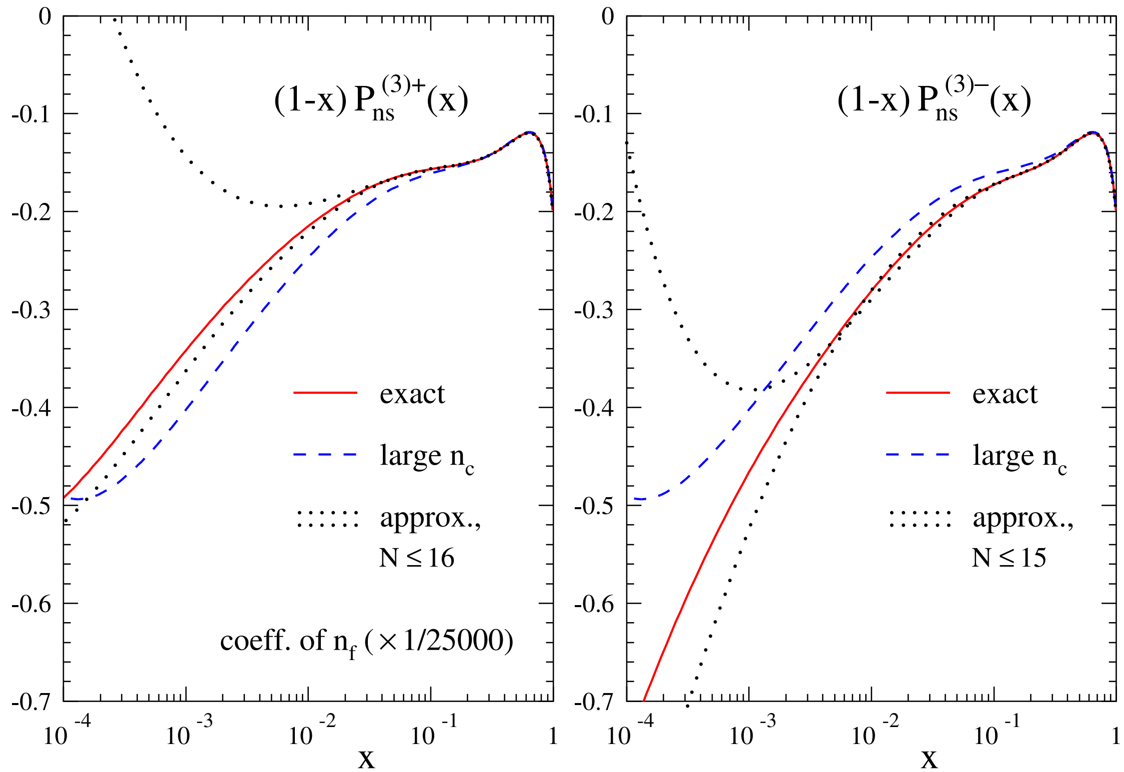


- Splitting functions $P_{\rm ns}^{(3),+}$ (left) and $P_{\rm ns}^{(4),+}$ (right) Davies, Kom, S.M., Vogt '22
  - small-x approximations in large-n_c limit for $P^{(n)}_{\rm ns,\,L}(x)$ with $n_f = 4$
  - resummation to N^{7}LL accuracy (modified Bessel functions)

Davies, Kom, S.M., Vogt '22

Four-loop non-singlet splitting functions (I)

- Four-loop $P_{\rm ns}^{(3)\pm}(x)$ (n_f^1 terms)
- Comparison to large-n_c limit and uncertainty bands from approximations

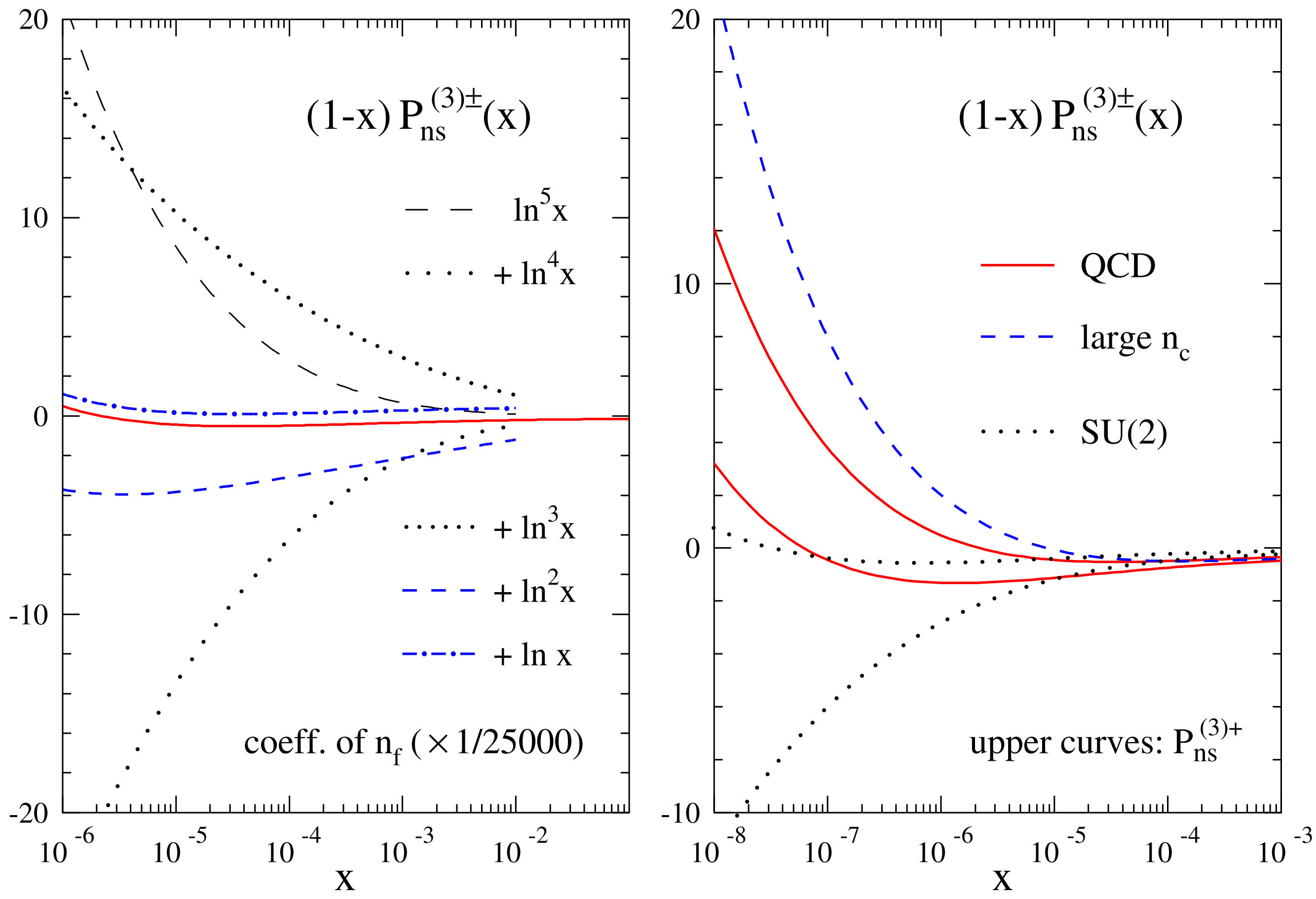


Analytic results

- Contributions to non-singlet splitting functions
  - n_f-terms (n_f^3 Gracey '94; n_f^2 Davies, Vogt, Ruijl, Ueda, Vermaseren '16)
  - leading n_c terms S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - $n_f C_F^3$ terms Gehrmann, von Manteuffel, Sotnikov, Yang '23
  - all n_f terms Kniehl, S.M., Velizhanin, Vogt '25

    analytic reconstruction based on moments $N \leq 24$ for $n_f d_{FF}^{(4)}$ and $N \leq 32$ for $n_f C_F^2 C_A$

Four-loop non-singlet splitting functions (II)



- n_f^1 terms of $P_{\rm ns}^{(3)\pm}(x)$
  - left: exact result and small-x logarithms: $\ln^k x$ with $k = 5, 4, 3, 2, 1$
  - right: illustration of large-n_c approximation with QCD $n_c = 3$ and $SU(2)$ gauge theory

Splitting functions at large x

The large x-limit: $x \to 1$

- Structure of diagonal splitting functions $P_{\rm ii}$ (for ${\rm i = q, g}$) at large x

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_n^{\rm i}}{(1-x)_+} + B_n^{\rm i}\,\delta(1-x) + \ldots$$

- Cusp anomalous dimension $A_n^{\rm i}$
  - known from $1/\epsilon^2$-poles of QCD form factor
- Four loop results in QCD

Large-n_c (Henn, Lee, Smirnov, Smirnov, Steinhauser ‘16; S. M., Ruijl, Ueda, Vermaseren, Vogt ‘17); n_f (Grozin ‘18; Henn, Peraro, Stahlhofen, Wasser ‘19); n_f^2 (Davies, Ruijl, Ueda, Vermaseren, Vogt ‘16; Lee, Smirnov, Smirnov, Steinhauser ‘17); n_f^3 (Gracey ‘94; Beneke, Braun, ‘95); quartic colour (Lee, Smirnov, Smirnov, Steinhauser ‘19; Henn, Korchemsky, Mistlberger ‘19)

- Virtual anomalous dimension $B_n^{\rm i}$
  - parts related to $1/\epsilon$-poles of QCD form factor

Quark virtual anomalous dimension

- Four loop result (up to one unknown $b^{\mathrm{q}}_{4,FA}$) Kniehl, S.M., Velizhanin, Vogt '25

$$
\begin{aligned}
B^{\mathrm{q}}_4 \quad = \\
& C_F^4 \left(\frac{4873}{24} - 450\zeta_2 - \frac{684}{5}\zeta_2^2 - \frac{16888}{35}\zeta_2^3 + 2004\zeta_3 - 120\zeta_3\zeta_2 + \frac{128}{5}\zeta_3\zeta_2^2 - 1152\zeta_3^2 - 2520\zeta_5 - 384\zeta_5\zeta_2 + 5880\zeta_7\right) \\
& + C_F C_A^3 \left(-\frac{371201}{648} - \frac{1}{24} b^{\mathrm{q}}_{4,FA} + \frac{4582}{3}\zeta_2 - \frac{22388}{135}\zeta_2^2 + \frac{48368}{315}\zeta_2^3 - \frac{153670}{81}\zeta_3 + \frac{472}{3}\zeta_3\zeta_2 + \frac{16}{5}\zeta_3\zeta_2^2 + 528\zeta_3^2 \right. \\
& \left. + \frac{11372}{9}\zeta_5 + 504\zeta_5\zeta_2 - 2870\zeta_7\right) + C_F^2 C_A^2 \left(\frac{29639}{36} - \frac{46771}{27}\zeta_2 - \frac{24340}{27}\zeta_2^2 - \frac{21988}{35}\zeta_2^3 + \frac{129662}{27}\zeta_3 + \frac{2096}{9}\zeta_3\zeta_2 \right. \\
& \left. - \frac{64}{5}\zeta_3\zeta_2^2 - \frac{7102}{3}\zeta_3^2 + \frac{5354}{9}\zeta_5 - 2104\zeta_5\zeta_2 + 8610\zeta_7\right) + C_F^3 C_A \left(-\frac{2085}{4} + 1167\zeta_2 + \frac{4334}{5}\zeta_2^2 + \frac{317188}{315}\zeta_2^3 \right. \\
& \left. - 3260\zeta_3 - \frac{1988}{3}\zeta_3\zeta_2 + \frac{256}{5}\zeta_3\zeta_2^2 + 3220\zeta_3^2 - 976\zeta_5 + 2064\zeta_5\zeta_2 - 10920\zeta_7\right) + \frac{d_F^{abcd} d_A^{abcd}}{n_F}\left(b^{\mathrm{q}}_{4,FA}\right) \\
& + n_f C_F^3 \left(32 + 162\zeta_2 - \frac{408}{5}\zeta_2^2 - \frac{51472}{315}\zeta_2^3 - 308\zeta_3 - \frac{256}{3}\zeta_3\zeta_2 + 224\zeta_3^2 + 912\zeta_5\right) \\
& + n_f C_F^2 C_A \left(-\frac{7751}{54} - \frac{3892}{27}\zeta_2 + \frac{55708}{135}\zeta_2^2 + \frac{2808}{35}\zeta_2^3 - \frac{15400}{27}\zeta_3 + \frac{2672}{9}\zeta_3\zeta_2 - \frac{1232}{3}\zeta_3^2 - \frac{7432}{9}\zeta_5\right) \\
& + n_f C_F C_A^2 \left(\frac{20027}{108} - \frac{41092}{81}\zeta_2 + \frac{2468}{45}\zeta_2^2 - \frac{4472}{135}\zeta_2^3 + \frac{9554}{27}\zeta_3 - \frac{580}{3}\zeta_3\zeta_2 + \frac{416}{3}\zeta_3^2 + \frac{1130}{9}\zeta_5\right) \\
& + n_f \frac{d_F^{abcd} d_F^{abcd}}{n_F} \left(-192 + \frac{1888}{3}\zeta_2 - \frac{704}{15}\zeta_2^2 + \frac{2048}{45}\zeta_2^3 - \frac{992}{3}\zeta_3 + 64\zeta_3\zeta_2 + 256\zeta_3^2 - 1120\zeta_5\right) \\
& + n_f^2 C_F^2 \left(-\frac{188}{27} + \frac{1244}{27}\zeta_2 - \frac{4208}{135}\zeta_2^2 + \frac{56}{27}\zeta_3 - \frac{160}{9}\zeta_3\zeta_2 + \frac{368}{9}\zeta_5\right) + n_f^2 C_F C_A \left(-\frac{193}{54} + \frac{3170}{81}\zeta_2 - \frac{32}{9}\zeta_2^2 \right. \\
& \left. - \frac{320}{9}\zeta_3 + \frac{80}{3}\zeta_3\zeta_2 - \frac{88}{9}\zeta_5\right) + n_f^3 C_F \left(-\frac{131}{81} + \frac{32}{81}\zeta_2 - \frac{64}{135}\zeta_2^2 + \frac{304}{81}\zeta_3\right)
\end{aligned}
$$

Summary

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N^{3}LO (and even N^{4}LO)
  - evolutions equations expected to achieve percent-level
  - massive use of computer algebra
- Four-loop splitting functions approximated from moments $N = 2, \ldots 20$
  - residual uncertainties negligible in wide kinematic range of x probed at current and future colliders
  - $P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps}$, $P_{\rm qg}$ $P_{\rm gq}$ and $P_{\rm gg}$ all done
- More all-N results to come
- Novel structural insights into QCD from integrabilty and conformal symmetry
  - Key parts of QCD inherited from $N = 4$ Super Yang-Mills theory
  - Conformal symmetry in parts of QCD evolution equations