Conformal moments of two-loop coefficient functions in DVCS

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- **Deeply Virtual Compton Scattering**: $\gamma^* \; N \longrightarrow \gamma \; N'$

Müller 94, Ji 96, Radyushkin 96

$$\mathcal{A}_{\mu\nu}(q,q',p) = i \int d^4x \, e^{-iqx} \langle p'|T\{j_\mu^{\rm em}(x) j_\nu^{\rm em}(0)\}|p\rangle \, .$$

- **The leading twist approximation**

$$\mathcal{A}_{\mu\nu} = -g^{\perp}_{\mu\nu} V + \epsilon^{\perp}_{\mu\nu} A + \dots$$

- **Parametrize vector amplitude in terms of Compton Form Factors (CFFs)**
  Diehl 03, Belitsky, Radyushkin 05

$$V = \frac{1}{2P^+} \bar{u}(p') \left[\gamma^+ \mathcal{H}(\xi, t, Q) + \frac{i\sigma^{+\alpha}}{2M} \mathcal{E}(\xi, t, Q) \right] u(p).$$

$$\xi = \frac{x_B}{2 - x_B} + \mathcal{O}(1/Q^2), \qquad x_B = \frac{Q^2}{2p \cdot q}, \qquad \Delta_\mu = (q' - q)_\mu, \qquad P_\mu = (p + p')_\mu / 2$$

CFFs can be factorized in terms of GPDs

GPD factorization

$$\mathcal{H}(\xi, t, Q) = \sum_q \int_{-1}^{1} \frac{\mathrm{d}x}{\xi} C_q(x/\xi, \mu^2/Q^2, \alpha_s(\mu))\, H_q(x, \xi, t, \mu) + \int_{-1}^{1} \frac{\mathrm{d}x}{\xi^2} C_g(x/\xi, \mu^2/Q^2, \alpha_s(\mu))\, H_g(x, \xi, t, \mu),$$

and similarly for $\mathcal{E}$

$$H_q(x, 0, 0, \mu) = q(x, \mu), \qquad H_g(x, 0, 0, \mu) = xg(x, \mu)$$

GPDs normalized to PDFs in forward limit

CFs at LO and NLO

Coefficient functions for quark and gluon

$$C_q = C_q^{(0)} + a_s C_q^{(1)} + a_s^2 C_q^{(2)} + \mathcal{O}(a_s^3),$$

$$C_g = a_s C_g^{(1)} + a_s^2 C_g^2 + \mathcal{O}(a_s), \qquad a_s = \alpha_s/4\pi,$$

LO and NLO CFs

$$C_q^{(0)} = \frac{e_q^2(1-2z)}{2z\bar{z}},$$

$$C_q^{(1)} = \frac{e_q^2 C_F}{2z\bar{z}}\left\{4Lz\ln(\bar{z}) + \bar{z}\ln^2(z) + 3\bar{z}\ln(z) - (z \leftrightarrow \bar{z}) - 3(1-2z)(L+3)\right\}$$

$$C_g^{(1)} = \frac{\left(\sum_q e_q^2\right)T_F}{4z^2\bar{z}^2}\left\{2Lz^2\ln(z) - z^2\ln^2(z) + 2z(1+z)\ln(z) + (z \leftrightarrow \bar{z})\right\}$$

Ji, Osborne, 98, Noritzsch, 04

$$z = \tfrac{1}{2}(1 - x/\xi), \qquad L = \ln(\mu^2/Q^2), \qquad \bar{z} = 1 - z.$$

CFs at NNLO

Braun, Ji, Schoenleber, 2022

$$C_q^{(2)} = \frac{1}{2z\bar{z}}\left[e_q^2 C_F\left(C_F C_{NS}^{(F)} + C_A C_{NS}^{(A)} + \beta_0 C_{NS}^{(\beta_0)}\right) + \left(\sum_{q'} e_{q'}^2\right) T_F C_F C_{PS}\right]$$

$$C_g^{(2)} = \frac{\left(\sum_q e_q^2\right)}{4z^2\bar{z}^2} T_F\left(C_F C_g^{(F)} + C_A C_g^{(A)}\right).$$

$H_{\vec{m}}$ – Harmonic Polylogarithms

$$\begin{aligned}
C_{\rm PS} = L^2\Big[&-8(z-1)H_{1,0} + 4H_1\left(4z^2-5z+1\right) + 4H_0 z(4z-3) - 8H_2 z + 8\zeta_2 z\Big]\\
&+ 8L\Big[z(4z-3)(H_2 - H_{0,0}) + (z-1)(4z-1)(H_{1,1} - H_{1,0}) - 2zH_{2,1}\\
&+ 2(z-1)(H_{1,0,0} - H_{1,1,0}) - \left(2\zeta_2 - 3\right)H_1(z-1) - 3H_0 z + 2H_3 z - \zeta_2 z(4z-3)\Big]\\
&+ 16(z-1)\left(-H_{1,0,0,0} + H_{1,1,0,0} + \tfrac{1}{2}H_{1,1,1,0}\right) + 8z(4z-3)(H_{0,0,0} + H_{2,1})\\
&+ 8(z-1)(4z-1)\left(H_{1,0,0} + H_{1,1,1}\right) - 4(z-1)(4z+5)H_{1,1,0}\\
&+ 16z\left(-H_{2,1,1} + \tfrac{3}{2}H_{0,0} + H_{3,1} - zH_{2,0}\right) - 8\left(\zeta_2 - 3\right)(z-1)H_{1,1}\\
&- 16(z-1)^2 H_{1,2} + 4(1-z)H_{1,0} - 8H_0 z\left(2\zeta_2 z + 5\right) - 4H_3 z(4z-9) - 8H_4 z\\
&- 4H_1(z-1)\left(2\left(\zeta_3 - 5\right) + \zeta_2(4z+5)\right) - 4H_2 z + 4z\left(2\zeta_2^2 + \zeta_2 - 3\zeta_3(4z+1)\right).
\end{aligned}$$

Mellin-Barnes representation for CFFs
Müller, 2006, Müller, Kumerički, Passek-Kumerički 2007

$$V(\xi,t,Q^2) \simeq \sum_{a\in\{q,g\}} \int_{c-i\infty}^{c+i\infty} dj\, \xi^{-j-1} \left[i + \tan(\pi j/2)\right] C_{a,j}(Q/\mu) H_{a,j}(\xi,t,\mu)$$

$$C_{q,j}(Q/\mu) = \int_0^1 dz\, z\bar{z}\, C_{j-1}^{(3/2)}(1-2z)\, C_q(z,Q/\mu) = 1 + a_s\, C_{q,j}^{(1)} + a_s^2\, C_{q,j}^{(2)} + \dots$$

$$C_{g,j}(Q/\mu) = \int_0^1 dz\, z^2\bar{z}^2 C_{j-2}^{(5/2)}(1-2z)\, C_g(z,Q/\mu) = a_s C_{g,j}^{(1)} + a_s^2 C_{g,j}^{(1)} + \dots$$

$C_j^{(\alpha)}$ - Gegenbauer Polynomials

Consider only quark pure singlet contributions

Two special sl$(2,\mathbf{R})$ invariant operators: $\widehat{\mathrm{H}},\ \mathrm{H}_+$

Their eigenvalues $\widehat{\mathrm{H}} \mapsto 2S_1(j)$**,** $\qquad \mathrm{H}_+ \mapsto 1/j(j+1)$

Linked to one-loop evolution kernel $\leftrightarrow$ anomalous dimension

Action of operator $[\mathrm{H}_\omega f](z_1,z_2) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \omega(\tau) f(z_{12}^\alpha, z_{21}^\beta), \quad \tau = \frac{\alpha\beta}{\bar\alpha\bar\beta},\ z_{12}^\alpha = z_1\bar\alpha + \alpha z_2$

Choosing $f(z_1,z_2) = z_{12}^{j-1}$ **gives eigenvalues** $\mathrm{E}_\omega(j) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \omega(\tau)\,(1-\alpha-\beta)^{j-1}$

Invariant kernels for sl$(2$,R$)$ invariant operators $\delta_+(\tau) \leftrightarrow \widehat{\mathrm{H}}$**,** $\qquad 1 \leftrightarrow \mathrm{H}_+$

$$\ln\bar\tau \leftrightarrow -\frac{1}{j^2(j+1)^2}, \qquad \ln\tau \leftrightarrow -\frac{S_1(j)}{j(j+1)}, \qquad \bar\tau \leftrightarrow (-1)^j(2S_{-2}(j)+\zeta_2), \quad \text{etc.}$$

Invariant kernels approach

Eigenfunctions:

$$\Psi_j^p(z_1, z_2) \sim \int_0^1 dx e^{-ip(xz_1+\bar{x}z_2)} x\bar{x} C_{j-1}^{(3/2)}(1-2x)$$

1. **Normalization**

$$\int_0^\infty dz_1 \Psi_j^p(z_1, 0) \sim \int_0^1 dx \frac{1}{x} \left(x\bar{x} C_{j-1}^{(3/2)}(1-2x) \right) = 1$$

2.

$$\int_0^\infty dz_1 [\mathrm{H}_\omega \Psi_j^p](z_1, 0) = \mathrm{E}_\omega(j) = \int_0^1 dx f_\omega(x) \left(x\bar{x} C_{j-1}^{(3/2)}(1-2x) \right)$$

$$f_\omega(x) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \frac{\omega(\tau)}{x\bar{\alpha} + \beta\bar{x}}$$

Eigenvalues and conformal moments are related using the following statement

If $f_\omega(x) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \omega(\tau) f(x\bar{\alpha} + \beta\bar{x})$ then $M_j(f_\omega) = M_j(f)\mathrm{E}_\omega(j)$

From the CFs we obtain $f(x) = \mathrm{H}_{\vec{m}}(x) \times \left(1, \frac{1}{x}, \frac{1}{1-x}\right)$ **HyperInt , E. Panzer, 2014**

We want to calculate conformal moments of functions $\frac{1}{x}\mathrm{H}_{\vec{m}}(x)$ up to weight 4,
$\vec{m} = \left(\{0\}, \{1\}, \{0,0\}, \dots, \{0,0,0,0\} \dots, \{1,1,1,1\}\right)$

- Level 0 one function: $1/x.$; Its conformal moment $M(j) = 1$
- Level 1 Two HPL functions: $\mathrm{H}_0, \mathrm{H}_1$

  and two invariant kernels $\widehat{H} \mapsto S_1(j), \quad H_+ \mapsto \eta(j) = 1/j(j+1).$

$$
\begin{aligned}
\widehat{H}\frac{1}{x} &= \frac{1}{x}\Big(A\,H_0(x) + BH_1(x)\Big),\\
H_+\frac{1}{x} &= \frac{1}{x}\Big(C\,H_0(x) + D\,H_1(x)\Big)
\end{aligned}
$$

$$
\begin{pmatrix} S_1(j) \\ \eta(j) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} M_0(j) \\ M_1(j) \end{pmatrix}
$$

- Level 2: Four HPLs $\mathrm{H}_{00}, \mathrm{H}_{01}, \mathrm{H}_{10}, \mathrm{H}_{11}$. Kernels S_1^2, $\eta\, S_1$, η^2 plus $(-1)^j S_{-2}(j)$

$$\begin{pmatrix} S_1^2(j) \\ \vdots \\ S_{-2}(j) \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{14} \\ \vdots & \ddots & \vdots \\ A_{41} & \cdots & A_{44} \end{pmatrix} \begin{pmatrix} M_{00}(j) \\ \vdots \\ M_{11}(j) \end{pmatrix} + \begin{pmatrix} F_{00}(j) \\ \vdots \\ F_{11}(j) \end{pmatrix}$$

- Level 3: Eight HPLs $\mathrm{H}_{000}, \ldots, \mathrm{H}_{111}$. Eight kernels $S_1^3, \ldots \eta^3$, $(-1)^j S_{-2}(j) \times (\eta, S_1)$ plus S_3 and $\Omega_{1,-2} = (-1)^j (S_{1,-2} - 1/2 S_{-3})$

- Level 4: Sixteen HPLs. Three new kernels $(-1)^j S_{-4}$, $\Omega_{1,3} = S_{1,3} - \frac{1}{2} S_4$, $(-1)^j \left(\Omega_{1,1,-2} = S_{1,1,-2} - \frac{1}{2} S_{2,-2} - \frac{1}{2} S_{1,-3} + \frac{1}{4} S_4 \right)$

$$
\begin{aligned}
C_{\mathrm{PS}}(j) = \Bigg[& -8S_{-4}(j) - 4S_{-2}(j) - 16\left(-\tfrac{3}{2} + \zeta_2 + 2S_{-2}(j)\right)S_1(j)^2 - 4\zeta_2 S_2(j) \\
& - 16S_{1,-3}(j) + 16\zeta_2 S_{1,1}(j) - 16S_{2,-2}(j) + 32S_{1,1,-2}(j) \\
& + \frac{2}{(1+2j)(3+2j)}\Big(6(8j^2+8j-7)S_{-3}(j) - 6\Big(10 - \zeta_2 + 4j(1+j)(\zeta_2 + \tfrac{2}{3}(\zeta_3 - 5)) \\
& - 12\zeta_3 + 12(8j^2+8j-5)S_{-2}(j)\Big)S_1(j) + 8S_1(j)^3 - 4(6j^2+6j-7)S_3(j) \\
& + 12(8j^2+8j-7)S_{-2,1}(j)\Big)\Bigg] + L\Bigg[-\tfrac{8}{3}\Big(-\tfrac{3}{2} + \zeta_2 + 6S_{-2}(j)\Big)S_1(j) \\
& + \frac{16j(1+j)}{(-1+2j)(1+2j)(3+2j)}\Big((3+2j)(S_{-2}(j-2) - S_1(j-2)^3) \\
& - \tfrac{3}{2}(1+2j)(S_{-2}(j) - S_1(j)^2) - (-1+2j)S_1(j+2)\Big)\Bigg] + L^2\Bigg[-8S_{-2}(j) \\
& + \frac{4}{(-1+2j)(1+2j)(3+2j)}\Big(12(1+2j)S_1(j) + j(1+j)\Big(S_1(j-2) - 2S_1(j+2)\Big)\Big)\Bigg]
\end{aligned}
$$

Gluon contributions

What's different for the gluon?

- Gluon CF at NLO and conformal moment representation in terms of Gegenbauer polynomials

$$C_g^{(1)} = \frac{\left(\sum_q e_q^2\right) T_F}{4z^2\bar{z}^2}\left\{2Lz^2\ln(z) - z^2\ln^2(z) + 2z(1+z)\ln(z) + (z \leftrightarrow \bar{z})\right\}$$

$$C_g(j) = \int_0^1 dz\, z^2\bar{z}^2 C_{j-2}^{(5/2)}(1-2z)\, C_g(z) = a_s C_g^{(1)}(j) + a_s^2 C_g^{(1)}(j) + \ldots$$

- $f_\omega(x) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \omega(\tau) f(x\bar{\alpha} + \beta\bar{x})$ still holds, but

$$f_\omega(x) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \frac{\bar{\alpha}\bar{\beta}\,\omega(\tau)}{(x\bar{\alpha}+\beta\bar{x})^2}$$

- Eigenvalues now calculated through

$$E_\omega(j) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta (1-\alpha-\beta)^{j-2}\, \bar{\alpha}\bar{\beta}\,\omega(\tau)$$

  Can use previous kernels by replacement $\omega(\tau) = (1-\tau)\widetilde{\omega}(\tau)$.