Constraints for alien operators in QCD

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Introduction

At HL-LHC: Statistical/systematic uncertainties $\sim 1\%$

$\Rightarrow$ Theory needs to keep up!

	Q [GeV]	$\delta\sigma^{\text{N}^3\text{LO}}$	δ(scale)	δ(PDF-TH)
$gg \to$ Higgs	m_H	3.5%	$^{+0.21\%}_{-2.37\%}$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	-2.3%	$^{+3.0\%}_{-4.8\%}$	$\pm 2.5\%$
NCDY	30	-4.8%	$^{+1.53\%}_{-2.54\%}$	$\pm 2.8\%$
	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm 2.5\%$
CCDY(W^+)	30	-4.7%	$^{+2.5\%}_{-1.7\%}$	$\pm 3.2\%$
	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm 2.1\%$
CCDY(W^-)	30	-5.0%	$^{+2.6\%}_{-1.6\%}$	$\pm 3.2\%$
	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm 2.13\%$

Table: [Baglio et al., 2022]

$$\delta(\text{PDF-TH}) = \frac{1}{2}\frac{\left|\sigma^{\text{NNLO}}(\text{NNLO PDF}) - \sigma^{\text{NNLO}}(\text{NLO PDF})\right|}{\sigma^{\text{NNLO}}(\text{NNLO PDF})}$$

PDF scale dependence

Scale evolution of PDFs is set by the DGLAP equation [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d}f_i(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P_{ij}(y) f_j\left(\frac{x}{y},\mu^2\right)$$

with P_{ij} the QCD splitting functions. These are perturbative quantities and can be computed as the anomalous dimensions of the leading-twist operators that define the PDFs

$$\frac{\mathrm{d}[\mathcal{O}_i]}{\mathrm{d}\ln\mu^2} = \gamma_{ij}[\mathcal{O}_j], \quad \gamma_{ij} \equiv a_s\gamma_{ij}^{(0)} + a_s^2\gamma_{ij}^{(1)} + \ldots$$

$$\gamma_{ij} = -\int_0^1 \mathrm{d}x\, x^N P_{ij}(x)$$

Operator renormalization

The leading-twist operators of interest are

$$\mathcal{O}^{(N)}_{g;\mu_1\ldots\mu_N}(x) = \frac{1}{2}\mathcal{S}\left[\, F^{a_1}{}_{\mu\mu_1}\, D^{a_1 a_2}_{\mu_2} \ldots D^{a_{N-2}a_{N-1}}_{\mu_{N-1}}\, F^{a_{N-1};\mu}{}_{\mu_N}\,\right]$$

$$\mathcal{O}^{(N)}_{q\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

$$\mathcal{O}^{(N)}_{q\,\mathrm{NS};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\lambda^{\alpha}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

with

$$D^{ac}_{\mu} = \partial_\mu \delta^{ac} + g_s f^{abc} A^b_\mu$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$

In practice the operators are contracted with N copies of a lightlike vector Δ. The anomalous dimensions are now computed from the renormalization of the off-shell matrix elements of these operators.

Construction of the alien operators

When doing so, it is well-known that mixing with non-gauge-invariant (alien) operators needs to be taken into account [Dixon and Taylor, 1974, Kluberg-Stern and Zuber, 1975a, Kluberg-Stern and Zuber, 1975b, Joglekar and Lee, 1976, Joglekar, 1977a, Joglekar, 1977b].

The corresponding operators involve

- ghost fields
- the field equations of motion (EOM)

The EOM operators can be reconstructed using generalized gauge symmetry [Falcioni and Herzog, 2022]

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu \,.$$

We write the perturbative expansion of the operator as

$$\mathcal{O}^{(N)}_{\text{EOM}} = \mathcal{O}^{(N),I}_{\text{EOM}} + \mathcal{O}^{(N),II}_{\text{EOM}} + \mathcal{O}^{(N),III}_{\text{EOM}} + \mathcal{O}^{(N),IV}_{\text{EOM}} + \ \ldots$$

Construction of the alien operators

$$\mathcal{O}_{\text{EOM}}^{(N),I} = \eta(N)\ \left(D\cdot F^a + g_s\overline{\psi}\,\Delta\!\!\!/\, T^a\psi\right)\ \left(\partial^{\,N-2}A^a\right),$$
$$\mathcal{O}_{\text{EOM}}^{(N),II} = g_s\left(D\cdot F^a + g_s\overline{\psi}\,\Delta\!\!\!/\, T^a\psi\right)\sum_{i+j=N-3} C_{ij}^{abc}(\partial^i A^b)(\partial^j A^c),$$
$$\mathcal{O}_{\text{EOM}}^{(N),III} = g_s^2\ \left(D\cdot F^a + g_s\overline{\psi}\Delta\!\!\!/ T^a\psi\right)\sum_{i+j+k=N-4} C_{ijk}^{abcd}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d),$$
$$\mathcal{O}_{\text{EOM}}^{(N),IV} = g_s^3\ \left(D\cdot F^a + g_s\overline{\psi}\Delta\!\!\!/ T^a\psi\right)\sum_{i+j+k+l=N-5} C_{ijkl}^{abcde}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e)$$

At leading order, the EOM coupling η is simply a function of N. The one-loop value of this quantity was computed in [Dixon and Taylor, 1974, Hamberg and van Neerven, 1992] to be

$$\eta(N) = -\frac{C_A}{N(N-1)}.$$

Construction of the alien operators

The higher-order couplings however are non-trivial functions of both colour and the indices $i, j, \ldots$. We have

$$
\begin{aligned}
C^{abc}_{ij} &= f^{abc}\kappa_{ij}, \\
C^{abcd}_{ijk} &= (f\,f)^{abcd}\kappa^{(1)}_{ijk} + d^{abcd}_{4}\kappa^{(2)}_{ijk} + d^{abcd}_{\widehat{4ff}}\kappa^{(3)}_{ijk}, \\
C^{abcde}_{ijkl} &= (f\,f\,f)^{abcde}\kappa^{(1)}_{ijkl} + d^{abcde}_{4f}\kappa^{(2)}_{ijkl}.
\end{aligned}
$$

The κ couplings are chosen to inherit the behaviour of the colour structure they multiply, e.g. $\kappa_{ij} = -\kappa_{ji}$. The ghost operators can now be derived from the EOM ones by promoting the generalized gauge transformation

$$
A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta_\omega{}^\Delta A^a_\mu
$$

to a generalized BRST (gBRST) transformation

$$
A^a_\mu \to A^a_\mu + \delta_c A^a_\mu + \delta_c{}^\Delta A^a_\mu
$$

Construction of the alien operators

$$\mathcal{O}_c^{(N)} = \mathcal{O}_c^{(N),I} + \mathcal{O}_c^{(N),II} + \mathcal{O}_c^{(N),III} + \mathcal{O}_c^{(N),IV} + \ldots$$

$$\mathcal{O}_c^{(N),I} = -\eta(N)(\partial\overline{c}^a)(\partial^{N-1}c^a),$$

$$\mathcal{O}_c^{(N),II} = -g_s \sum_{i+j=N-3} \tilde{C}_{ij}^{abc}(\partial\overline{c}^a)(\partial^i A^b)(\partial^{j+1}c^c),$$

$$\mathcal{O}_c^{(N),III} = -g_s^2 \sum_{i+j+k=N-4} \tilde{C}_{ijk}^{astu}(\partial\overline{c}^a)(\partial^i A^s)(\partial^j A^t)(\partial^{k+1}c^u),$$

$$\mathcal{O}_c^{(N),IV} = -g_s^3 \sum_{i+j+k+l=N-5} \tilde{C}_{ijkl}^{abcde}(\partial\overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1}c^e)$$

The colour decomposition of the ghost couplings is similar to the EOM ones

$$\tilde{C}_{ij}^{abc} = f^{abc}\eta_{ij},$$

$$\tilde{C}_{ijk}^{abcd} = (f\,f)^{abcd}\eta_{ijk}^{(1)} + d_4^{abcd}\eta_{ijk}^{(2)} + d_{\widehat{4ff}}^{abcd}\eta_{ijk}^{(3)},$$

$$\tilde{C}_{ijkl}^{abcde} = (f\,f\,f)^{abcde}\eta_{ijkl}^{(1)} + d_{4f}^{abcde}\eta_{ijkl}^{(2a)} + d_{4f}^{aebcd}\eta_{ijkl}^{(2b)}.$$

Identities between alien couplings

- As the ghost operators were constructed directly from the EOM ones using gBRST, the η couplings are connected to the κ ones.
- An equivalent approach to generate the ghost operators would be to start from anti-gBRST, for which $\omega^a(x)$ in the generalized gauge transformation should be replaced by the anti-ghost field $\overline{c}^a(x)$

$$A_\mu^a \rightarrow A_\mu^a + \delta_{\overline{c}} A_\mu^a + \delta_{\overline{c}}{}^{\Delta} A_\mu^a$$

$\rightarrow$ the functional form of the resulting operators is different from those derived from gBRST
$\Rightarrow$ non-trivial identities for the η-couplings!

- These identities allow one to restrict the function space of the couplings and hence constrain their generic N-dependence.
- During this talk: Focus on couplings coming with a string of f's

Class II couplings

$$\mathcal{O}_{\text{EOM}}^{(N),II} = g_s \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi \right) \sum_{i+j=N-3} C_{ij}^{abc} (\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_c^{(N),II} = -g_s \sum_{i+j=N-3} \tilde{C}_{ij}^{abc} (\partial \overline{c}^a)(\partial^i A^b)(\partial^{j+1} c^c)$$

$$\kappa_{ij} + \kappa_{ji} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N) \binom{i+j+1}{i}, \qquad \text{[gBRST]}$$

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0 \qquad \text{[anti-gBRST]}$$

Class II couplings

Note that the anti-gBRST relation is an example of a conjugation relation, in the sense that a second application of the sum leads to

$$\sum_{t=0}^{i}(-1)^{t+j}\binom{t+j}{j}\eta_{(i-t)(j+t)} = -\sum_{t=0}^{i}(-1)^{t+j}\binom{t+j}{j}\sum_{s=0}^{i-t}(-1)^{s+j+t}\binom{s+j+t}{j+t}\eta_{(i-t-s)(j+t+s)}$$

and hence

$$\eta_{ij} = \sum_{t=0}^{i}\binom{t+j}{j}\sum_{s=0}^{i-t}(-1)^{s}\binom{s+j+t}{j+t}\eta_{(i-t-s)(j+t+s)}.$$

- Already encountered in the computation of the anomalous dimensions of leading-twist operators in non-forward kinematics, see e.g. [Moch and Van Thurenhout, 2021, Van Thurenhout, 2024]
- Great predictive power!
- Valuable information about the function space

Class II couplings

$$\kappa_{ij} + \kappa_{ji} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i}, \qquad \text{[gBRST]}$$

$$\eta_{ij} + \sum_{s=0}^{i}(-1)^{s+j}\binom{s+j}{j}\eta_{(i-s)(j+s)} = 0 \qquad \text{[anti-gBRST]}$$

Combining anti-symmetry with gBRST we have

$$\eta_{ij} + \eta_{ji} = \eta(N)\left[\binom{i+j+1}{i} + \binom{i+j+1}{j}\right]$$

which gives an idea about the function space of η_{ij}.

Class II couplings

Using the RHS of the previous equation as an Ansatz for η_{ij} gives

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = (-1)^j c_1 - c_2 \binom{i+j+1}{j}$$

for even values of N. Hence, only the trivial solution $c_1 = c_2 = 0$ obeys the anti-gBRST relation. The RHS however suggests the inclusion of a new structure: $(-1)^j$. With

$$\eta_{ij} = c_1 (-1)^j + c_2 \binom{i+j+1}{i} + c_3 \binom{i+j+1}{j}$$

we find

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = (c_1 + c_2) \left[\binom{i+j+1}{i} + (-1)^j \right]$$

and hence $c_1 = -c_2$.

Class II couplings

Assuming that κ_{ij} lives in the same function space as η_{ij}, the full set of relations fixes both couplings up to 1 free parameter

$$\eta_{ij} = \eta(N)\left\{(1+2c)\left[\binom{i+j+1}{i} - (-1)^j\right] - 2c\binom{i+j+1}{j}\right\}$$
$$\kappa_{ij} = \eta(N)\left\{c\left[\binom{i+j+1}{i} - \binom{i+j+1}{j}\right] - \frac{1}{2}(1+2c)(-1)^j\right\}$$

The unknown c can be determined by the computation of 1 fixed-N matrix element computation. E.g. for $N=6$ we have $\kappa_{30} = 1/24$ which sets $c = -3/8$

$$\eta_{ij} = -\frac{\eta(N)}{4}\left[(-1)^j - 3\binom{N-2}{i+1} - \binom{N-2}{i}\right]$$
$$\kappa_{ij} = -\frac{\eta(N)}{8}\left[(-1)^j + 3\binom{i+j+1}{i} - 3\binom{i+j+1}{i+1}\right]$$

The solution above exactly agrees with the known solution [Hamberg and van Neerven, 1992, Falcioni and Herzog, 2022].

Class III couplings

$$\mathcal{O}^{(N),III}_{\text{EOM}} = g_s^2 \left(D\cdot F^a + g_s\overline{\psi}\not{\Delta} T^a\psi\right) \sum_{i+j+k=N-4} C^{abcd}_{ijk}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d),$$

$$\mathcal{O}^{(N),III}_{c} = -g_s^2 \sum_{i+j+k=N-4} \tilde{C}^{astu}_{ijk}(\partial\overline{c}^a)(\partial^i A^s)(\partial^j A^t)(\partial^{k+1} c^u)$$

$$\kappa^{(1)}_{ijk} + \kappa^{(1)}_{ikj} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\kappa^{(1)}_{ijk} + \kappa^{(1)}_{jki} + \kappa^{(1)}_{kij} = 0, \qquad \text{[Jacobi identity]}$$

$$\eta^{(1)}_{ijk} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2[\kappa^{(1)}_{ijk} + \kappa^{(1)}_{kji}], \qquad \text{[gBRST]}$$

$$\eta^{(1)}_{ijk} = \sum_{m=0}^{i}\sum_{n=0}^{j}\frac{(m+n+k)!}{m!\,n!\,k!}(-1)^{m+n+k}\eta^{(1)}_{(j-n)(i-m)(k+m+n)}. \qquad \text{[anti-gBRST]}$$

Class III couplings

The combination of the Jacobi identity with gBRST leads to

$$\eta^{(1)}_{ijk} + \eta^{(1)}_{kij} + \eta^{(1)}_{jki} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2\kappa_{k(i+j+1)}\binom{i+j+1}{i} + 2\kappa_{j(i+k+1)}\binom{i+k+1}{k}.$$

$\rightarrow$ relates the class III coupling $\eta^{(1)}_{ijk}$ to the class II coupling κ_{ij}, at one order lower in perturbation theory!

$\Rightarrow$ use it to determine the function space of the all-N expression of $\eta^{(1)}_{ijk}$

$\rightarrow$ leads to 18-dimensional function space

$$\left\{(-1)^{i+j}\binom{i+j+1}{i}, \binom{N-2}{k+1}\binom{i+j+1}{i}, \binom{N-2}{k}\binom{i+j+1}{i}, (-1)^{j+k}\binom{j+k+1}{j}, \binom{N-2}{i+1}\binom{j+k+1}{j}, \binom{N-2}{i}\binom{j+k+1}{j}, (-1)^{i+k}\binom{i+k+1}{k}, \binom{N-2}{j+1}\binom{i+k+1}{k}, \binom{N-2}{j}\binom{i+k+1}{k} + \text{ independent permutations of } i, j \text{ and } k\right\}.$$

Class III couplings

We assume $\kappa_{ijk}^{(1)}$ to live in the same function space. Hence in total we have 36 free parameters. Using the relations described above we are able to fix 34 of these. The final 2 free parameters are then fixed using $\kappa_{110}^{(1)} = 0$ and $\kappa_{121}^{(1)} = 13/336$, which follow from the explicit operator renormalization for $N = 6$ and $N = 8$ respectively. Our final result for $\kappa_{ijk}^{(1)}$ then becomes [new!]

$$\kappa_{ijk}^{(1)} = \frac{\eta(N)}{48}\Bigg\{2(-1)^{i+j}\binom{i+j+1}{i} + (-1)^{i+k}\binom{i+k+1}{k} + 3(-1)^{j+k+1}\binom{j+k+1}{j} + \binom{i+k+1}{i}\Bigg[2(-1)^{i+k+1} + 5\binom{N-1}{j+1}\Bigg] + \binom{j+k+1}{k}\Bigg[3(-1)^{j+k} - 10\binom{N-2}{i} + 4\binom{N-2}{i+1}\Bigg] + \binom{i+j+1}{j}\Bigg[(-1)^{i+j+1} + 5\binom{N-2}{k} - 9\binom{N-2}{k+1}\Bigg]\Bigg\}.$$

Class III couplings

We have checked that the above expression agrees with explicitly computed values, following from the renormalization of the operators, up to $N = 20$. Substituting this expression into the gBRST relation allows one to also reconstruct the full N-dependence of $\eta^{(1)}_{ijk}$ [new!]

$$\eta^{(1)}_{ijk} = -\frac{\eta(N)}{24}\Bigg\{5(-1)^{i+j+1}\binom{i+j+1}{i} + (-1)^{i+k}\binom{i+k+1}{k} + 2(-1)^{j+k+1}\binom{j+k+1}{j} + \binom{i+k+1}{i}\left[(-1)^{i+k} + 4\binom{N-2}{j+1}\right] + \binom{j+k+1}{k}\left[5(-1)^{j+k+1} - 3\binom{N-2}{i} + \binom{N-2}{i+1}\right] + \binom{i+j+1}{j}\left[4(-1)^{i+j} - 15\binom{N-2}{k} - 5\binom{N-2}{k+1}\right]\Bigg\}.$$

Class IV couplings

$$\mathcal{O}^{(N),IV}_{\text{EOM}} = g_s^3 \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi\right) \sum_{i+j+k+l=N-5} C^{abcde}_{ijkl} (\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e),$$

$$\mathcal{O}^{(N),IV}_{c} = -g_s^3 \sum_{i+j+k+l=N-5} \tilde{C}^{abcde}_{ijkl} (\partial \overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1} c^e)$$

$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ijlk} = 0, \qquad \text{[anti-symmetry]}$$

$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{iklj} + \kappa^{(1)}_{iljk} = 0, \qquad \text{[Jacobi]}$$

$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{jilk} + \kappa^{(1)}_{lkji} + \kappa^{(1)}_{klij} = 0, \qquad \text{[double Jacobi]}$$

$$\eta^{(1)}_{ijkl} = 2[\kappa^{(1)}_{ij(l+k+1)} + \kappa^{(1)}_{(l+k+1)ji}]\binom{l+k+1}{k} + 2[\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ilkj} + \kappa^{(1)}_{likj} + \kappa^{(1)}_{lkij}], \qquad \text{[gBRST]}$$

$$\eta^{(1)}_{ijkl} = -\sum_{s_1=0}^{i}\sum_{s_2=0}^{j}\sum_{s_3=0}^{k} \frac{(s_1+s_2+s_3+l)!}{s_1!\,s_2!\,s_3!\,l!}(-1)^{s_1+s_2+s_3+l}\eta^{(1)}_{(k-s_3)(j-s_2)(i-s_1)(s_1+s_2+s_3+l)} \qquad \text{[anti-gBRST]}$$

Combining the double Jacobi identity with the gBRST one allows one to write $\eta^{(1)}_{ijkl}$ in terms of $\kappa^{(1)}_{ijk}$ appearing already in the class III operators at one order lower in perturbation theory!

$$\eta^{(1)}_{ijkl} + \eta^{(1)}_{jilk} + \eta^{(1)}_{lkji} + \eta^{(1)}_{klij} = 2[\kappa^{(1)}_{ij(k+l+1)} + \kappa^{(1)}_{(k+l+1)ji}]\binom{k+l+1}{k} + 2[\kappa^{(1)}_{ji(k+l+1)} + \kappa^{(1)}_{(k+l+1)ij}]\binom{k+l+1}{l}$$
$$+ 2[\kappa^{(1)}_{lk(i+j+1)} + \kappa^{(1)}_{(i+j+1)kl}]\binom{i+j+1}{j} + 2[\kappa^{(1)}_{kl(i+j+1)} + \kappa^{(1)}_{(i+j+1)lk}]\binom{i+j+1}{i}.$$

Class IV couplings

Again this tells us something about the function space for $\eta^{(1)}_{ijkl}$. Taking into account all the independent permutations of the indices i, k, j and l this space is now 264-dimensional. Assuming that the functional form of $\kappa^{(1)}_{ijkl}$ is similar to the one of $\eta^{(1)}_{ijkl}$ then implies that in total we now have 528 parameters to fix. However, after implementing all of the above relations, only 8 remain in the end!

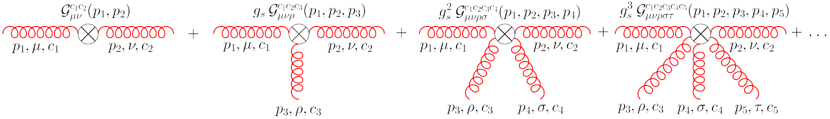
Application: Alien Feynman rules

With the couplings known, one can derive the Feynman rules of the alien operators

- The Feynman rules for the gauge-invariant quark and gluon operators, up to the four-loop level, can be found e.g. in [Falcioni and Herzog, 2022, Gehrmann et al., 2023, Floratos et al., 1977, Floratos et al., 1979, Mertig and van Neerven, 1996, Kumano and Miyama, 1997, Hayashigaki et al., 1997, Bierenbaum et al., 2009, Klein, 2009, Blümlein, 2001, Velizhanin, 2012, Velizhanin, 2020, Moch et al., 2017, Moch et al., 2022, Falcioni et al., 2023b, Falcioni et al., 2023a, Falcioni et al., 2024, Moch et al., 2024, Gehrmann et al., 2024, Kniehl and Velizhanin, 2023] and references therein. The generalization to arbitrary orders in perturbation theory can be found in [Somogyi and Van Thurenhout, 2024] [1]
- The alien rules were computed up to two loops in [Hamberg and van Neerven, 1992],[Matiounine et al., 1998],[Blümlein et al., 2022], and an extension to the three-loop level was recently presented in [Gehrmann et al., 2023]

[1] Note that the latter also presents the corresponding rules for the operators with total derivatives, relevant for non-zero momentum flow through the operator vertex.

Application: Alien Feynman rules



$$
\begin{aligned}
\mathcal{G}^{c_1c_2c_3c_4c_5}_{\mu\nu\rho\sigma\tau}(p_1,p_2,p_3,p_4,p_5) = {} & \frac{1+(-1)^N}{2}\, i^{N-1} f^{c_1c_2x} f^{xc_3y} f^{yc_4c_5} \Big\{ \\
& - g_{\mu\rho}\Delta_\nu\Delta_\sigma\Delta_\tau \sum_{i+j=N-3} \tilde{\kappa}_{ij}(\Delta\cdot p_4)^i(\Delta\cdot p_5)^j + \Delta_\rho\Delta_\sigma\Delta_\tau[(p_1+2p_2)_\mu\Delta_\nu \\
& - (\Delta\cdot p_2)g_{\mu\nu}] \sum_{i+j+k=N-4} \tilde{\kappa}^{(1)}_{ijk}(\Delta\cdot p_3)^i(\Delta\cdot p_4)^j(\Delta\cdot p_5)^k + [p_1^2\Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \tilde{\kappa}^{(1)}_{ijkl}(\Delta\cdot p_2)^i(\Delta\cdot p_3)^j(\Delta\cdot p_4)^k(\Delta\cdot p_5)^l \Big\} \\
& + \frac{1+(-1)^N}{2}\, i^{N-1} d_{4f}^{c_1c_2c_3c_4c_5} \Big\{ \Delta_\mu\Delta_\nu\Delta_\rho[(p_4+2p_5)_\sigma\Delta_\tau \\
& - (\Delta\cdot p_5)g_{\sigma\tau}] \sum_{i+j+k=N-4} \kappa^{(2)}_{ijk}(\Delta\cdot p_1)^i(\Delta\cdot p_2)^j(\Delta\cdot p_3)^k + [p_1^2\Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \kappa^{(2)}_{ijkl}(\Delta\cdot p_2)^i(\Delta\cdot p_3)^j(\Delta\cdot p_4)^k(\Delta\cdot p_5)^l \Big\} \\
& + \text{permutations}
\end{aligned}
$$

Application: Alien Feynman rules

- Ghost vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0- and 1-gluon vertices and $(f\,f)$, d_4 parts of the 2-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 2-gluon vertex new!
  (c) 3-gluon vertex new!
- Alien gluon vertices:
  (a) Agreement with [Blümlein et al., 2022, Gehrmann et al., 2023] for 2- and 3-gluon vertices; agreement with [Gehrmann et al., 2023] for $(f\,f)$, d_4 parts of the 4-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 4-gluon vertex new!
  (c) 5-gluon vertex new!
- Alien quark vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0-, 1- and 2-gluon vertices
  (b) 3- and 4-gluon vertices new!

Conclusions and outlook

- One way to reconstruct the functional form of the alien operators is based on the use of generalized gauge symmetry, which is then promoted to a generalized (anti)-BRST symmetry
- One then finds classes of EOM and ghost operators, the couplings of which obey interesting consistency relations
- These relations allow one to build up the function space of the couplings and constrain their all-N dependence
- Next steps:
  (a) Publication of the paper
  (b) Generalization to higher orders

End [2]

Thank you for your attention!

[2]Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

Colour structures

f^{abc} are the QCD structure constants. The other colour structures are in turn defined as

$$(f\,f)^{abcd} = f^{abe} f^{cde},$$
$$(f\,f\,f)^{abcde} = f^{abm} f^{mcn} f^{nde},$$
$$d_4^{abcd} = \frac{1}{4!}[\mathrm{Tr}(T_A^a T_A^b T_A^c T_A^d) + \text{symmetric permutations}],$$
$$d_{4ff}^{abcd} = d_4^{abmn} f^{mce} f^{edn},$$
$$d_{\widehat{4ff}}^{abcd} = d_{4ff}^{abcd} - \frac{1}{3} C_A d_4^{abcd},$$
$$d_{4f}^{abcde} = d_4^{abcm} f^{mde}.$$

Solving conjugation relations

- To take full advantage of the anti-gBRST conjugation relations, one needs to be able to evaluate them analytically
- Use principles of symbolic summation!
- Creative telescoping [Zeilberger, 1991]: evaluate the sum of interest by rewriting it as a recursion relation using Gosper's algorithm [Gosper, 1978]
- The closed-form expression of the sum then corresponds to the linear combination of the solutions of the recursion that has the same initial values as the sum.

$\rightarrow$ For single sums: `Sigma` [Schneider, 2004, Schneider, 2007]

$\rightarrow$ For multiple sums: `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Classical telescoping and Gosper's algorithm

The telescoping algorithm is a well-known method for evaluating finite sums. Suppose we want to evaluate the following sum

$$\sum_{k=a}^{N} f(k)$$

with $a, N \in \mathbb{N}$ and $a \leq N$. Now, if we can find a function $g(N)$ such that

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

then

$$\begin{aligned} \sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a). \end{aligned}$$

Here, Δ represents the finite difference operator. The telescoping function $g(N)$ can be found by application of Gosper's algorithm [Gosper, 1978].

Classical telescoping and Gosper's algorithm

Suppose

$$\frac{g(N)}{g(N-1)}$$

is a rational function in N. The algorithm consists of three main steps. Assume we want to calculate the telescoping function for some sequence $\{a_N\}$

$$a_N = \Delta b(N).$$

It is assumed that $\{a_N\}$ is a hypergeometric sequence, that is

$$\frac{a_{N+1}}{a_N} = q(N)$$

with $q(N)$ a rational function of N. The steps of Gosper's algorithm can then be summarized as follows

Classical telescoping and Gosper's algorithm

1. Determine three functions $f(x), g(x)$ and $h(x)$ such that

$$q(x) = \frac{f(x+1)}{f(x)} \frac{g(x)}{h(x+1)}$$

   and

$$\gcd[g(x), h(x+n)] = 1 \quad (n \in \mathbb{N}_0).$$

2. Solve the so-called Gosper equation,

$$f(x) = g(x)y(x+1) - h(x)y(x),$$

   for the polynomial $y(x)$.

3. If such a polynomial solution does not exist, it means that the sum in question does not have a hypergeometric closed form. Otherwise, the telescoping function is determined by

$$t(x) = \frac{h(x)}{f(x)} y(x) \qquad \text{with } b(N) = t(N)a(N)$$

More details can e.g. be found in [Kauers and Paule, 2011]

Creative telescoping

Classical telescoping works when dealing with sequences that depend on one variable only. When we want to determine a closed form for a summation of a sequence depending on two variables, we can use the creative telescoping algorithm by Zeilberger [Zeilberger, 1991]. The idea is similar to that of classical telescoping. Suppose we want to evaluate

$$\sum_{k=a}^{b} f(N,k) \equiv S(N).$$

The way to go about this is by attempting to find d functions $c_0(N), \ldots, c_d(N)$ and a function $g(N,k)$ such that

$$g(N,k+1) - g(N,k) = c_0(N) f(N,k) + ... + c_d(N) f(N+d,k).$$

Summing both sides, and applying classical telescoping to the left-hand side then gives

$$g(N,b+1) - g(N,a) = c_0(N) \sum_{k=a}^{b} f(N,k) + ... + c_d(N) \sum_{k=a}^{b} f(N+d,k).$$

Creative telescoping

This leads to an inhomogeneous recursion relation for the original sum of the form

$$q(N) = c_0(N)S(N) + ... + c_d(N)S(N+d).$$

Typically, one starts this procedure at $d = 0$, which is equivalent to classical telescoping. The value of d is then increased stepwise until a solution is found. The creative telescoping algorithm can be applied when the sequence under consideration is holonomic. A sequence $\{a_N\}$ is said to be holonomic if there exist polynomials $p_0(x), \ldots, p_r(x)$ such that the following recursion relation is obeyed [Kauers and Paule, 2011]

$$p_0(N)a_N + p_1(N)a_{N+1} + \cdots + p_r(N)a_{N+r} = 0 \quad (N \in \mathbb{N}, p_r(N) \neq 0).$$

For example, the harmonic numbers $\{S_1(N)\}$ form a holonomic sequence as they obey

$$(N+1)S_1(N) - (2N+3)S_1(N+1) + (N+2)S_1(N+2) = 0.$$

More details on the summation algorithms reviewed here can e.g. be found in the excellent books [Graham et al., 1989, Petkovŝek et al., 1996].

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