Asymptotic behavior of angular integrals and expansion by regions

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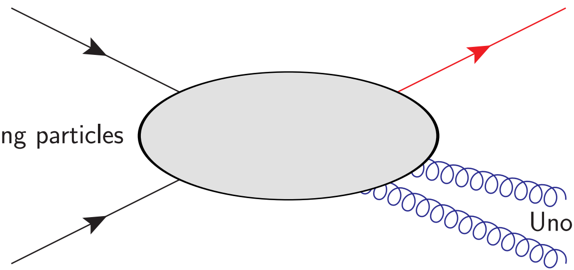
Outline

1. Introduction
2. Extracting the massless limit of angular integrals
3. Expansion by regions meets ang. int.
4. Conclusion

1. Introduction

What integrals are we talking about?



Two particle phase space ($d = 4 - 2\varepsilon$):

$$\int \mathrm{dPS}_{2,P} = \int \frac{\mathrm{d}^{d-1} k_1}{(2\pi)^{d-1} 2k_1^0} \int \frac{\mathrm{d}^{d-1} k_2}{(2\pi)^{d-1} 2k_2^0} (2\pi)^d \delta^d (P - k_1 - k_2)$$

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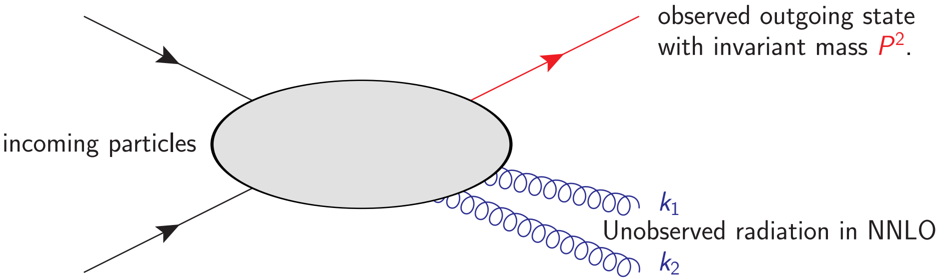

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Angular integration measure:

$$\mathrm{d}\Omega_{3-2\varepsilon} \equiv \mathrm{d}\theta_1 \sin^{1-2\varepsilon}\theta_1 \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2 \ldots \mathrm{d}\theta_n \sin^{2-n-2\varepsilon}\theta_n \mathrm{d}\Omega_{3-n-2\varepsilon}$$

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Typical integral from the literature (2→3 kinematics, 2 denominators, e.g. Van Neerven 1986):

$$\int \frac{\mathrm{d}\theta_1 \mathrm{d}\theta_2 \sin^{1-2\varepsilon}\theta_1 \sin^{-2\varepsilon}\theta_2}{(a + b\cos\theta_1)^{j_1}(A + B\cos\theta_1 + C\sin\theta_1\cos\theta_2)^{j_2}}$$

Definition of Angular Integrals (Somogyi 2011)

$$\Omega_{j_1,j_2,\ldots,j_n}(v_1,\ldots,v_n;d) \equiv \int \mathrm{d}\Omega_{d-1}(k) \prod_{i=1}^{n} \frac{1}{(v_i \cdot k)^{j_i}}$$

with normalized d-vectors

$$v_i = (1, \mathbf{v}_i)\,,$$

$$k = (1, \mathbf{k}) = (1, \ldots, \cos\theta_n \prod_{i=1}^{n-1} \sin\theta_i, \ldots, \cos\theta_2 \sin\theta_1, \cos\theta_1)\,.$$

Common integral normalization ($v_{ij} = v_i \cdot v_j$ with $i \leq j$):

$$I_{j_1,\ldots,j_n}^{(\#\text{non-zero masses})}(v_{ij};\varepsilon) \equiv \frac{\Omega_{j_1,j_2,\ldots,j_n}(v_1,\ldots,v_n;4-2\varepsilon)}{\Omega_{1-2\varepsilon}}$$

Divergences in the massless limit

- Example: (two denominators, two masses)

$$I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{\pi}{\sqrt{X}} \log\left(\frac{v_{12} + \sqrt{X}}{v_{12} - \sqrt{X}}\right) + \mathcal{O}(\varepsilon)\,,$$

  with $X = v_{12}^2 - v_{11}v_{22}$. Diverges in limit $v_{11} \to 0$ since $v_{12} - \sqrt{X} \to 0$.

- problem if we were to consider

$$\int_0^{v_{11}^{\max}} \mathrm{d}v_{11}\; v_{11}^{-1-\varepsilon}\; I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon)$$

  and want to use

$$v_{11}^{-1-n\varepsilon} = -\frac{1}{n\varepsilon}\,\delta(v_{11}) + \sum_{n=0}^{\infty} \frac{(-n\varepsilon)^n}{n!}\left[\frac{\log^n v_{11}}{v_{11}}\right]_+$$

- Have to extract behavior of $I_{1,1}^{(2)}$ for $v_{11} \to 0$ beforehand!

2. Extracting the massless limit of ang. int.

Two point splitting lemma

- Notation for propagators

$$\Delta_k(v_i, v_j) \equiv \frac{1}{v_i \cdot k \; v_j \cdot k},$$

- For any two vectors v_1 and v_2, we can choose any scalar λ and construct the linear combination $v_3 = (1-\lambda)v_1 + \lambda v_2$ to obtain the identity

Two-point splitting lemma

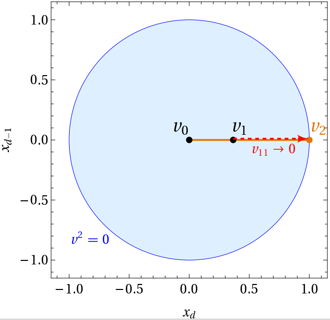
$$\Delta_k(v_1, v_2) = \lambda\, \Delta_k(v_1, v_3) + (1-\lambda)\, \Delta_k(v_2, v_3)\,.$$

Example: One denominator - one mass

- $$I_1^{(1)}(v_{11};\varepsilon) = \int \frac{\mathrm{d}\Omega_{3-2\varepsilon}}{\Omega_{1-2\varepsilon}} \frac{1}{v_1 \cdot k}\,.$$
- ε-expansion:

$$I_1^{(1)}(v_{11};\varepsilon) = \frac{\pi}{\sqrt{1-v_{11}}} \log\left(\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right) + \mathcal{O}(\varepsilon)$$

- Add denominator $1 = 1/(v_0 \cdot k)$ with $v_0 = (1, \mathbf{0})$ and use two-point splitting lemma with $v_2 = (1-\lambda)v_0 + \lambda v_1$
- choose λ s.t. $v_{22} = 0$

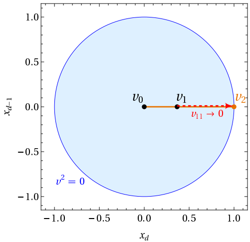


Example: One denominator - one mass

- Two-point splitting:

$$\Delta_k(v_0, v_1) = \lambda\, \Delta_k(v_0, v_2) + (1-\lambda)\, \Delta_k(v_1, v_2)$$

with $\lambda = 1/\sqrt{1-v_{11}}$.



  - Integrate:

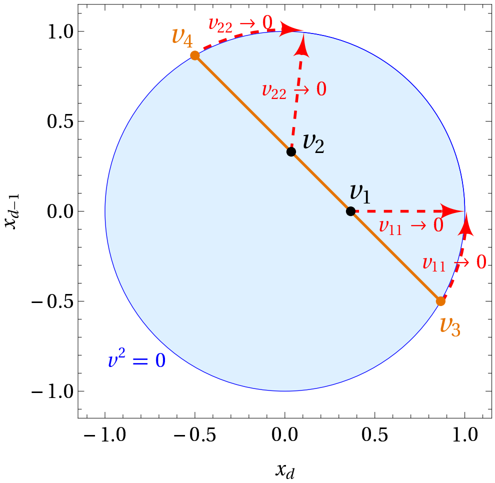
$$I_1^{(1)}(v_{11};\varepsilon) = \frac{I_1^{(0)}(\varepsilon)}{\sqrt{1-v_{11}}} - \frac{1-\sqrt{1-v_{11}}}{\sqrt{1-v_{11}}} I_{1,1}^{(1)}(v_{12}, v_{11};\varepsilon) \text{ with } v_{12} = 1-\sqrt{1-v_{11}}$$

  - Use known ε-expansion on $I_1^{(0)}$ and $I_{1,1}^{(1)}$:

$$I_1^{(1)}(v_{11};\varepsilon) = -\frac{\pi}{\sqrt{1-v_{11}}} \left\{ \frac{1}{\varepsilon} + v_{11}^{-\varepsilon}(1+\sqrt{1-v_{11}})^{2\varepsilon} \times \left[-\frac{1}{\varepsilon} - 2\varepsilon\, \mathrm{Li}_2\left(\frac{2\sqrt{1-v_{11}}}{1+\sqrt{1-v_{11}}}\right) + \mathcal{O}(\varepsilon^2)\right]\right\} \xrightarrow[v_{11}\to 0]{} -\frac{\pi}{\varepsilon}$$

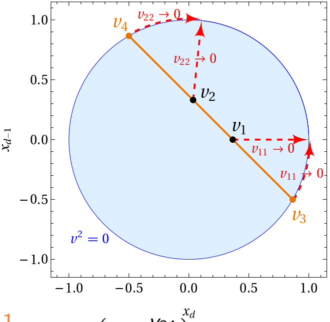
Two denominators - two masses

$$I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \int \frac{d\Omega_{3-2\varepsilon}}{\Omega_{1-2\varepsilon}} \frac{1}{(v_1 \cdot k)(v_2 \cdot k)}$$



Two denominators - two masses

- Splitting of double-massive integral:



$$I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{1}{\sqrt{X}} \Big[v_{34}\, I_{1,1}^{(0)}(v_{34}; \varepsilon) - v_{13}\, I_{1,1}^{(1)}(v_{13}, v_{11}; \varepsilon) - v_{24}\, I_{1,1}^{(1)}(v_{24}, v_{22}; \varepsilon) \Big] .$$

  - Expand in ε:

$$I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{\pi}{\sqrt{X}} \Bigg\{ 2 \left(\frac{v_{34}}{2}\right)^{-\varepsilon} \left[-\frac{1}{\varepsilon} - \varepsilon\, \mathrm{Li}_2\left(1 - \frac{v_{34}}{2}\right) + \varepsilon^2 f_0(v_{34}) + \mathcal{O}(\varepsilon^3) \right]$$
$$- v_{11}^{-\varepsilon} \left(\frac{v_{11}}{v_{13}}\right)^{2\varepsilon} \left[-\frac{1}{\varepsilon} - 2\varepsilon \left(\mathrm{Li}_2(\omega_{13}^+) + \mathrm{Li}_2(\omega_{13}^-) \right) + \varepsilon^2 f_1(\omega_{13}^+, \omega_{13}^-) + \mathcal{O}(\varepsilon^3) \right]$$
$$- v_{22}^{-\varepsilon} \left(\frac{v_{22}}{v_{24}}\right)^{2\varepsilon} \left[-\frac{1}{\varepsilon} - 2\varepsilon \left(\mathrm{Li}_2(\omega_{24}^+) + \mathrm{Li}_2(\omega_{24}^-) \right) + \varepsilon^2 f_1(\omega_{24}^+, \omega_{24}^-) + \mathcal{O}(\varepsilon^3) \right] \Bigg\} .$$

  - relevant e.g. for quarkonium production at NLO (Butenschön)

3. Expansion by regions meets ang. int.

Recap: Expansion by Regions

- method developed to expand Feynman integrals in small parameters, e.g. masses (Beneke & Smirnov 1998)
- result is sum of different "regions", where specific expansions of the integrand can be applied
- each "region integral" is simpler to calculate than the original integral
- formalized in terms of parametric integrals of the form

$$I = \int_0^\infty \mathrm{d}t_1 \cdots \int_0^\infty \mathrm{d}t_n \prod_{i=1}^{N} P_i^{a_i+\varepsilon b_i}(t_1, \ldots, t_n)$$

Idea

Construct such parametric integral representations for angular integrals

Parametric integral representations for angular integrals

General strategy:

1. Derive a suitable integral representation for the one-denominator integral $I_j(v^2)$
2. For $I_{j_1,\ldots,j_n}$ use a Feynman parametrization to combine denominators to just one with mass $v = \left(\sum_{i=1}^{n} x_i v_i\right)^2$.
3. Use delta function to integrate out x_n
4. Change variables $x_i \to t_i = \frac{x_i}{1-\sum_{l=1}^{i} x_l}$ such that integrals run from 0 to ∞
5. Plug in the integral representation from 1.

Parametric integral representations for angular integrals

For example for 3 denominators:

$$I_{j_1,j_2,j_3}(v_{12},v_{13},v_{23},v_{11},v_{22},v_{33};\varepsilon) = \frac{2\pi}{1-2\varepsilon}\frac{\mathrm{B}^{-1}(j_1,j_2,j_3)\Gamma\left(\frac{3}{2}-\varepsilon\right)}{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(1-\frac{j}{2}-\varepsilon\right)}$$
$$\times\int_0^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t_2\int_0^\infty \mathrm{d}t_3\, t_1^{j_1-1}t_2^{j_2-1}(1+t_2)^{j_1}t_3^{\frac{j-1}{2}}(1+t_3)^{\frac{j-3}{2}+\varepsilon}\,w_3^{-\frac{j}{2}},$$

where $j=j_1+j_2+j_3$, $\mathrm{B}(j_1,j_2,j_3)=\frac{\Gamma(j_1)\Gamma(j_2)\Gamma(j_3)}{\Gamma(j)}$, and w_3 denotes the polynomial

$$w_3 \equiv (1+t_1)^2(1+t_2)^2$$
$$+\left[t_1^2(1+t_2)^2v_{11}+2t_1(1+t_2)(v_{13}+t_2v_{12})+t_2^2v_{22}+2t_2v_{23}+v_{33}\right]$$

Two denominators - Regions

- Parametric integral representation:

$$I_{1,1} = \frac{4\sqrt{\pi}\,\Gamma(3/2-\varepsilon)}{(1-2\varepsilon)\,\Gamma(-\varepsilon)} \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \, \frac{(1+t_2)^{-1/2+\varepsilon}\sqrt{t_2}}{(1+t_1)^2 + t_2(t_1^2 v_{11} + 2t_1 v_{12} + v_{22})}$$

- Expansion by regions:

$$I_{1,1} \sim \sum_{r_i \in r} \mathcal{M}_{r_i} I_{1,1}$$

with "regions" $r = \{\{-1,-1\},\{1,-1\},\{0,0\}\}$ ($\texttt{asy.m}$).
$\mathcal{M}_{r_i}$ acts on the integrand in the following way:

1. scale by $t_j \to y^{(r_i)_j} t_j$
2. multiply by $y^{\sum_{j=1}^{3}(r_j)_i}$
3. expand in y at $y=0$ and set $y=1$ in the end.

Two denominators - Results

Three contributions:

$$\mathcal{M}_{r_1} I_{1,1} \overset{\text{LP}}{\approx} C_{2,1}^{\text{LP}} = \frac{\pi}{\varepsilon}\left(\frac{v_{11}}{4}\right)^{-\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)}{v_{12}\,\Gamma(1-\varepsilon)}$$

$$\mathcal{M}_{r_2} I_{1,1} \overset{\text{LP}}{\approx} C_{2,2}^{\text{LP}} = \frac{\pi}{\varepsilon}\left(\frac{v_{22}}{4}\right)^{-\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)}{v_{12}\,\Gamma(1-\varepsilon)}$$

$$\mathcal{M}_{r_3} I_{1,1} \overset{\text{LP}}{\approx} C_{2,3}^{\text{LP}} = -\frac{2\pi}{v_{12}\,\varepsilon} + \frac{2\pi}{v_{12}} \log \frac{v_{12}}{2} + \mathcal{O}(\varepsilon)$$

Pole cancels in the sum:

$$I_{1,1} \overset{\text{LP}}{\approx} C_{2,1}^{\text{LP}} + C_{2,2}^{\text{LP}} + C_{2,3}^{\text{LP}} = \frac{\pi}{v_{12}}\left(2\log v_{12} - \log\frac{v_{11}}{2} - \log\frac{v_{22}}{2}\right) + \mathcal{O}(\varepsilon)$$

Compare with algebraic decomposition:

$$I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{1}{\sqrt{X}}\left[v_{34}\, I_{1,1}^{(0)}(v_{34};\varepsilon) - v_{13}\, I_{1,1}^{(1)}(v_{13}, v_{11};\varepsilon) - v_{24}\, I_{1,1}^{(1)}(v_{24}, v_{22};\varepsilon)\right]$$

Regions for n denominators

- n-denominator angular integral:

$$I_{1,\ldots,1} \overset{\text{LP}}{\approx} \sum_i C^{\text{LP}}_{n,i} \quad \text{with} \quad \mathcal{M}_{r_i} I_{\underbrace{1,\ldots,1}_{n}} \overset{\text{LP}}{\approx} C^{\text{LP}}_{n,i}$$

- results checked explicitly for $n \leq 4$, conjectured for all n
- One region for each massive vector:

$$C^{\text{LP}}_{n,i} = \frac{\pi}{\varepsilon} \left(\frac{v_{ii}}{4}\right)^{-\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{v_{ij}}$$

- One mass-independent region equal to the massless angular integral:

$$C^{\text{LP}}_{n,n+1} = I^{(0)}_{1,\ldots,1} \, .$$

Pole part for n denominators

- known massive regions + pole cancellation for fully massive integral $\longrightarrow$ massless pole:

$$I^{(0)}_{1,\ldots,1}(v_1,\ldots,v_n) = -\frac{\pi}{\varepsilon}\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{v_{ij}} + \mathcal{O}\big(\varepsilon^0\big)\,.$$

- For $v_{ii}=0$, drop corresponding massive region $\rightarrow$ pole of angular integral with n denominators, m masses:

$$I^{(m)}_{1,\ldots,1}(v_1,\ldots,v_n) \overset{\text{LP}}{\approx} C^{\text{LP}}_{n,n+1} + \sum_{i=1}^{m} C^{\text{LP}}_{n,i} = -\frac{\pi}{\varepsilon}\sum_{i=m+1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{v_{ij}} + \mathcal{O}\big(\varepsilon^0\big)\,.$$

4. Conclusion

Conclusion

- Explicit asymptotic behavior for 2-denominator angular integral (full analytic form, all orders in ε)
- constructed parametric integral representations for n denominator angular integrals similar to representations of loop integrals
- employed expansion by regions to study asymptotics for n denominators
- Analytic results for massive regions (leading power), leading-log contribution to massive integral
- pole part of angular integral with n denominators and m masses