Recap: the Ising model

Let $\sigma = \pm$ be binary variables

the classical Ising model is denoted by:

$$H = J_{12}\sigma_1\sigma_2 + J_{13}\sigma_1\sigma_3 + \dots = \sum_{i<j} J_{ij}\sigma_i\sigma_j$$

Hamiltonian of a spin system coupling N binary

variables $\sigma_0 \dots \sigma_{N-1}$

J_{ij} denotes the interaction coefficients

by conv. $H = -\sum_{i<j} J_{ij}\sigma_i\sigma_j$

The quantum Ising model Hamiltonian

$$\hat{H} = \sum_{i<j} J_{ij} \; \hat{Z}_i \otimes \hat{Z}_j$$

$\sigma_i \longrightarrow \hat{\sigma}_i$ (operator)

$|1\rangle \Leftrightarrow |\downarrow\rangle \longrightarrow \sigma = -1$, negative z direction

Both states are eigenvectors of the magnetic momentum operator

denoted by $\hat{Z}$

if $|\underline{z}\rangle = |z_0, \ldots, z_{N-1}\rangle$, the eigenstates are config. of spins

$$\hat{z}_j |z_j\rangle = (-1)^{z_j} |z_j\rangle = \sigma_j |z_j\rangle$$

the energy of the ground state

$$\langle z|\hat{H}|z\rangle = \sum_{i<j} J_{ij}\,\sigma_i \sigma_j$$

is the minimum energy of the classical model

The Quantum Ising Model (transverse-field)

$$\hat{H}_{QI} = -\sum_{i<j} J_{ij}\, \hat{Z}_i \otimes \hat{Z}_j - \sum_i h_i \hat{X}_i$$

here, $\hat{Z}_\alpha$ & $\hat{X}_\alpha$ are the Pauli operators on qubit at index α.

* $\hat{H}_{QI}$ = terms in z-direction + terms in x-direction

h_i: excites superposition of qubits

$$\hat{H}_{QI} = \sum_{i<j} J_{ij}\, \hat{Z}_i \hat{Z}_j - \sum_i h_i \hat{X}_i$$

One qubit Quantum Ising model

$$\hat{H}_{QI} = -J\hat{Z} - h\hat{X}$$

in matrix form

$$\hat{H}_{QI} = \begin{pmatrix} -J & -h \\ -h & J \end{pmatrix}$$

eigenstates

$$\hat{H}_{QI} |E\rangle = E |E\rangle$$

eigenvalues

$$E_{\pm} = \pm\sqrt{J^2 + h^2}$$

eigen vectors (normalized)

$$|E_{\mp}\rangle = \frac{1}{\sqrt{h^2 + \left(\pm J \mp \sqrt{h^2+J^2}\right)^2}} \begin{pmatrix} \pm J \pm \sqrt{h^2+J^2} \\ h \end{pmatrix}$$

Quantum ML with Quantum Ising Model

recall: Quantum feature maps the classical data
to quantum data

target: define a varcational quantum model to
minimize the quantum Ising Hamiltonian
with single qubit

$$|0\rangle \longrightarrow \boxed{\hat{H}} \longrightarrow \boxed{\hat{Z}_{\theta_z}} \longrightarrow \boxed{\hat{Y}_{\theta_y}} \longrightarrow \boxed{\hat{X}_{\theta_x}} \longrightarrow |\psi\rangle$$

$$|\psi\rangle = e^{j\theta_x \hat{X}} e^{j\theta_y \hat{Y}} e^{j\theta_z \hat{Z}} \hat{H} |0\rangle$$

Variational anzatz is determined by $\hat{U}$

$$\hat{U} = e^{j\theta_x \hat{X}} e^{j\theta_y \hat{Y}} e^{j\theta_z \hat{Z}} \hat{H}$$

θ_x, θ_y & $\hat{\theta}_z$ $\longrightarrow$ parameters of the variational model

$$\underline{\theta} = (\theta_x, \theta_y, \theta_z)$$

$\hat{U}$ is a quantum feature map that maps the classical $\underline{\theta}$ into a one qubit Hilbert space

Implentation with Tensorflow

A simple N.N. Layer is implemented computing the expected value of the Hamiltonian $\langle \hat{H} \rangle$ at the initial state

the model is optimized to compute $\hat{\Theta}^{opt}$ that minimize the output of the Hamilt. Layer the model is trained by minimizing the mean value of the Hamiltonian - a loss function

* Tests

* the model is trained where the final state is compared to

$$|E_-\rangle = \frac{1}{\sqrt{h^2 + \left(J + \sqrt{h^2 + J^2} \right)^2}} \begin{pmatrix} J + \sqrt{h^2 + J^2} \\ h \end{pmatrix}$$

for diff. values of J & h

$\underline{h = 0, \quad J = 1}$ $\longrightarrow |E_-\rangle = |0\rangle = |\downarrow\rangle$ $\left\{ \begin{array}{l} \text{eigenvalue of } \hat{Z} \\ \text{with } E_- = -J \\ \qquad\quad = -1 \end{array} \right.$

$\underline{h = 0, \quad J = 1.0}$ $\longrightarrow \langle \hat{H} \rangle = -1$

$h = 0.5, \quad J = 0$ $\longrightarrow \langle \hat{H} \rangle = -0.5$ i.e. $\left\{ \begin{array}{l} E_- = -h \\ |E_-\rangle = H|0\rangle \end{array} \right.$

$h = 1.7, \quad J = 1 \quad \rightarrow \langle \hat{H} \rangle = -1.97$ (as expected