Off-forward anomalous dimensions for the transvesity operators

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Introduction

Inclusive processes

Forward kinematics $\longrightarrow$ PDFs $\longrightarrow$ DGLAP equation

Exclusive processes

Off-forward kinematics $\longrightarrow$ GPDs $\longrightarrow$?

Using Wilson's OPE we can address hadronic part of the interaction to the

$$\langle P'|\mathcal{O}(z_1, z_2)|P\rangle. \tag{1}$$

Light-ray operators

- Scale dependence of GPD $\Rightarrow$ Renormalization of $\mathcal{O}(z_1, z_2)$.

Light-ray transversity operators

Light-ray non-local operator

$$\mathcal{O}(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n]\sigma_{\perp +} q(z_2 n), \tag{2}$$

where $z_1, z_2 \in \mathbb{R}$, $n^2 = 0$ is a light-like vector and $[z_1 n, z_2 n]$ is a Wilson line. Quark fields $q, \bar{q}$ are assumed to be of different flavors.

$$[z_1 n, z_2 n] = \text{Pexp}\left(igz_{12} \int_0^1 du\, n^\mu A_\mu(z_{21}^u n) \right), \tag{3}$$

where $z_{12}^u = z_1 \bar{u} + z_2 u$, $\bar{u} = 1 - u$ and $z_{12} = z_1 - z_2$.

Dirac structure

Introducing the second light-like vector $\bar{n}$, such that $\bar{n} \cdot n = 1$, we expand the arbitrary vector x as

$$x^\mu = x\_ n^\mu + x_+ \bar{n}^\mu + x_\perp^\mu, \tag{4}$$

so $\sigma_{\perp +}$ stands for the projection onto transverse subspace of the

$$\sigma_{\mu\nu} = \frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right]. \tag{5}$$

Renormalization

Renormalization of light-ray operators

$$[\mathcal{O}]\,(z_1, z_2) = Z\mathcal{O}(z_1, z_2), \tag{6}$$

where Z is an integral operator, which acts on the sample function f in the form

$$Zf(z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, z(\alpha, \beta) f(z_{12}^{\alpha}, z_{21}^{\beta}) \tag{7}$$

RG-equation

Renormalization group equation for the light-ray operators takes the form

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \mathbb{H}(a)\right)[\mathcal{O}](z_1, z_2) = 0, \tag{8}$$

where $a = \alpha_s/4\pi$ and $\mathbb{H}(a)$ is an integral operator called **evolution kernel**.

Recent progress

Different Dirac structure

- Three-loop result in [V. M. Braun, A. N. Manashov, S. Moch, M. Strohmaier'2017] for the vector case

$$\mathcal{O}^V(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n]\gamma_+ q(z_2 n); \tag{9}$$

- Three-loop result in [V. M. Braun, A. N. Manashov, S. Moch, M. Strohmaier'2021] for the vector-axial case

$$\mathcal{O}^A(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n]\gamma_+ \gamma_5 q(z_2 n). \tag{10}$$

This work

- Three-loop result in [A. N. Manashov, S. Moch, **LS**'2024] for the transversity case

Usage

Transversity operators are connected to the polarized processes:

- Semi-inclusive DIS;
- Polarized Drell-Yan.

Promising direction for the future **Electron-Ion Collider**

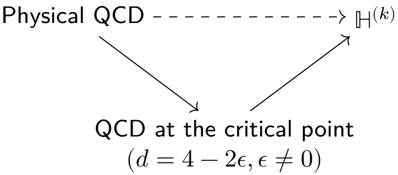
Method

Perturbative series

We can expand evolution kernel in the form

$$\mathbb{H}(a) = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + \dots, \tag{11}$$

where $\mathbb{H}^k$ are d-independent quantities.



Conformal invariance

We can separate evolution kernel in the two parts

$$\mathbb{H} = \mathbb{H}_{\text{inv}} + \mathbb{H}_{\text{non-inv}}, \tag{12}$$

where $\mathbb{H}_{\text{inv}}$ is invariant under canonical transformation form the Collinear subgroup of the conformal group $(SL(2,\mathbb{R}))$.

Method

Non-invariant part

Restore the three-loop result using **conformal anomaly**

$$\mathbb{H}^{(2)} \;\Rightarrow\; \Delta^{(2)} \;\Rightarrow\; \mathbb{H}^{(3)}_{\text{non-inv}}. \tag{13}$$

Conformal Ward identity at the critical point can be used as an equation for $\mathbb{H}^{(3)}$, including $\Delta^{(2)}$.

Invariant part

Restore the invariant part using three-loop **forward anomalous dimensions**

$$\gamma^{(3)} \;\Rightarrow\; \gamma^{(3)}_{\text{inv}} \;\Rightarrow\; \mathbb{H}_{\text{inv}}. \tag{14}$$

Forward anomalous dimensions are the eigenvalues of the evolution kernel.

Work is complete

Plan:

- Derive all the formalism in the transversity case;
- Calculate the one-loop evolution kernel $\mathbb{H}^{(1)}$;
- Calculate the one-loop conformal anomaly $\Delta_+^{(1)}$;
- Calculate the two-loop evolution kernel $\mathbb{H}^{(2)}$;
- Check that $\mathbb{H}^{(2)}$ is consistent with $\Delta_+^{(1)}$ prediction;

*** Progress at the previous FOR meeting in Tübingen ***

- Calculate the two-loop conformal anomaly $\Delta_+^{(2)}$;
- Restore the $\mathbb{H}^{(3)}_{\text{non-inv}}$.
- Restore the $\mathbb{H}^{(3)}_{\text{inv}}$.

*** Work is done! ***

Calculation of the non-invariant part

Modification of QCD action

Conformal anomaly can be calculated in the framework of the adjusted QCD action

$$S_{QCD} \mapsto S_\omega = S_{QCD} + \delta^\omega S = S_{QCD} - 2\omega \int d^d y (\bar{n}y) \left(\frac{1}{4} F^2 + \frac{1}{2\xi} (\partial A)^2 \right) . \qquad (15)$$

The renoramlization operator then takes the form

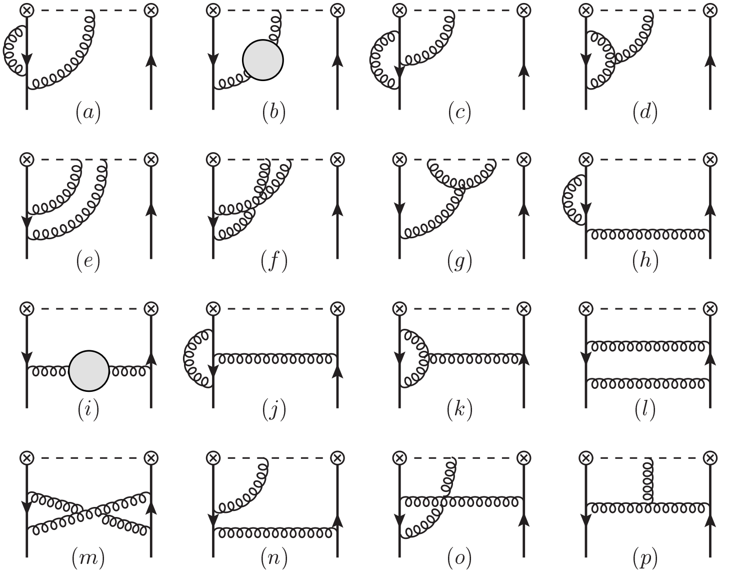
$$Z \mapsto Z_\omega = Z + \omega (n\bar{n}) \widetilde{Z}, \qquad \widetilde{Z} = \frac{1}{\epsilon} \widetilde{Z}_1(a) + \frac{1}{\epsilon^2} \widetilde{Z}_2 + \dots . \qquad (16)$$

Conformal anomaly and residues

Connection between conformal anomaly and renormalization operator has the form

$$\widetilde{Z}_1(a) = z_{12} \Delta_+(a) + \frac{1}{2} \left[\mathbb{H}(a) - 2\gamma_q(a) \right] (z_1 + z_2) . \qquad (17)$$

Two-loop diagrams for the evolution kernel



Two-loop conformal anomaly

The kernel $\Delta_+^{(2)}$ can be written in the following form

$$[\Delta_+^{(2)} f](z_1, z_2) = \int_0^1 du \int_0^1 dt\, \varkappa(t) \Big[f(z_{12}^{ut}, z_2) - f(z_1, z_{21}^{ut}) \Big] + \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \Big[\omega(\alpha,\beta) + \overline{\omega}(\alpha,\beta)\mathbb{P}_{12} \Big] \Big[f(z_{12}^{\alpha}, z_{21}^{\beta}) - f(z_{12}^{\beta}, z_{21}^{\alpha}) \Big]. \quad (18)$$

Two-loop conformal anomaly

$$\varkappa(t) = C_F^2 \varkappa_P(t) + \frac{C_F}{N_C} \varkappa_{FA}(t) + C_F \beta_0 \varkappa_{bF}(t), \tag{19}$$

$$\begin{aligned}
\varkappa_{bF}(t) =& -2\frac{\bar{t}}{t}\Big(\ln\bar{t} + \frac{5}{3}\Big), \\
\varkappa_{FA}(t) =& \frac{2\bar{t}}{t}\Big\{(2+t)\Big[\mathrm{Li}_2(\bar{t}) - \mathrm{Li}_2(t)\Big] - (2-t)\Big(\frac{t}{\bar{t}}\ln t + \ln\bar{t}\Big) \\
& - \frac{\pi^2}{6}t - \frac{4}{3} - \frac{t}{2}\left(1 - \frac{t}{\bar{t}}\right)\Big\}, \\
\varkappa_P(t) =& 4\bar{t}\Big[\mathrm{Li}_2(\bar{t}) - \mathrm{Li}_2(1)\Big] + 4\left(\frac{t^2}{\bar{t}} - \frac{2\bar{t}}{t}\right)\Big[\mathrm{Li}_2(t) - \mathrm{Li}_2(1)\Big] - 2t\ln t\ln\bar{t} \\
& - \frac{\bar{t}}{t}(2-t)\ln^2\bar{t} + \frac{t^2}{\bar{t}}\ln^2 t - 2\left(1 + \frac{1}{t}\right)\ln\bar{t} - 2\left(1 + \frac{1}{\bar{t}}\right)\ln t \\
& - \frac{16}{3}\frac{\bar{t}}{t} - 1 - 5t\,.
\end{aligned} \tag{20}$$

Two-loop conformal anomaly

$$\overline{\omega}(\alpha,\beta)=\frac{C_F}{N_C}\overline{\omega}_{NP}(\alpha,\beta), \tag{21}$$

$$\overline{\omega}_{NP}(\alpha,\beta)=-2\left\{\frac{\alpha}{\bar\alpha}\left[\mathrm{Li}_2\left(\frac{\alpha}{\bar\beta}\right)-\mathrm{Li}_2(\alpha)\right]-\alpha\bar\tau\ln\bar\tau-\frac{1}{\bar\alpha}\ln\bar\alpha\ln\bar\beta-\frac{\beta}{\bar\beta}\ln\bar\alpha-\frac{1}{2}\beta\right\}. \tag{22}$$

$$\omega(\alpha,\beta)=C_F^2\omega_P(\alpha,\beta)+\frac{C_F}{N_C}\omega_{NP}(\alpha,\beta), \tag{23}$$

$$\begin{aligned}\omega_P(\alpha,\beta)&=\frac{4}{\alpha}\left[\mathrm{Li}_2(\bar\alpha)-\zeta_2+\frac{1}{4}\bar\alpha\ln^2\bar\alpha+\frac{1}{2}(\beta-2)\ln\bar\alpha\right]\\&\quad+\frac{4}{\bar\alpha}\left[\mathrm{Li}_2(\alpha)-\zeta_2+\frac{1}{4}\alpha\ln^2\alpha+\frac{1}{2}(\bar\beta-2)\ln\alpha\right],\\\omega_{NP}(\alpha,\beta)&=2\left\{\frac{\bar\alpha}{\alpha}\left[\mathrm{Li}_2\left(\frac{\beta}{\bar\alpha}\right)-\mathrm{Li}_2(\beta)-\mathrm{Li}_2(\alpha)+\mathrm{Li}_2(\bar\alpha)-\zeta_2\right]-\ln\alpha-\frac{1}{\alpha}\ln\bar\alpha\right.\\&\quad\left.+\alpha\left(\frac{\bar\tau}{\tau}\ln\bar\tau+\frac{1}{2}\right)\right\}.\end{aligned} \tag{24}$$

Set of equations for the non-invariant part

Set of canonical generators

$$
\begin{aligned}
S_{-}^{(0)} &= -\partial_{z_1} - \partial_{z_2}; \\
S_{0}^{(0)} &= z_1\partial_{z_1} + z_2\partial_{z_2} + 2; \\
S_{+}^{(0)} &= z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1 + z_2).
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\left[S_{+}^{(0)}, \mathbb{H}^{(1)}\right] =& 0, \\
\left[S_{+}^{(0)}, \mathbb{H}^{(2)}\right] =& \left[\mathbb{H}^{(1)}, z_1 + z_2\right]\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + \left[\mathbb{H}^{(1)}, z_{12}\Delta^{(1)}\right], \\
\left[S_{+}^{(0)}, \mathbb{H}^{(3)}\right] =& \left[\mathbb{H}^{(1)}, z_1 + z_2\right]\left(\beta_1 + \frac{1}{2}\mathbb{H}^{(2)}\right) + \left[\mathbb{H}^{(2)}, z_1 + z_2\right]\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) \\
&+ \left[\mathbb{H}^{(2)}, z_{12}\Delta^{(1)}\right] + \left[\mathbb{H}^{(1)}, z_{12}\Delta^{(2)}\right].
\end{aligned} \tag{26}
$$

Invariant part

Form of the invariant part

Invariant part of the evolution kernel $\mathbb{H}_{\text{inv}}$ can be represented in the following form

$$\mathbb{H}_{\text{inv}}(a) = \Gamma_{\text{cusp}}(a)\widehat{\mathcal{H}} + \mathcal{A}(a) + \mathcal{H}(a), \tag{27}$$

where $\Gamma_{\text{cusp}}(a)$ is a cusp anomalous dimension, $\mathcal{A}(a)$ is a constant and operators have the form

$$\left[\widehat{\mathcal{H}} f\right](z_1, z_2) = \int_0^1 \frac{d\alpha}{\alpha} \left(2f(z_1, z_2) - \bar{\alpha}(f(z_{12}^\alpha, z_2) + f(z_1, z_{21}^\alpha))\right). \tag{28}$$

and

$$\left[\mathcal{H}(a) f\right](z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \left(h(\tau) + \overline{h}(\tau)\mathbb{P}_{12}\right) f(z_{12}^\alpha, z_{21}^\beta). \tag{29}$$

Consequences of $SL(2,\mathbb{R})$ invariance

The crucial point is that $h(\tau)$ and $\overline{h}(\tau)$ are functions of only one variable $\tau = \frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$.

Forward anomalous dimension

Connection with the forward anomalous dimensions

Eigenvalues of the evolution kernel $\psi_N(z_1, z_2)$ correspond with the **forward** anomalous dimensions

$$\mathbb{H}\psi_N(z_1, z_2) = \gamma(N)\psi_N(z_1, z_2), \tag{30}$$

the same is valid for the invariant part

$$\mathbb{H}_{\text{inv}}\psi_N(z_1, z_2) = \gamma_{\text{inv}}(N)\psi_N(z_1, z_2) \tag{31}$$

Moments of integral operator

Applying integral operator to the eigenfunctions ψ_N we get the following relation

$$\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \left(h(\tau) + (-1)^{N-1}\,\overline{h}(\tau)\right)(1-\alpha-\beta)^{N-1} = m(N), \tag{32}$$

so if we know $m(N)$ for arbitrary N we can restore the integral kernel

$$m(N) \;\Rightarrow\; h(\tau), \overline{h}(\tau). \tag{33}$$

How to find the moments?

Forward anomalous dimensions can be divided into invariant and non-invariant part as well

$$\gamma(N) = \gamma_{\text{inv}}(N) + \gamma_{\text{non-inv}}(N). \tag{34}$$

Structure of the invariant part

Invariant part has the particular structure

$$\gamma_{\text{inv}}(N) = 2\Gamma_{\text{cusp}}(a) S_1(N) + \mathcal{A}(a) + m(N), \tag{35}$$

where $S_1(N)$ is a Harmonic sum.

Using forward anomalous dimensions

We use the result for the $\gamma^{(3)}(N)$ [V. N. Velizhanin'2012]

$$\gamma^{(3)}(N) \;\Rightarrow\; m^{(3)}(N). \tag{36}$$

Naive way

Analytic continuation

Let us consider function $f(z)$ for the $z \in \mathbb{C}$

$$f(z) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\tau)(1-\alpha-\beta)^{z-1}. \tag{37}$$

Inverse formula

We can obtain the inverse formula for the kernel $h(\tau)$ in the form

$$h(\tau) = \int_C \frac{dz}{2\pi i}(2z+1)f(z)P_z\left(\frac{1+\tau}{1-\tau}\right), \tag{38}$$

where P_z are the Legendre function of the first kind, and integration contour C goes along a line parallel to the imaginary axis such that all singularities of $f(z)$ lie to the left of the contour.

This result gives a solution to the problem but it is not enough for the practical use.

Reciprocity relation

Generalization of the Gribov-Lipatov reciprocity relation takes the form

$$\gamma(N) = \gamma_{\text{inv}}\left(N + \bar{\beta}(a) + \frac{1}{2}\gamma(N)\right). \tag{39}$$

Asymptotic expansion of the $\gamma_{\text{inv}}(N)$ for large N is invariant under reflection $N \mapsto -N-1$.

Reciprocity respecting harmonic sums

Expression for the $m(N)$ includes only special combinations of harmonic sums $\Omega_{a,b,c,\dots}(N)$ with asymptotic expansion invariant under

$$N \mapsto -N-1, \tag{40}$$

and we additionally ask $\Omega_{a,b,c,\dots}(\infty) = 0$.

Two-loop example

Tow-loop result for the moments $m(N)$ has the form

$$m_{\pm}^{(2)}(N) = \frac{2C_F}{N_c}\left(32S_1\left(S_{-2}+\frac{\zeta_2}{2}\right)+16\left(S_3-\zeta_3\right)-32\left(S_{-2,1}-\frac{1}{2}S_{-3}+\frac{1}{3}\zeta_3\right)\right.$$
$$\left.+\frac{2\left(1-(-1)^N\right)}{N(N+1)}\right), \quad (41)$$

$$\Omega_1(N) = S_1(N), \qquad \Omega_{-2}(N) = (-1)^N\left(S_{-2}(N)+\frac{\zeta_2}{2}\right),$$
$$\Omega_3(N) = S_3(N)-\zeta_3, \quad \Omega_{-2,1}(N) = (-1)^N\left(S_{-2,1}(N)-\frac{1}{2}S_{-3}(N)+\frac{1}{3}\zeta_3\right), \quad (42)$$

Parity invariant form of the answer

Using these sums we can rewrite two-loop result in the form

$$m_{\pm}^{(2)}(N) = \frac{2C_F}{N_c}\left(16\Omega_3 \pm 32\left(\Omega_1\Omega_{-2}+\Omega_{-2,1}\right)+\frac{2(1\mp 1)}{N(N+1)}\right). \quad (43)$$

Two-loop example

Construction of the integration kernel

Kernels corresponding to the parity invariant sums can be effectively constructed using [Y. Ji, A. Manashov, S. Moch, 2023]

$$\Omega_{-2}(N) \to \frac{\bar{\tau}}{2} \qquad \Omega_3(N) \to \frac{\bar{\tau}}{2\tau}\ln\tau, \tag{44}$$

$$\Omega_{-2,1}(N) \to -\frac{\bar{\tau}}{4}\left(\mathrm{H}_1(\tau) + \mathrm{H}_0(\tau)\right), \tag{45}$$

where $\mathrm{H}_{a,b,c,\ldots}$ are HPLs.

Result for the two-loop

Result fot the integral kernels then reads

$$h^{(2)}(\tau) = \frac{8C_F}{N_c}\left(\frac{\bar{\tau}}{\tau}\ln\bar{\tau} + \frac{1}{2}\right), \qquad \overline{h}^{(2)}(\tau) = \frac{8C_F}{N_c}\left(-\bar{\tau}\ln\bar{\tau} + \frac{1}{2}\right), \tag{46}$$

Results for the invariant part

We present result for the invariant part

$$\mathbb{H}^{(3)}_{\text{inv}}(a) = \Gamma^{(3)}_{\text{cusp}}(a)\widehat{\mathcal{H}} + \mathcal{A}^{(3)}(a) + \mathcal{H}^{(3)}(a), \tag{47}$$

$$\begin{aligned}
\mathcal{A}^{(3)} = {} & C_F n_f^2 \left(\frac{34}{9} - \frac{160}{27}\zeta_2 + \frac{32}{9}\zeta_3\right) + C_F^2 n_f \left(-34 + \frac{4984}{27}\zeta_2 - \frac{512}{15}\zeta_2^2 + \frac{16}{9}\zeta_3\right) \\
& + \frac{C_F n_f}{N_c}\left(-40 + \frac{2672}{27}\zeta_2 - \frac{8}{5}\zeta_2^2 - \frac{400}{9}\zeta_3\right) \\
& + C_F^3 \left(\frac{1694}{9} - \frac{22180}{27}\zeta_2 + \frac{2464}{15}\zeta_2^2 + \frac{1064}{9}\zeta_3 - 320\zeta_5\right) \\
& + \frac{C_F^2}{N_c}\left(\frac{5269}{18} - \frac{28588}{27}\zeta_2 + \frac{2216}{15}\zeta_2^2 + \frac{7352}{9}\zeta_3 - 32\zeta_2\zeta_3 - 560\zeta_5\right) \\
& + \frac{C_F}{N_c^2}\left(\frac{1657}{18} - \frac{8992}{27}\zeta_2 + 4\zeta_2^2 + \frac{3104}{9}\zeta_3 - 80\zeta_5\right).
\end{aligned} \tag{48}$$

Results for the invariant part

$$
\begin{aligned}
h^{(3)}(\tau) = &-C_F n_f^2 \frac{16}{9} + C_F^2 n_f \left(\frac{352}{9} - \frac{8}{3}\mathrm{H}_0 + \frac{16}{3}\frac{\bar\tau}{\tau}\left(\mathrm{H}_2 - \mathrm{H}_{10}\right)\right) \\
&+ \frac{C_F n_f}{N_c}\left(8 - \frac{8}{3}\mathrm{H}_1 - \frac{4}{3}\mathrm{H}_0 + \frac{\bar\tau}{\tau}\left(8\mathrm{H}_2 - \frac{8}{3}\mathrm{H}_{10} + \frac{16}{3}\mathrm{H}_{11} + \frac{160}{9}\mathrm{H}_1\right)\right) \\
&+ C_F^3\left(-\frac{1936}{9} + \frac{88}{3}\mathrm{H}_0 + 32\frac{\bar\tau}{\tau}\left(\mathrm{H}_3 + \mathrm{H}_{12} - \mathrm{H}_{110} - \mathrm{H}_{20} - \frac{1}{3}\mathrm{H}_2 + \frac{1}{3}\mathrm{H}_{10} + \frac{1}{2}\mathrm{H}_1\right)\right) \\
&+ \frac{C_F^2}{N_c}\Bigg(-\frac{152}{3} - 96\zeta_3 - \left(\frac{8}{3} - 48\zeta_2\right)\mathrm{H}_0 + \frac{76}{3}\mathrm{H}_1 - 32\mathrm{H}_{10} + 4\mathrm{H}_2 - 48\mathrm{H}_{20} - 16\mathrm{H}_{11} \\
&- 24\mathrm{H}_{21} + \frac{\tau}{\bar\tau}\left(-24\zeta_2 - 48\zeta_3 + 64\mathrm{H}_0\right) + \frac{\tau+1}{\bar\tau}\Big(-(32 - 16\zeta_2)\mathrm{H}_0 \\
&+ 12\mathrm{H}_2 - 16\mathrm{H}_{20} - 8\mathrm{H}_{21}\Big) + \frac{\bar\tau}{\tau}\Big(-\left(\frac{2000}{9} + 16\zeta_2\right)\mathrm{H}_1 + \frac{32}{3}\mathrm{H}_{10} - \frac{208}{3}\mathrm{H}_2 \\
&- 64\mathrm{H}_{20} - \frac{32}{3}\mathrm{H}_{11} - 32\mathrm{H}_{110} + 64\mathrm{H}_3 + 80\mathrm{H}_{12} + 64\mathrm{H}_{21} + 96\mathrm{H}_{111}\Big)\Bigg) \\
&+ \frac{C_F}{N_c^2}\Bigg(\frac{544}{9} + 16\zeta_2 - 96\zeta_3 - \left(\frac{68}{3} - 36\zeta_2\right)\mathrm{H}_0 + \frac{68}{3}\mathrm{H}_1 - 24\mathrm{H}_{10} + 4\mathrm{H}_2 - 36\mathrm{H}_{20} \\
&+ \frac{\tau}{\bar\tau}\left(-8\zeta_2 - 48\zeta_3 + 48\mathrm{H}_0\right) + \frac{\tau+1}{\bar\tau}\left((-24 + 12\zeta_2)\mathrm{H}_0 + 4\mathrm{H}_2 - 12\mathrm{H}_{20}\right) \\
&+ \frac{\bar\tau}{\tau}\Big(-\left(\frac{1072}{9} + 16\zeta_2\right)\mathrm{H}_1 + \frac{44}{3}\mathrm{H}_{10} - 44\mathrm{H}_2 - 32\mathrm{H}_{20} - \frac{16}{3}\mathrm{H}_{11} - 16\mathrm{H}_{110} \\
&+ 32\mathrm{H}_3 + 32\mathrm{H}_{12} + 48\mathrm{H}_{21} + 32\mathrm{H}_{111}\Big)\Bigg).
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\overline{h}^{(3)}(\tau) = &-\frac{C_F n_f}{N_c}\left(\frac{104}{9}+\frac{8}{3}\mathrm{H}_0+\frac{8}{9}\Big(23-20\tau\Big)\mathrm{H}_1+\frac{16}{3}\bar{\tau}\Big(\mathrm{H}_{11}+\mathrm{H}_{10}\Big)\right)\\
&+\frac{C_F^2}{N_c}\Bigg(\frac{1480}{9}-40\zeta_2-48\zeta_3+\left(\frac{28}{3}+24\zeta\right)\mathrm{H}_0+\frac{76}{3}\mathrm{H}_1+16\mathrm{H}_{10}-4\mathrm{H}_2-24\mathrm{H}_{20}\\
&-16\mathrm{H}_{11}+24\mathrm{H}_{21}+\frac{\tau}{\bar{\tau}}\Big(-24\zeta_2+48\zeta_3-32\mathrm{H}_0\Big)+\frac{\tau+1}{\bar{\tau}}\Big(\Big(16-8\zeta_2\Big)\mathrm{H}_0+12\mathrm{H}_2\\
&+8\mathrm{H}_{20}-8\mathrm{H}_{21}\Big)+\bar{\tau}\Big(-24+48\zeta_2+48\zeta_3-16\zeta_2\mathrm{H}_0+\left(\frac{2144}{9}+16\zeta_2\right)\mathrm{H}_1+\frac{104}{3}\mathrm{H}_{10}\\
&-24\mathrm{H}_2+16\mathrm{H}_{20}+\frac{32}{3}\mathrm{H}_{11}-16\mathrm{H}_{110}-32\mathrm{H}_{12}-32\mathrm{H}_{21}-96\mathrm{H}_{111}\Big)\Bigg)\\
&+\frac{C_F}{N_c^2}\Bigg(\frac{1028}{9}-24\zeta_2-48\zeta_3+\left(\frac{44}{3}+36\zeta_2\right)\mathrm{H}_0+\frac{68}{3}\mathrm{H}_1+24\mathrm{H}_{10}-4\mathrm{H}_2-36\mathrm{H}_{20}\\
&+\frac{\tau}{\bar{\tau}}\Big(-8\zeta_2+48\zeta_3-48\mathrm{H}_0\Big)+\frac{\tau+1}{\bar{\tau}}\Big(\Big(24-12\zeta_2\Big)\mathrm{H}_0+4\mathrm{H}_2+12\mathrm{H}_{20}\Big)\\
&+\bar{\tau}\Big(-24+24\zeta_2+48\zeta_3-32\zeta_2\mathrm{H}_0+\left(\frac{1072}{3}+16\zeta_2\right)\mathrm{H}_1+\frac{88}{3}\mathrm{H}_{10}\\
&-24\mathrm{H}_2+32\mathrm{H}_{20}+\frac{16}{3}\mathrm{H}_{11}-32\mathrm{H}_{110}+16\mathrm{H}_{12}+16\mathrm{H}_{21}-32\mathrm{H}_{111}\Big)\Bigg).
\end{aligned}
\tag{50}
$$

Back to the local

Gegenbauer basis

Local twist-2 operators can be expressed in the Gegenbauer basis as follows

$$\mathcal{O}^G_{n,k} = (\partial_{z_1} + \partial_{z_2})^k C_n^{3/2}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right)\mathcal{O}(z_1, z_2)\Big|_{z_1=z_2=0}, \qquad (51)$$

where $C_N^\nu(x)$ is the Gegenbauer polynomial.

RG-equation then takes the form

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a}\right)\left[\mathcal{O}^G_{n,k}\right] = -\sum_{n'=0}^{n}\gamma^G_{n,n'}\left[\mathcal{O}^G_{n',k}\right]. \qquad (52)$$

Anomalous dimension matrix

- $\gamma^G_{n,n}$ — forward anomalous dimensions, contribute to the $\langle P|\mathcal{O}(z_1, z_2)|P\rangle$
- $\gamma^G_{n,n'}$ — **off-forward** anomalous dimensions, contribute to the $\langle P'|\mathcal{O}(z_1, z_2)|P\rangle$

Results for the matrix

We consider $N_c = 3$ and $0 \leq n, n' \leq 5$ and separating $\gamma^{(3)}_{\text{off}} = \gamma^{(3)}_1 + n_f \gamma^{(3)}_{n_f} + n_f^2 \gamma^{(3)}_{n_f^2}$ we get

$$\gamma^{(3)}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{44992}{81} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1316680}{2187} & 0 & 0 & 0 & 0 \\ \frac{1977808}{10125} & 0 & \frac{54669748}{91125} & 0 & 0 & 0 \\ 0 & \frac{68848018}{273375} & 0 & \frac{443231668}{759375} & 0 & 0 \end{pmatrix} \tag{53}$$

and

$$\gamma^{(3)}_{n_f} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{21008}{243} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{200060}{2187} & 0 & 0 & 0 & 0 \\ \frac{998842}{30375} & 0 & \frac{898436}{10125} & 0 & 0 & 0 \\ 0 & \frac{745418}{18225} & 0 & \frac{4266496}{50625} & 0 & 0 \end{pmatrix}, \tag{54}$$

$$\gamma^{(3)}_{n_f^2} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{160}{81} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{520}{243} & 0 & 0 & 0 & 0 \\ \frac{1012}{2025} & 0 & \frac{4088}{2025} & 0 & 0 & 0 \\ 0 & \frac{3268}{3645} & 0 & \frac{416}{225} & 0 & 0 \end{pmatrix}. \tag{55}$$

Result for the matrix

The diagonal elements have the form

$$\begin{aligned}
\gamma_{00}^{(3)} &= \frac{105110}{81} - \frac{1856}{27}\zeta_3 - \left(\frac{10480}{81} + \frac{320}{9}\zeta_3\right) n_f - \frac{8}{9}n_f^2 \\
\gamma_{11}^{(3)} &= \frac{19162}{9} - \left(\frac{5608}{27} + \frac{320}{3}\zeta_3\right) n_f - \frac{184}{81}n_f^2 \\
\gamma_{22}^{(3)} &= \frac{17770162}{6561} + \frac{1280}{81}\zeta_3 - \left(\frac{552308}{2187} + \frac{4160}{27}\zeta_3\right) n_f - \frac{2408}{729}n_f^2 \\
\gamma_{33}^{(3)} &= \frac{206734549}{65610} + \frac{560}{27}\zeta_3 - \left(\frac{3126367}{10935} + \frac{5120}{27}\zeta_3\right) n_f - \frac{14722}{3645}n_f^2 \\
\gamma_{44}^{(3)} &= \frac{144207743479}{41006250} + \frac{9424}{405}\zeta_3 - \left(\frac{428108447}{1366875} + \frac{5888}{27}\zeta_3\right) n_f - \frac{418594}{91125}n_f^2 \\
\gamma_{55}^{(3)} &= \frac{183119500163}{47840625} + \frac{3328}{135}\zeta_3 - \left(\frac{1073824028}{3189375} + \frac{2176}{9}\zeta_3\right) n_f - \frac{3209758}{637875}n_f^2.
\end{aligned} \tag{56}$$

The final three-loop result is consistent with the [S. Van Thurenhout, S.-O. Moch, 2022] large n_f limit prediction.