Phase Space Integrals through Mellin-Barnes Technique

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Based on work to appear on arxiv with Taushif Ahmed, Saurav Goyal, Syed Mehedi Hasan, Roman N. Lee, Sven-Olaf Moch, Vaibhav Pathak, Narayan Rana and V. Ravindran

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Outline

- Method of radial-angular decomposition
- Mellin-Barnes technique for angular part
- Resolution of singularities
- Radial integration

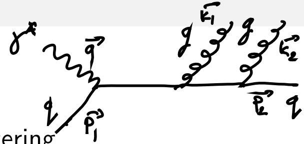
Why Phase Space Integrals?

- They appear in QFT cross sections and other observables
- KLN theorem

Different Methods

- Radial-Angular decomposition
- Differential equation method
- numerical techniques (example: slicing)

Radial-Angular Part Decomposition



Example: $2 \to 3$ Semi-Inclusive Deep Inelastic Scattering

$$\int dPS_3' \frac{1}{(k_1 - p_1)^2 (k_2 - p_1 - q)^2 (k_1 + k_2 - p_1)^2}$$

$$\int dPS_{3,\mathrm{r}}' \frac{1}{r} \int dPS_{3,\mathrm{ang}}' \frac{1}{(a + b\cos\theta)(A + B\cos\theta + C\sin\theta\cos\phi)}$$

Our first task: solve this angular part of the integral

Somogyi and Neerven parametrizations

General two denominator case in Neerven parametrization:

$$\int d\Omega_{d-1} \frac{1}{(a + bcos\theta_1)^j (A + Bcos\theta_1 + Csin\theta_1 cos\theta_2)^l}$$

´92, van Neerven

General two denominator case in Somogyi parametrization:

$$\frac{1}{a^j A^l} \int d\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j (p_2 \cdot q)^l}$$

2011, Somogyi

We categorize it

Massless: $a^2 = b^2, A^2 = B^2$

Single massive: $a^2 = b^2$ or $A^2 = B^2 + C^2$

Double massive: $a^2 \neq b^2, A^2 \neq B^2 + C^2$

$p_1 = (1, 0_{d-2}, -\frac{b}{a})$, $p_2 = (1, 0_{d-3}, -\frac{C}{A}, -\frac{B}{A})$

$q = (1, \ldots, sin\theta_1 cos\theta_2, cos\theta_1)$

State-of-the-art

Two-denominator: 2021, Lyubovitskij, Wunder, Zhevlakov

Three denominator: Asymptotoic behaviour 2024, Smirnov, Wunder

Our Goal:

- Computation of 3-denominator case in dimensional regularization using MB 2408.xxxxx Ahmed, Hasan, AR
- Calculate the phase-space integrals for NNLO SIDIS using our technique 2408.xxxxx Ahmed, Goyal, Hasan, N. Lee, Moch, Pathak, Rana, AR, Ravindran
  See recent NNLO QCD SIDIS cross-section

  2024, Goyal, N.Lee, Moch, Pathak, Rana, Ravindran

  2024, Bonino, Gehrmann, Stagnitto
  Bonino, Gehrmann, Löchner, Schönwald, Stagnitto

Three denominator massless

$$\int d\Omega_{d-1}(q)\frac{1}{(p_1\cdot q)^{j_1}(p_2\cdot q)^{j_2}(p_3\cdot q)^{j_3}}$$

$$p_1^\mu = (1,0,0,1)$$
$$p_2^\mu = (1,0,sinx_2^{(1)},cosx_2^{(1)})$$
$$p_3^\mu = (1,sinx_3^{(2)}sinx_3^{(1)},cosx_3^{(2)}sin(x_3^{(1)}),cosx_3^{(1)})$$

$$q^\mu = (1,cos\theta_3 sin\theta_2 sin\theta_1,cos\theta_2 sin\theta_1,cos\theta_1)$$

Three denominator massless

In Mellin Barnes representation

$$
\begin{aligned}
&2^{2-j-k-l-2\epsilon}\pi^{1-\epsilon}\frac{1}{\Gamma(j)\Gamma(k)\Gamma(l)\Gamma(2-j-k-l-2\epsilon)} \\
&\times\int_{-i\infty}^{+i\infty}\frac{dz_{12}dz_{13}dz_{23}}{(2\pi i)^3}\Gamma(-z_{12})\Gamma(-z_{13})\Gamma(-z_{23})\Gamma(j+z_{12}+z_{13})\Gamma(k+z_{12}+z_{23}) \\
&\times\Gamma(l+z_{13}+z_{23})\Gamma(1-j-k-l-\epsilon-z_{12}-z_{13}-z_{23})(u_{12})^{z_{12}}(u_{13})^{z_{13}}(u_{23})^{z_{23}}
\end{aligned}
\tag{1}
$$

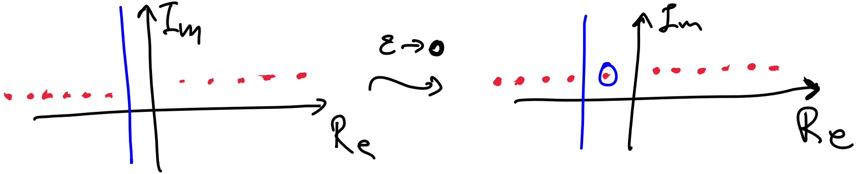
- integral in the complex plane
- involves several gamma functions → contains infinitely many poles

Objective: solve this integral in powers of dimensional regularization parameter ϵ $(d = 4 - 2\epsilon)$

Expansion in ϵ

- Analytically continue to $\epsilon = 0$ by deforming the integration contour and pick up residues.

  original MB integral $\rightarrow$ sum of several MB integrals
- Expand each of the MB integrals around $\epsilon = 0$ and collect coefficients of the same order of ϵ
- Calculate the MB itegrals that appear in the ϵ coefficients



How do we compute multifold MB integrals?

It is not at all an easy task to compute multifold MB integrals in analytic form:

- Three denominator case massless: Threefold MB integrals
- Single mass three denominator case: Fourfold MB integrals

How do we compute multifold MB integrals?

- Convert the MB integrals in real integration integrals
- Express these integrals in terms of Multiple Polylogarithms

How do we compute multifold MB integrals?

Gamma functions $\rightarrow$ Beta functions

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Real integral representations:

$$B(a,b) = \int_0^1 dx x^{a-1}(1-x)^{b-1}, \quad \mathcal{R}(a), \mathcal{R}(b) > 0$$

$$B(a,b) = \int_0^\infty dx x^{a-1}(1+x)^{-a-b}, \quad \mathcal{R}(a), \mathcal{R}(b) > 0$$

How do we compute multifold MB integrals?

we get:

$$\int_0^1 \left(\prod_{k=1}^{K} dx_k \right) R_0(x,u) \int_{-i\infty}^{+i\infty} \prod_{l=1}^{L} \frac{dz_l}{2\pi i} \Big[R_l(x,u) \Big]^{z_l}$$

How do we compute multifold MB integrals?

Use the formula:

$$\int_{-i\infty+z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} A^z = \delta(1-A), \quad A > 0$$

Then:

$$\int_0^1 \left(\prod_{k=1}^{K} dx_k \right) R_0(x,u) \prod_{l=1}^{L} \delta\left[1 - R_l(x,u) \right]$$

How do we compute multifold MB integrals?

Real Integrations over rational functions $\rightarrow$ Multiple Polylogarithms (MPLs):

$$G(a_1, ..., a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_n; t)$$

How do we compute multifold MB integrals?

Digamma terms:

$$\psi(z) = \int_0^1 \frac{t^{z-1} - 1}{t - 1} dt - \gamma, \quad \mathcal{R}(z) > 0$$

or equivalently:

$$\psi(z) = \int_0^\infty \left[(1+x)^{-1} - (1+x)^{-z} \right] \frac{dx}{x} - \gamma, \quad \mathcal{R}(z) > 0$$

Resolution of Singularities

- At the end, we combine the angular with the parametric part
- To perform the onefold parametric integration, first we need to resolve the singularities in the integration domain (Pole subtraction)

Resolution of Singularities

Consider $g(z)$ singular at $z = b$. Then separate the singular part $S(z)$ and the finite part $F(z)$ as:

$$g(z) = S(z)F(z)$$

Next we define:

$$\widetilde{F} \equiv \lim_{z \to b}(F(z))$$

So we can separate the singularity as:

$$\int_a^b g(z)dz = \int_a^b [S(z)F(z) - \tilde{F}S(z)]dz + \int_a^b \tilde{F}S(z)dz$$

The integrand $S(z)F(z) - \tilde{F}S(z)$ in non singular and $\int_a^b \tilde{F}S(z)dz$ can be integrated easily.

Resolution of Singularities

Why can we expand the angular part?

- Angular part $\rightarrow$ Collinear Singularities
- Parametric Part $\rightarrow$ Soft Singularities

Therefore we can perform an all order resummation and factorization of the soft singularity contribution that enters the angular part

Final result

- In many cases (Example: SIDIS NNLO) the full result can be expressed it terms of Multiple Polylogarithms
- Alternatively, one can keep the final result as a single onefold numerical integral (Sufficient for all phenomenological purposes)
- we have checked with both ways and they match at many numerical digit accuracy
- Our NNLO results are verified also with the differential equation method

Conclusion

Benefits of the Radial-Angular part decomposition:

- The angular part admits a universal structure
- Onefold real integration on top
- Transparent Singularity Structure

Conclusion

SIDIS:

- NNLO SIDIS is now available:

  2024, Goyal, N.Lee, Moch, Pathak, Rana, Ravindran

  2024, Bonino, Gehrmann, Stagnitto

  Bonino, Gehrmann, Löchner, Schönwald, Stagnitto
- If someone wants to go to higher order, then one requires to compute this kind of integrals.

Conclusion

Mellin-Barnes Technique:

- General technique, that can give results for arbitrary number of denominators up to any order in ϵ.
- It can be applied to compute any phase-space integrals appearing in the context of LHC as well
- Multifold MB integrals appear in many other scenarios and this method can also be helpfull there.

Thank you for listening!