REGGE TRAJECTORIES AND BRIDGES BETWEEN THEM IN ABJM THEORY FROM INTEGRABILITY

Andrea Cavaglià, University of Torino

UNIVERSITÀ DI TORINO

Talk at DESY, July 2024

Based on work in progress with
N. Brizio, R. Tateo, V. Tripodi

Plan of the talk

Introduction to Regge trajectories in CFT

Review of the Quantum Spectral Curve

Building the QSC for non-integer spin in ABJM theory

Results and new symmetry of the spin Riemann surface

Continuation in spin of the mass spectrum plays important role in high energy scattering

$$\mathscr{A}(s,t) \sim s^{j(t)},\ s \to \infty$$

controlled by unphysical value of spin where $m^2(j(t)) \equiv t$ [Regge]

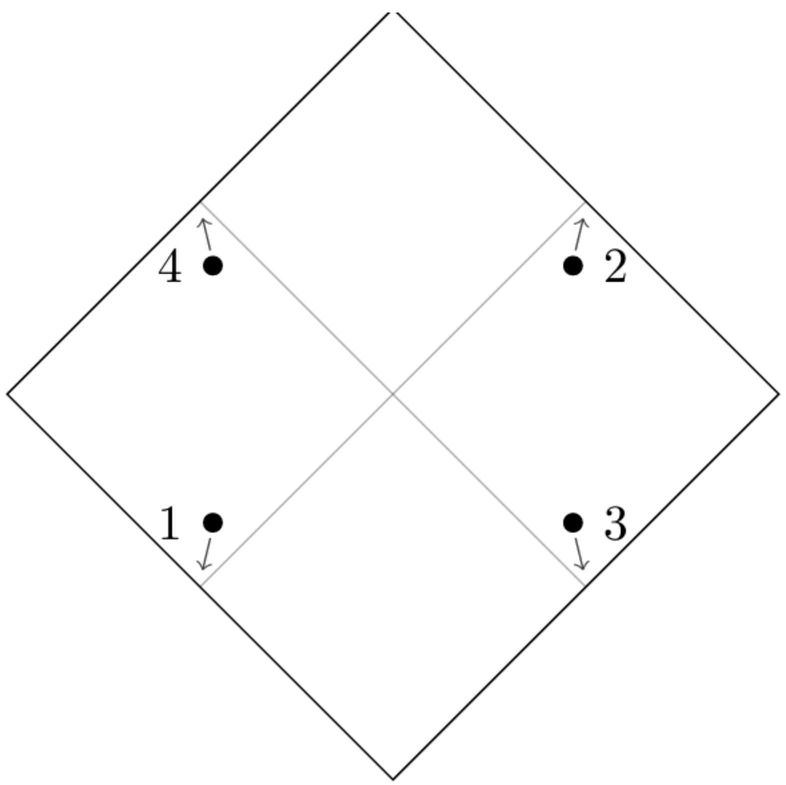
CFT spectra $\Delta(S)$ are also analytic in spin. [Caron-Huot]

This continuation is important for Lorentzian physics

$$\lim_{\text{boost } \eta\to\infty} (G-1) \propto e^{(j_*-1)\eta}$$

"pomeron"

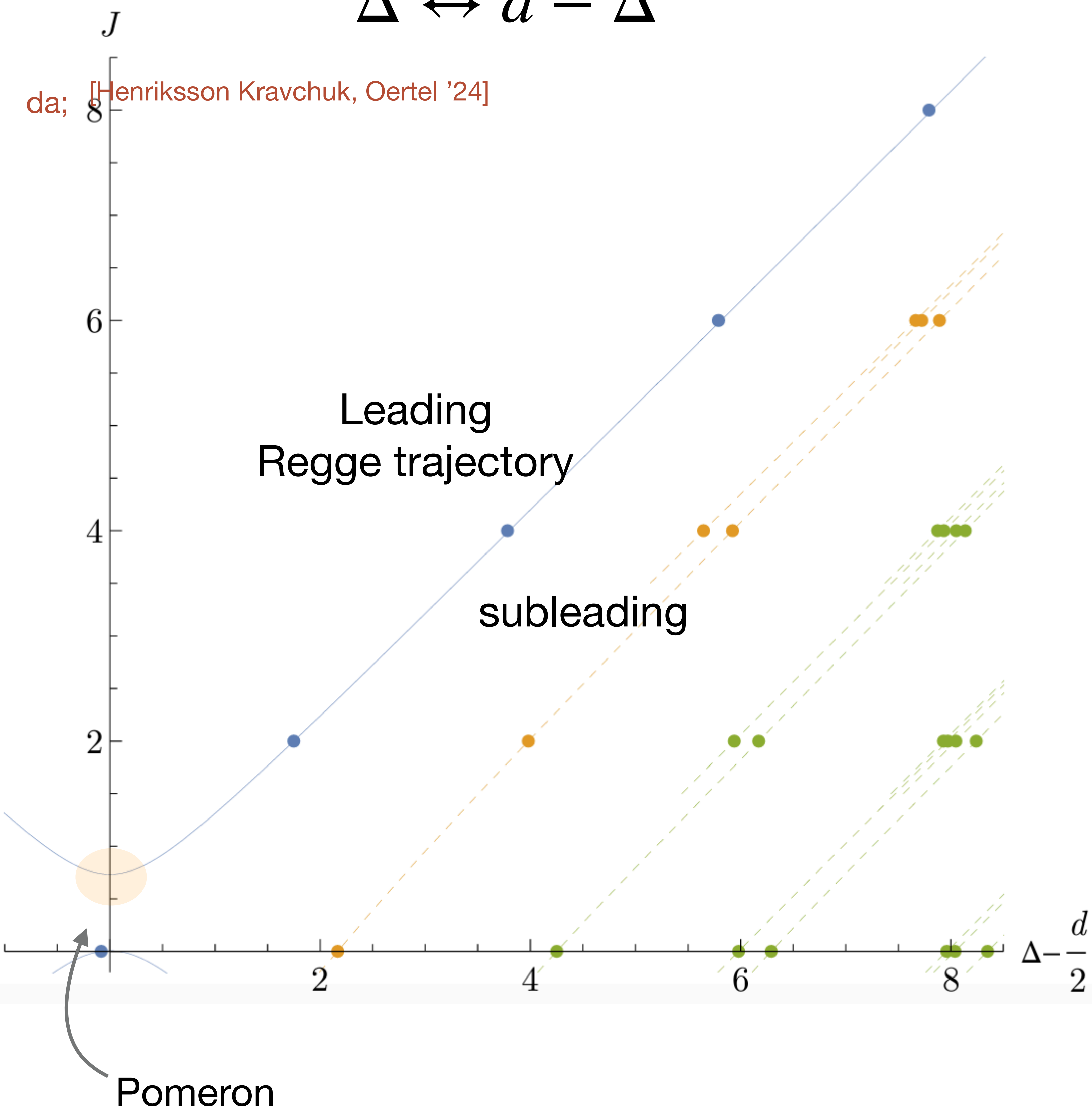
$$\Delta(j_*) \equiv d/2$$



Costa-Goncalves,Penedones]

“Chew-Frautschi plot” J vs Δ

$$\Delta \leftrightarrow d - \Delta$$



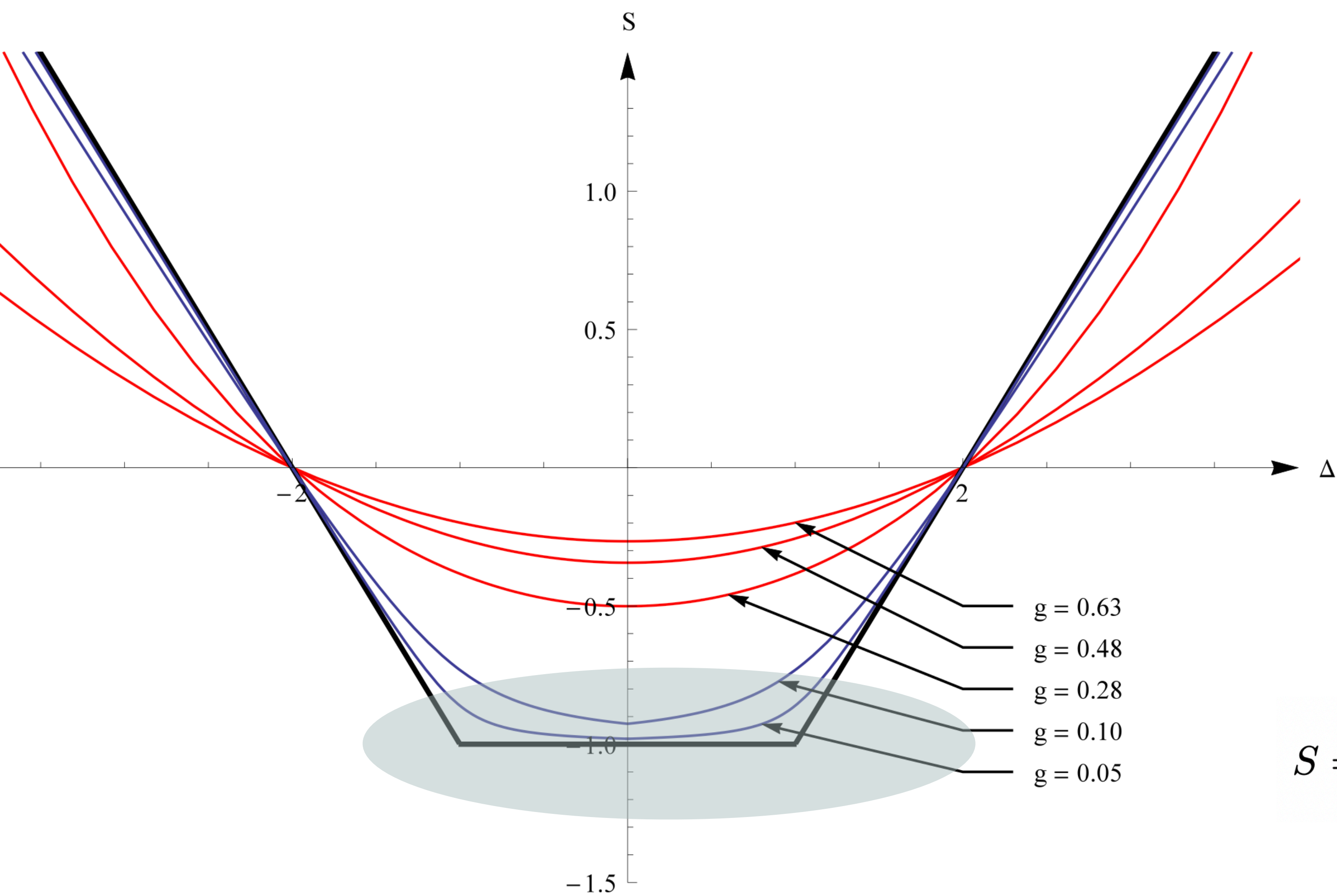
Generic points on the trajectories: non-local “light-ray” observables (can be seen as detectors at null infinity)

[`Kravchuk, Simmons-Duffin]

[`Caron-Huot, Kologlu,Kravchuk, Meltzer, Simmons-Duffin]

In planar N=4 SYM many results thanks to integrability!

ubiquity of BFKL-type behaviour at weak coupling



For leading trajectory in SL(2) sector,
leading BFKL behaviour
is the same as in QCD!

[Balitsky, Lipatov '78]
[Kuraev, Lipatov, Fadin '77]

$$S = -1 - 4g^2 \left(\psi\left(\frac{1+n-\Delta}{2}\right) + \psi\left(\frac{1+n+\Delta}{2}\right) - 2\psi(1) \right) + \mathcal{O}(g^4)$$

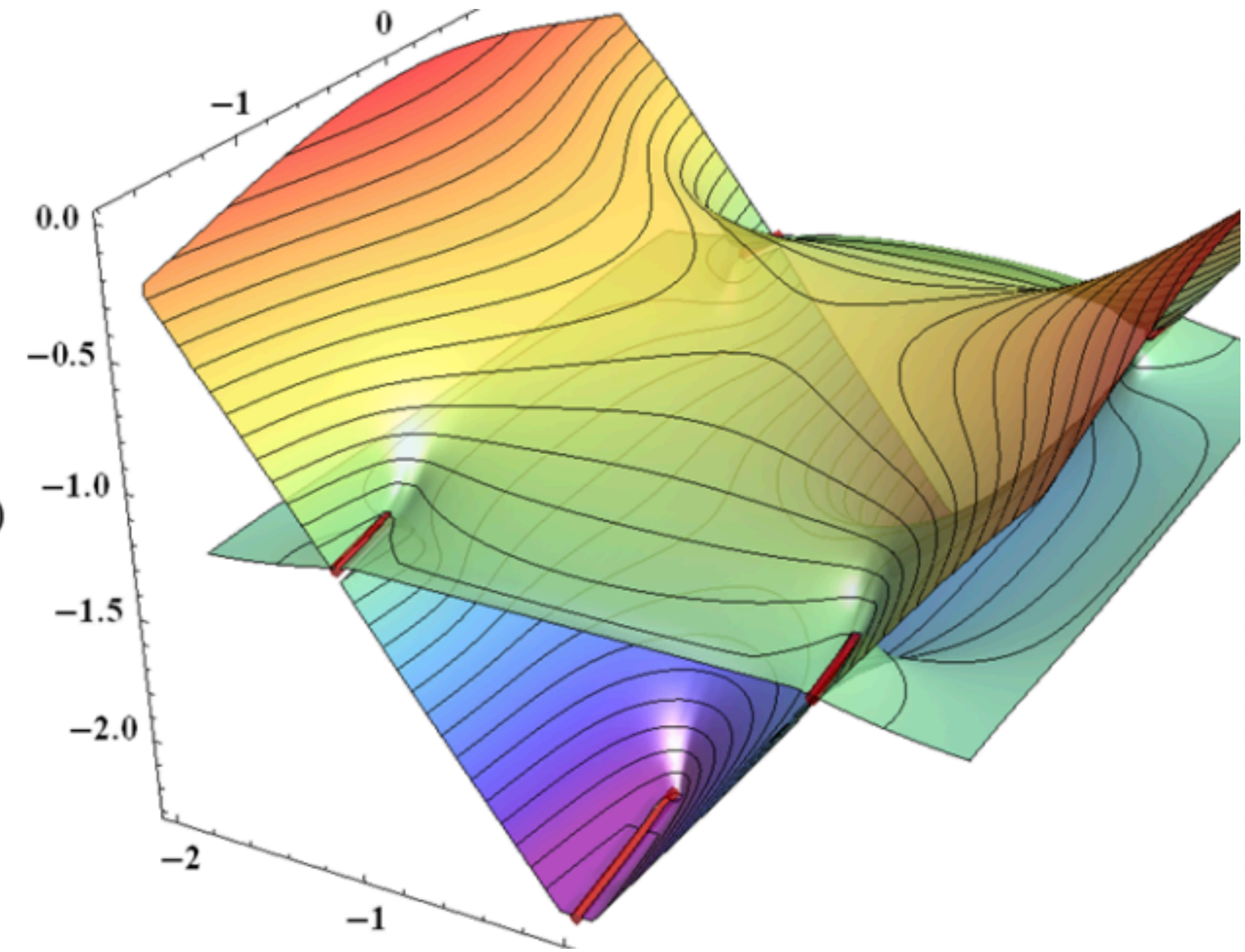
$$S = S_0 + g^2 \chi_{LO}(\Delta) + g^4 \chi_{NLO}(\Delta) + + g^4 \chi_{NNLO}(\Delta) + \dots$$

[Alfimov Kazakov Gromov '15] [+ Sizov '15]
[Gromov Levkovich-Maslyuk Sizov '15]
[Preti Klabbers, Szécsenyi '23]

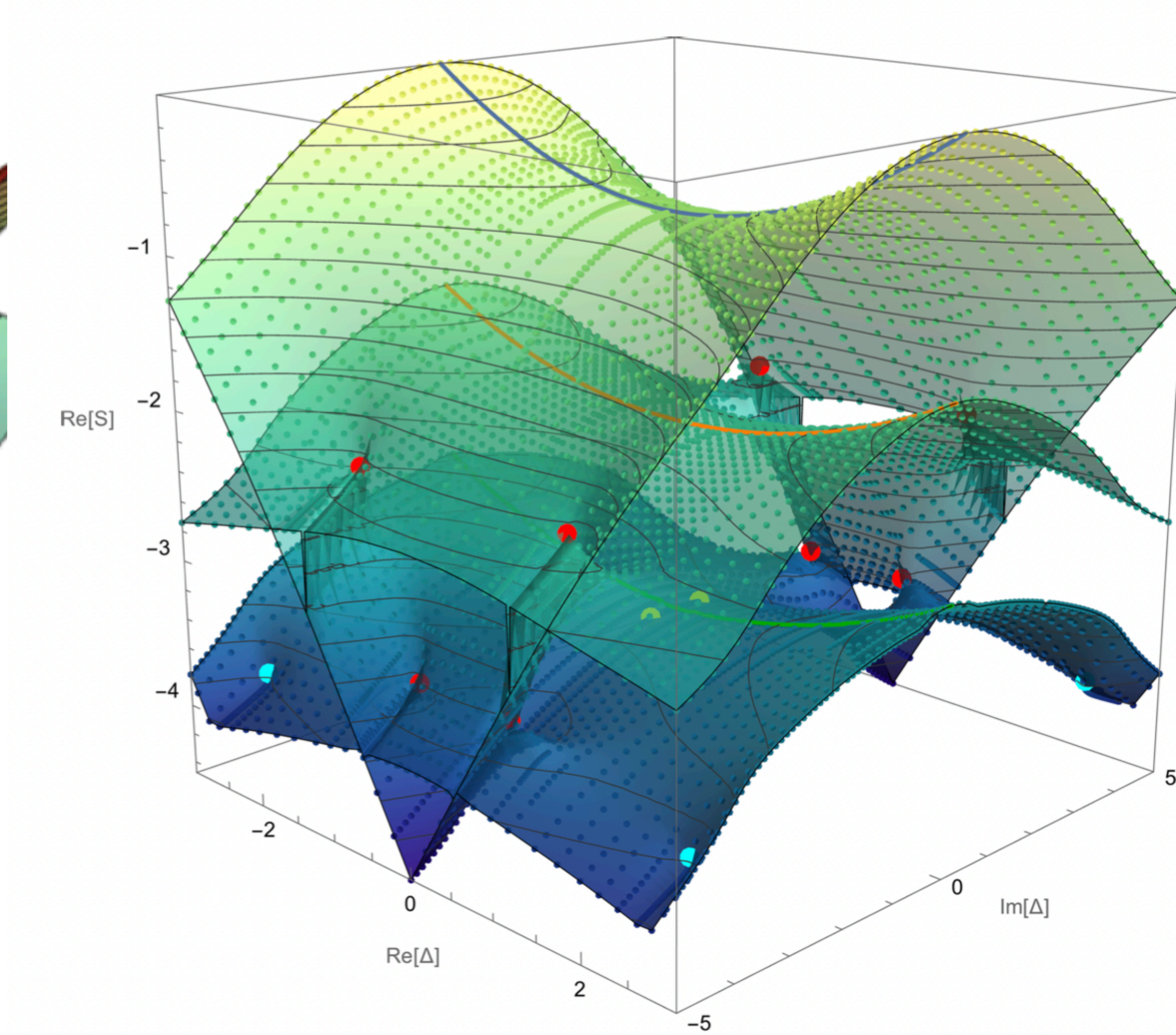
"ABBA": new method to get expansion [Ekhammar, Gromov, Preti '24]

Emerging picture: leading and subleading trajectories are all sheets of the same Riemann surface, compatibly with global symmetries

Picture from [Gromov, Levkovich-Maslyuk, Sizov ’15]



Picture from [Preti, Klabbers, Szecsenyi ’23]

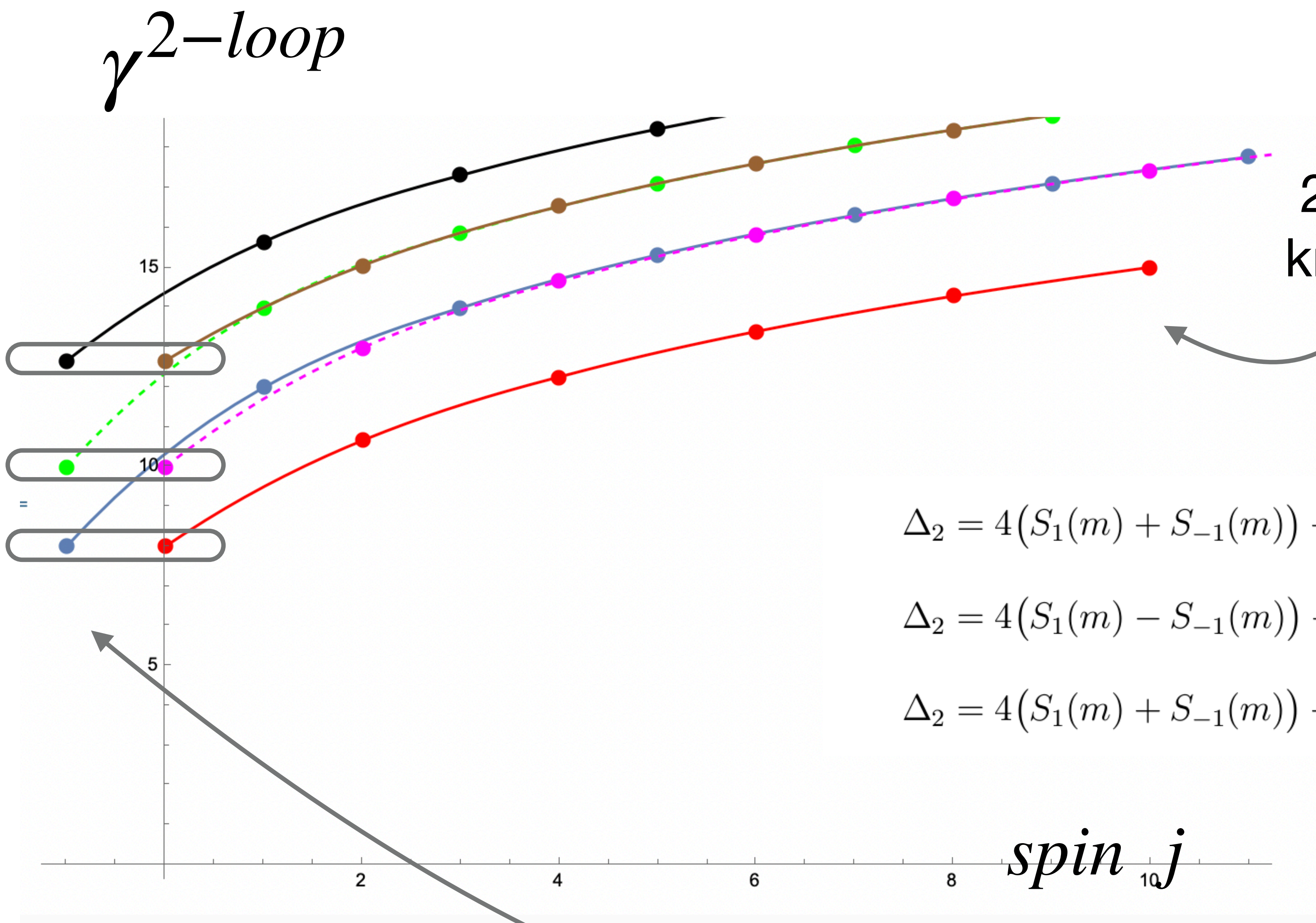


Connected via complex branch points in S

Inspired by these developments,
we worked out how to do the same for ABJM…

Before jumping in the details,
let me give a preview of an
observation which came as a surprise…

Preview: an **unexpected symmetry**



2-loop anomalous dimension in ABJM known analytically for some trajectories

twist $\Delta - j = 1{,}2{,}3{,}4$

$$\Delta_2 = 4\big(S_1(m) + S_{-1}(m)\big) + \frac{2(1-(-1)^m)}{m+1} + 8 \qquad \text{for } \mathcal{V}^4_{2m+1,3},$$

$$\Delta_2 = 4\big(S_1(m) - S_{-1}(m)\big) + \frac{1+(-1)^m}{m+1} - \frac{1-(-1)^m}{m} + 4 \qquad \text{for } \mathcal{V}^4_{2m,4},$$

$$\Delta_2 = 4\big(S_1(m) + S_{-1}(m)\big) + \frac{2(1-(-1)^m)}{m+1} + \frac{32}{3} \qquad \text{for } \mathcal{V}^4_{2m+1,5}\ (m \text{ odd}).$$

[Papathanasiou, Spradlin '09]

Same anomalous dimensions here! Not accidental

This is **valid at all loops**, and part of a **wider symmetry** connecting points on different trajectories with $j \to -j-1$

We will see how this **emerges from the QSC**

The Quantum Spectral Curve

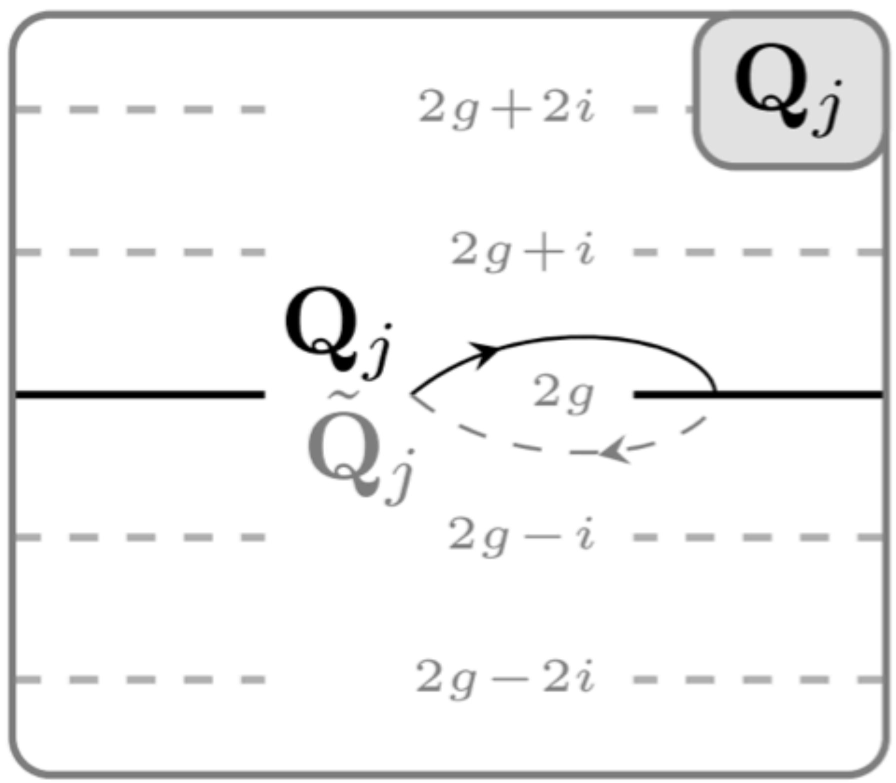
How do you pack the planar spectrum of N=4 SYM or ABJM in one page (almost) ?

A complex analysis problem for "Q-functions" $Q_i(u)$

N=4 SYM [Gromov, Kazakov, Leurent, Volin '13]

ABJM: [AC, Fioravanti, Gromov, Tateo '15]

spectral parameter



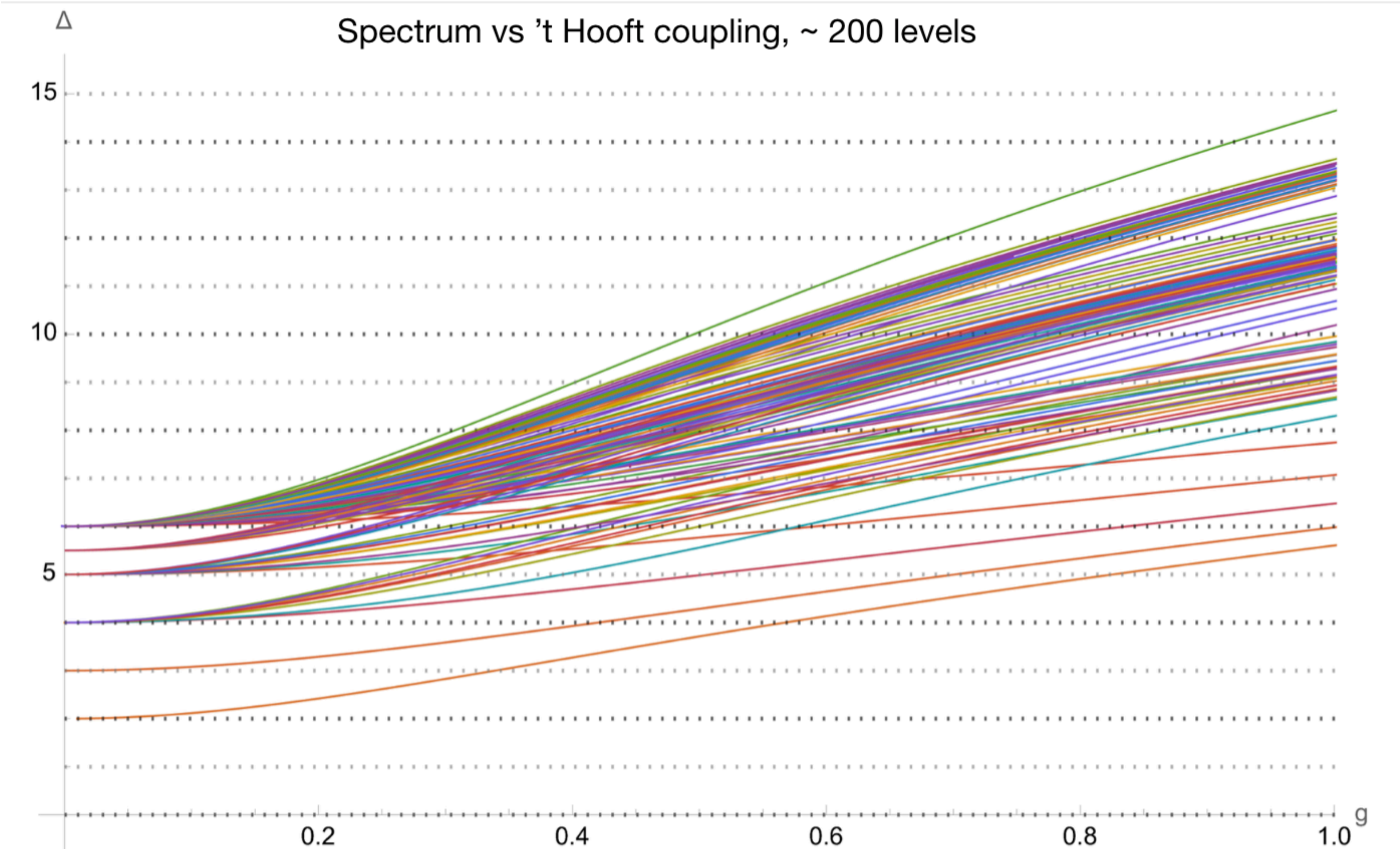
Global charges (incl. Δ)

$$\mathbf{Q}_i(u) \sim u^{\hat{M}_i}, \quad u \sim \infty$$

$\frac{R^2_{AdS}}{4\pi\alpha'} \equiv g \simeq$ position of branch points

Extremely powerful method to study spectrum.
Only limitations: need to study it "state by state", computing time…

The QSC can really do a lot



e.g. numerics

Numerics open source code (N=4 SYM)

(https://github.com/julius-julius/qsc)

[Gromov, Julius, Sokolova '23]

It provides sharp mathematical insights on the spectrum

e.g. what MZV numbers can appear in expansions?

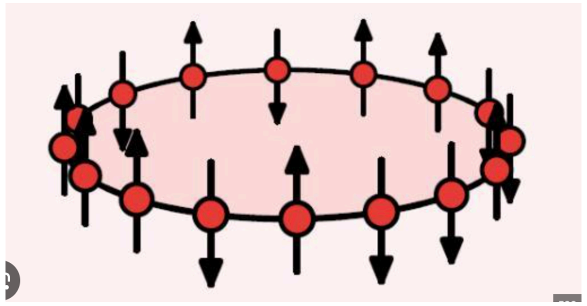
[Marboe, Volin '14]+..   [Anselmetti Bombardelli,AC,Tateo'15]…

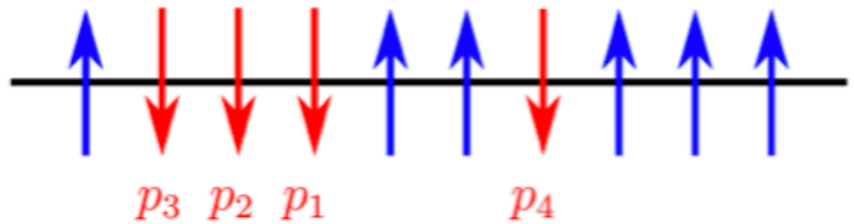
$$
\begin{aligned}
\Delta \;=\; & 4+12g^2-48g^4+336g^6+g^8\left(-2496+576\,\zeta_3-1440\,\zeta_5\right)\\
&+g^{10}\left(15168+6912\,\zeta_3-5184\,\zeta_3^2-8640\,\zeta_5+30240\,\zeta_7\right)\\
&+g^{12}\left(-7680-262656\,\zeta_3-20736\,\zeta_3^2+112320\,\zeta_5+155520\,\zeta_3\,\zeta_5+75600\,\zeta_7-489888\,\zeta_9\right)\\
&+g^{14}\left(-2135040+5230080\,\zeta_3-421632\,\zeta_3^2+124416\,\zeta_3^3-\;229248\,\zeta_5+411264\,\zeta_3\,\zeta_5\right.\\
&\qquad\left.-993600\,\zeta_5^2-1254960\,\zeta_7-1935360\,\zeta_3\,\zeta_7-835488\,\zeta_9+7318080\,\zeta_{11}\right)\\
&+g^{16}\Big(54408192-83496960\,\zeta_3+7934976\,\zeta_3^2+1990656\,\zeta_3^3-19678464\,\zeta_5-4354560\,\zeta_3\,\zeta_5
\end{aligned}
$$

… or, more enigmatically, position of many spectral singularities in the coupling

$$g_*^2 = -(n/4)^2$$

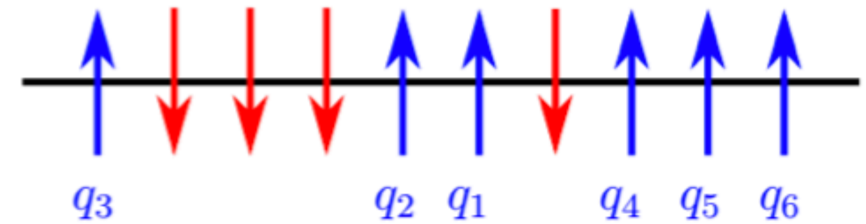
What is the QSC? Example in the SU(2) Heisenberg spin chain





L-spins, N-spins up, L-N down

$p_i = \mathscr{P}(w_i)$



$q_i = \mathscr{P}(u_i)$

Bethe equations

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L = e^{-2i\phi}\prod_{j\neq k}^{N}\frac{u_k - u_j+i}{u_k - u_j-i} \qquad E = \sum_{i=1}^{N} \mathscr{E}(u_i)$$

Reformulation:

$$Q_1(u) = e^{\phi u}\prod_{i=1}^{N}(u - u_i), \quad Q_2(u) = e^{-\phi u}\prod_{i=1}^{L-N}(u - w_i)$$

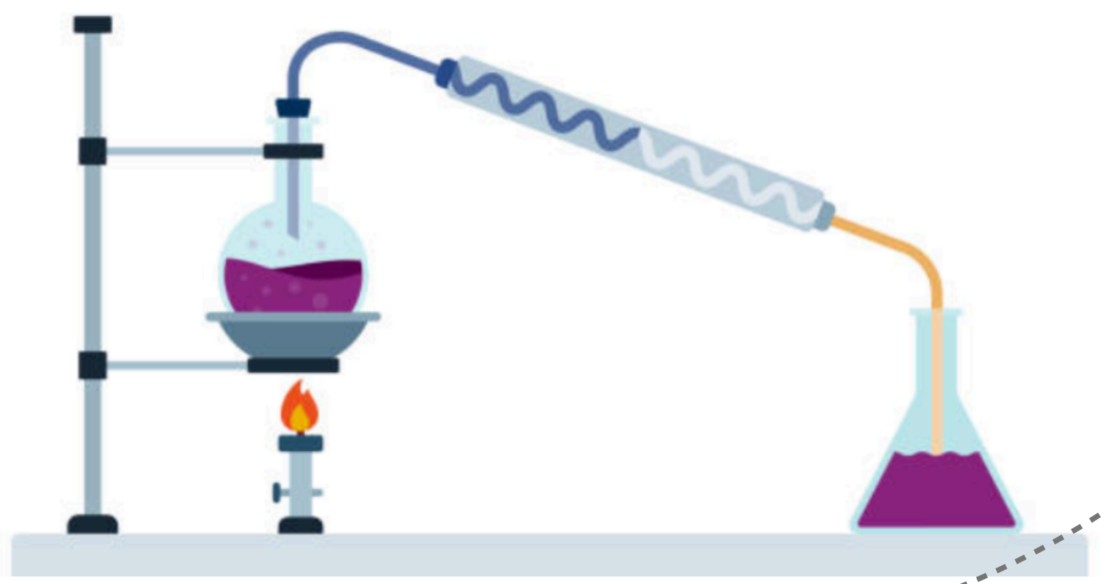
SU(2) QQ-relations:

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

With **polynomiality** requirement, implies Bethe equations and generates all (and only) physical solutions!

The paradigm holds **also in integrable systems without simple Bethe equations**

QSC



Symmetry + **Analyticity**

Q-system functional relations

Fixed by global symmetry of model

Analytic properties of Q-functions

Model dependent
In AdS/CFT Q's are no longer polynomial

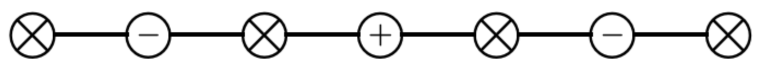
Symmetry

Here we know everything

Q-system

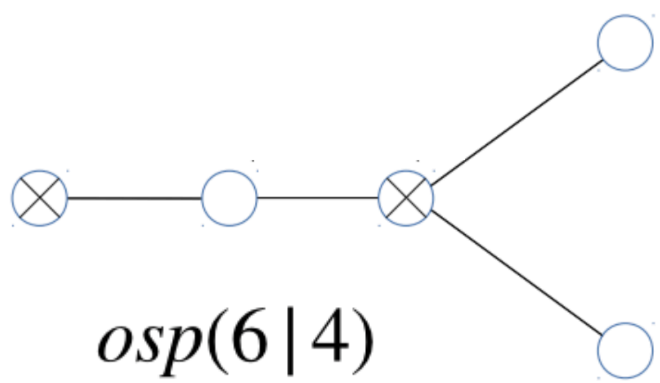
Known for all A-type superalgebras: [Tsuboi '09]

N=4 SYM



$psu(2,2|4)$

ABJM



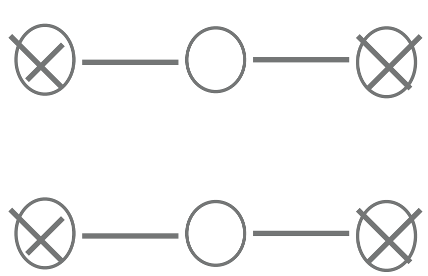
$osp(6|4)$

QQ-relations in [Bombardelli, AC, Fioravanti, Gromov, Tateo '17]

[Tsuboi '24]

wip with [Frassek, Primi, Szecsenyi - to appear]

AdS3



$psu(1,1|2)_L \oplus psu(1,1|2)_R$

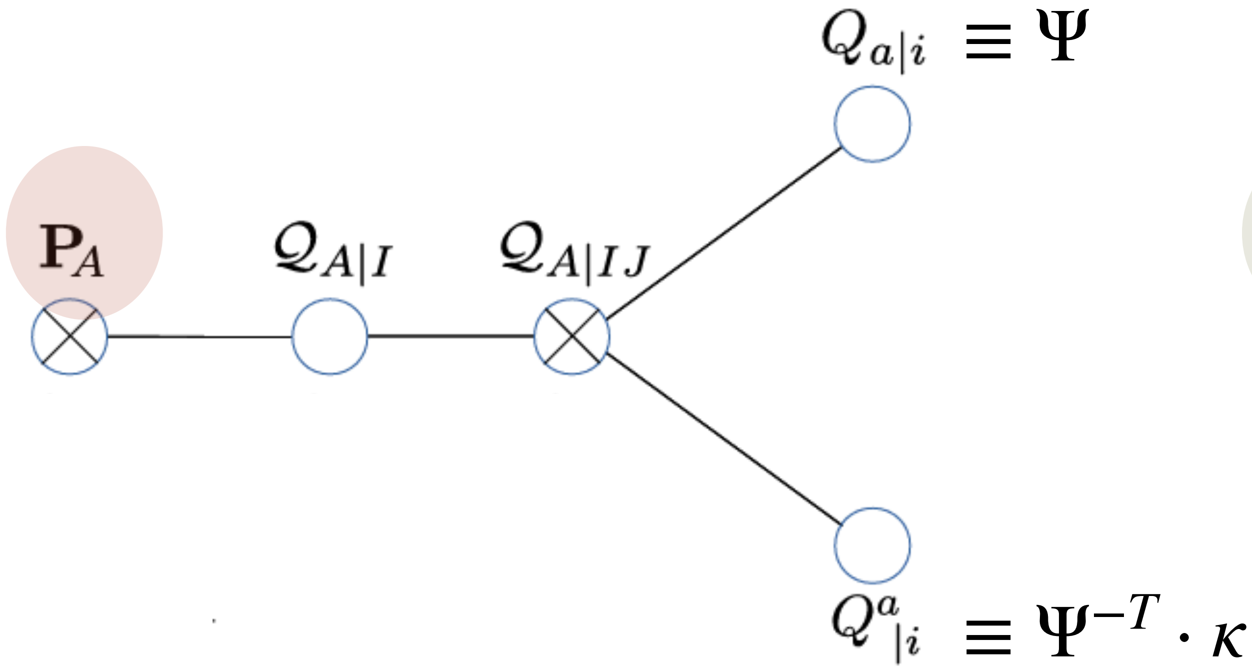
Two copies of a small-rank version of the AdS5 case

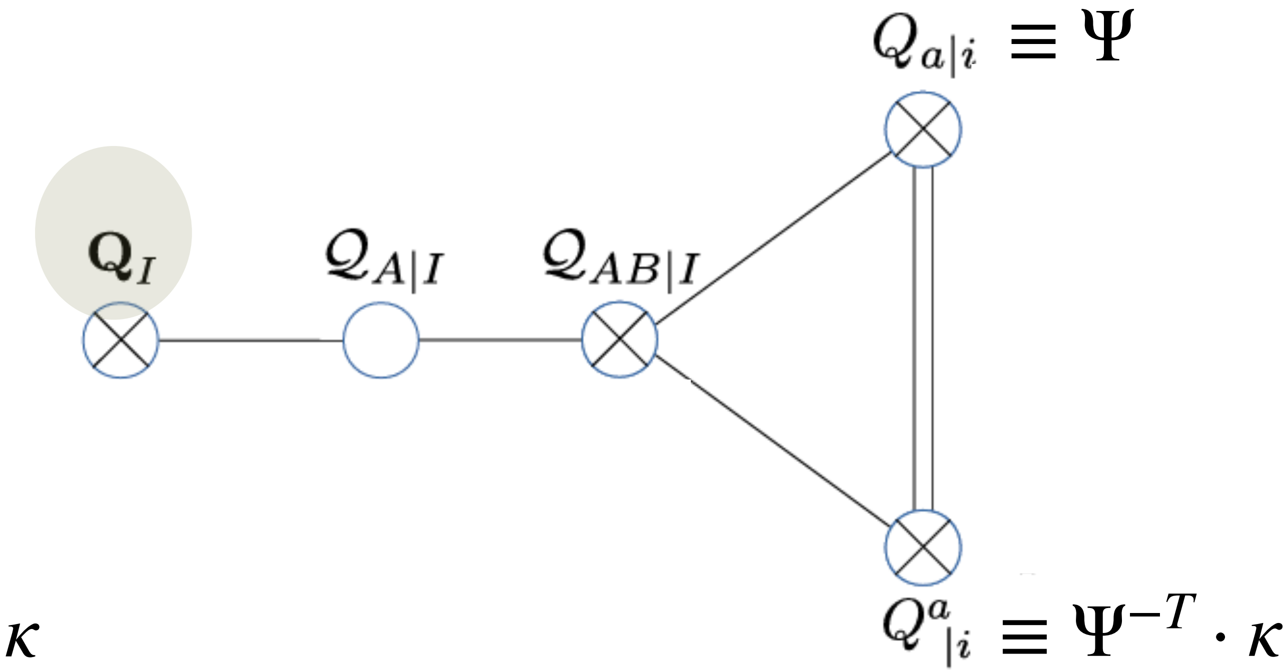
Analyticity

Here we don't really know the rules a priori…

…but the same minimal axioms seem to hold for N=4 SYM, ABJM & Ramond-Ramond AdS3*…

* note: AdS3 wants to be different and has one important difference (non quadratic branch points). This arises spontaneously, is not an extra axiom





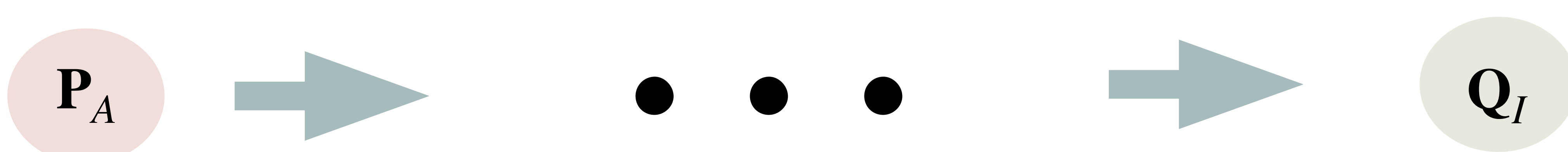
QQ relations essentials (ABJM)

$$\Psi(u + \frac{i}{2}) = \not{\mathbf{P}}(u) \cdot \Psi^{-T}(u - \frac{i}{2}) \cdot \kappa$$

$$\not{\mathbf{Q}}(u) = -\Psi(u - \frac{i}{2}) \cdot \not{\mathbf{P}}^{-1}(u) \cdot \Psi(u - \frac{i}{2})$$

$$\kappa \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Useful logic: given some of the functions, build the others



This is the scheme of numerics, once we add analytic properties

Analyticity

Some Q-functions play a special role.

Asymptotics at $u \to +\infty$

SuperConf. Charges

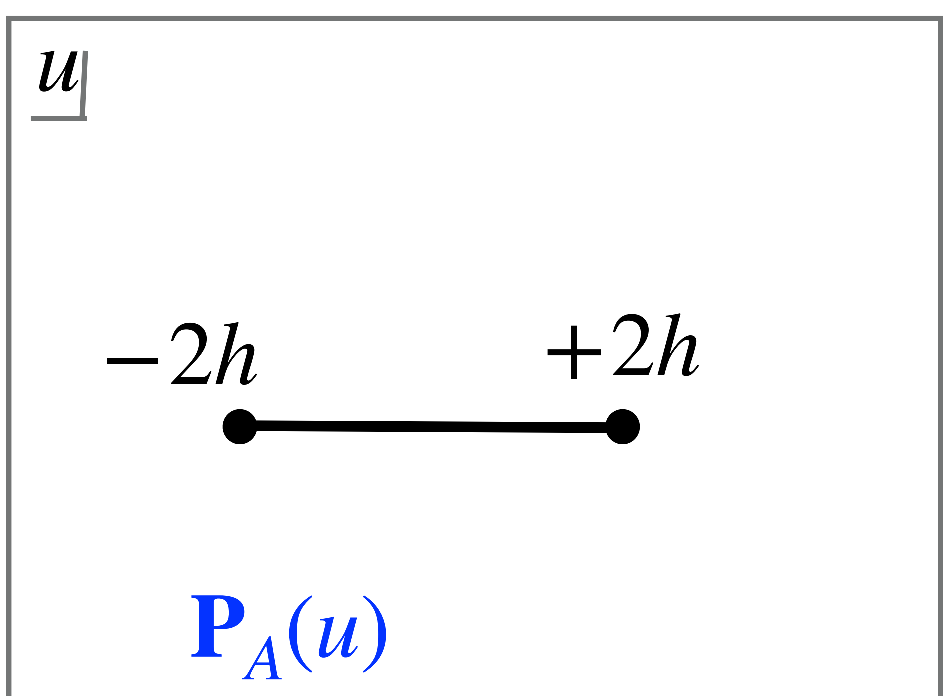
$$(\mathbf{P}_1, \ldots, \mathbf{P}_6) \simeq \left(u^{-1-r_2}, u^{-2-r_1}, u^{2+r_1}, u^{1+r_2}, u^{-r_3}, u^{r_3}\right)$$

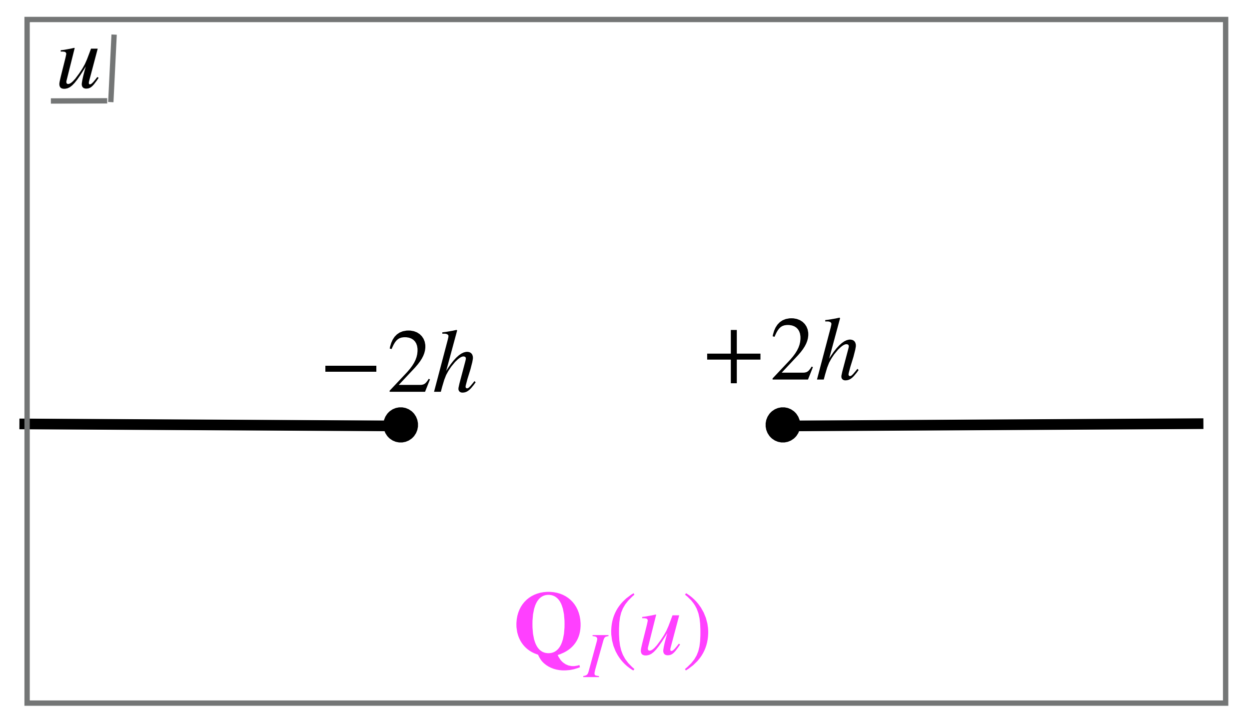
$[r_1, r_2, r_3]$: **R-symmetry** of top superprimary

$$(\mathbf{Q}_1, \ldots, \mathbf{Q}_4) \simeq \left(u^{\Delta+S+1}, u^{\Delta-S}, u^{S-\Delta-2}, u^{-\Delta-S-3}\right)$$

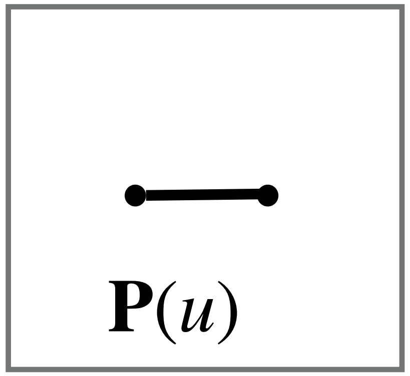
(Δ, S) **conformal charges** of top superprimary

Cuts (square root type)



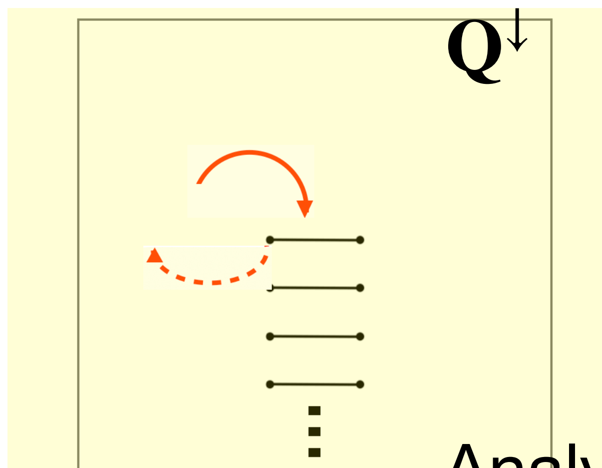


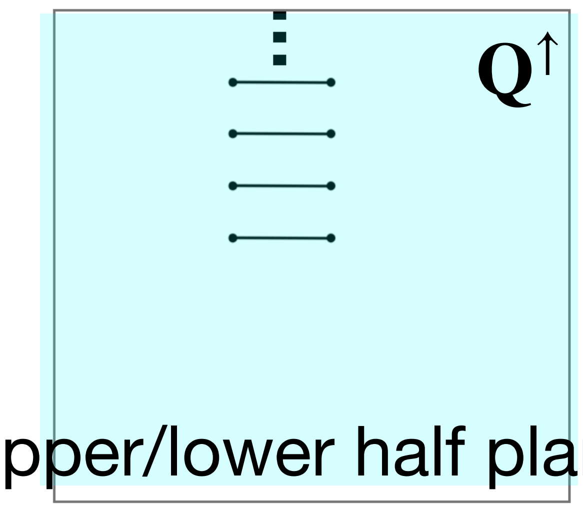
Symmetry & Analyticity... Resolving the tension



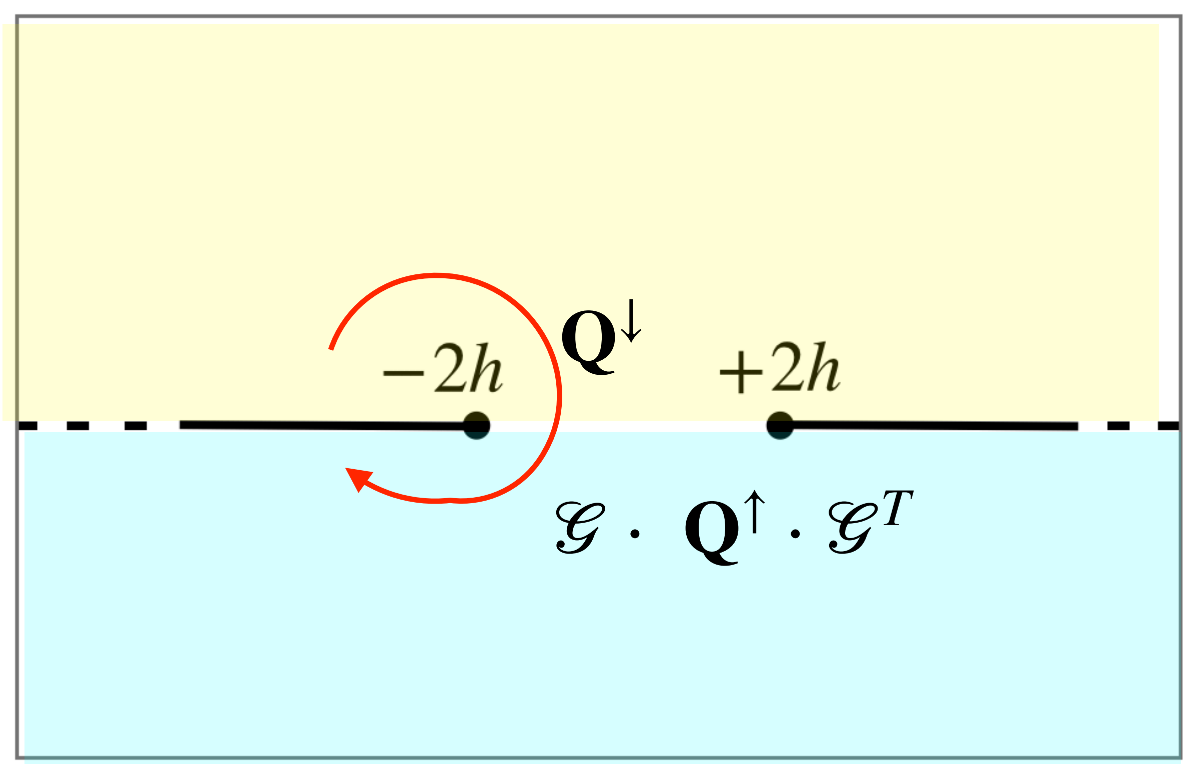
Start from single-cut functions and build the others solving the difference equations…

QQ relations have shifts of $u \pm i$: we can get **two alternative bases** of solutions for





Analytic in upper/lower half planes



two Riemann sheets need to be glued!

$\mathscr{G}(u)$ gluing matrix

Should be symmetry of Q-system

Should be entire periodic function

Compatible with quadratic cuts

Physical spectrum <-> $\mathscr{G}$ constant

Suppose $\mathscr{G}$ is constant. Then it should have the form

$$\mathcal{G} = \begin{pmatrix} \imath\sec(\pi\Delta) & 0 & 0 & \imath\,\delta_1 \\ 0 & \mp\imath & 0 & 0 \\ 0 & 0 & \pm\imath & 0 \\ -\imath\,\delta_2 & 0 & 0 & -\imath\sec(\pi\Delta) \end{pmatrix}$$

[Bombardelli, AC, Fioravanti, Gromov, Tateo ’17]

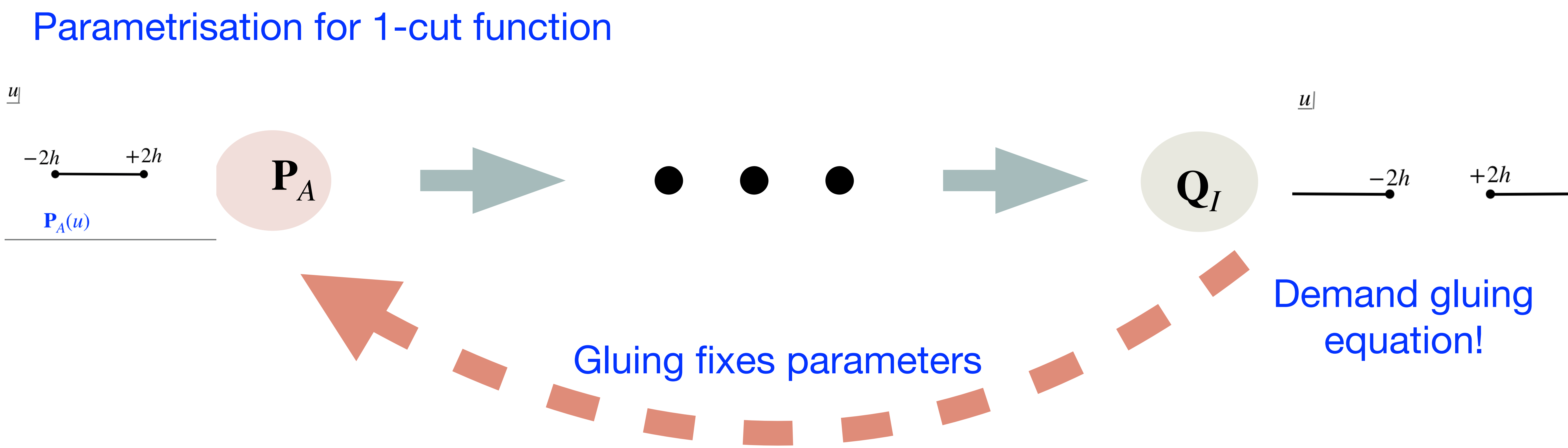
$$-\imath\,\mathrm{e}^{-\imath\pi S} = \mp\imath$$

and S is quantized automatically!

Local spectrum at finite coupling studied in

[Bombardelli, Conti AC, Tateo ’17]

Logic for solving the equations



This logic holds at any coupling,
and **for S integer or not**

The only difference will be in the gluing matrix

What we have

$$\mathcal{G} = \begin{pmatrix} \imath \sec(\pi\Delta) & 0 & 0 & \imath\,\delta_1 \\ 0 & \mp\imath & 0 & 0 \\ 0 & 0 & \pm\imath & 0 \\ -\imath\,\delta_2 & 0 & 0 & -\imath\sec(\pi\Delta) \end{pmatrix}$$

$$-\imath\, \mathrm{e}^{-\imath\pi S} = \mp\imath$$

Constant gluing matrix

Spin is quantised

Local operators

What we want

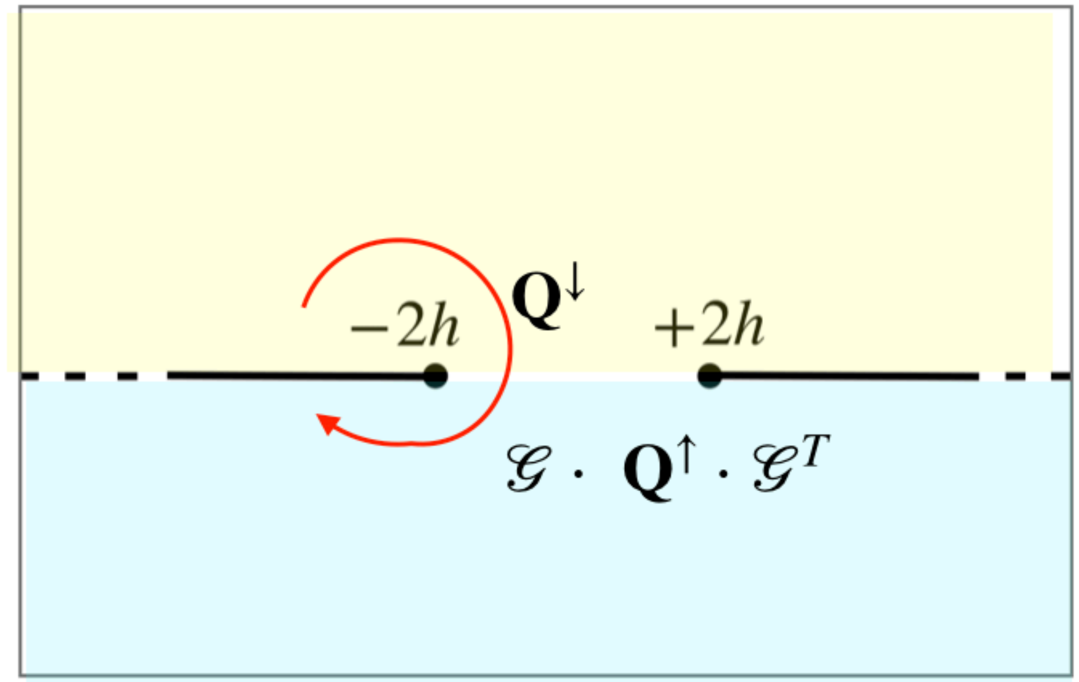
Gluing matrix should now depend on u

Any spin

Light-ray operators

Let’s take our QSC to the luthier…

Luthier instructions



Q-system is left invariant:

$$\mathcal{G}(u)\kappa\mathcal{G}^T(u+i) = \kappa$$

$\mathbb{Z}_2$ symmetries

we focus on this sector

$$\mathcal{G}(u+i) = \mathbb{K}\mathcal{G}(u)\mathbb{K}$$

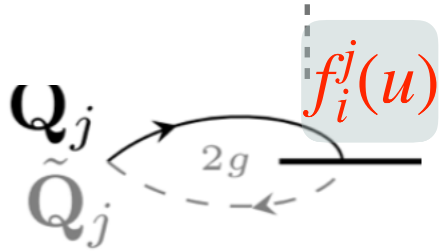
$$\mathcal{G}(-u) = -\kappa\,\mathcal{G}^T(u)\kappa^{-1}$$

$$\kappa \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad \mathbb{K} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

analytic properties

Entire, periodic $\rightarrow$ polyn in $e^{\pm\pi u}$

Ansatz: maximal degree $e^{2\pi|u|}$



"hidden" analytic constraint: quadratic b.p.'s

$$f(u) + \kappa f^T(u+i)\kappa^{-1} = 2\mathbb{1}$$

with $f^T(u) \equiv \Psi^{-1\,\downarrow}(u - \frac{i}{2})\,\Psi^{\uparrow}(u - \frac{i}{2})\,\mathscr{G}^T(u)$

This happens also in N=4 SYM - previously unnoticed

We find **2** distinct **non-constant** gluing matrices

$$\mathcal{G}_{Regge} \equiv \begin{pmatrix} * & 0 & *\,e^{-\pi u} + *\,e^{\pi u} & * \\ *\,e^{-\pi u} + *\,e^{\pi u} & -\imath & *\cosh 2\pi u + * & *e^{\pi u} + *e^{-\pi u} \\ 0 & 0 & \imath & 0 \\ * & 0 & *e^{\pi u} + *e^{-\pi u} & * \end{pmatrix}$$

$$\mathcal{G}_{bridge} \equiv \mathcal{G}^{T}_{Regge} \neq \mathcal{G}_{Regge}$$

Diagnostics of locality -
the gluing matrix reducing to a constant:

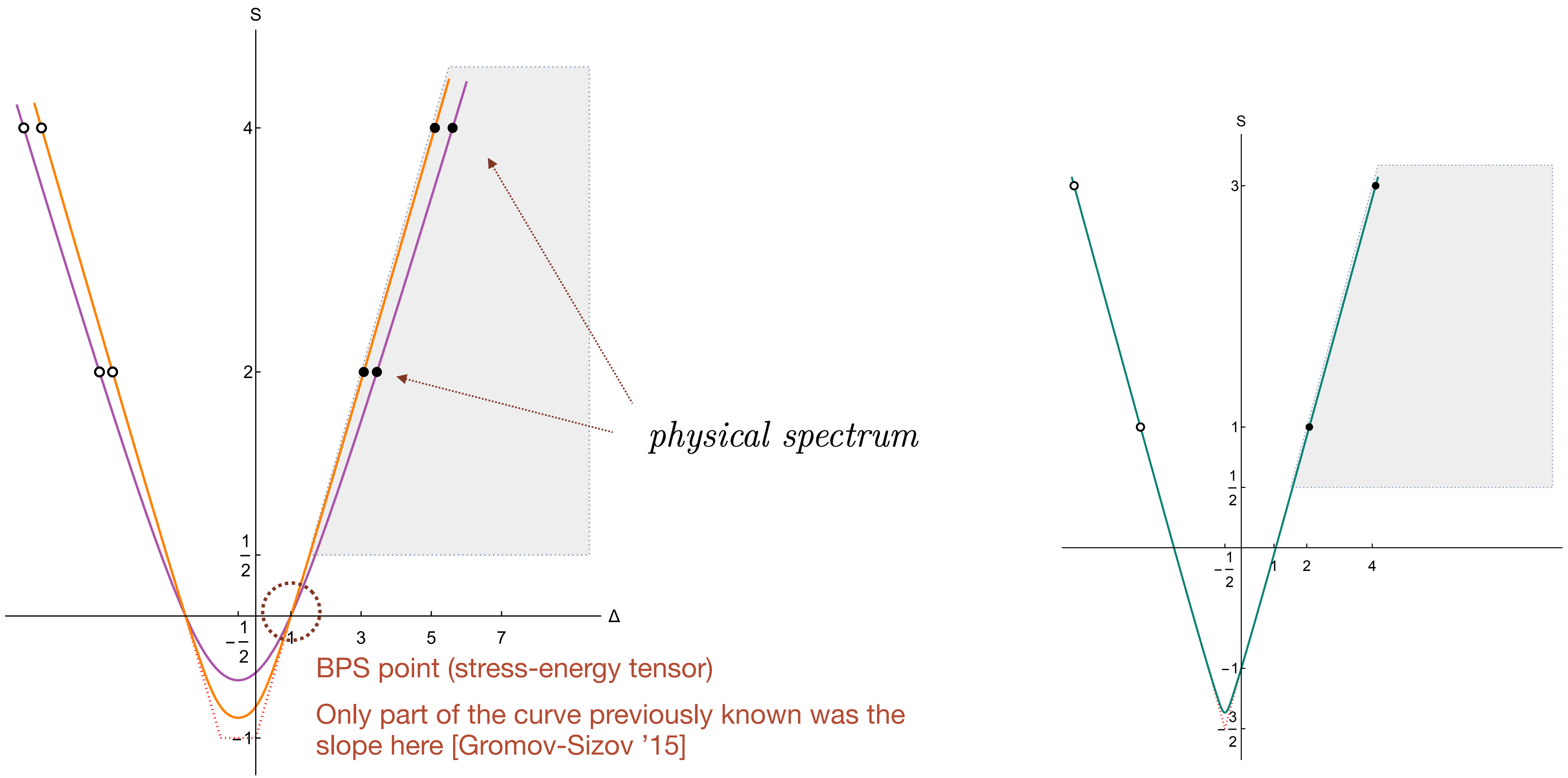
$$\boxed{\text{Local operator} \longleftrightarrow * = 0}$$

In this case, the two gluing matrices agree.

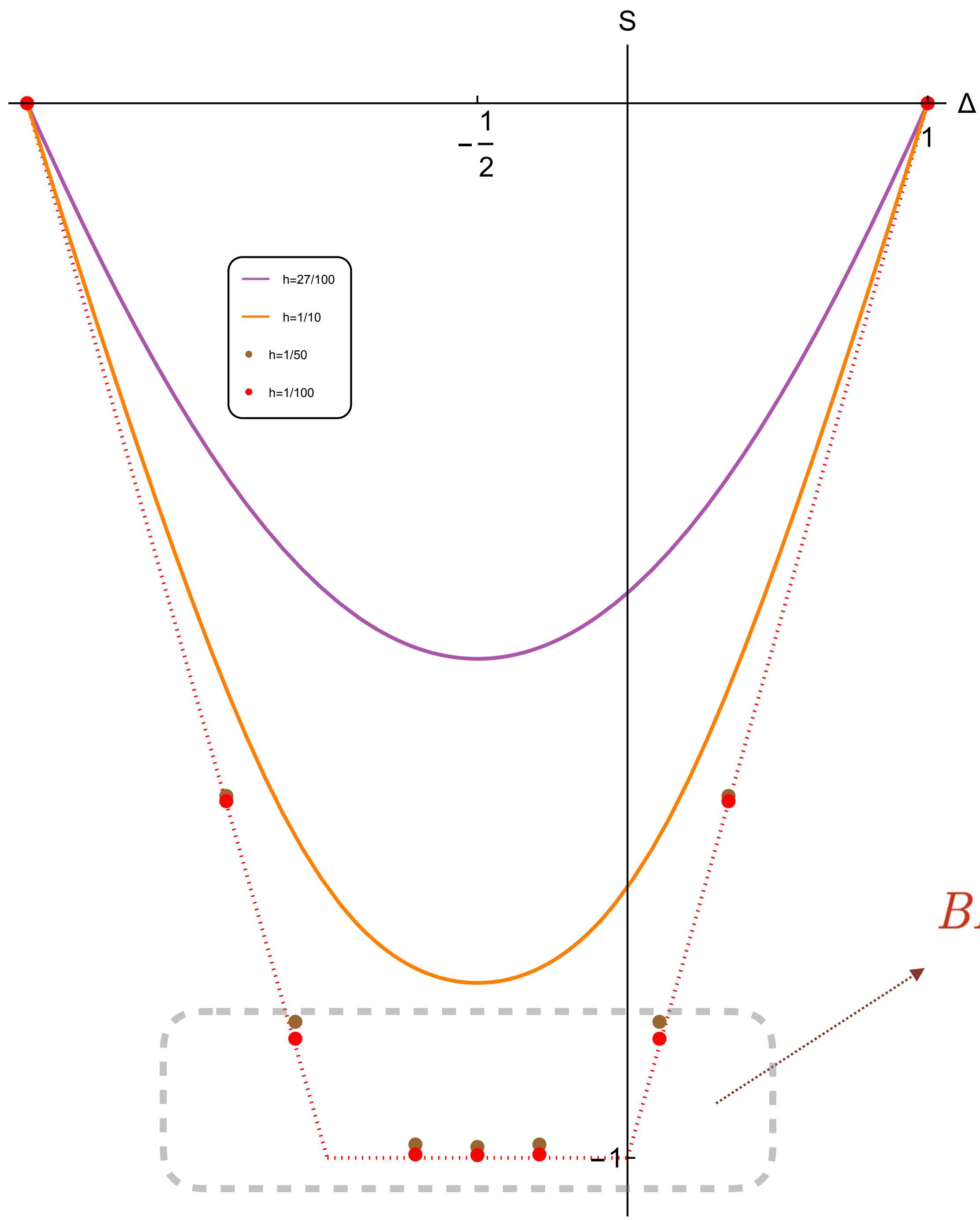
What do the 2 choices mean for non-local operators?

Results

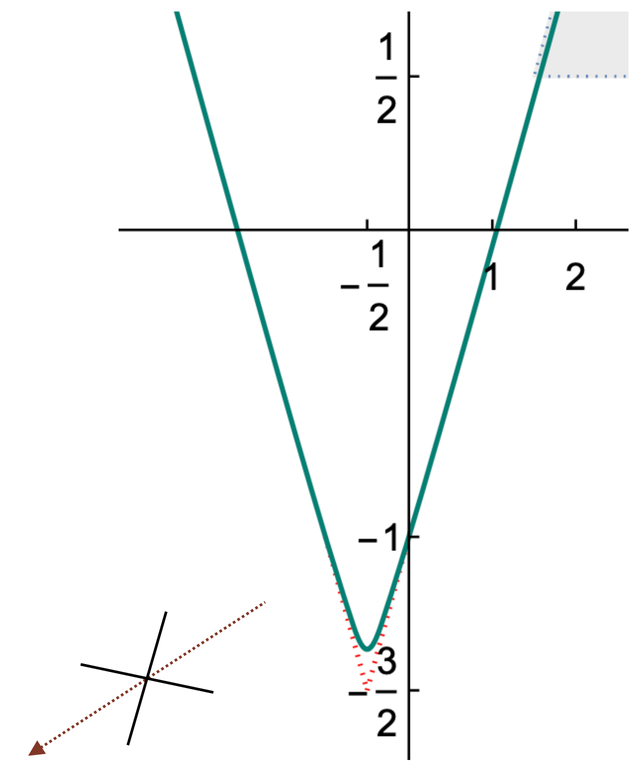
NUMERICS - LEADING TRAJECTORIES IN THE EVEN AND ODD SPIN CASE



Note: even and odd spins local operators live on separate trajectories.



odd plane

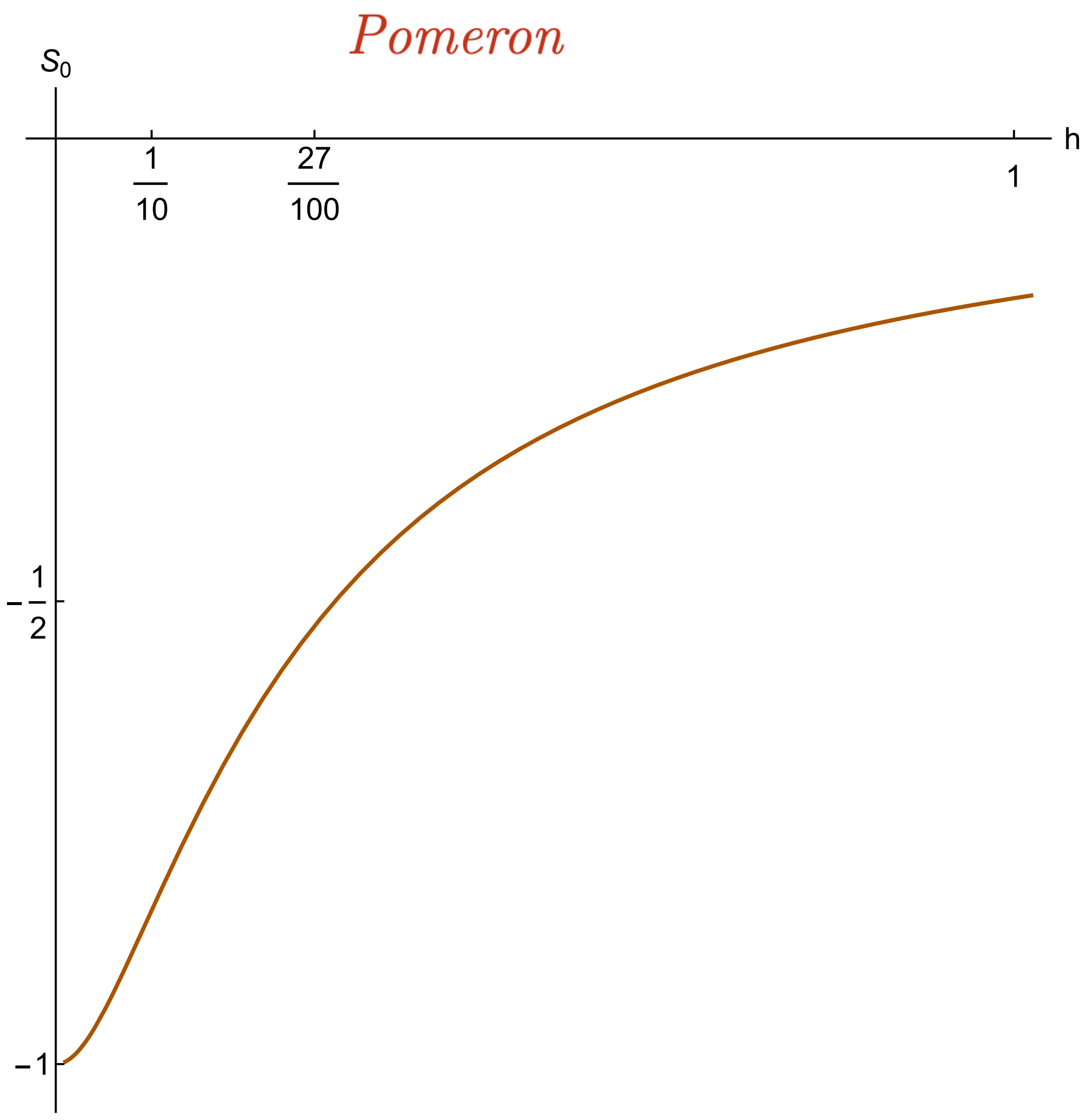


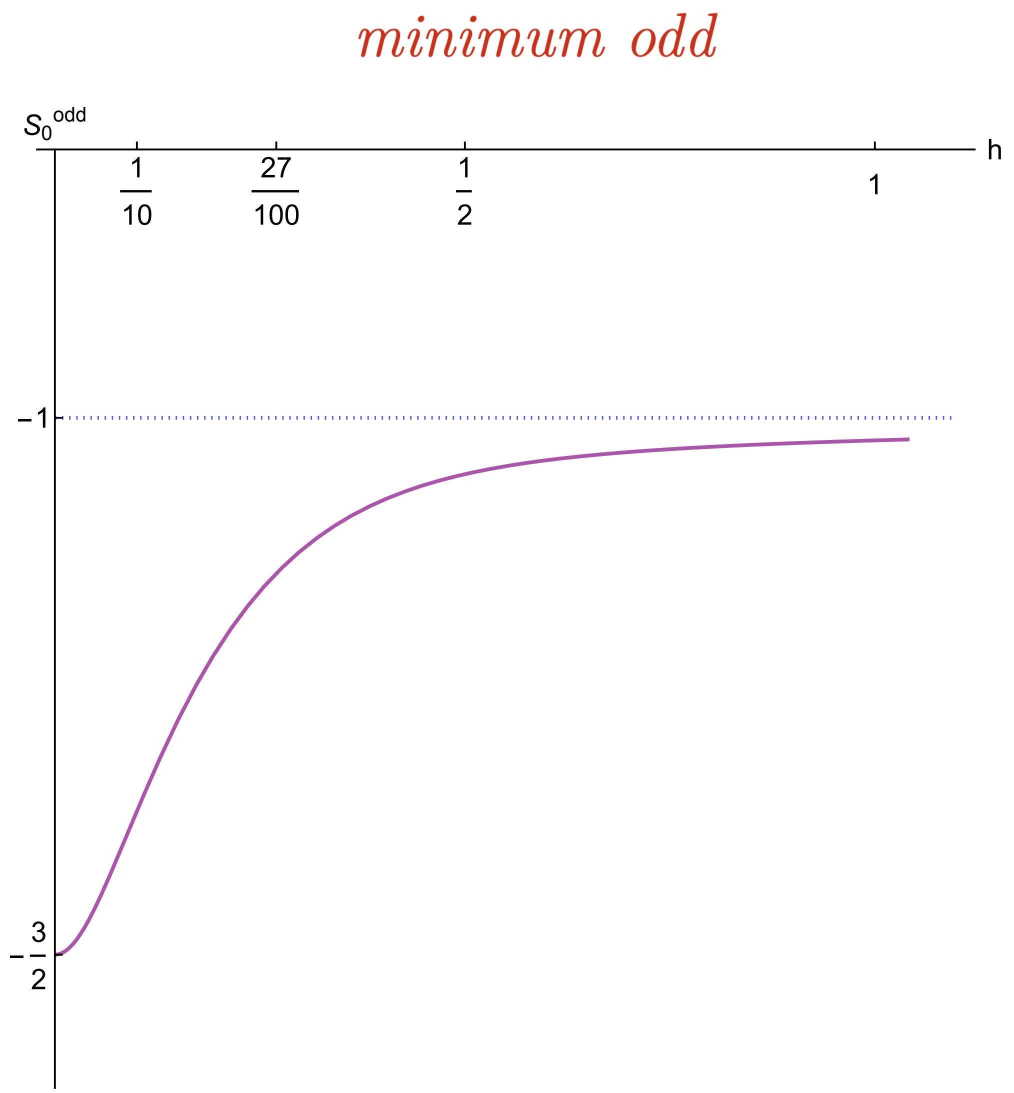
BFKL-like behaviour

As predicted in [Velizhanin '22]

An analytic description of the BFKL-type limit would be interesting

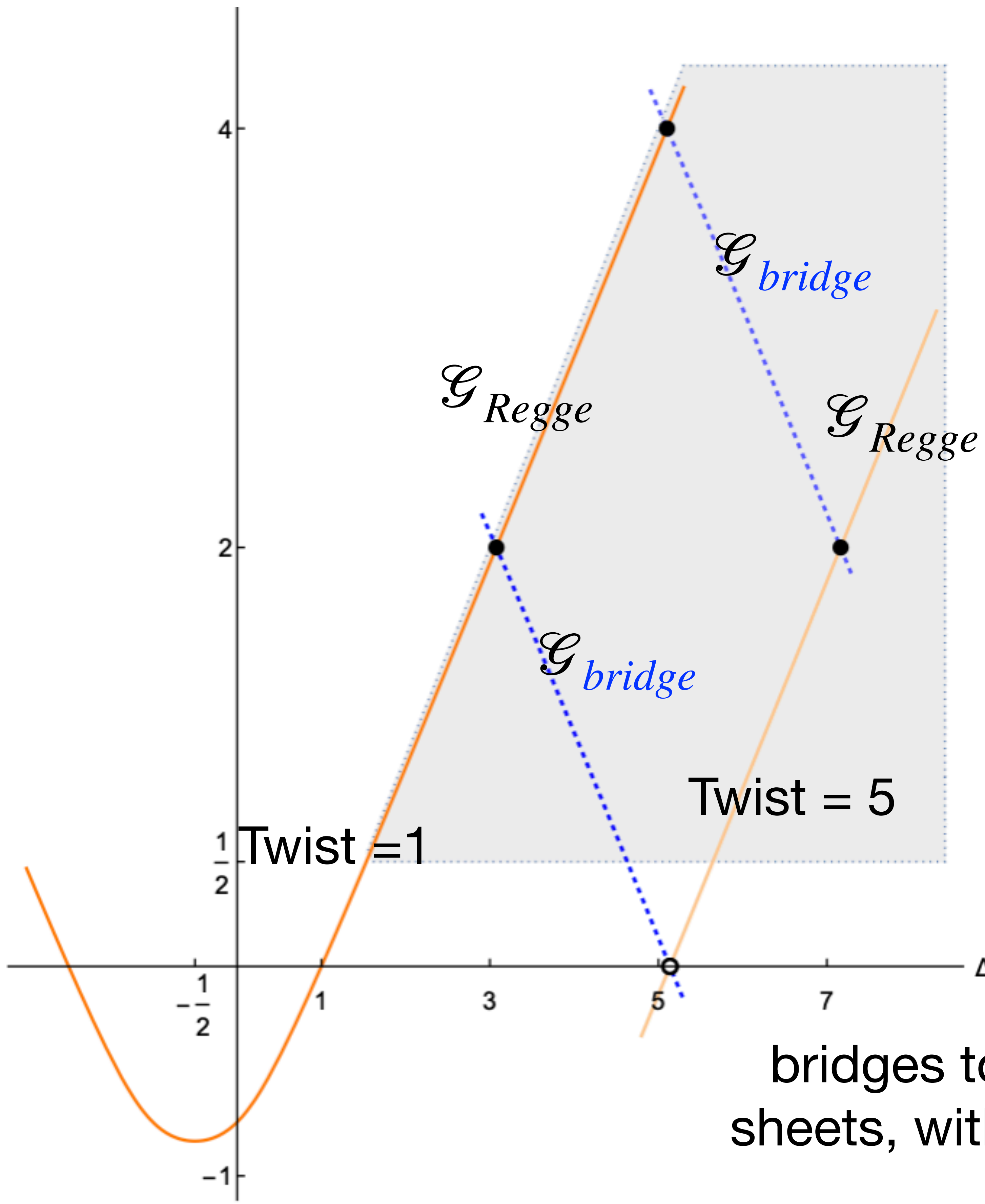
LEADING REGGE POLES





The dominant pole is related to the representative at the bottom of the supermultiplet and satisfies $1 \leq \mathcal{J}_0 \leq 2$ (chaos bound at large N)

What happens with the second gluing matrix?



Two gluing matrices:

there are 2 trajectories going through the same physical operator!

One is upside down.

It builds a bridge connecting different Regge trajectories!

We can use the bridges to reach (all?) the subleading sheets, without leaving the real spin axis

To interpret the flipped trajectory consider the **spin-shadow** map

$$W_{spin}: \qquad \Delta \to \Delta \qquad S \to 2-d-S$$

It generates the Weyl group, together with the standard shadow map

$$W_{\Delta}: \qquad \Delta \to d-\Delta \qquad S \to S$$

These reflections map QSC solutions to other solutions

$$(\mathbf{Q}_1, \ldots, \mathbf{Q}_4) \simeq \left(u^{\Delta+S+1}, u^{\Delta-S}, u^{S-\Delta-2}, u^{-\Delta-S-3}\right)$$

W_{Δ}

For W_{Δ} , gluing remains invariant

$$(\mathbf{Q}_1, \ldots, \mathbf{Q}_4) \simeq \left(u^{\Delta+S+1}, u^{\Delta-S}, u^{S-\Delta-2}, u^{-\Delta-S-3}\right)$$

W_{spin}

Relabeling indices changes gluing!

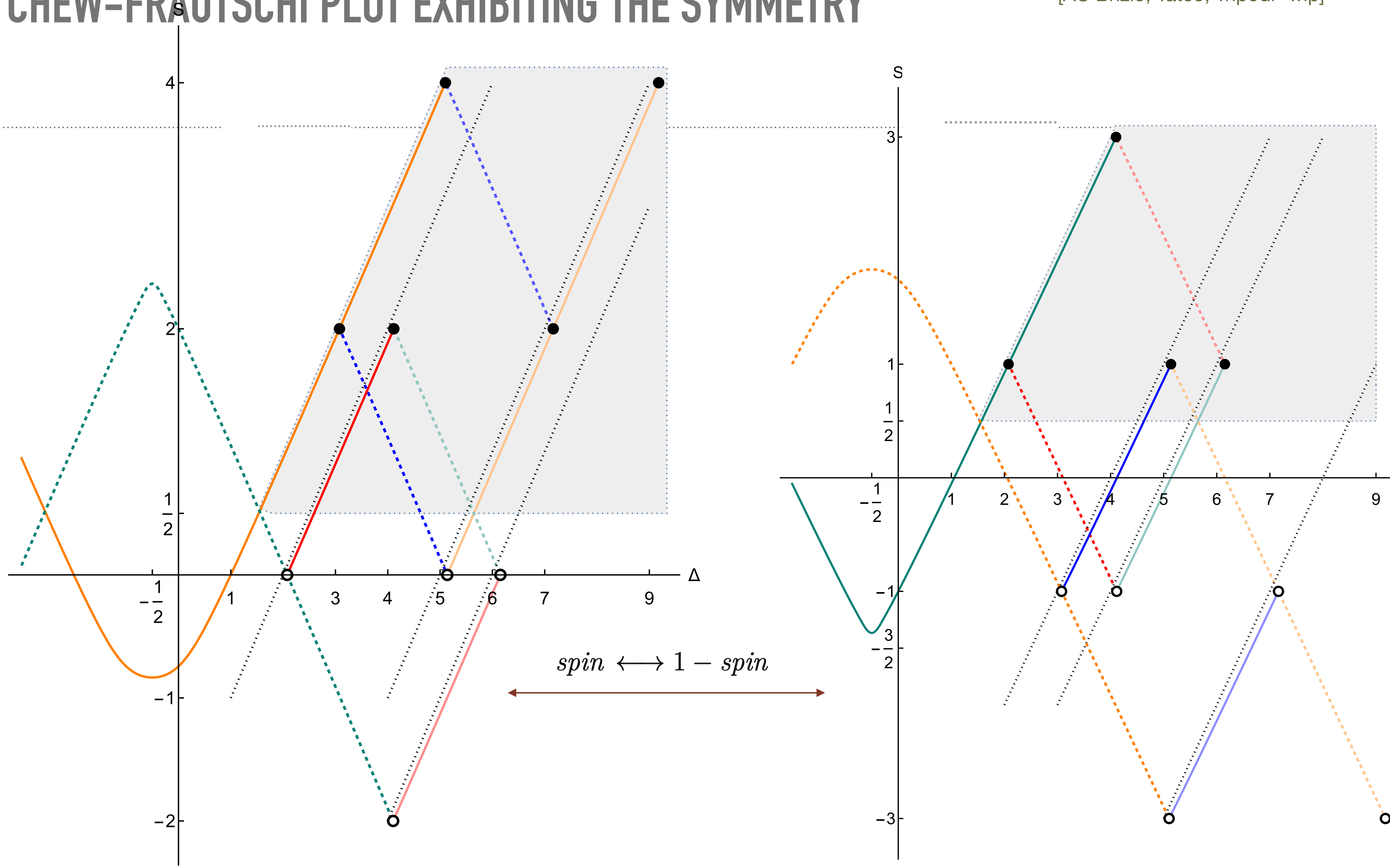
$$\mathscr{G}_{Regge} \underset{W_{spin}}{\leftrightarrow} \mathscr{G}_{bridge}$$

This **shows** that “bridging trajectories” are spin-flipped standard Regge trajectories

… the connection between sheets implied by the symmetry is nontrivial

CHEW-FRAUTSCHI PLOT EXHIBITING THE SYMMETRY

[AC Brizio, Tateo, Tripodi -wip]



Questions & outlook

What’s up with this symmetry?

Symmetry connecting operators on different Regge trajectories, in different supermultiplets (it’s not straightforward susy)

Different way to navigate the Riemann surface

We expect the same to occur in N=4 SYM

Compare with observations of
[Henriksson]
[Aprile Drummond Heslop Paul ’22]

It seems (to me) this is not a general CFT phenomenon.
What is the origin?

QSC & light-ray operators

There are ongoing attempts to connect the QSC and correlators

(ask Till, Carlos & Davide!)

Connect elements of the gluing matrix with $\langle \mathbb{O}\mathbb{O} \rangle$?
[Henriksson Kravchuk, Oertel ’24]

What happens when there are (possibly, infinitely) degenerate trajectories? More exotic gluing matrix? $e^{\pi u n}$ terms?

Other questions

Understand BFKL-type limit in ABJM analytically
(from QSC and from field theory)

Nice new method by [Ekhammar, Gromov, Preti ’24] ?

Can leverage data on Regge trajectories for Bootstrability?

Thank you for your attention!

What we studied: trajectories related to simplest 4-pt function

$$< \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) >$$

Four BPS scalars
(superprimaries related
to stress-energy tensor)
$\simeq Tr(C\bar{C})$

In this OPE:
non-protected operators have superprimaries
with **no R-symmetry charge** [Binder, Chester, Jerdee, Pufu '21]

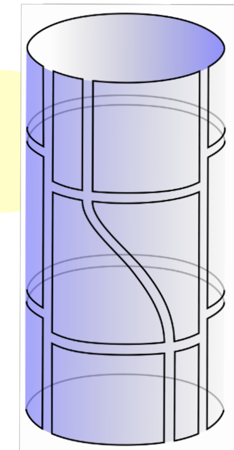
Focus on QSC solutions with $\left(\mathbf{P}_1, \ldots, \mathbf{P}_6\right) \simeq \left(u^{-1}, u^{-2}, u^2, u^1, u^0, u^0\right) \quad u \to +\infty$

(we look at states with two additional $\mathbb{Z}_2$ symmetries — can be relaxed)

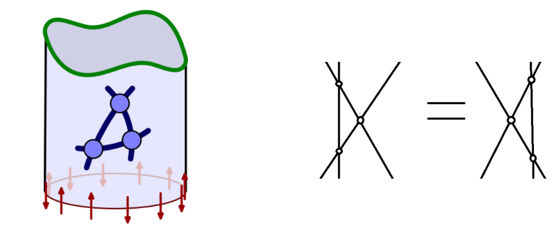
$$\mathbf{P}_5 = \mathbf{P}_6$$
$$\mathbf{P}_A(-u) = (-1)^{n_A}\,\mathbf{P}_A(u)$$

Simple description: we look at operators $Tr(CD^S\bar{C})$
and their analytic continuations in S

The QSC comes from a long story…



Large worldsheet

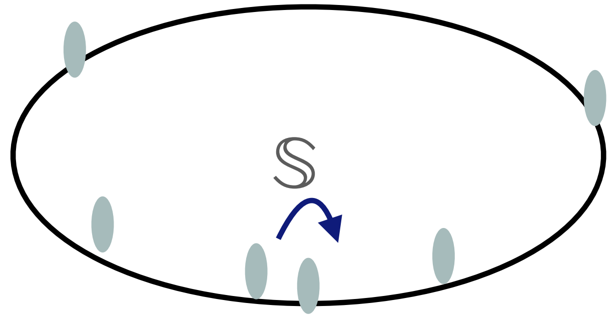


dispersion relation, worldsheet S-matrix

$\mathbb{S}(p_i, p_j)$

Large but periodic

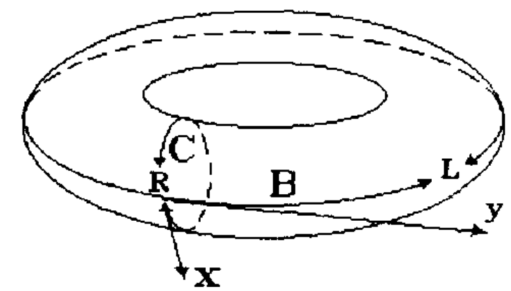
Asymptotic Bethe Ansatz



$\prod_j \mathbb{S}(p_i, p_j) = 1$

[Beisert, Staudacher ‘05]

Finite volume $\simeq$ finite T

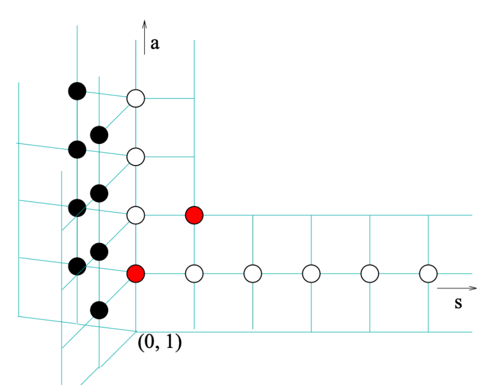


Thermodynamic Bethe Ansatz

Integral equations for quasiparticle densities in thermod limit

[Bombardelli, Fioravanti, Tateo ’09]
[Arutyunov, Frolov ’09] [Gromov, Kazakov, Vieira ’09]

Simplify (a lot)



Y- ,T- , Q-system, … QSC

Functional equations for “Q-functions”: QSC

[AC, Fioravanti, Tateo ’10]
[Gromov, Kazakov, Leurent, Volin ’11,’13]