An Introduction to Hadron Spectroscopy from lattice QCD

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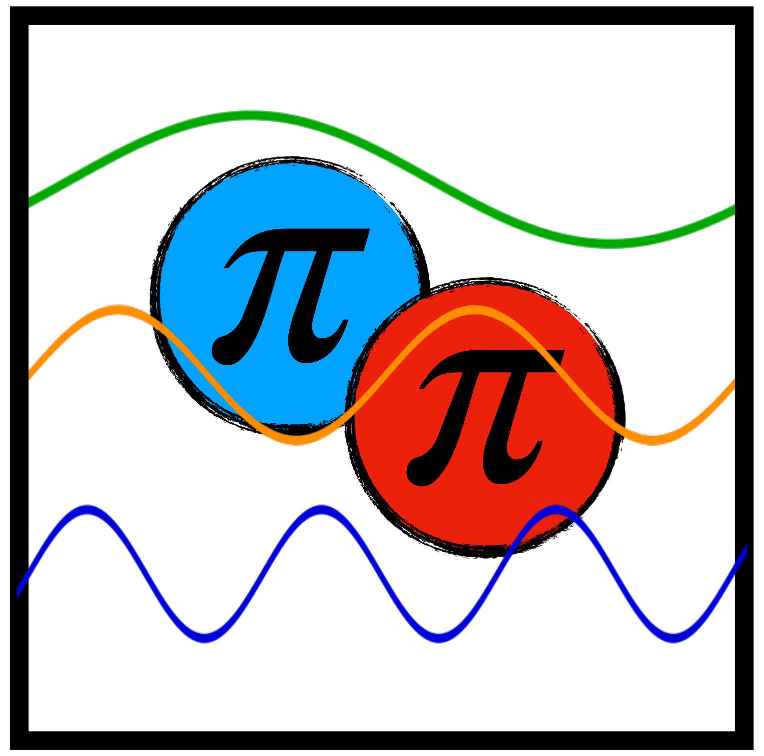
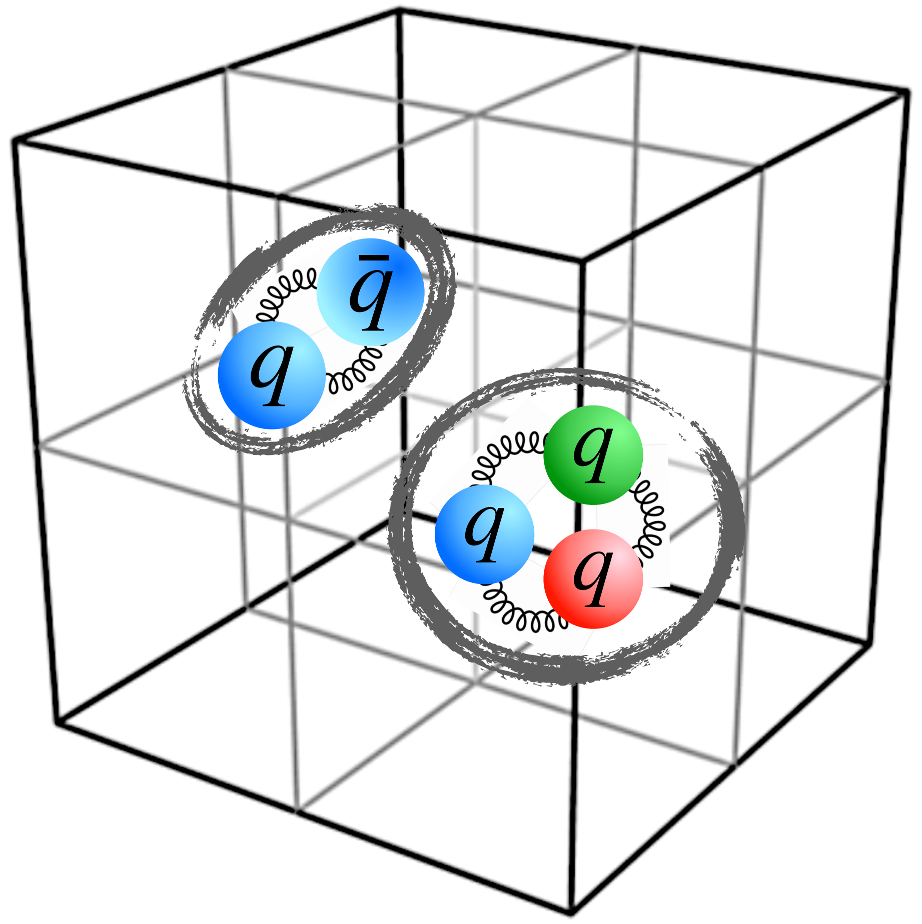


Lattice Practices 2024

The Cyprus Institute



Goals:

- How does the QCD spectrum look like?
- How can we investigate the QCD spectrum with lattice QCD?
- What are the finite-volume effects in QCD stable particles?
- How to obtain resonance properties from lattice QCD?
- What are the finite-volume effects in scattering states?

Outline

1. The Hadron Spectrum
2. Lattice QCD spectroscopy
3. Finite-volume effects: stable particles
4. Scattering processes and resonances
5. Finite-volume effects: multi-hadron states
6. Selection of recent results

The Hadron Spectrum

Quantum Chromodynamics

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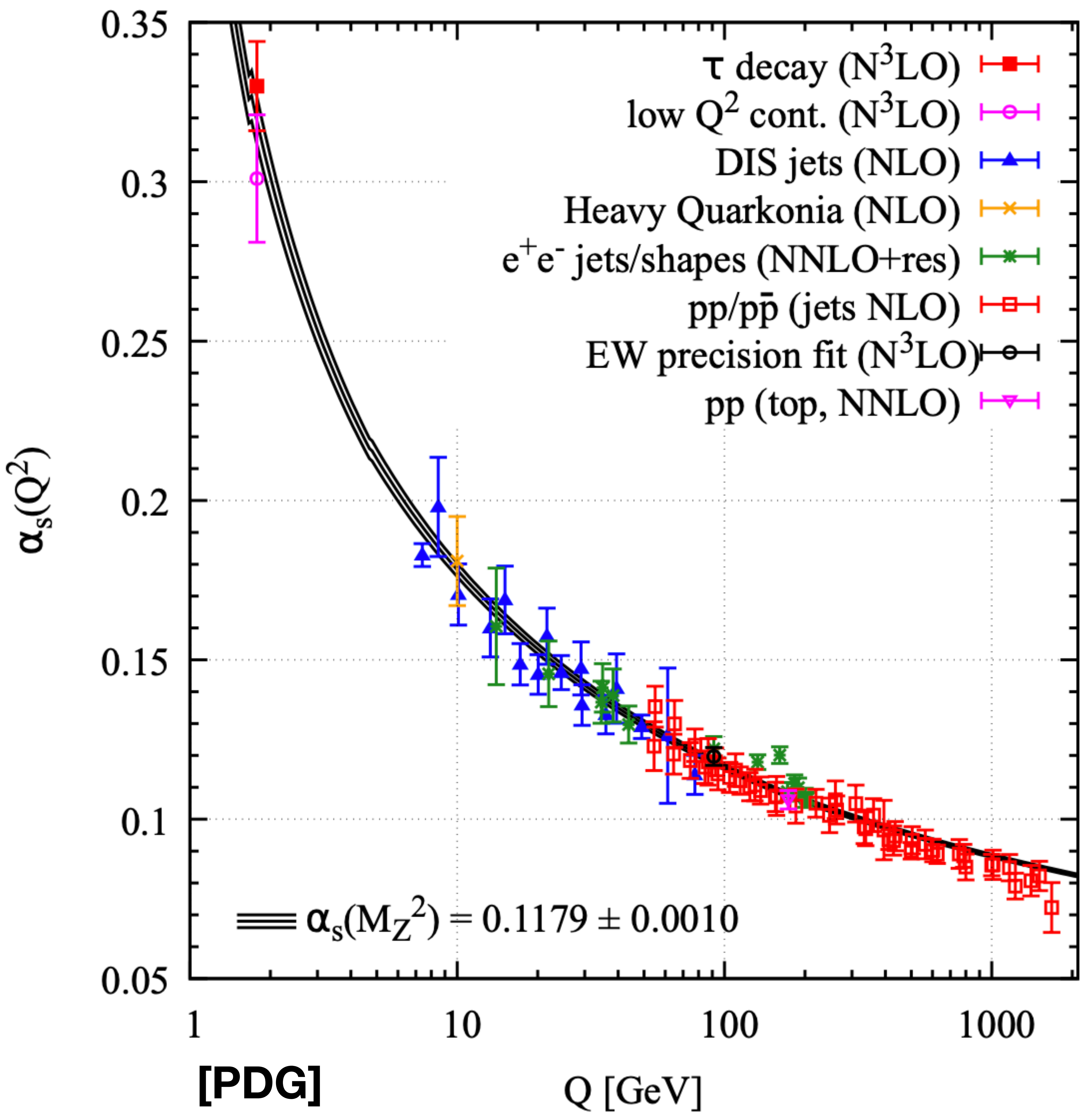
The QCD coupling



[PDG]

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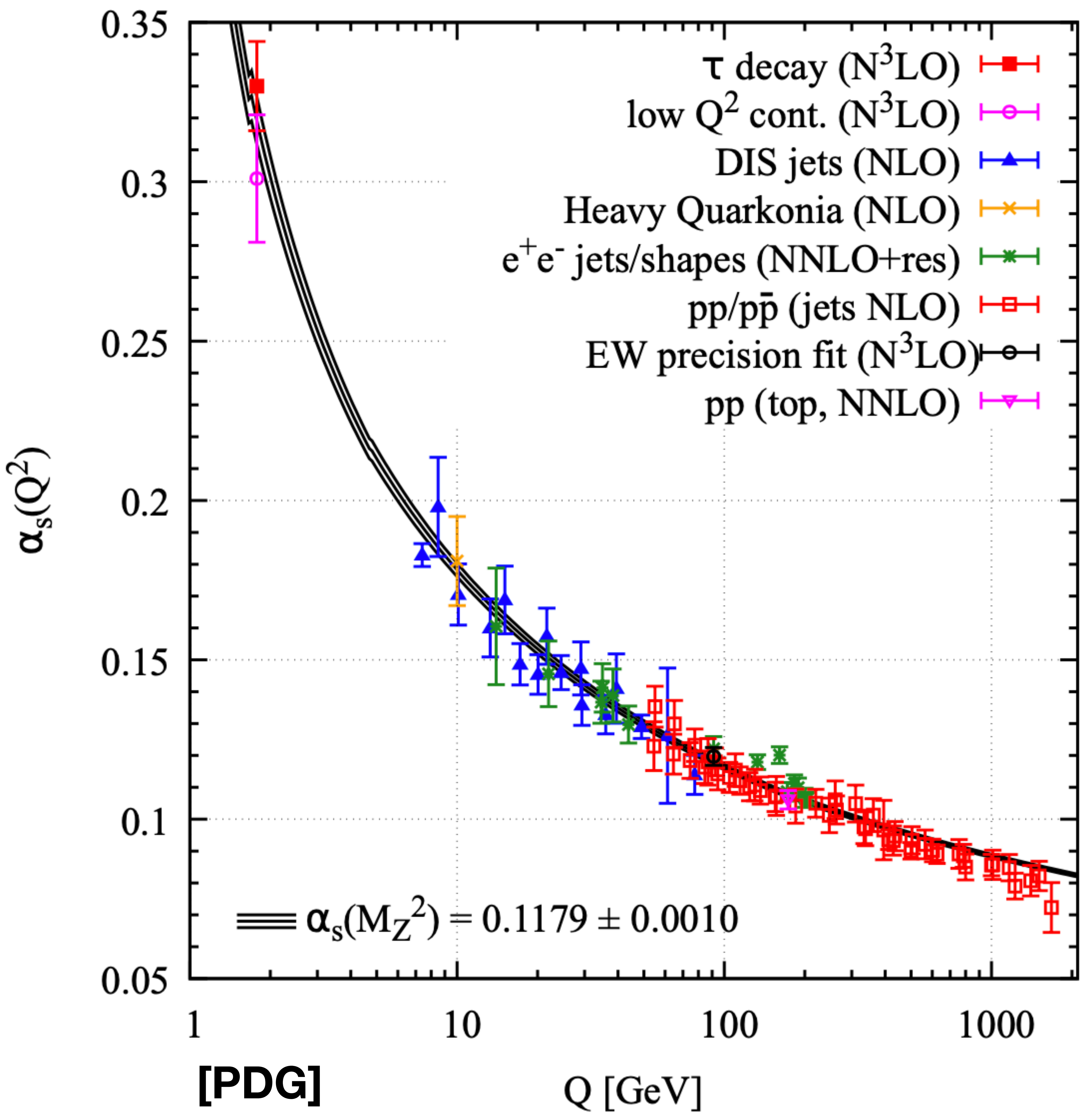


[PDG]

Asymptotic freedom

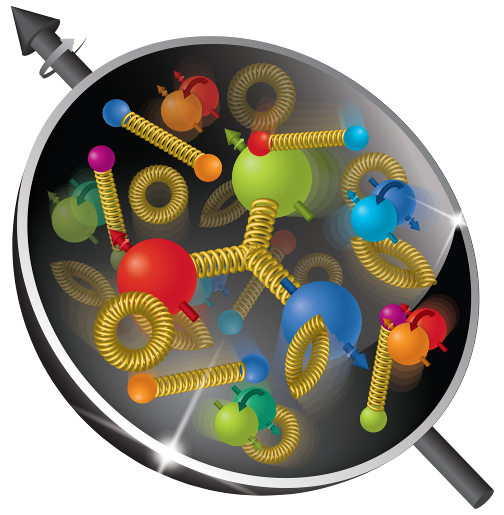
[Gross, Politzer, Wilczek, Nobel 2004]

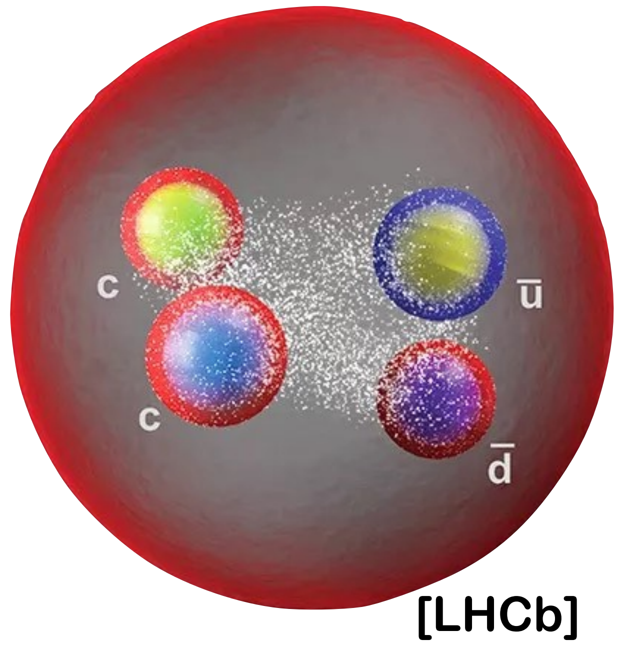
Perturbation theory:

$$\mathcal{O} = \mathcal{O}_0 + \alpha_s \mathcal{O}_1 + \alpha_s^2 \mathcal{O}_2 + \ldots$$

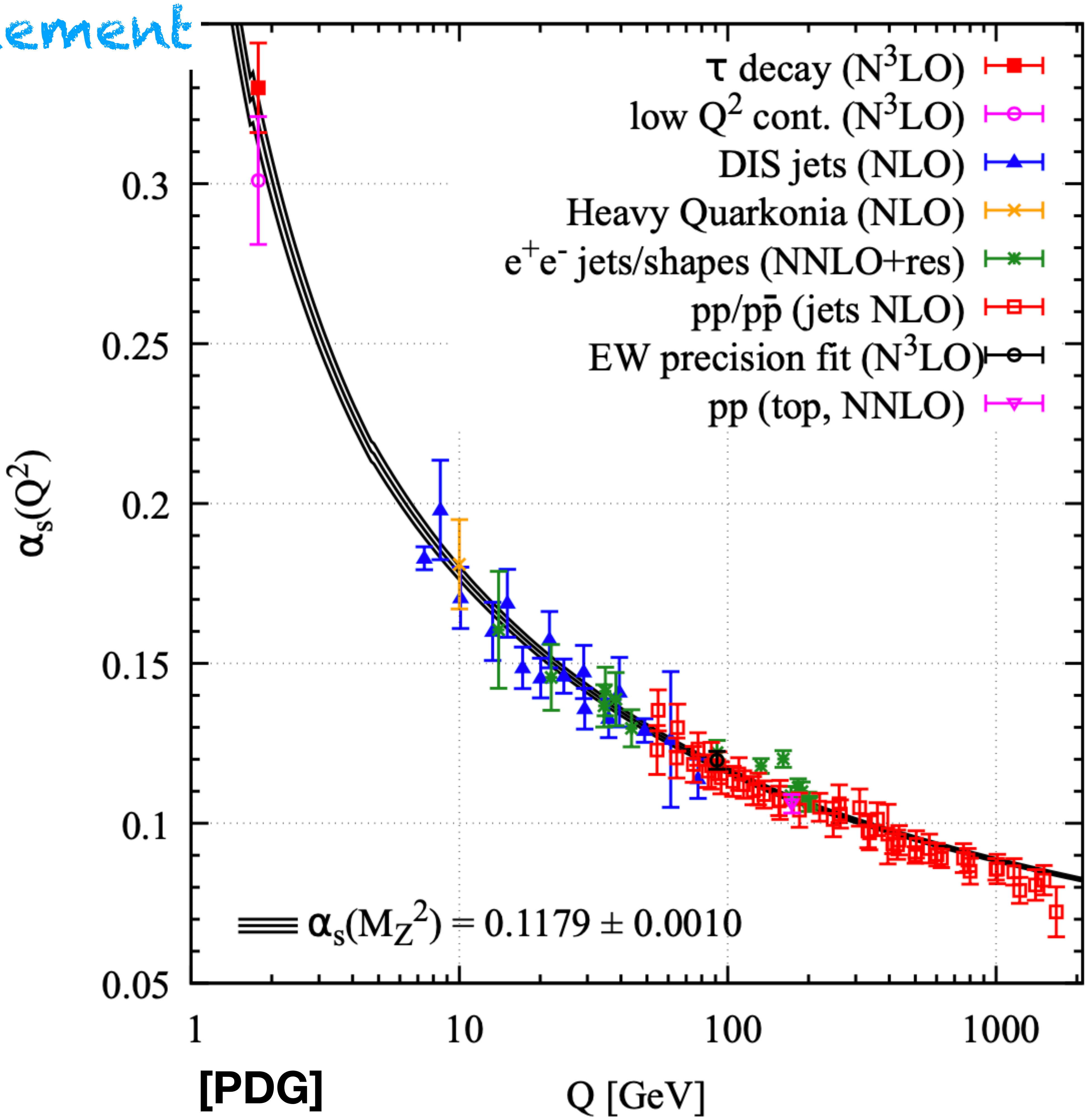
The QCD coupling

Confinement





[LHCb]



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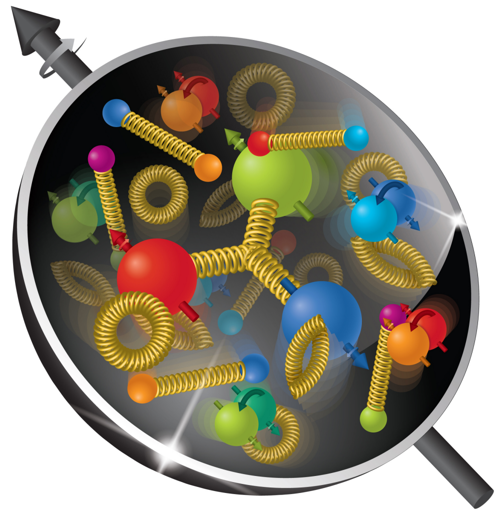
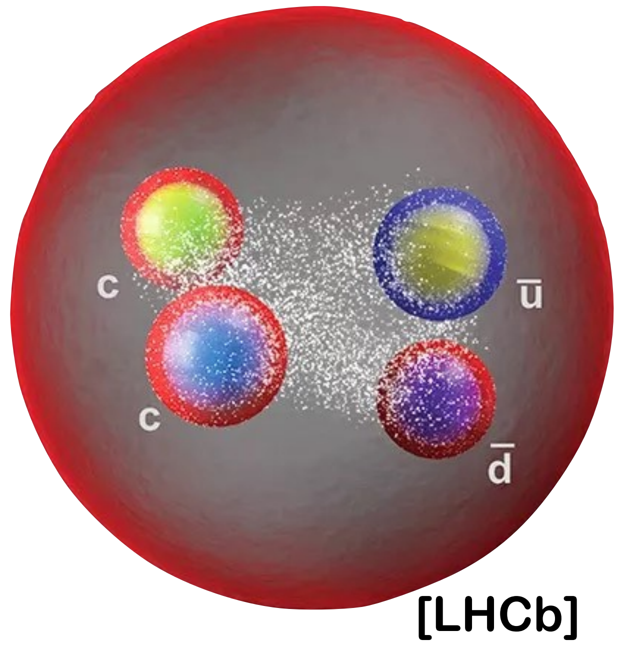
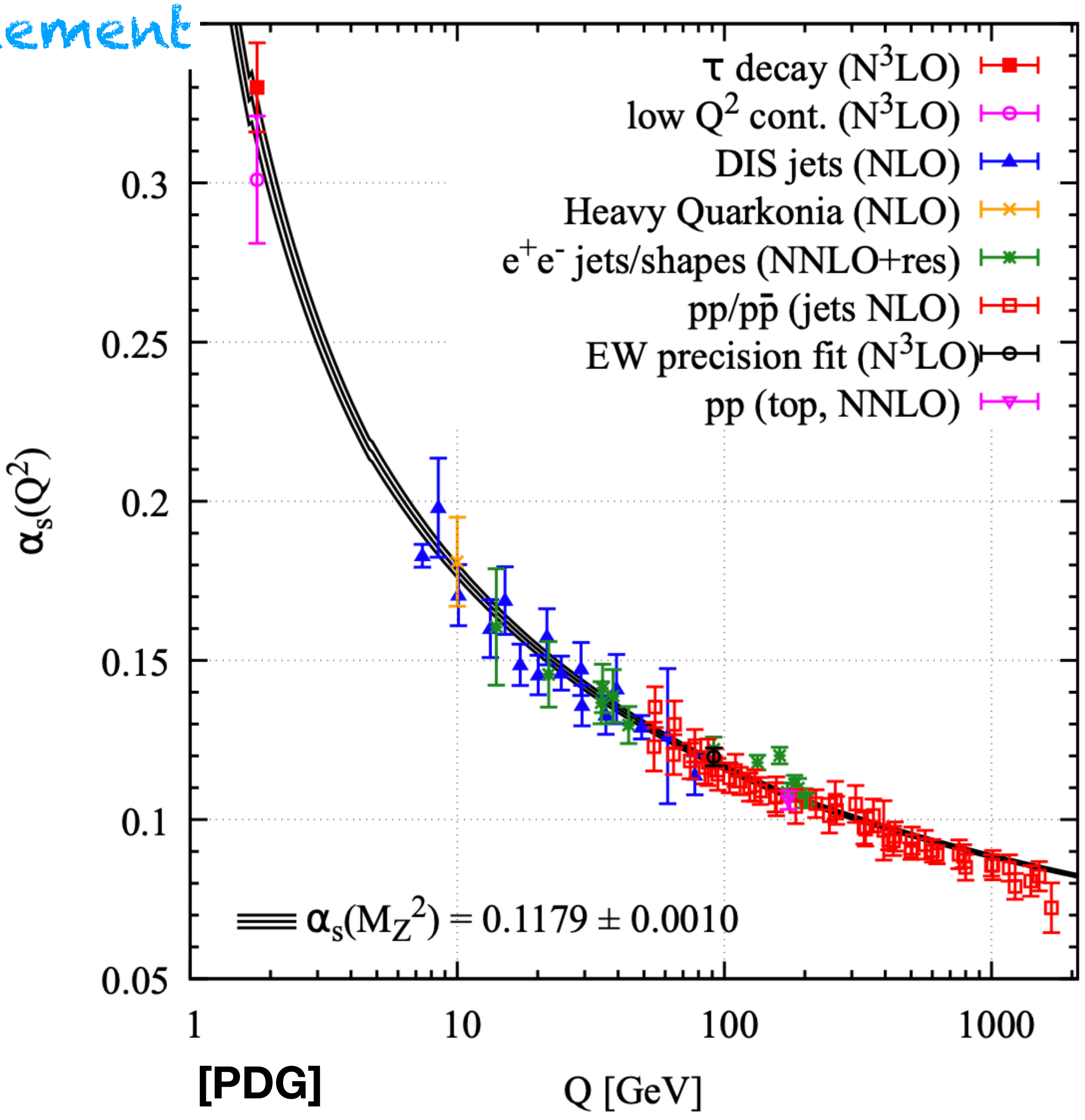




[LHCb]

Non-perturbative
at low energies
Lattice QCD



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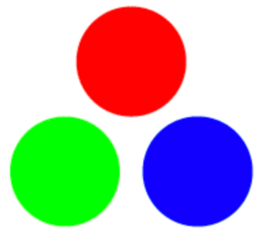
Hadrons & Confinement

Our understanding of the SM is limited by the complexity of the strong force

- **Quarks and gluons carry the strong charge: the so-called "color"** 

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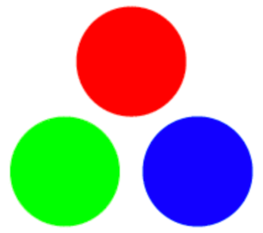
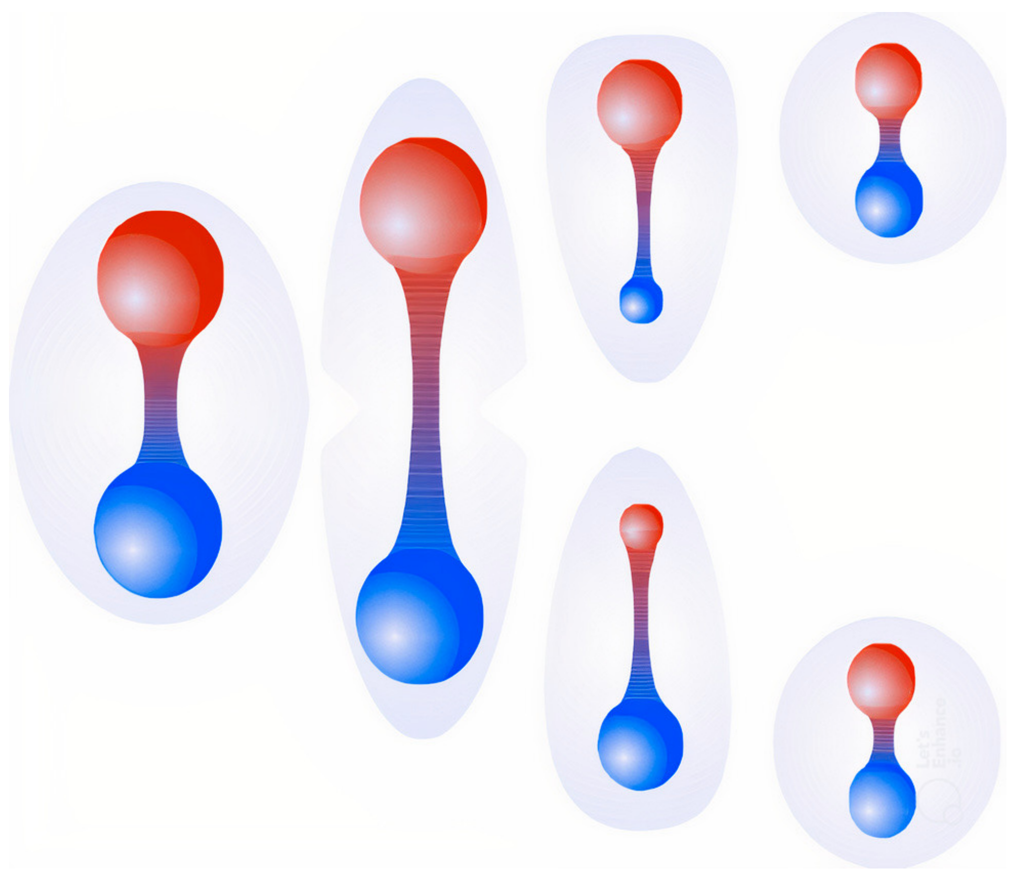


Confinement

Quarks and gluons can only be found within colorless composite states. These are called "hadrons".



source: IFIC

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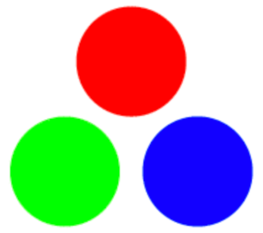

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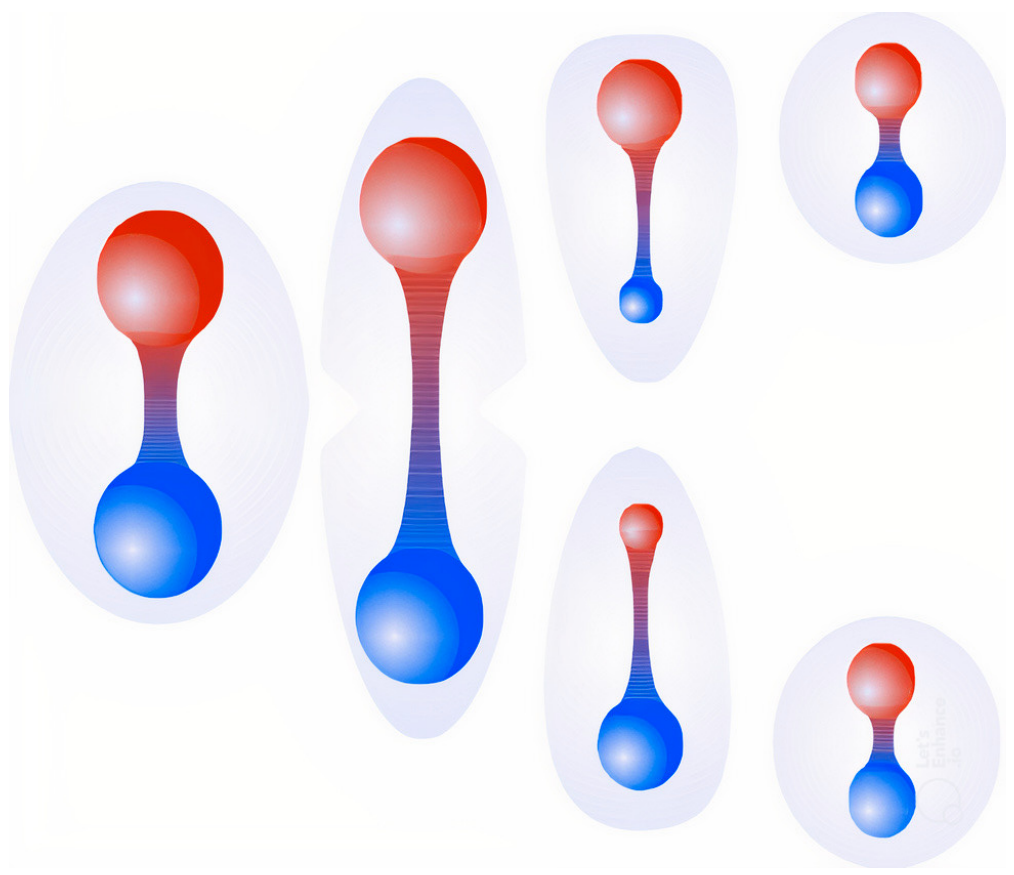

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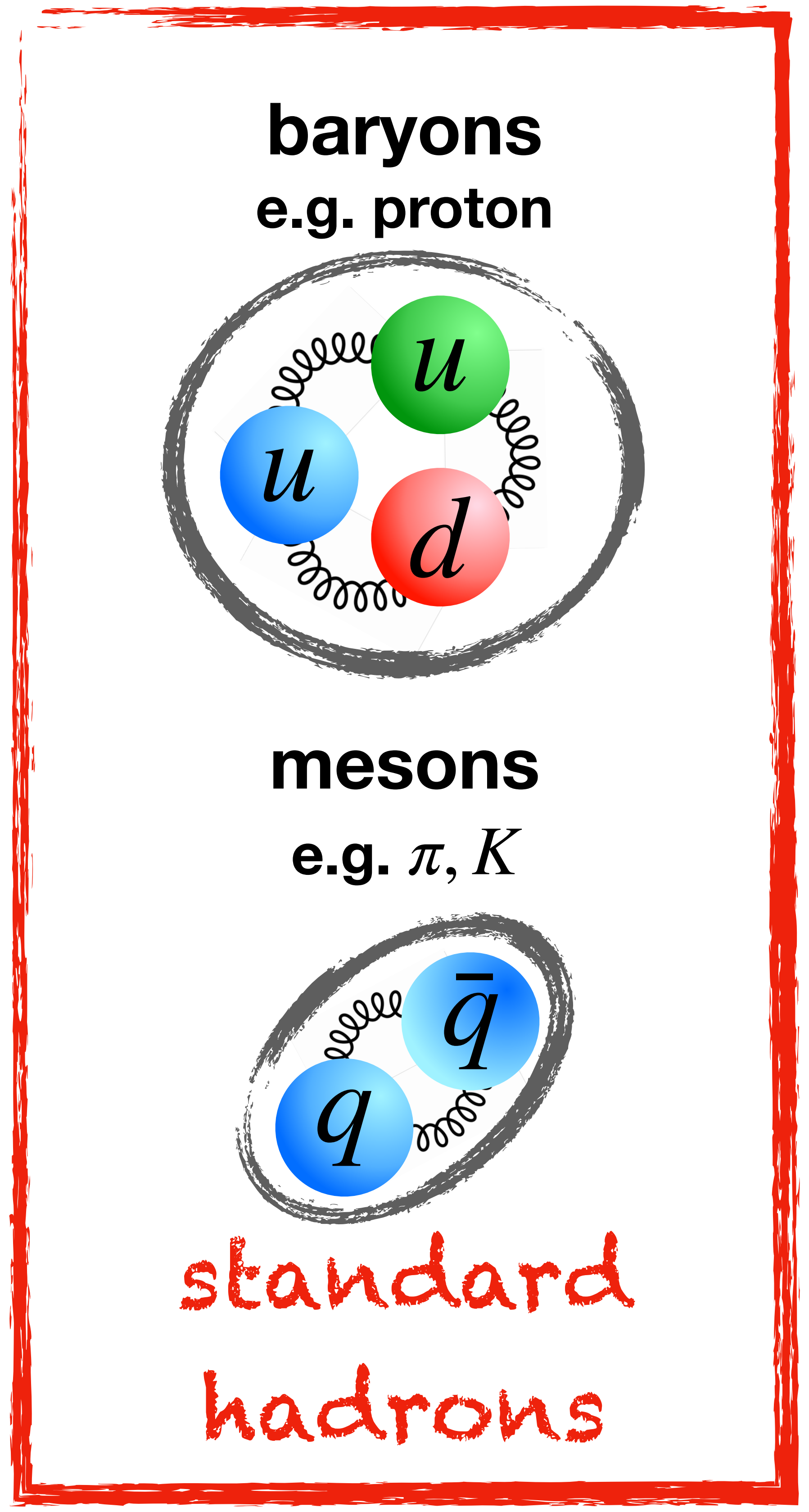

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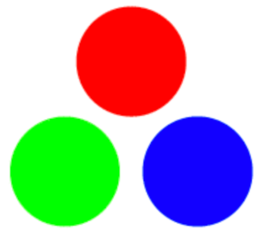

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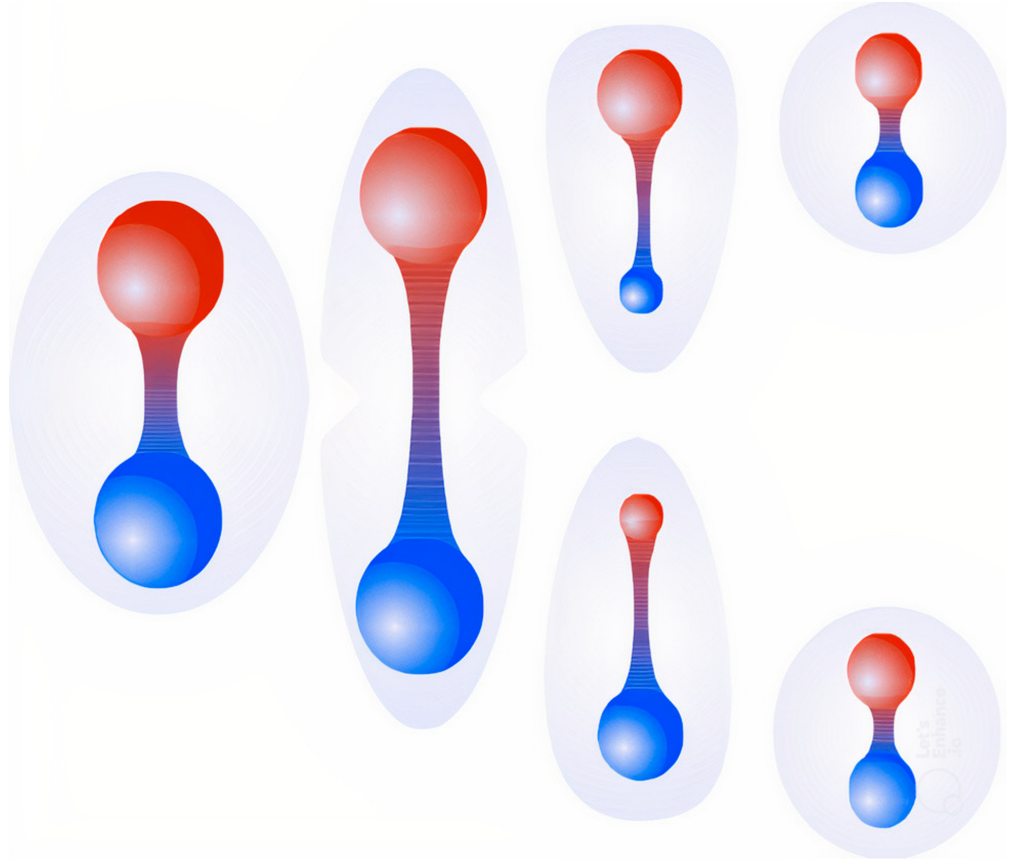

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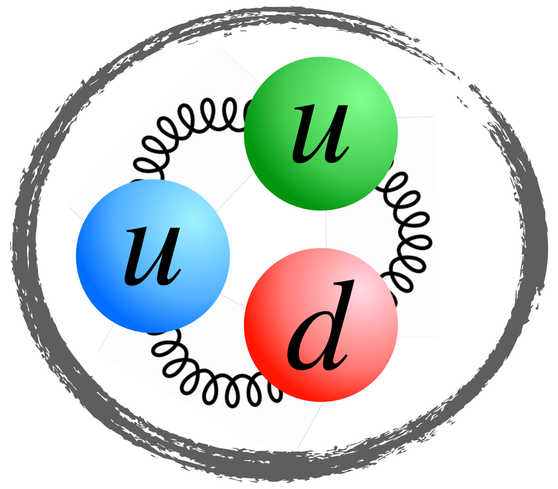
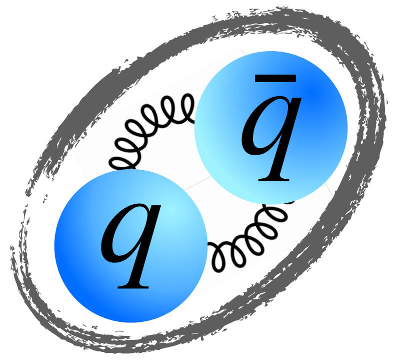
baryons
e.g. proton

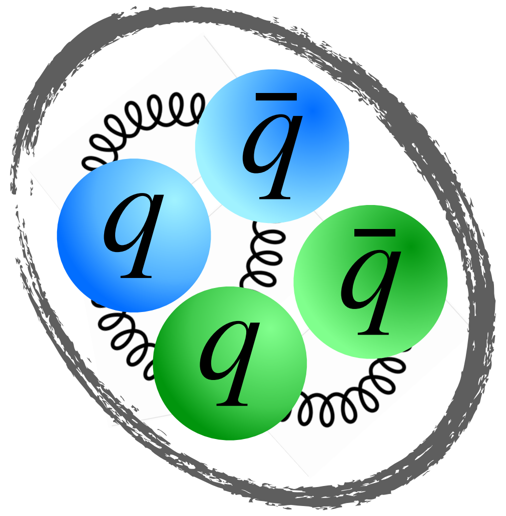


mesons
e.g. π, K



standard hadrons

tetraquarks



+ **other structures**
e.g. pentaquarks

"exotic" hadrons

The QCD Fock space

- The only stable hadron in Nature is the proton
- In Lattice QCD, we (typically) treat QCD in isolation. Several hadrons become stable: π, K, D, p, n, Σ

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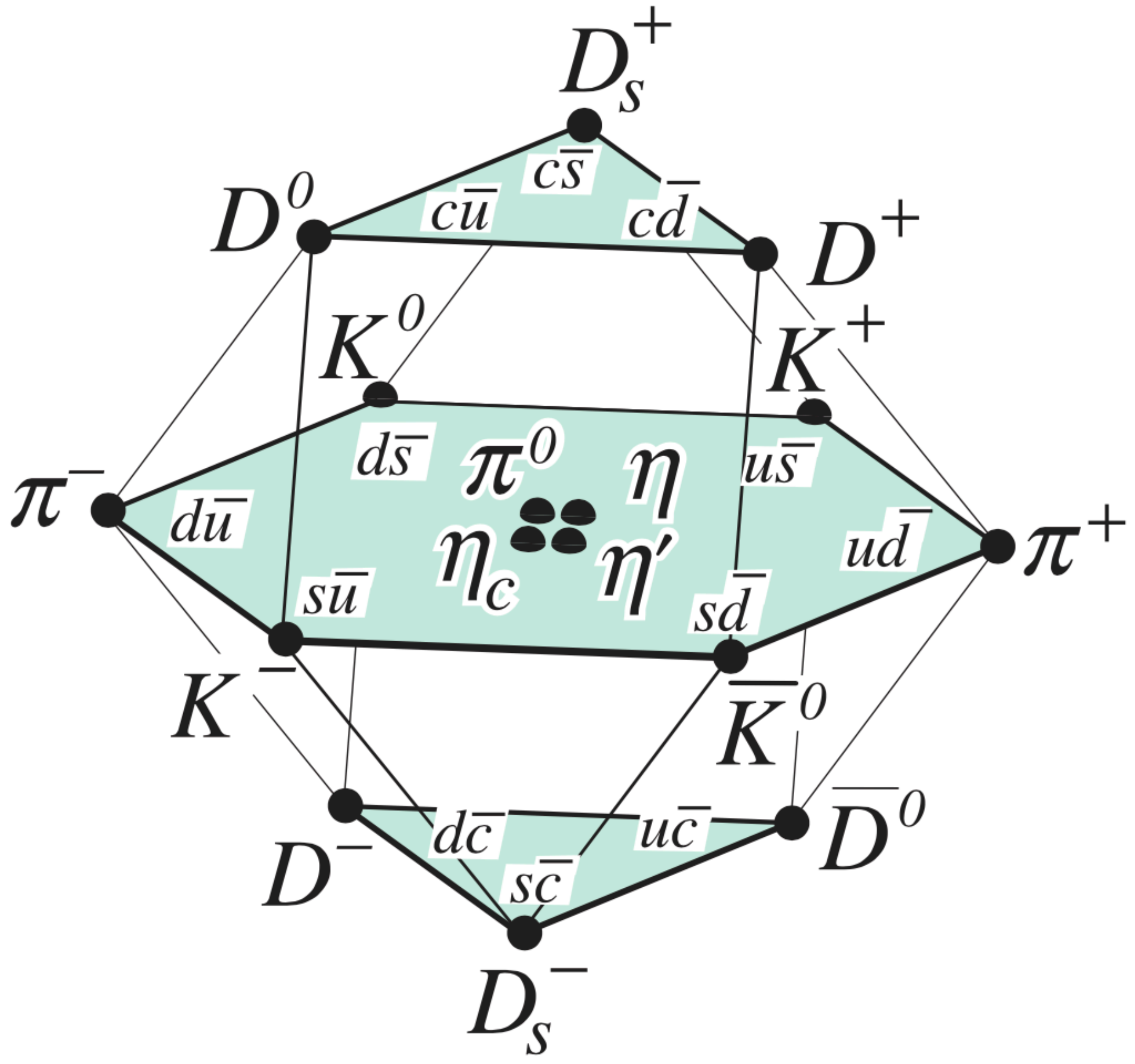

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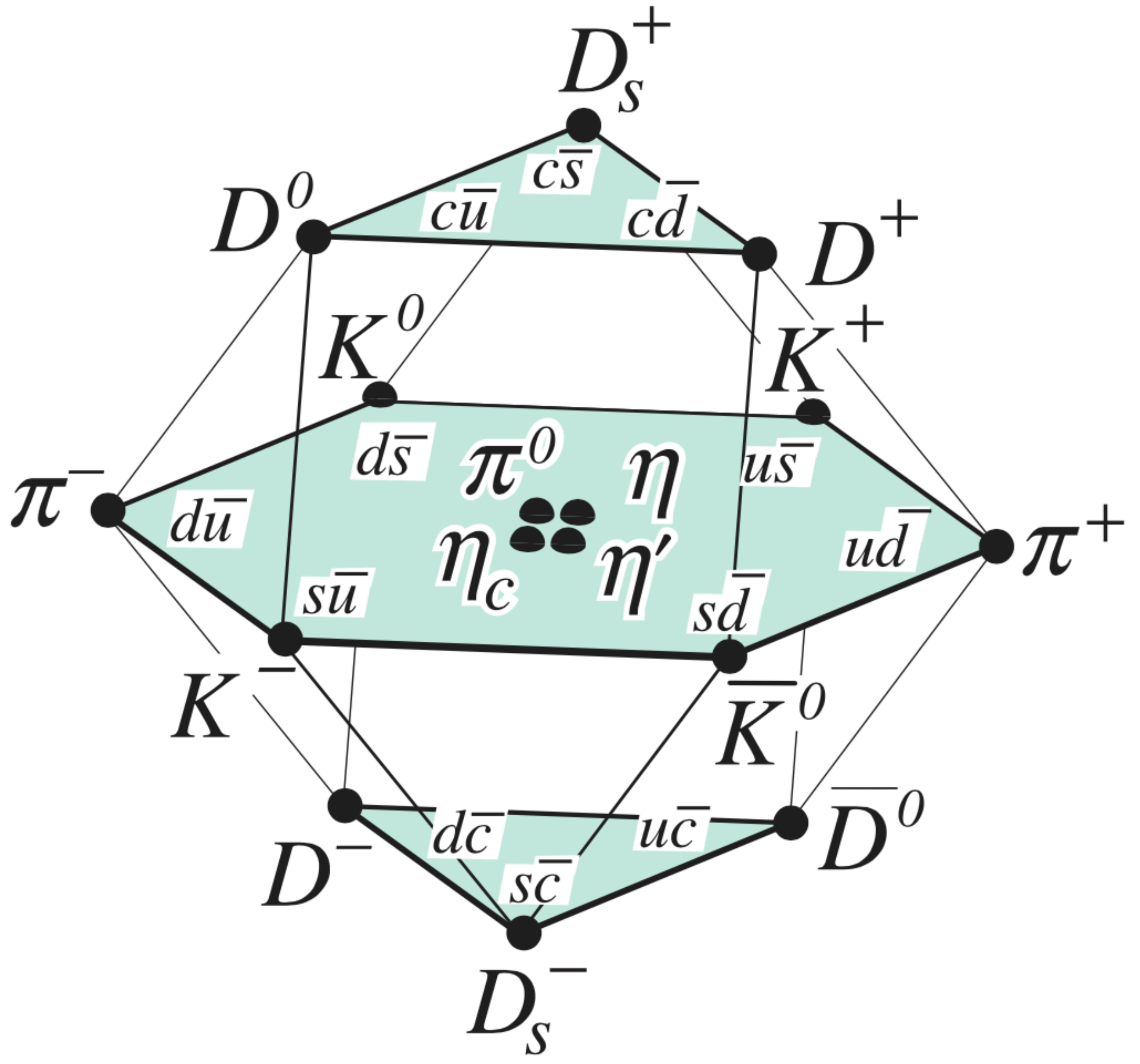
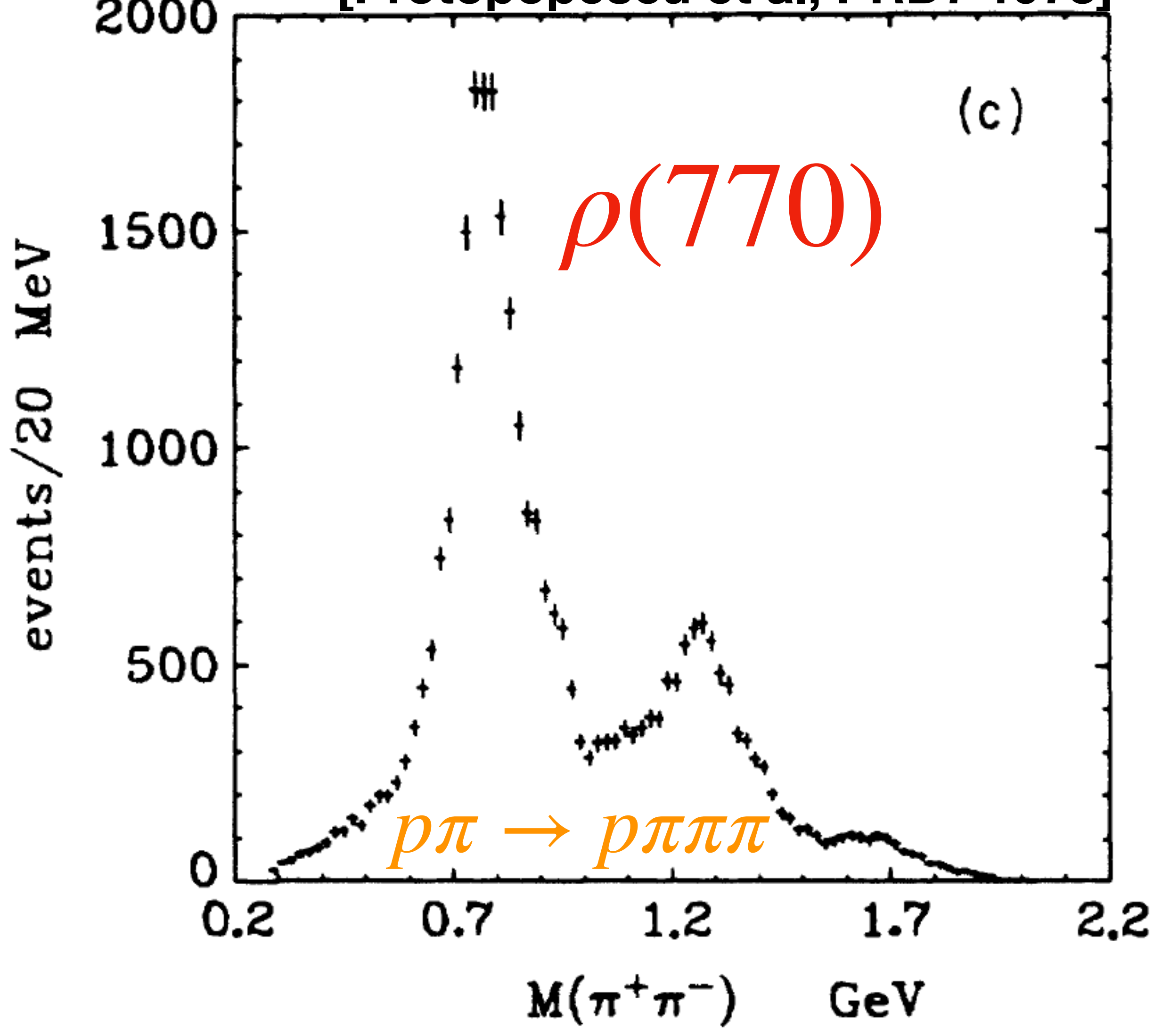


$|\pi\rangle, |K\rangle \in \text{QCD Fock}$

▶ Resonances show up in scattering processes

[Protopopescu et al, PRD7 1973]



$|\rho\rangle \notin \text{QCD Fock}$

The quark model picture

- Crude classification of the hadron spectrum but still useful for intuition
  [Gell-Mann ’64 & Zweig ’64]
- Construct color singlets combining quarks and antiquarks

$$q \to \mathbf{3} \text{ irrep of } \mathrm{SU}(3)_\mathrm{c}$$
$$\bar{q} \to \bar{\mathbf{3}} \text{ irrep of } \mathrm{SU}(3)_\mathrm{c}$$

Meson: $\mathbf{3} \otimes \bar{\mathbf{3}} = \boxed{\mathbf{1}} \oplus \mathbf{8}$

Baryon: $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \boxed{\mathbf{1}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$

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- Use approximate flavor symmetries to classify multiplets

$$m_u \simeq m_d \simeq m_s$$
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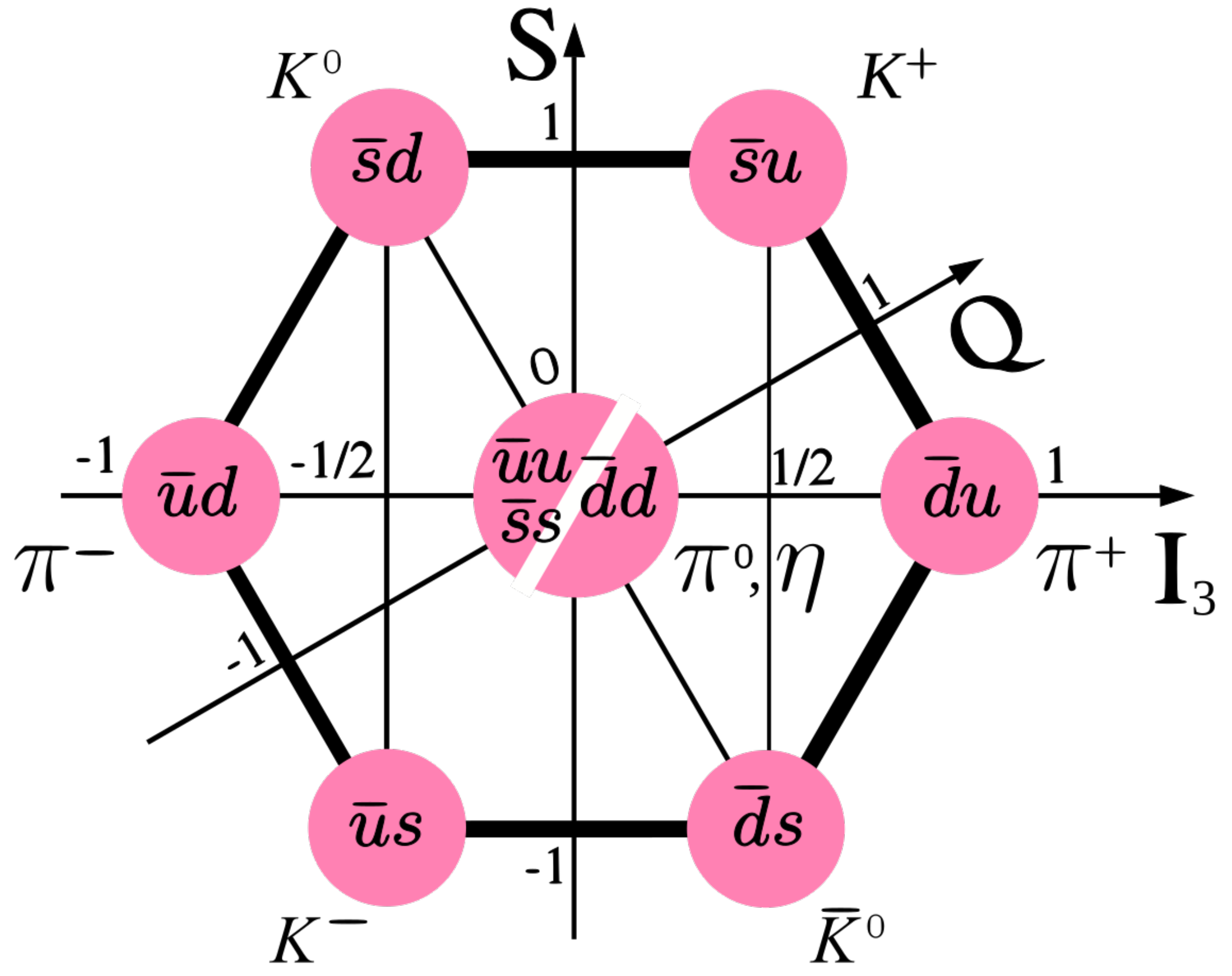
singlet + octet



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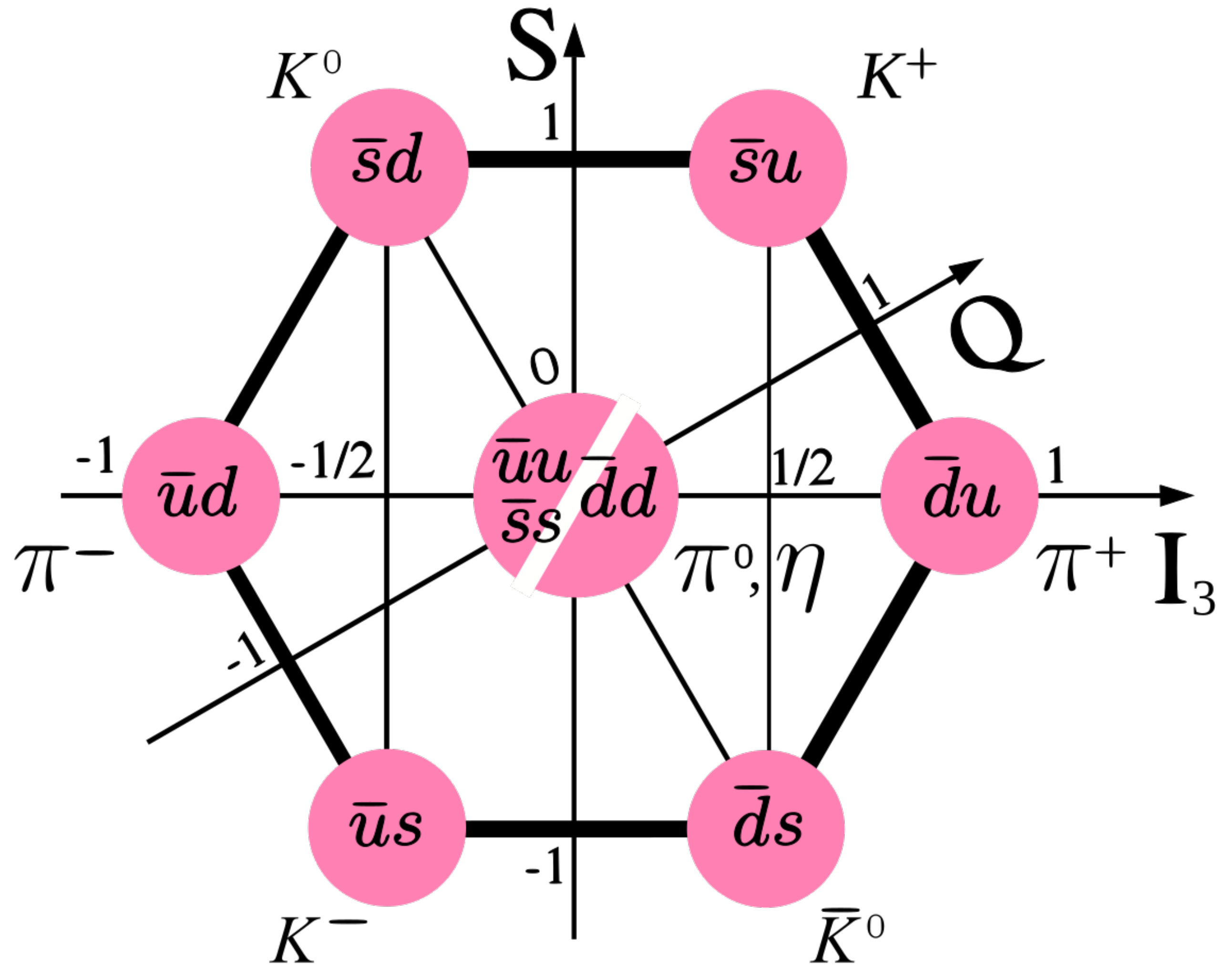
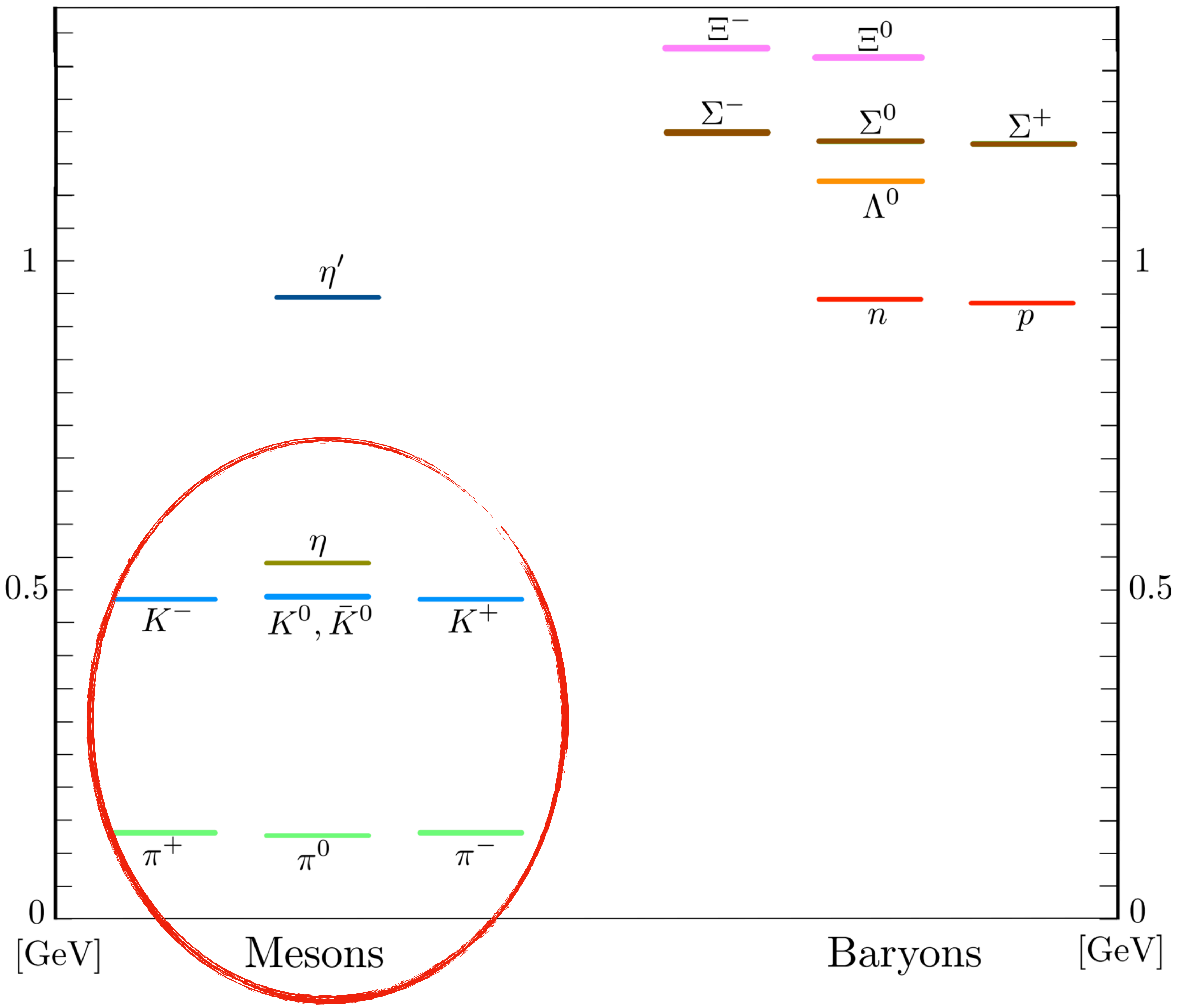


- Many hadrons do not fit in this picture: “exotics”
  - They are not mesons/baryons (e.g. tetraquarks) or cannot be explained through multi-quark objects

Low-lying hadron spectrum



Light meson octet

Pseudo-Nambu-Goldstone bosons from spontaneous chiral symmetry breaking

$$M_m^2 \propto m_q$$

Low-lying hadron spectrum

heavy singlet meson

Mass generated through the chiral anomaly

$$M_{\eta'}^2 \propto \chi_{\text{topo}}$$

[Witten, Veneziano]

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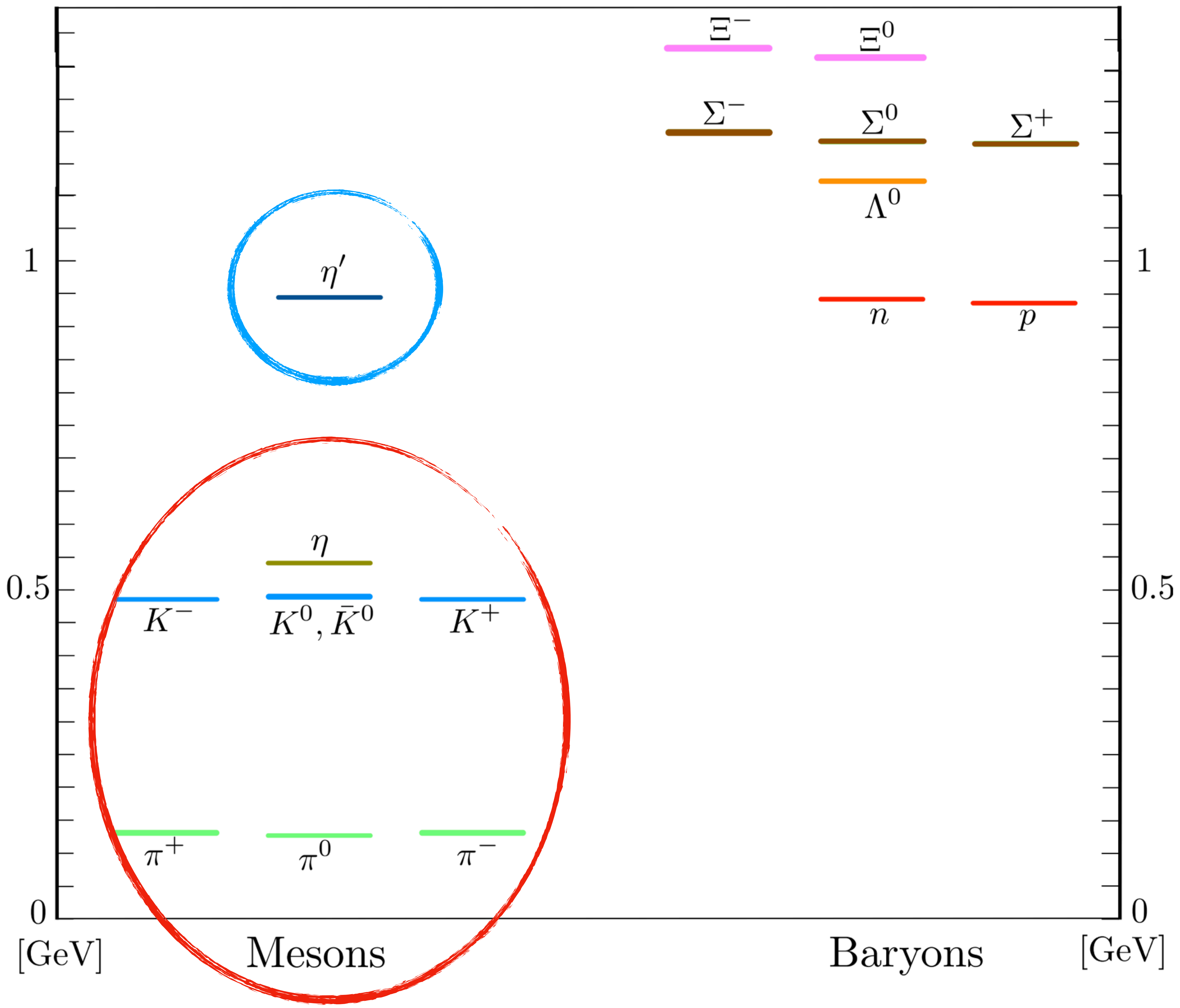

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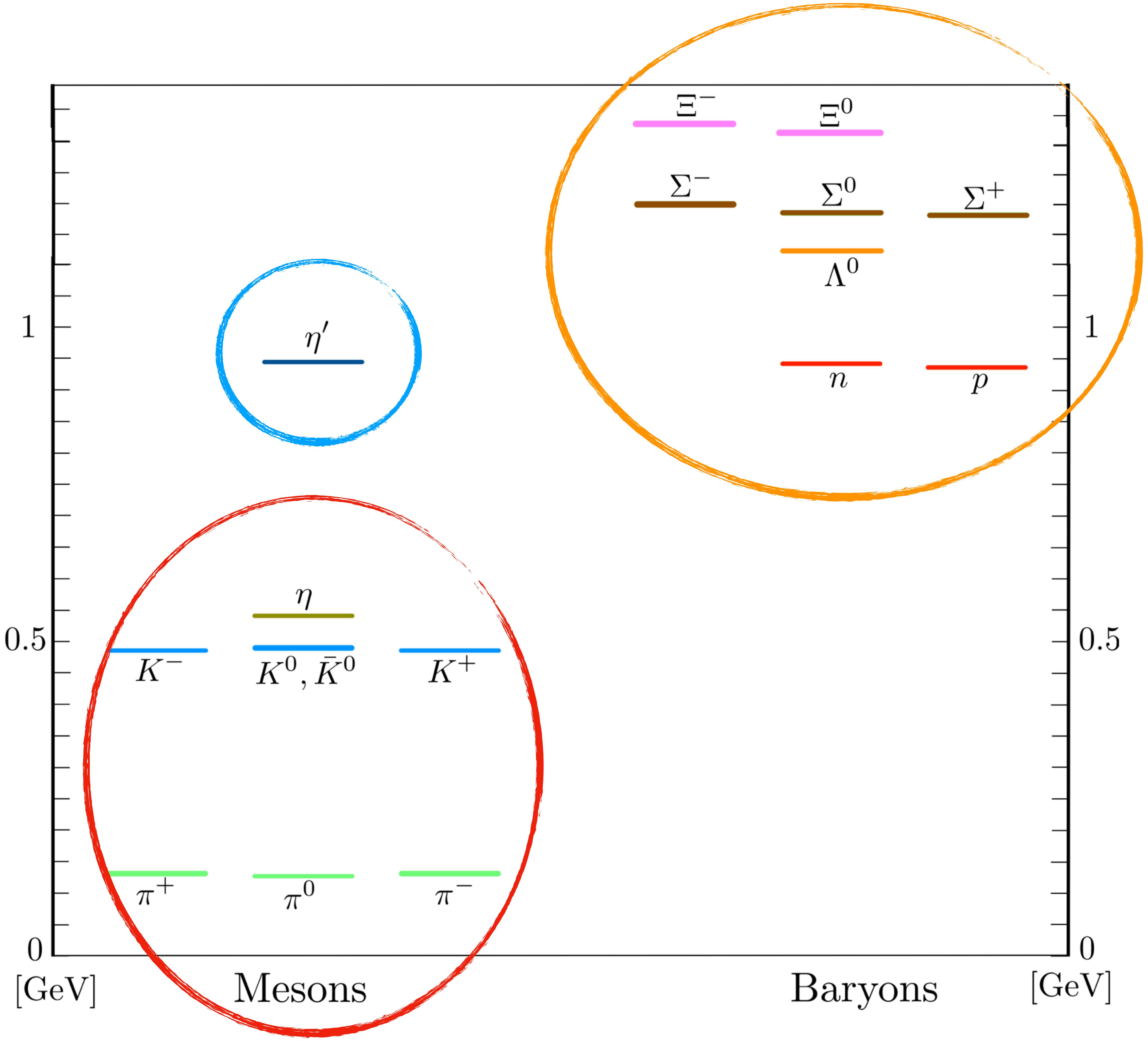


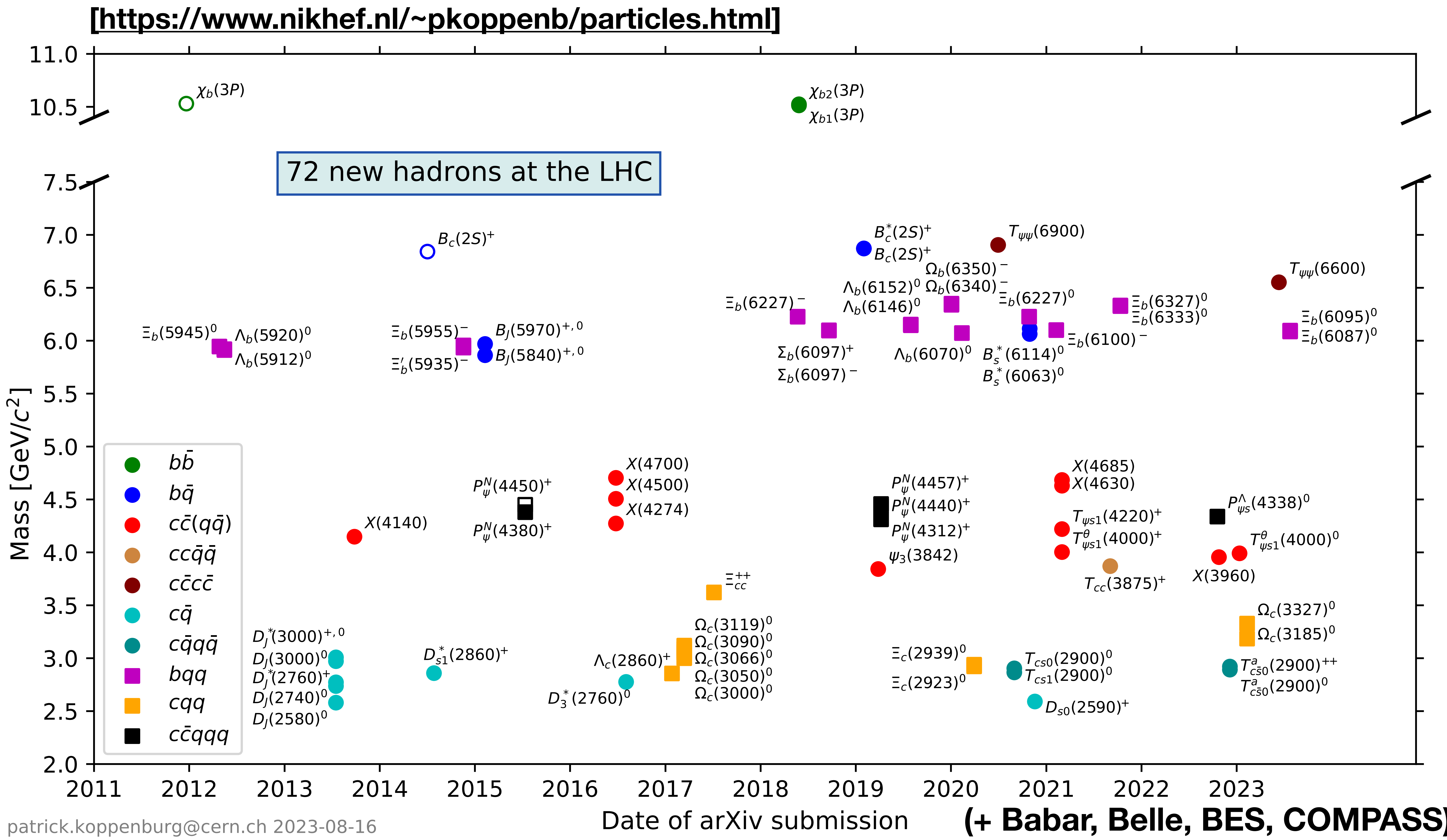
Baryon octet

Mass of the typical scale of QCD

$$M_B \sim \Lambda_{\mathrm{QCD}}$$

An experimental conundrum

A growing hadron spectrum still requires first principles understanding



(+ Babar, Belle, BES, COMPASS)

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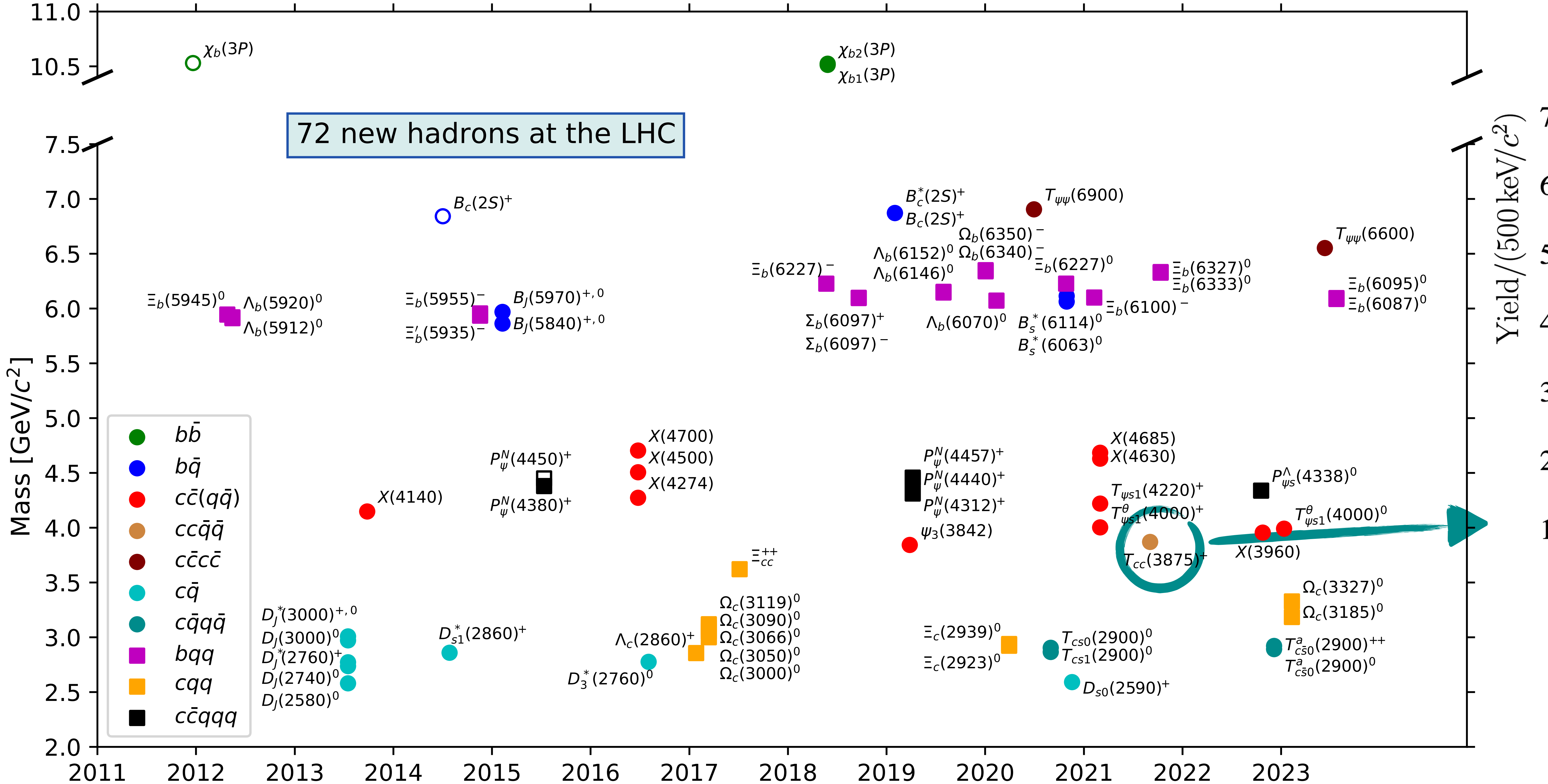
[https://www.nikhef.nl/~pkoppenb/particles.html]



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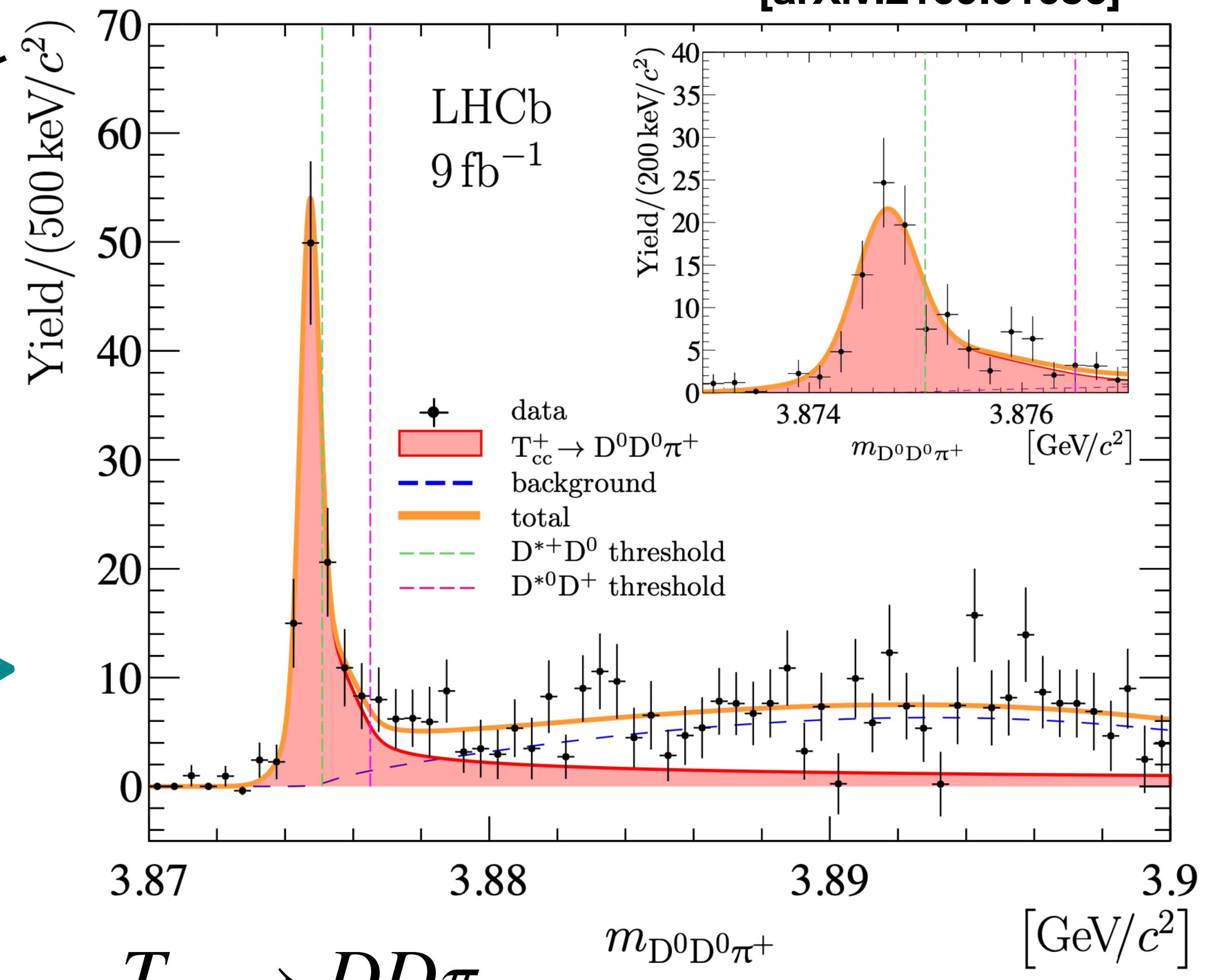
[arXiv:2109.01056]



$T_{cc} \to DD\pi$

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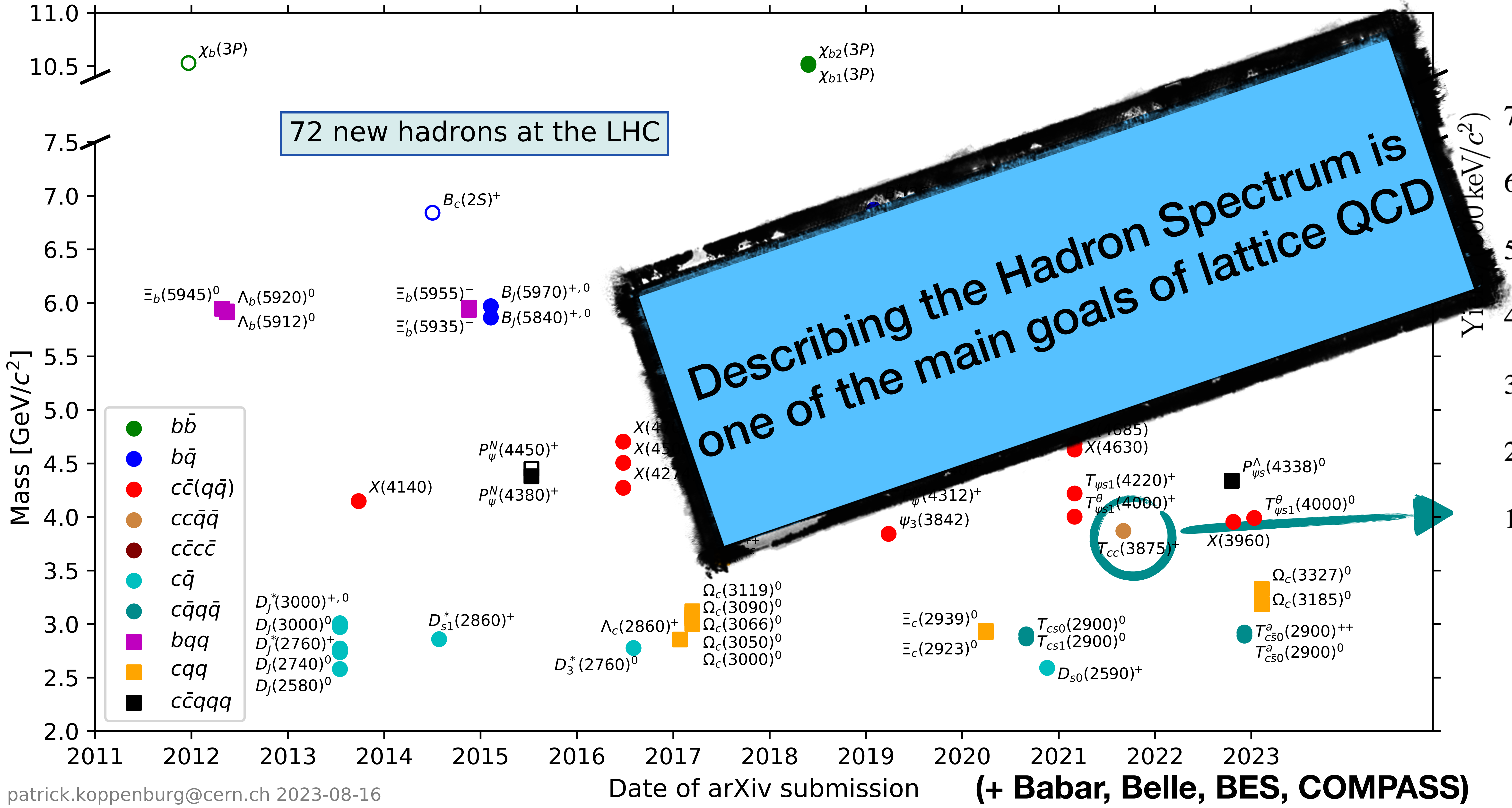

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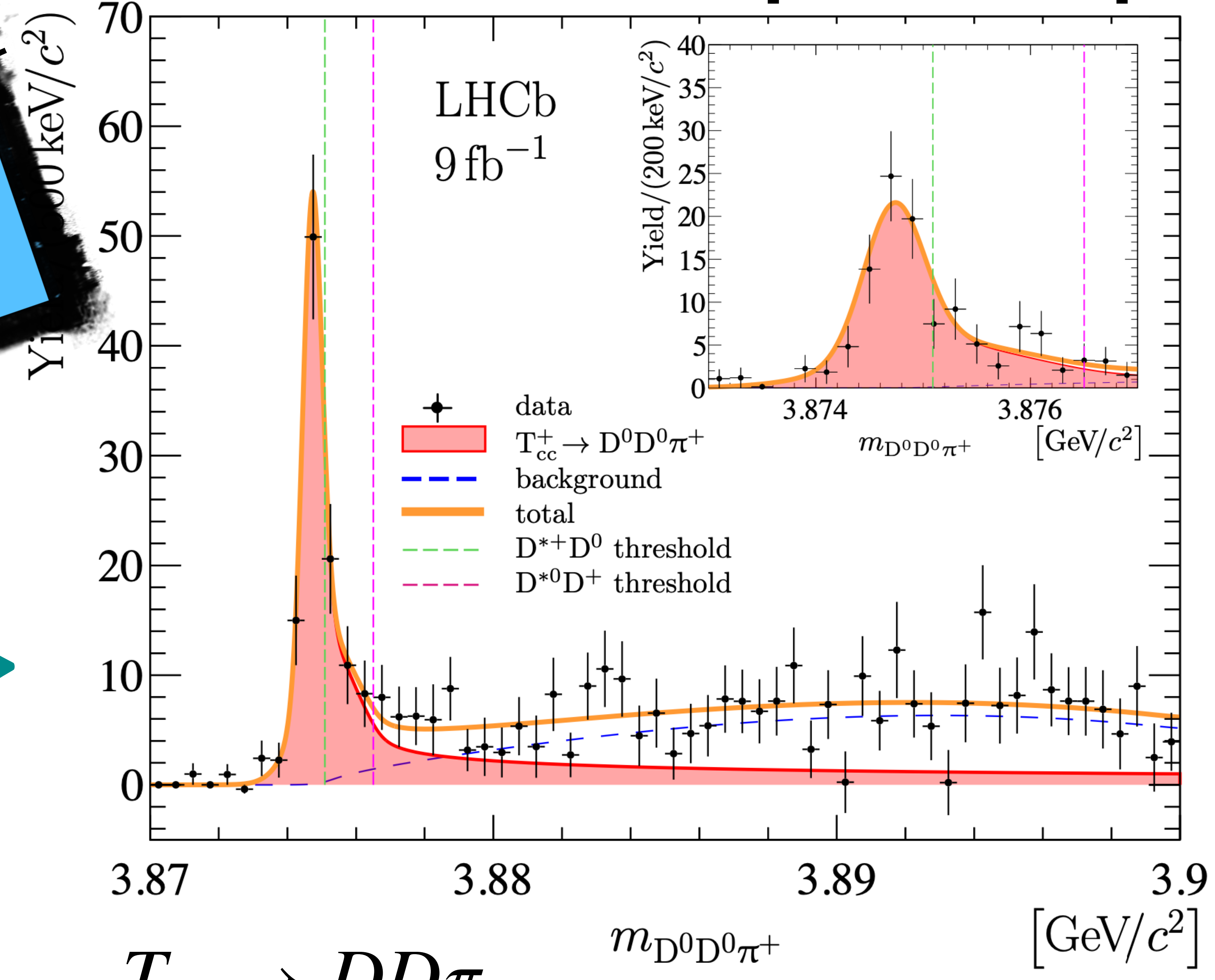
Describing the Hadron Spectrum is one of the main goals of lattice QCD

[arXiv:2109.01056]



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Lattice QCD spectroscopy

Energies from correlations

- We measure energy levels and matrix elements: "Spectral decomposition"

$$C(t) = \langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\rangle = \sum_n \langle 0|\mathcal{O}^\dagger(t)|n\rangle\langle n|\mathcal{O}(0)|0\rangle$$

$$= \sum_n \langle 0|e^{Ht}\mathcal{O}^\dagger(0)e^{-Ht}|n\rangle\langle n|\mathcal{O}(0)|0\rangle$$

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Ground state

$$\lim_{t\to\infty} C(t) = A_0 e^{-E_0 t}$$

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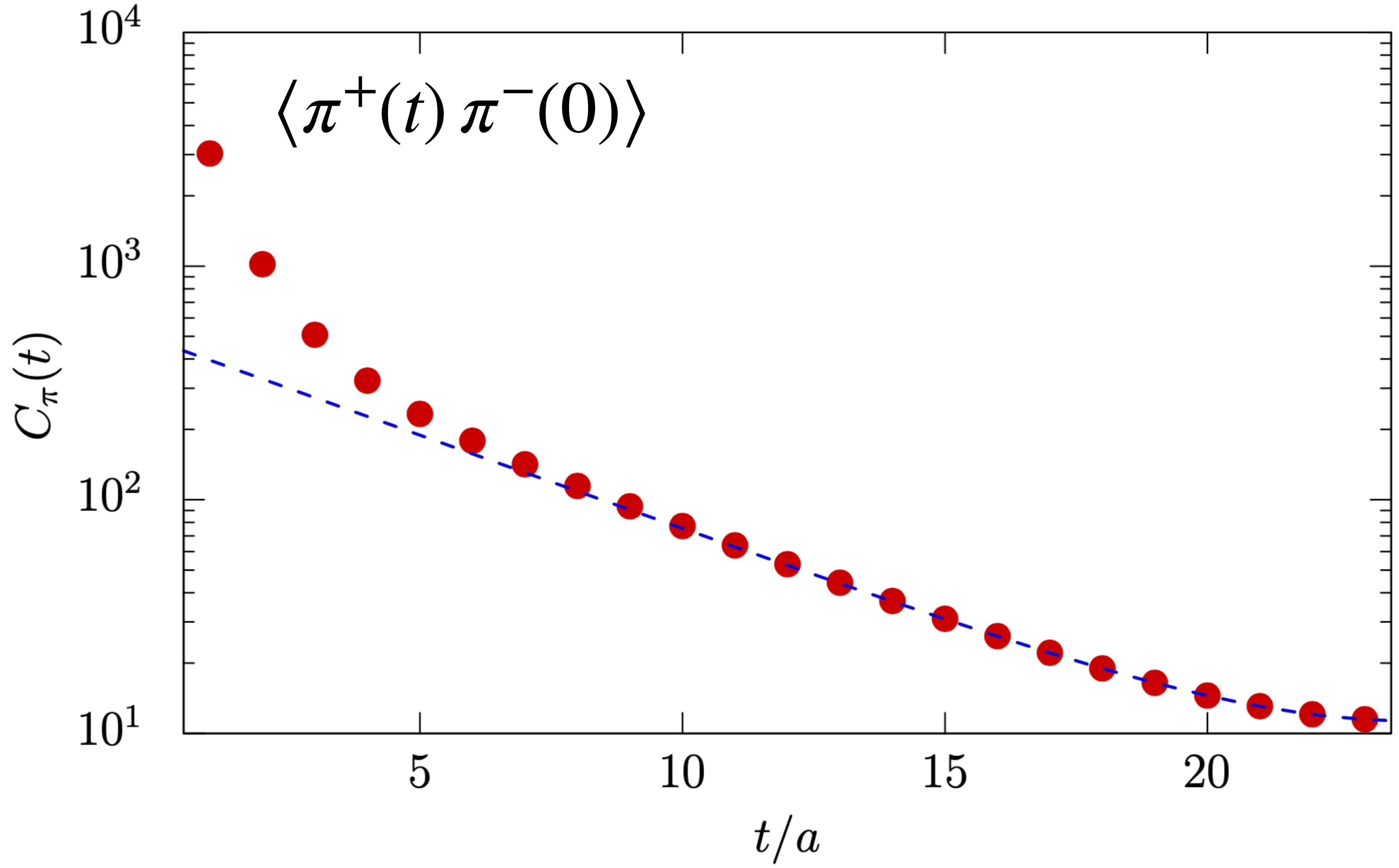
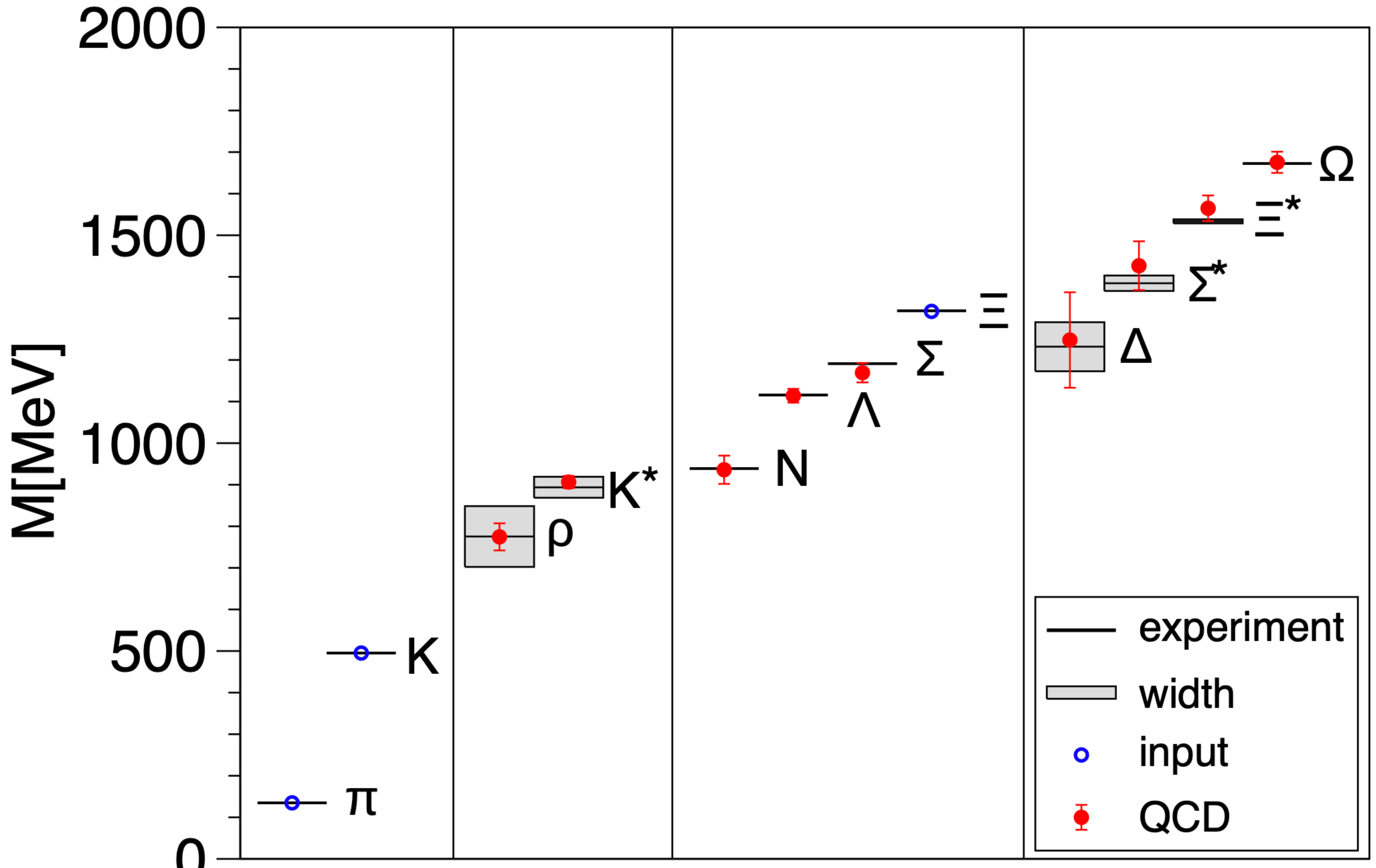


QCD spectrum from LQCD

- Consider isospin-symmetric QCD ($m_u = m_d$)
- Neglect QED effects in hadrons
- Need a 3 inputs to fix quark masses
  - Fix light and strange quark mass
  - Fix lattice spacing
- ☑ Reproduce the lowest-lying hadrons!



[https://www.science.org/doi/full/10.1126/science.1163233]

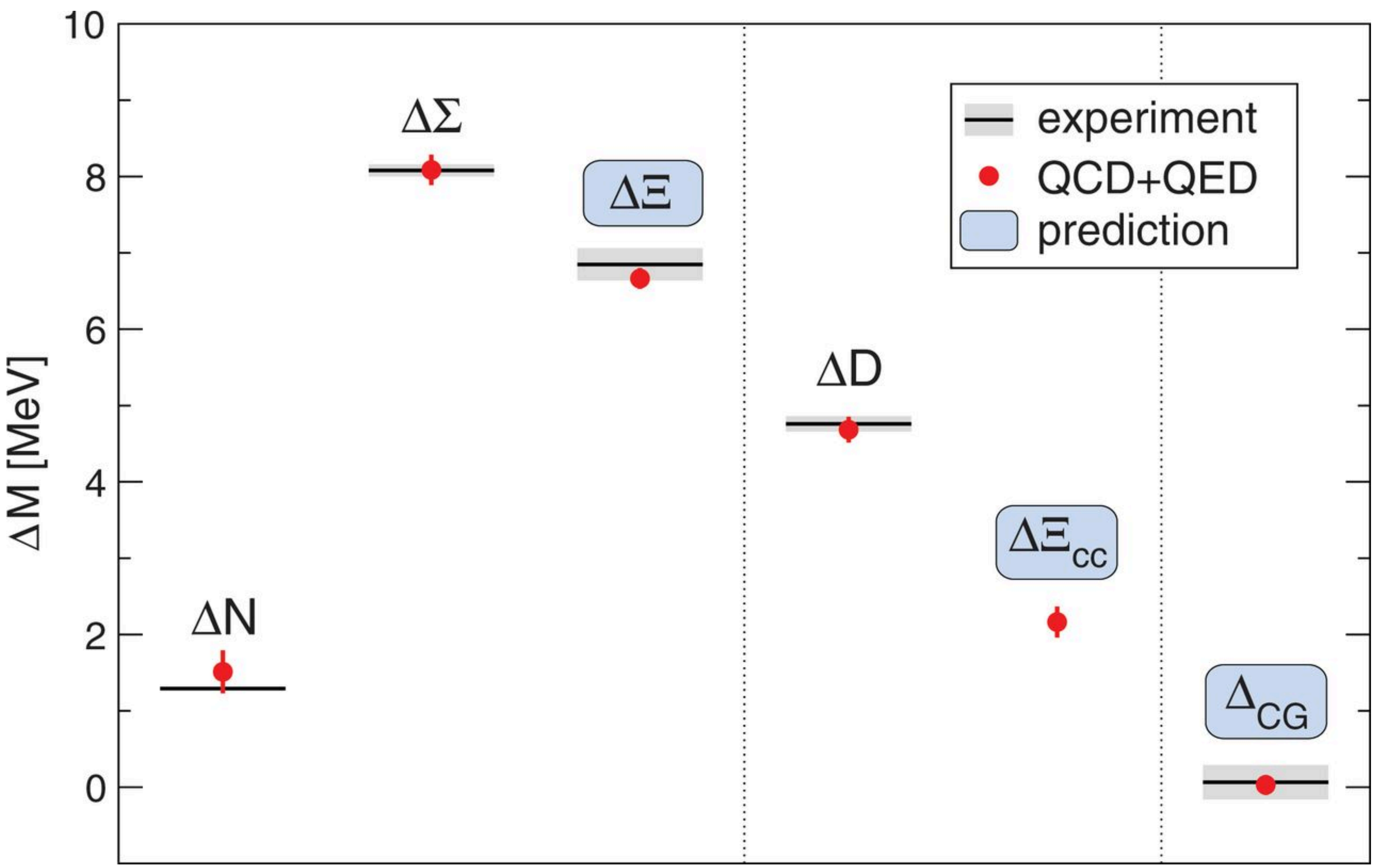
[BMW collaboration, 2008]

QCD spectrum from LQCD

- More recently: add QED and $m_u \neq m_d$

- ☑ Can reproduce neutron/proton mass difference

- ☑ More precise than some experimental results



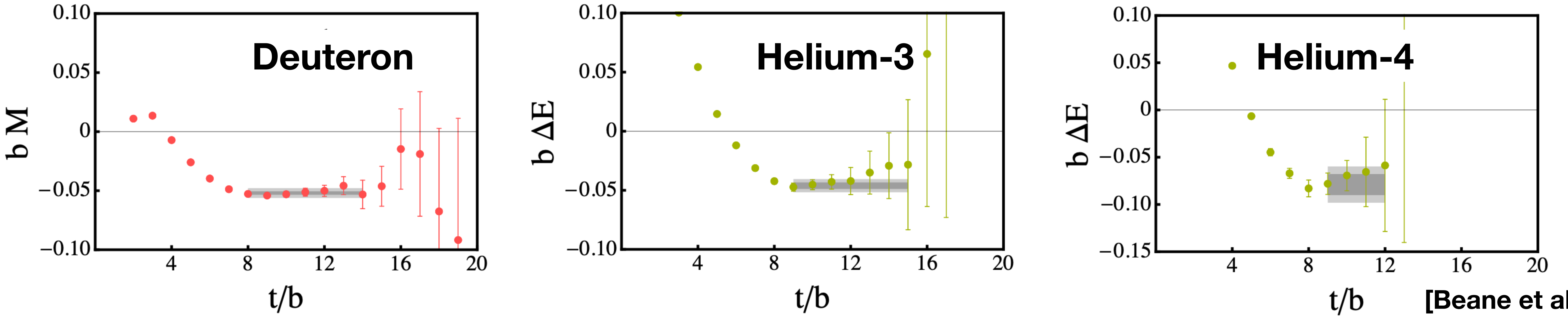
[BMW collaboration (2015)]

[https://www.science.org/doi/full/10.1126/science.1257050]

Towards Nuclear Physics

- Lattice QCD can in principle compute properties of nuclei. In practice, it is still very preliminary.

- Signal-to-noise problem grows rapidly with the baryon number: hard to control excited states.



[Beane et al (NPLQCD), 2012]

- Computational cost of Wick contractions grows (naively) factorially

  # contractions ~ (3A)!

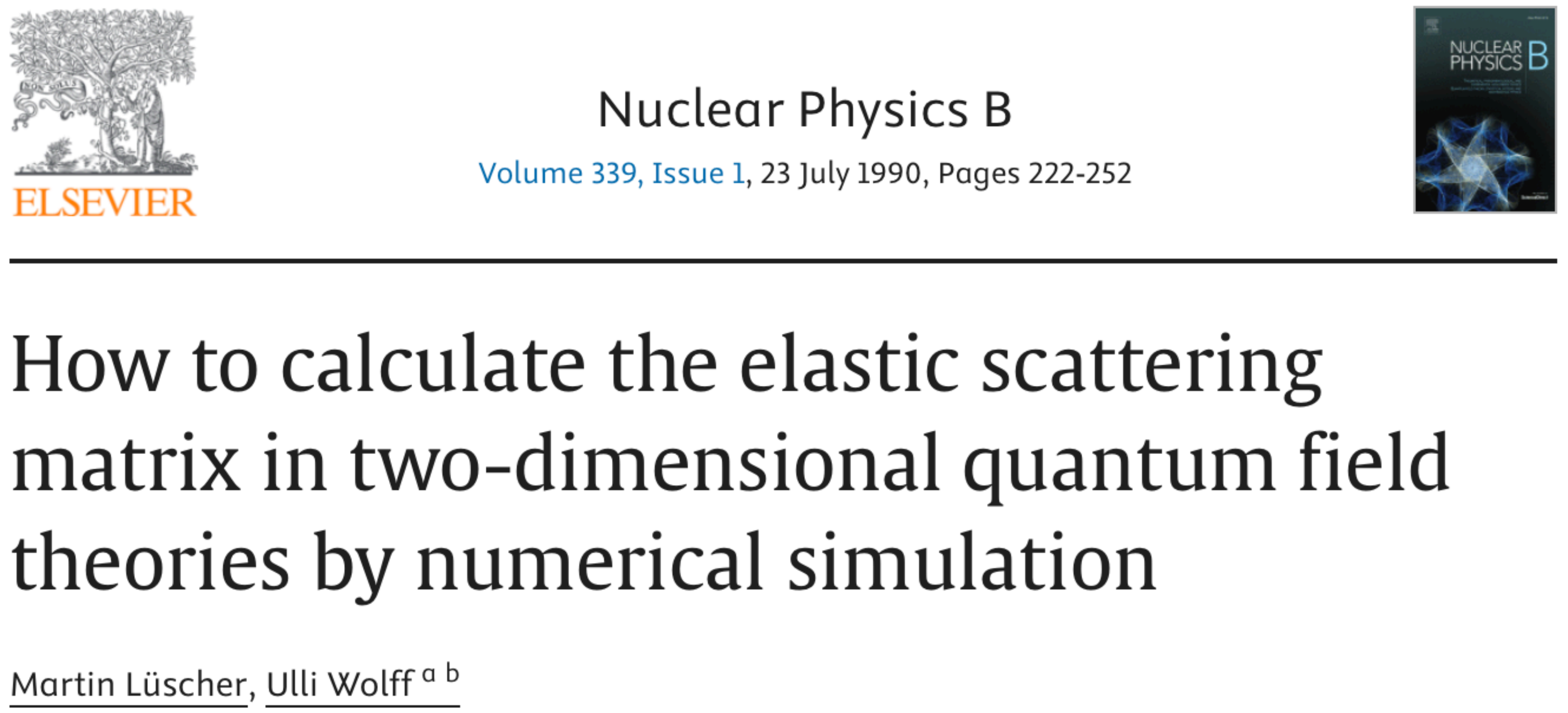
Beyond the ground state

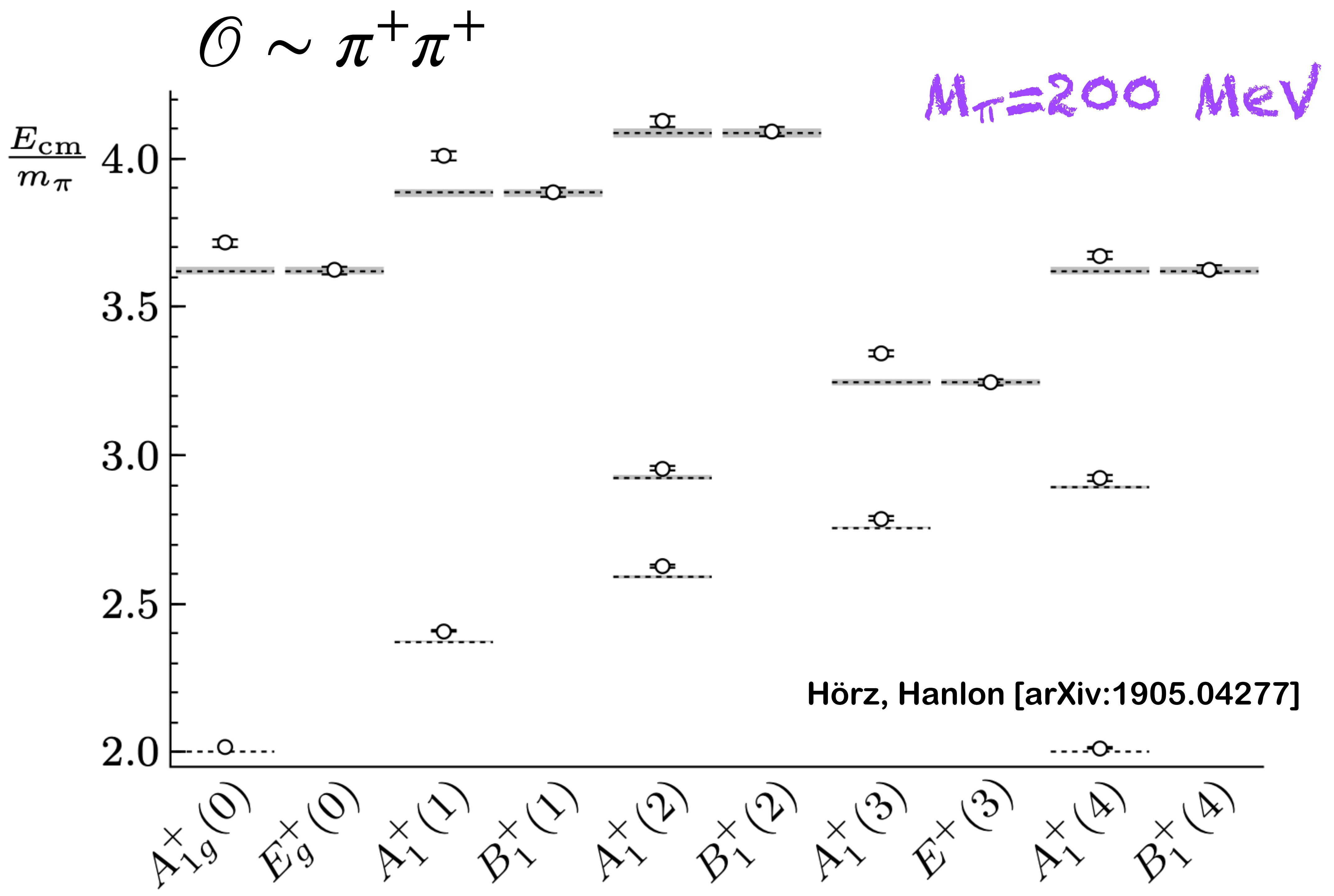
- Compute matrix of Euclidean correlation functions using operators with the same quantum numbers

$$C_{ij}(t) = \langle \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)\rangle$$

$$C_{ij}(t) = \langle 0|\mathcal{O}_i|n\rangle\langle n|\mathcal{O}_j|0\rangle^* e^{-E_n t}$$

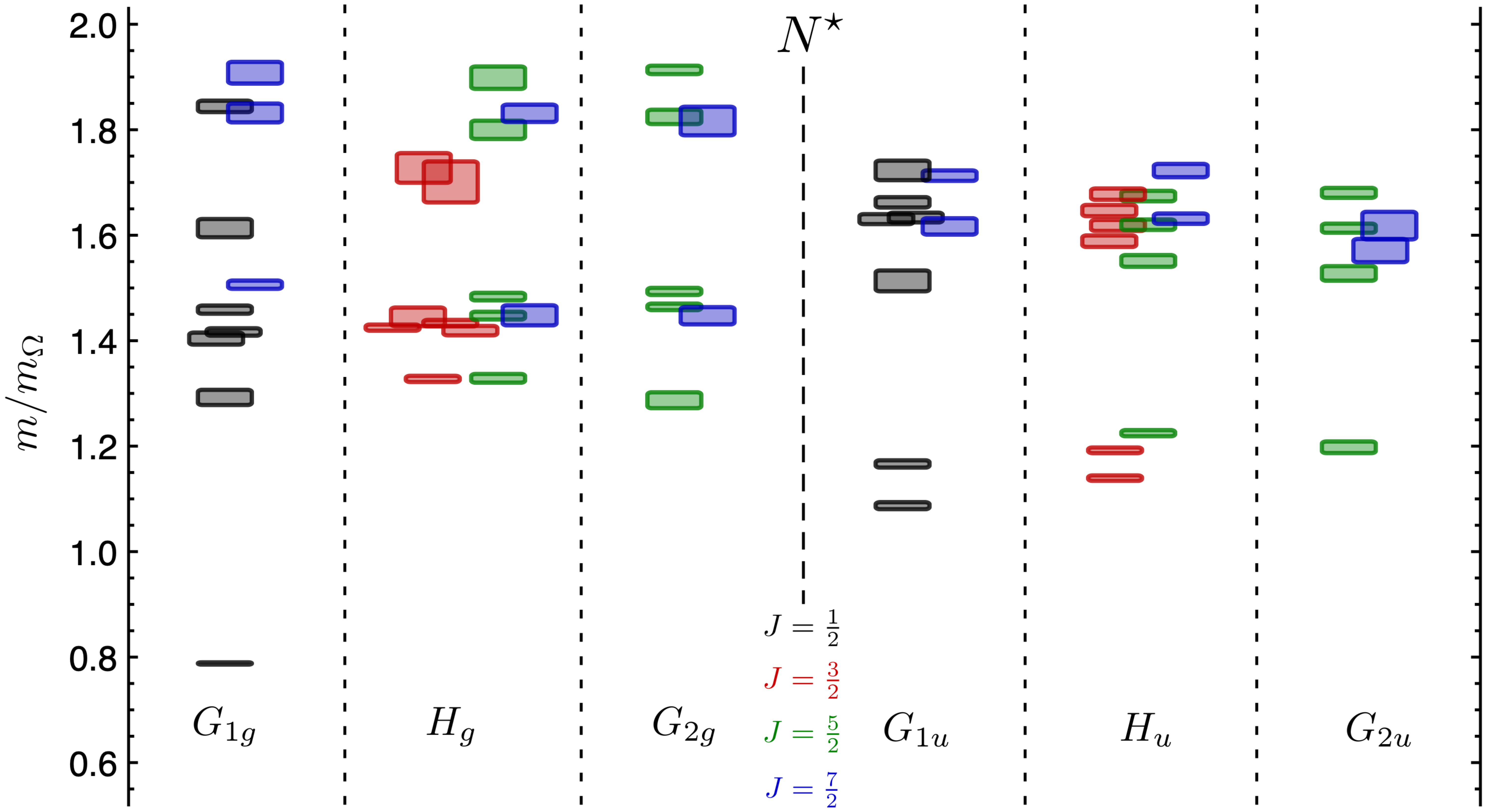
- **Variational techniques**
  (Generalized EigenValue Problem, GEVP)





Irreps = "finite-volume quantum numbers"

Hadron Spectrum



[HadSpec Collaboration, arXiv:1104.5152]

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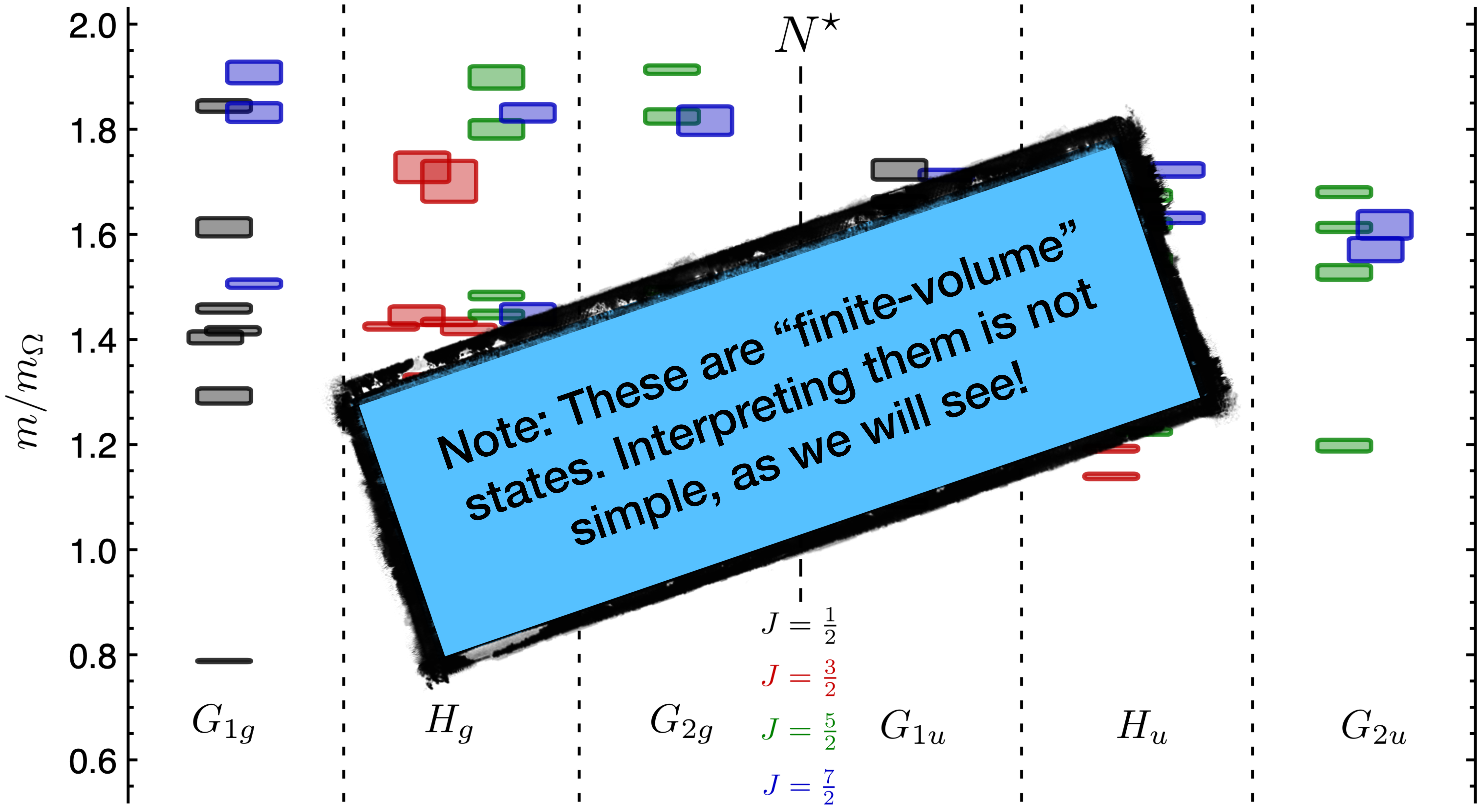


[HadSpec Collaboration, arXiv:1104.5152]

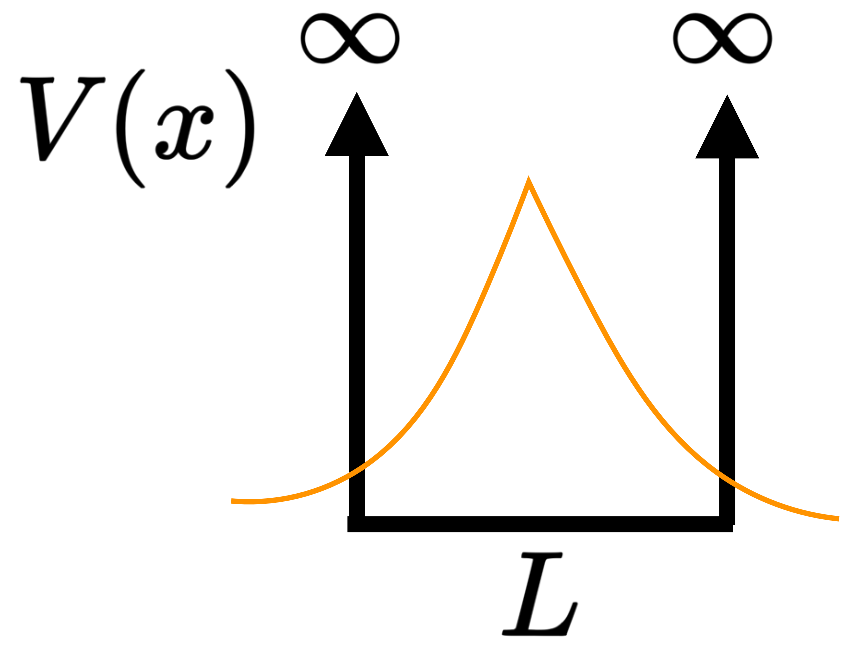
Finite-volume effects: stable particles

Stable particle in a box

- We want to compute the mass of a particle in lattice QCD.
  - Lattice QCD calculations are performed in a finite volume.
  - What are the finite-volume effects?

- Qualitative picture in Quantum Mechanics:
  - Finite volume is like a infinite potential well

$$\psi(x) = \frac{1}{\sqrt{\mathcal{N}}}\exp(-m\,|x|) \qquad m = \langle\psi|H|\psi\rangle$$



$$m(L) = \langle\psi_L|H|\psi_L\rangle \sim m \times \frac{1}{\mathcal{N}}\int_{-L/2}^{L/2} dx\,\exp\left(-2m\,x^2\right) = m + O(\exp(-mL))$$

Stable massive particle only "sees" boundary exponentially

Finite volume QFT

- In a QFT is formulated in finite-volume, loop integrals become sums:

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{dk^0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}\in(2\pi/L)\vec{n}}$$

- Mass of a particle given by the self-energy:

$$m^2 = m_0^2 + \Sigma(m^2) \xrightarrow{\text{Finite Volume}} m^2(L) = m_0^2 + \Sigma_L(m^2(L))$$

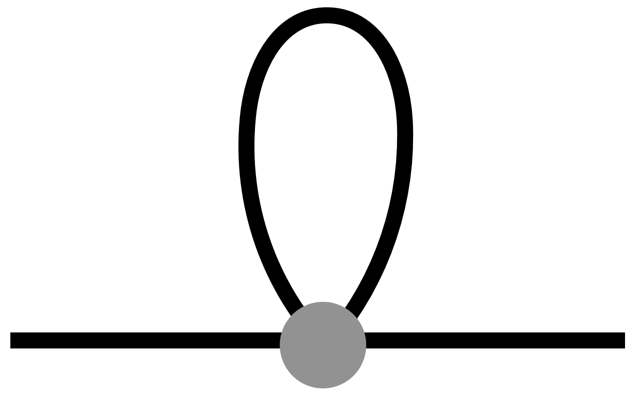
- Mass of a particle given by the self-energy:

$$m(L) - m = \frac{1}{2m}\left[\Sigma_L(m^2) - \Sigma(m^2)\right]$$

Example: ϕ^4 theory

- **Self-energy given by tad-pole diagram:**

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4$$



- **Finite-volume effects given as:**

$$\Sigma_L - \Sigma = \int \frac{2k^0}{2\pi}\left(\frac{1}{L^3}\sum_{\vec{n}} - \int \frac{d^3k}{(2\pi)^3}\right)\frac{-\lambda}{k^2-m^2} \simeq \text{const} \times m\frac{e^{-mL}}{L}$$

<u>**EXERCISE**</u> Demonstrate that the finite-volume corrections are: $\Delta m(L) \propto \frac{K_1(mL)}{mL} \approx \frac{e^{-mL}}{mL}$

1. Perform the temporal integral in the self energy
2. Use the first term in the Poisson summation formula: $\left(\frac{1}{L^3}\sum_{\vec{n}} - \int\frac{d^3p}{(2\pi)^3}\right)f(p^2) = \sum_{\vec{n}\neq\vec{0}}\int\frac{d^3p}{(2\pi)^3}f(p^2)e^{i(n\cdot\vec{p})L}$

Volume dependence

Non-perturbative volume dependence of stable particles

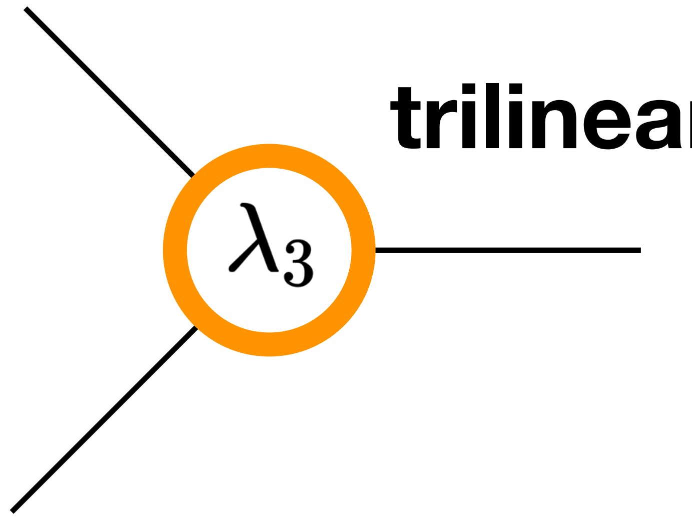
Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories

I. Stable Particle States

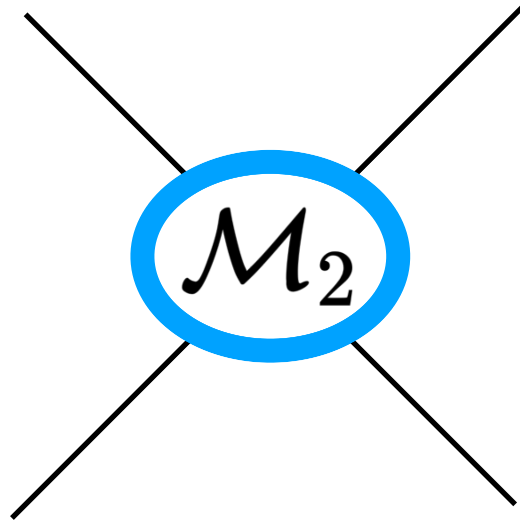
M. Lüscher

Theory Division, Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg 52, Federal Republic of Germany

$$\Delta m = -\frac{3}{16\pi m^2 L}\left\{\lambda_3^2 e^{-\frac{\sqrt{3}}{2}mL} + \frac{m}{\pi}\int_{-\infty}^{\infty} dy e^{-\sqrt{m^2+y^2}L} F(iy) + O\left(e^{-\overline{m}L}\right)\right\}$$



trilinear coupling

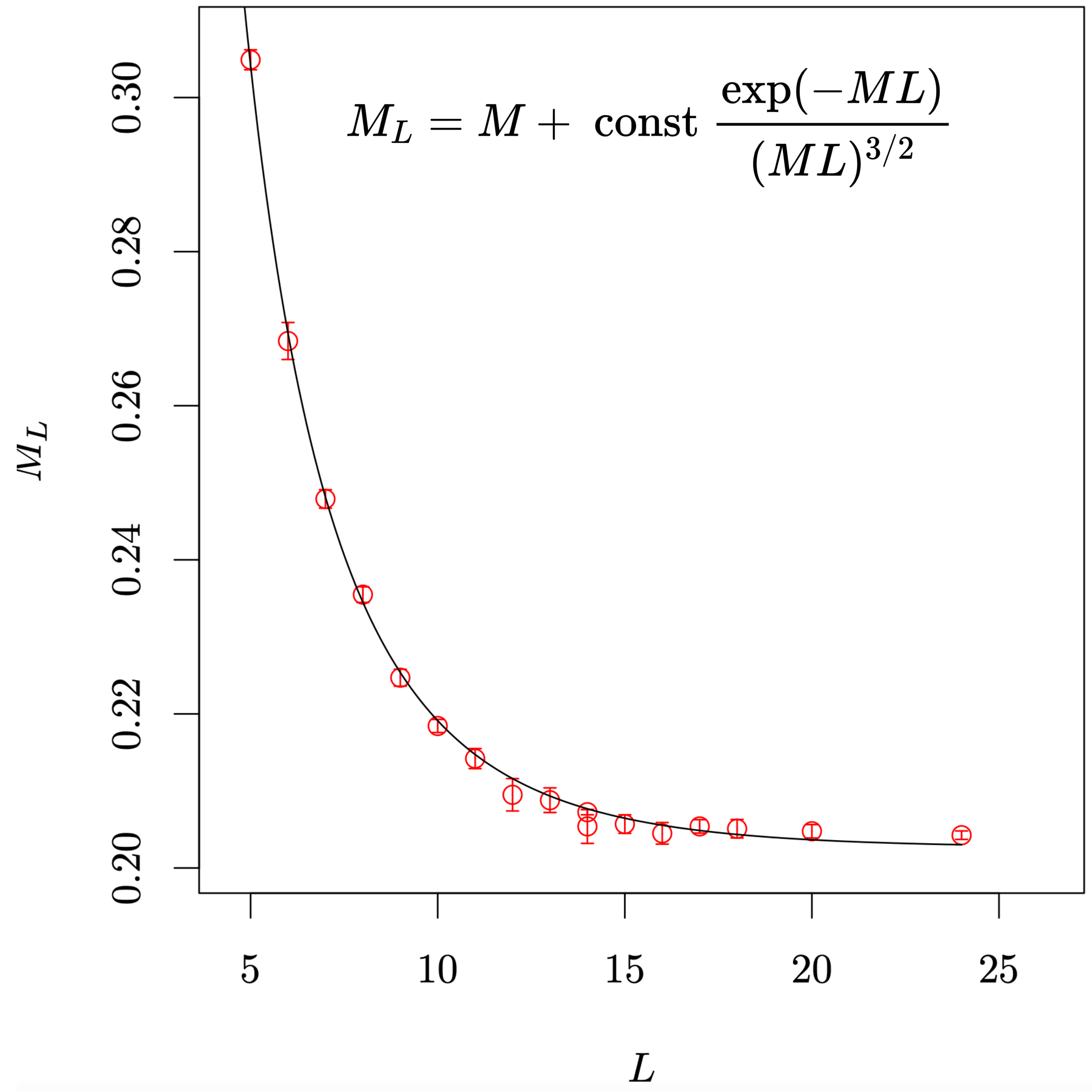


forward scattering amplitude

$F(s) = \mathcal{M}_2(s, t=0)$

Summary (1)

- Spectroscopy of stable particles leads only to exponentially suppressed effects
- Form of the FV effects is calculable in QFT
- Motivates choice: $M_\pi L > 4, \;\; \exp(-4) \sim 2\%$ error
- Example: Volume dependence in phi^4 theory
  [FRL, Rusetsky, Urbach, 1806.02367]
- Related ideas on the volume dependence of stable particles:
  - Volume dependence in Chiral Perturbation Theory
    [Colangelo et al, 0311023 & 0503014]
  - Volume dependence of bound states.
    [Hansen, Sharpe arXiv:1609.04317]
  - Finite-volume effects in g-2
    [Hansen, Patella arXiv:2004.03935]



Scattering processes and resonances

Scattering basics

- The "Scattering Matrix" is a unitary operator that connects asymptotic states

$$S_{ab}(E) \equiv \langle \text{ out } |\widehat{S}| \text{ in } \rangle$$

- It is related to the scattering amplitude as:

$$\langle \text{ out } |(\widehat{S} - 1)| \text{ in } \rangle = (2\pi)^4 \delta^{(4)}(P_{\text{in}} - P_{\text{out}}) i\mathcal{M}(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{p}_1, \boldsymbol{p}_2)$$

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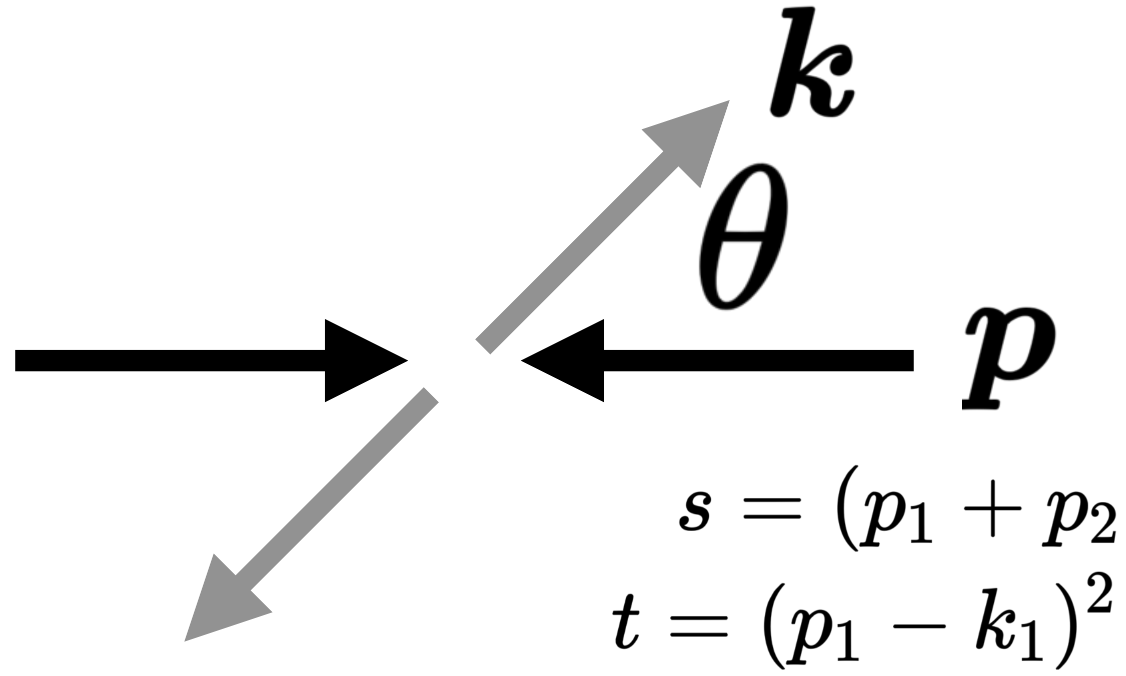
- Unitarity imposes important constraints: $\widehat{S}\widehat{S}^\dagger = 1$

$$\mathcal{M}_2(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{p}_1, \boldsymbol{p}_2) - \mathcal{M}_2^*(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{k}_1, \boldsymbol{k}_2) =$$
$$\frac{i}{2} \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6 4\omega(q_1)\omega(q_2)} \mathcal{M}_2(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{q}_1, \boldsymbol{q}_2) \mathcal{M}_2^*(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{q}_1, \boldsymbol{q}_2) \times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q_1 - q_2)$$

Two-particle scattering

- Consider two-hadron scattering in their CM frame
  - Amplitude depends on two kinematic variables:

$$\mathcal{M}(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{p}_1, \boldsymbol{p}_2) \equiv \mathcal{M}(s, \cos\theta)$$



$$s = (p_1 + p_2) = 4(m^2 + \boldsymbol{p}^2)$$
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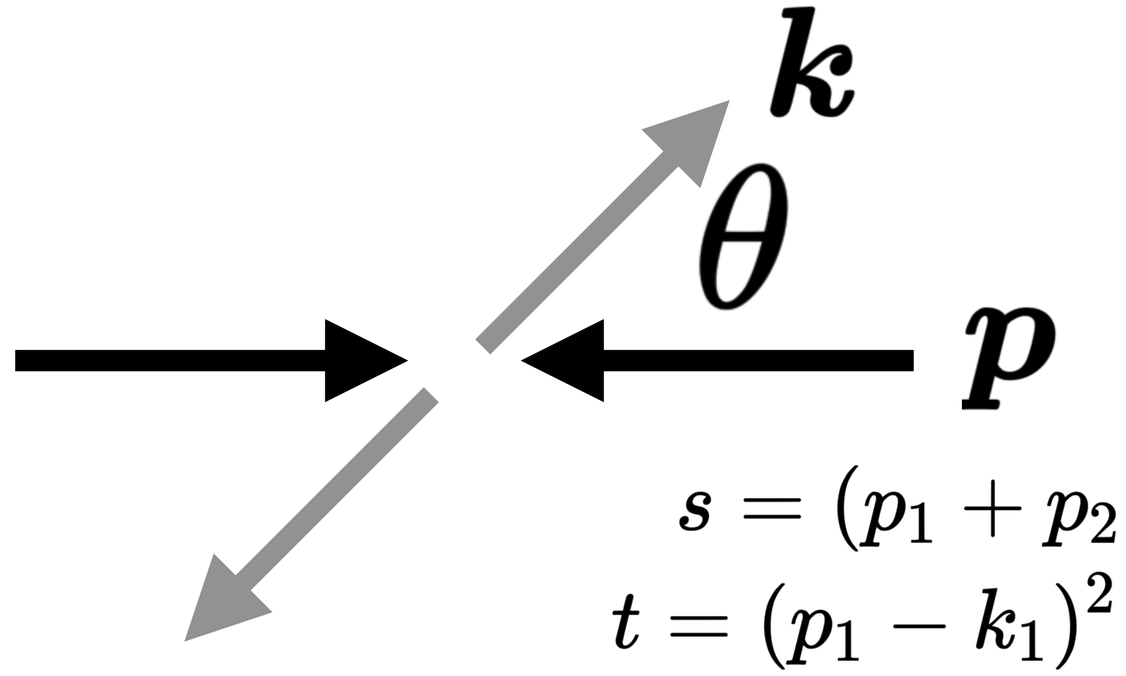

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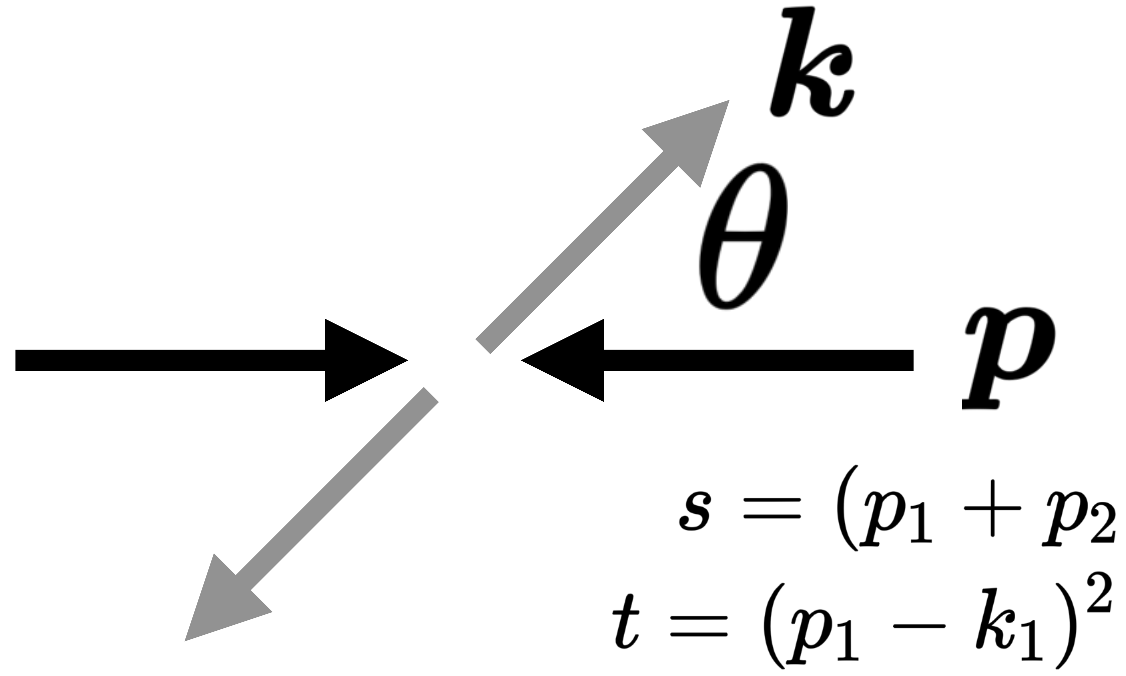

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- Unitarity implies that: $$\mathcal{M}_\ell = \frac{16\pi\sqrt{s}}{k\cot\delta_\ell - ik}$$

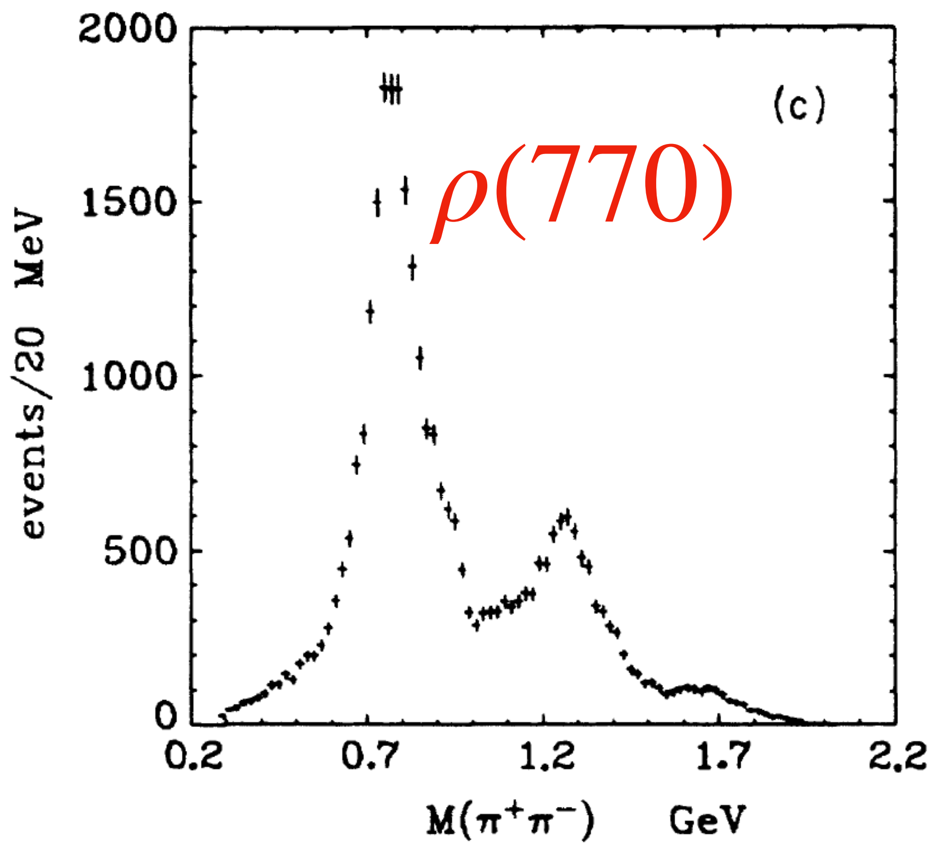
- Where $k^{2\ell+1}\cot\delta_\ell$ is a meromorphic function, e.g. simple polynomial or rational function
  (holomorphic up to isolated points)

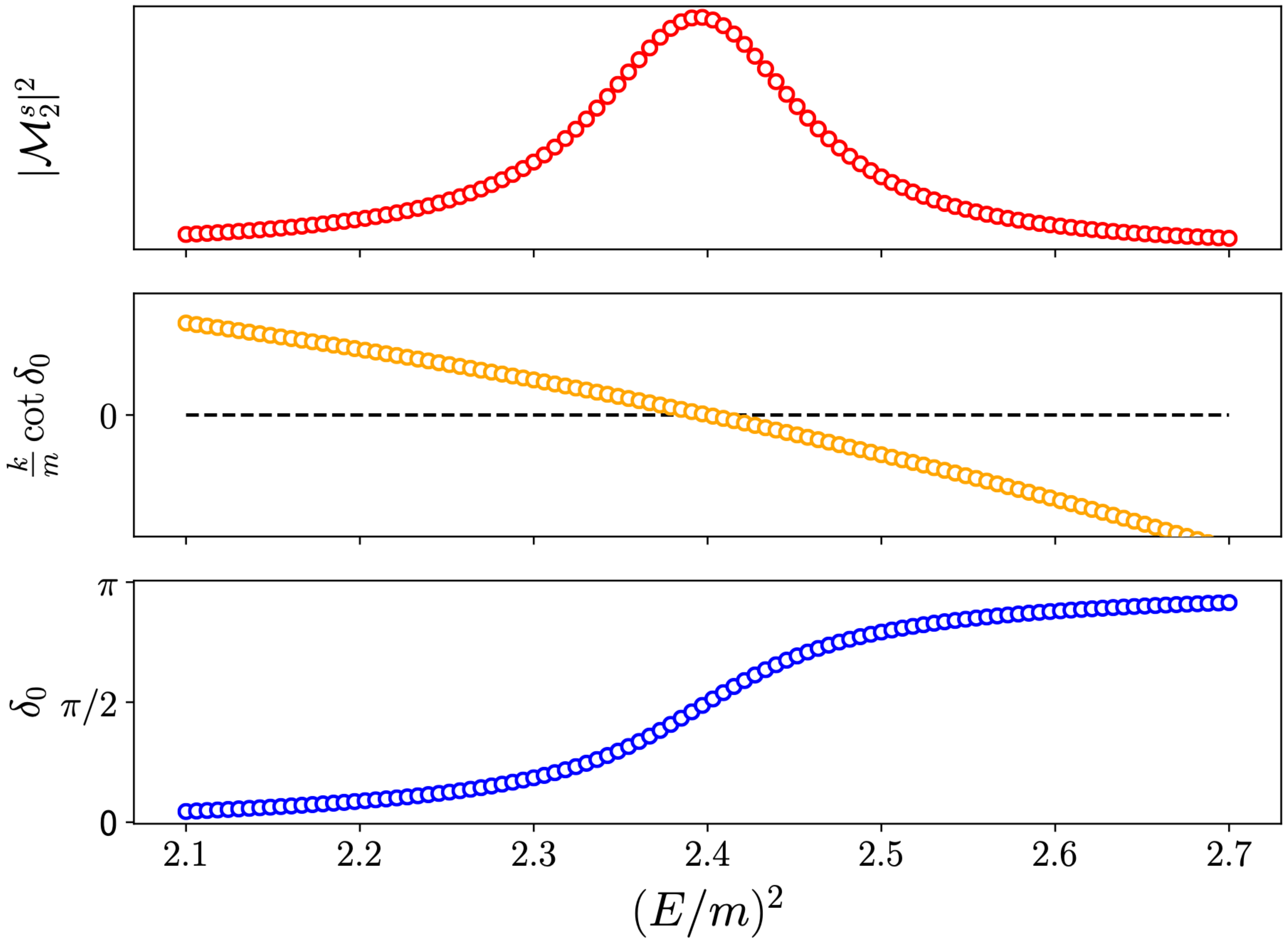
Scattering & Resonances

- Resonances typically show up as enhancements in the cross-section

$$\sigma_\ell \propto |\mathcal{M}_\ell|^2 = (16\pi\sqrt{s})^2 \frac{\sin^2\delta_\ell}{k}$$

  - Maximum when $\delta_\ell = 90^\circ$
  - Unitarity bound is $\sin\delta_\ell = 1$



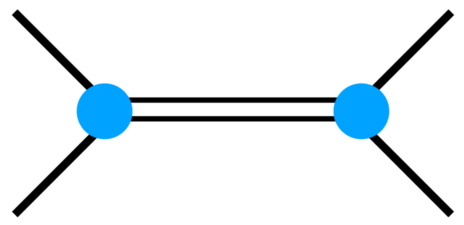


Poles in the complex plane

- The rigorous definition of a hadronic resonance is a pole in the complex plane

pole residue:
a.k.a coupling

$$\mathcal{M}_\ell \sim -\frac{g^2}{s - s_R}$$



$$\sqrt{s_R} = M_R - i\frac{\Gamma}{2}$$

width of
the resonance

mass of
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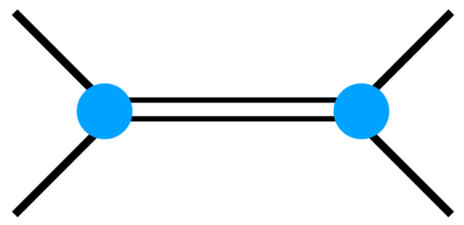

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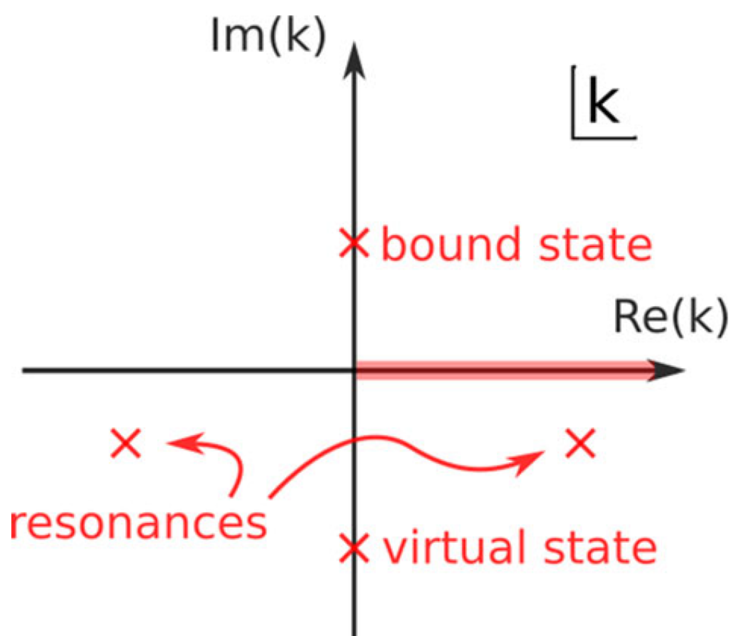
- Based on the location of the poles, they receive different names
  - Bound states: stable particles, e.g. the deuteron is an NN bound state
  - Resonances: unstable hadrons, e.g. the rho resonance
  - Virtual states: “non-renormalizable QM states”, e.g. “dineutron”



[Matuschek et al. EPJA 2021]

Fig. 1 Naming convention for the poles in the *k*-plane. The thick red line for positive real valued *k* marks the physical momenta in the scattering regime

Pole calculation

EXERCISE Compute the pole positions in the complex plane for the following cases

1. $k \cot \delta_0 = -\dfrac{1}{a_0}$
2. $k \cot \delta_0 = A(k_R^2 - k^2), \quad A \gg 1/k_R$

What kind of poles do these parametrizations lead to?

What is the residue of the pole? What sign does the residue have?

Summary (2)

The S-Matrix contains the physical information of the theory:

$$S_{ab}(E) \equiv \langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle$$

Lattice QCD $\longrightarrow$ QCD S-matrix

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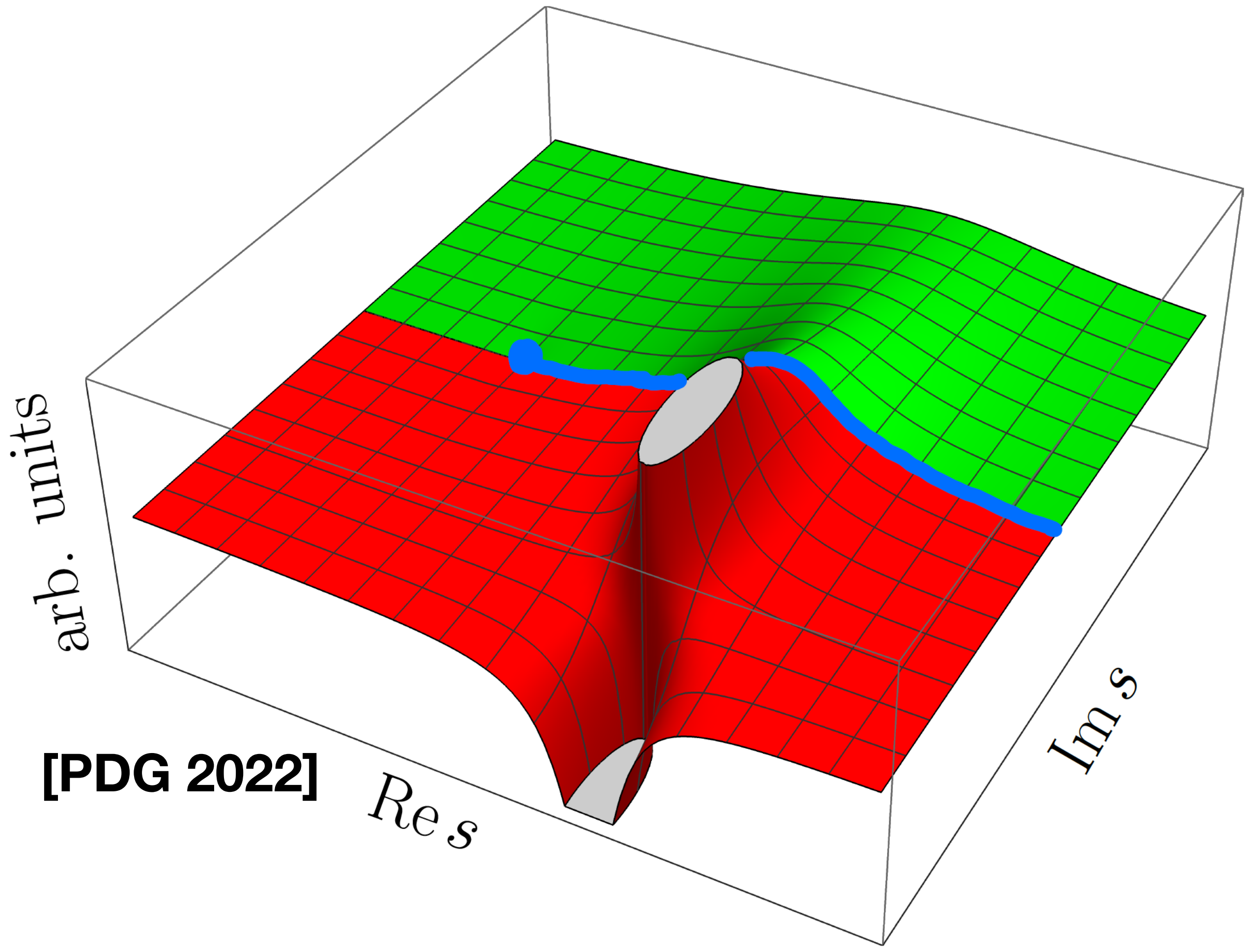

[PDG 2022]

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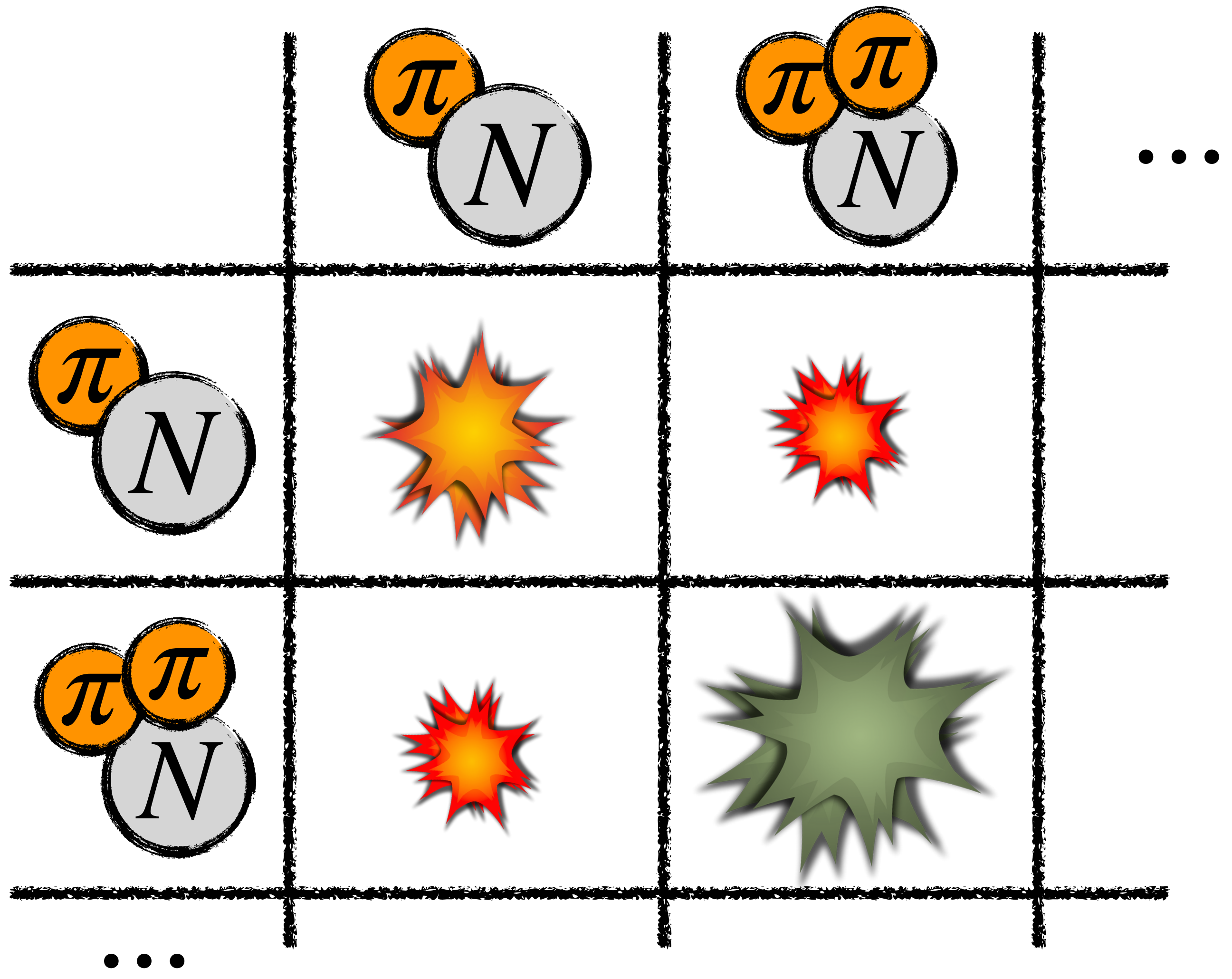

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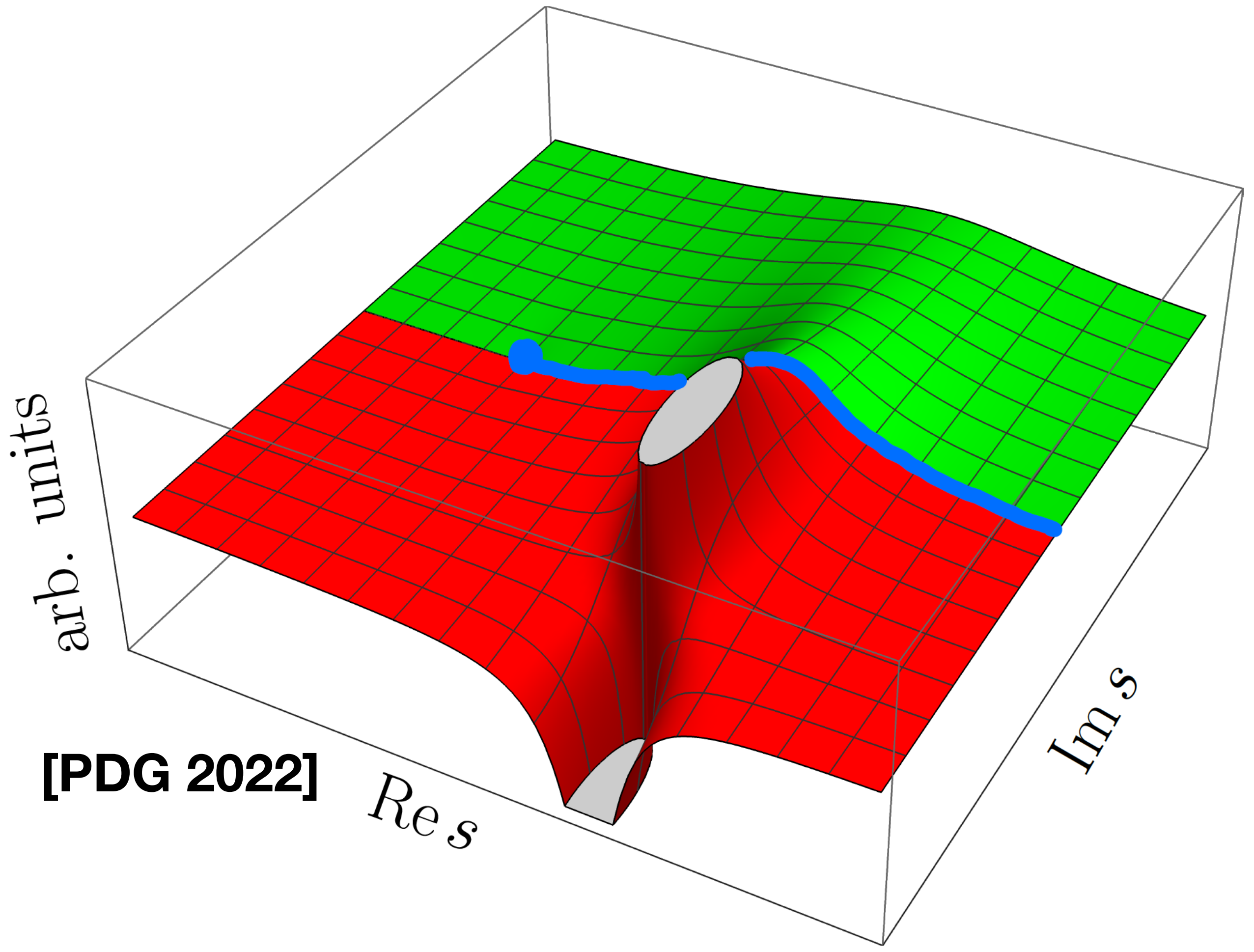

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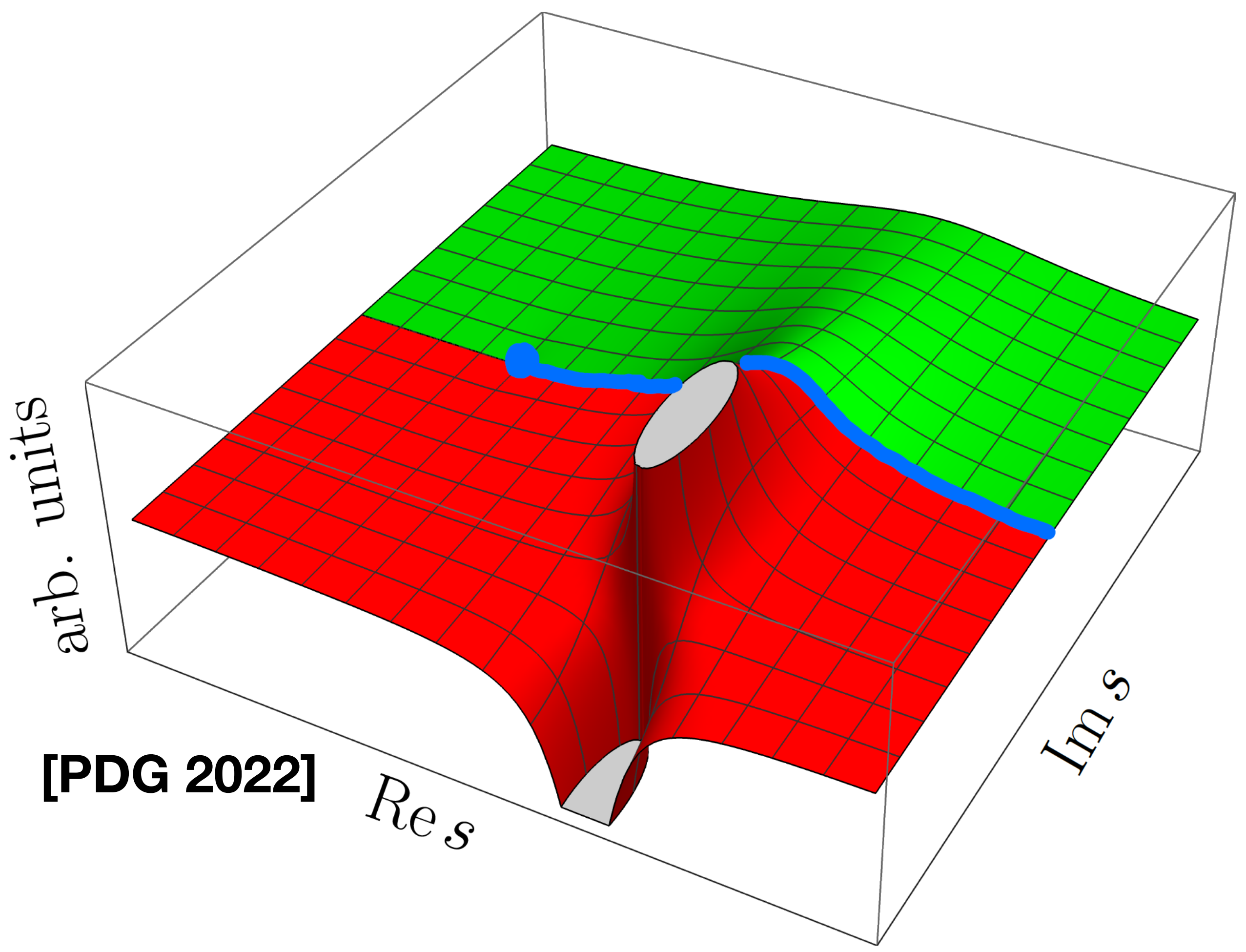

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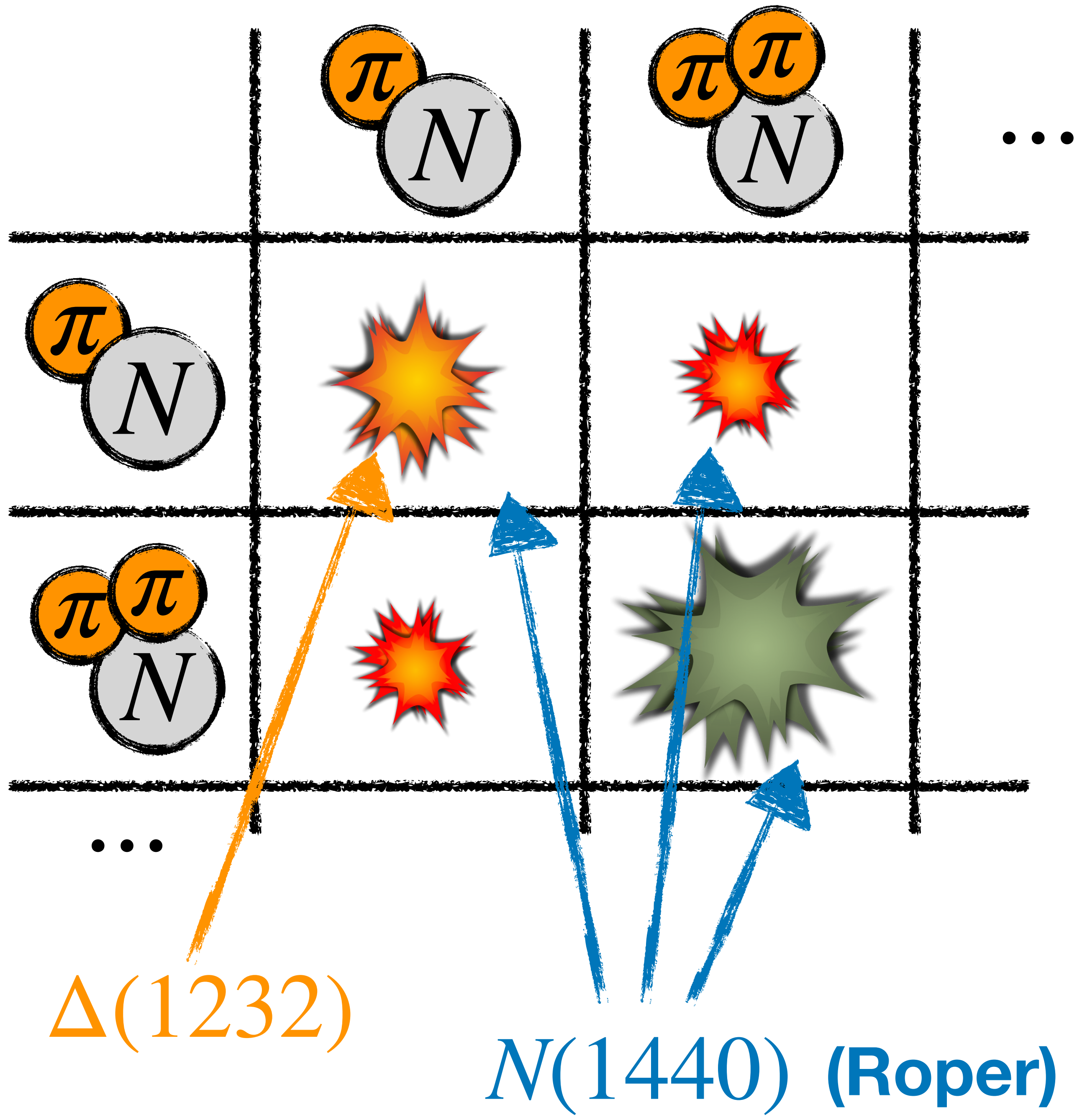
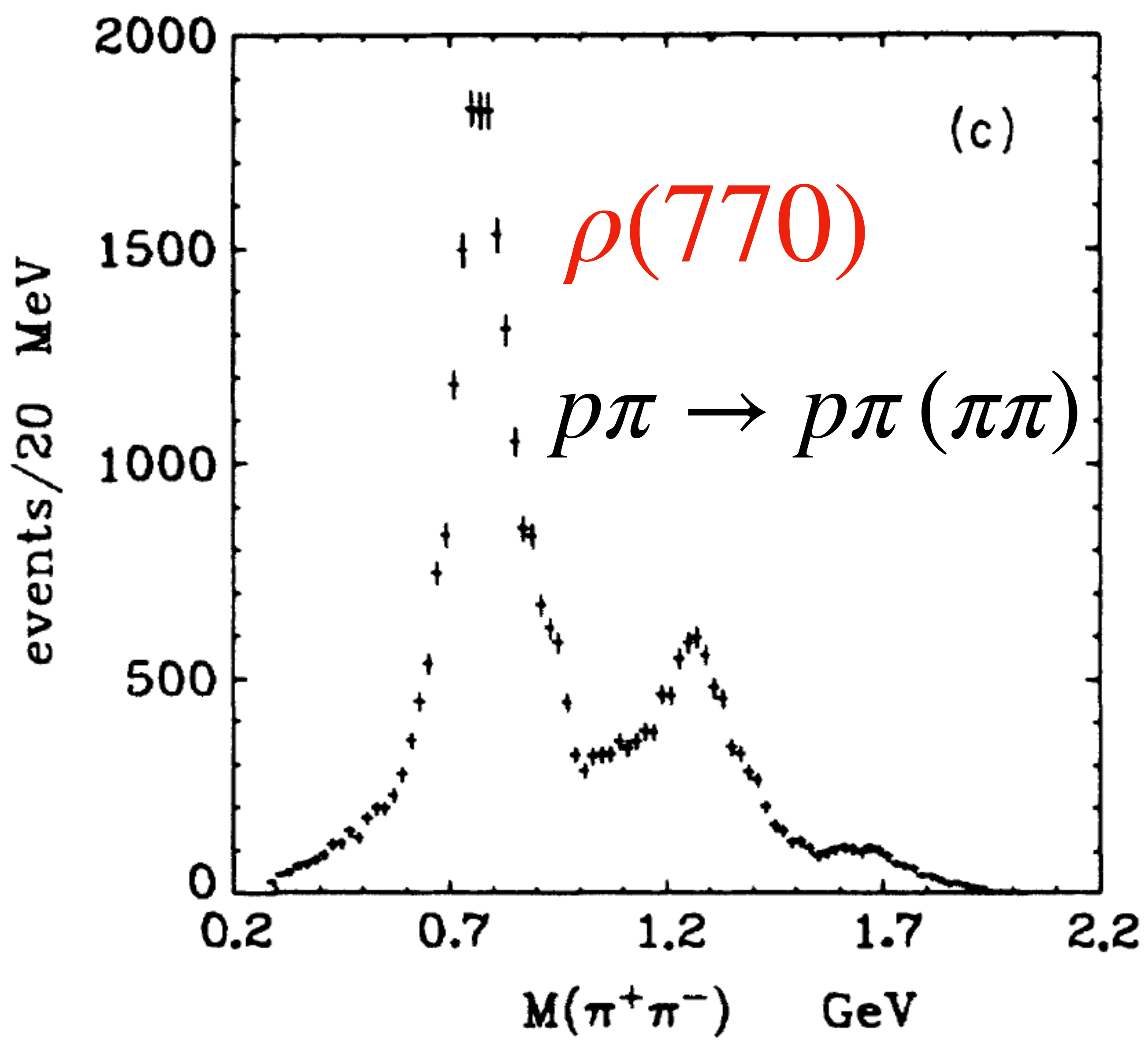


Finite-volume effects: scattering & resonances

Scattering amplitudes in LQCD

Experiments

- Asymptotic states
- Direct access to cross sections



[Protopopescu et al, PRD7 1973]

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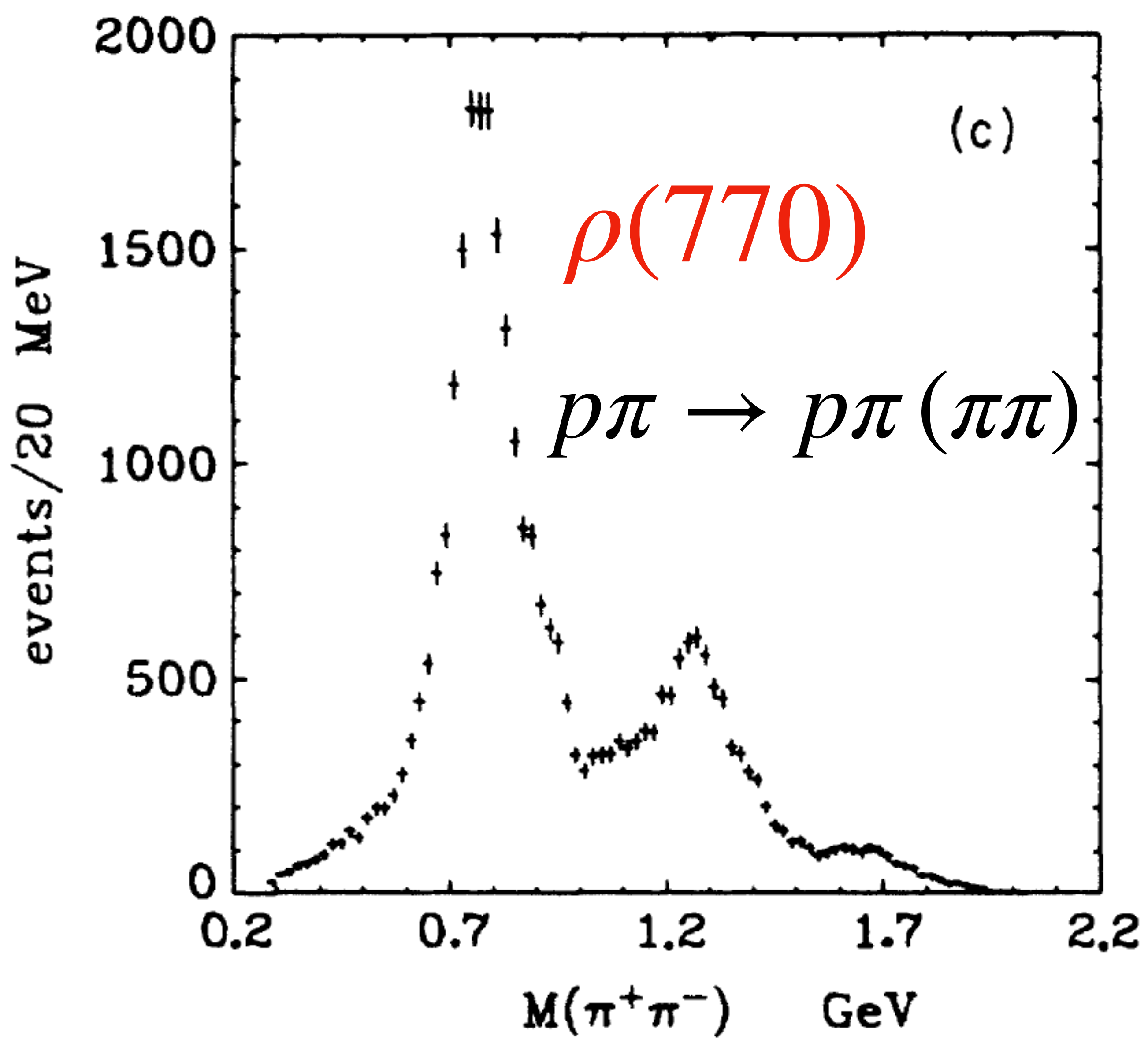

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Lattice QCD

- Euclidean time
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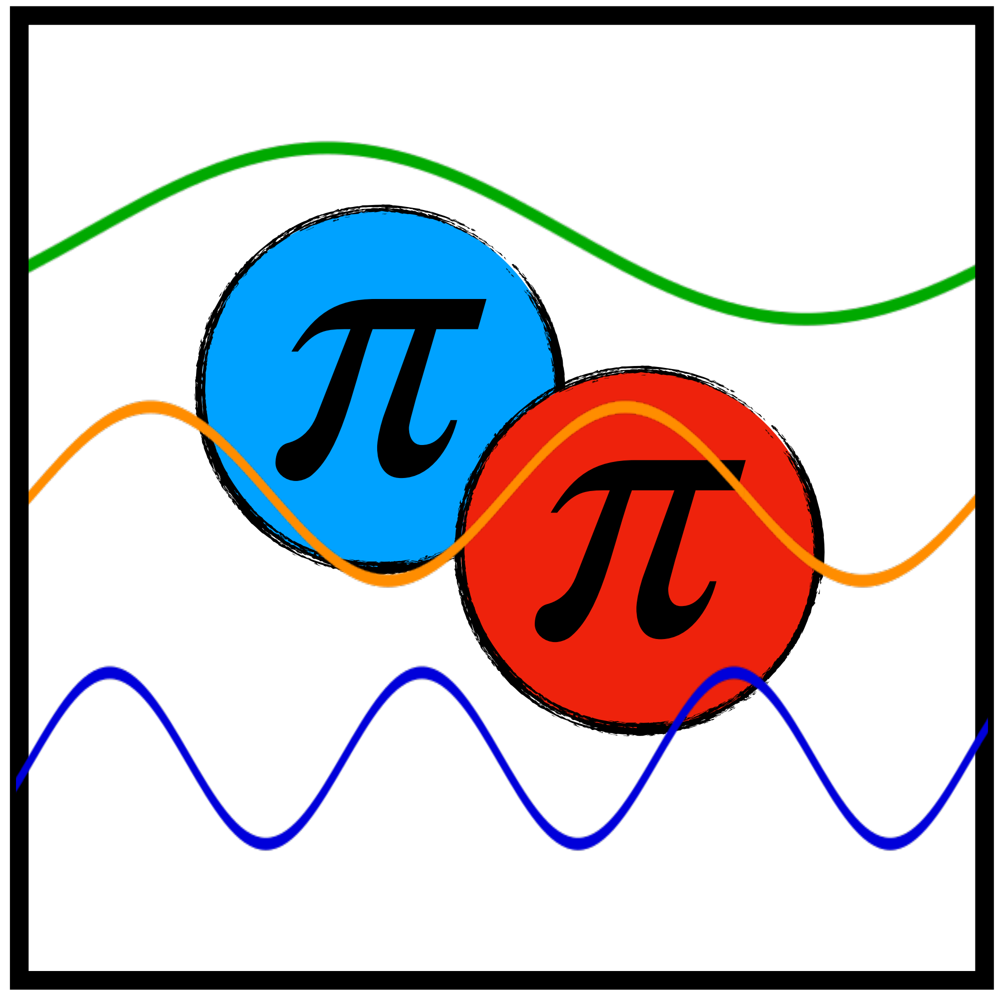

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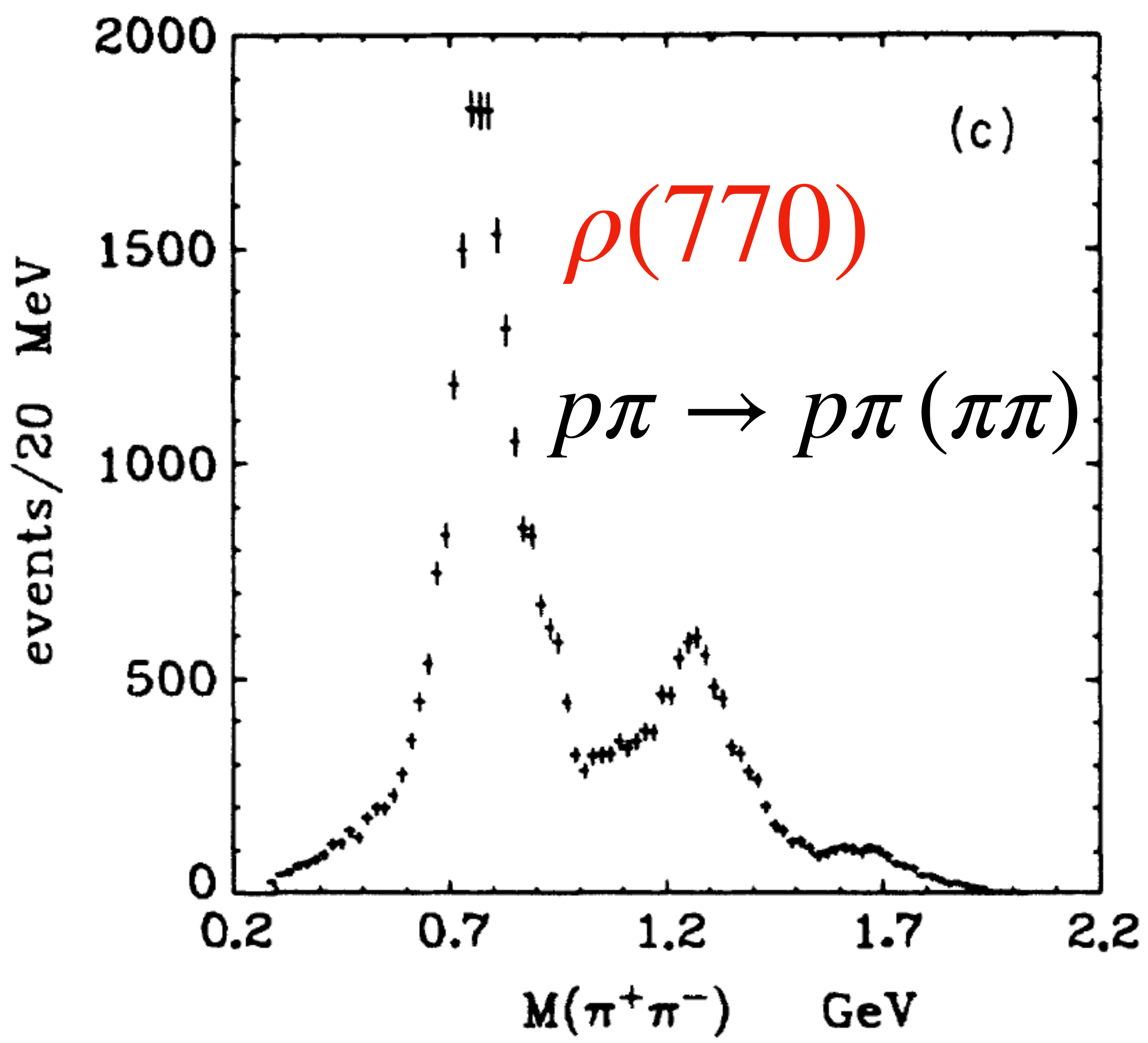

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Finite-volume formalism

[Lüscher, 89']

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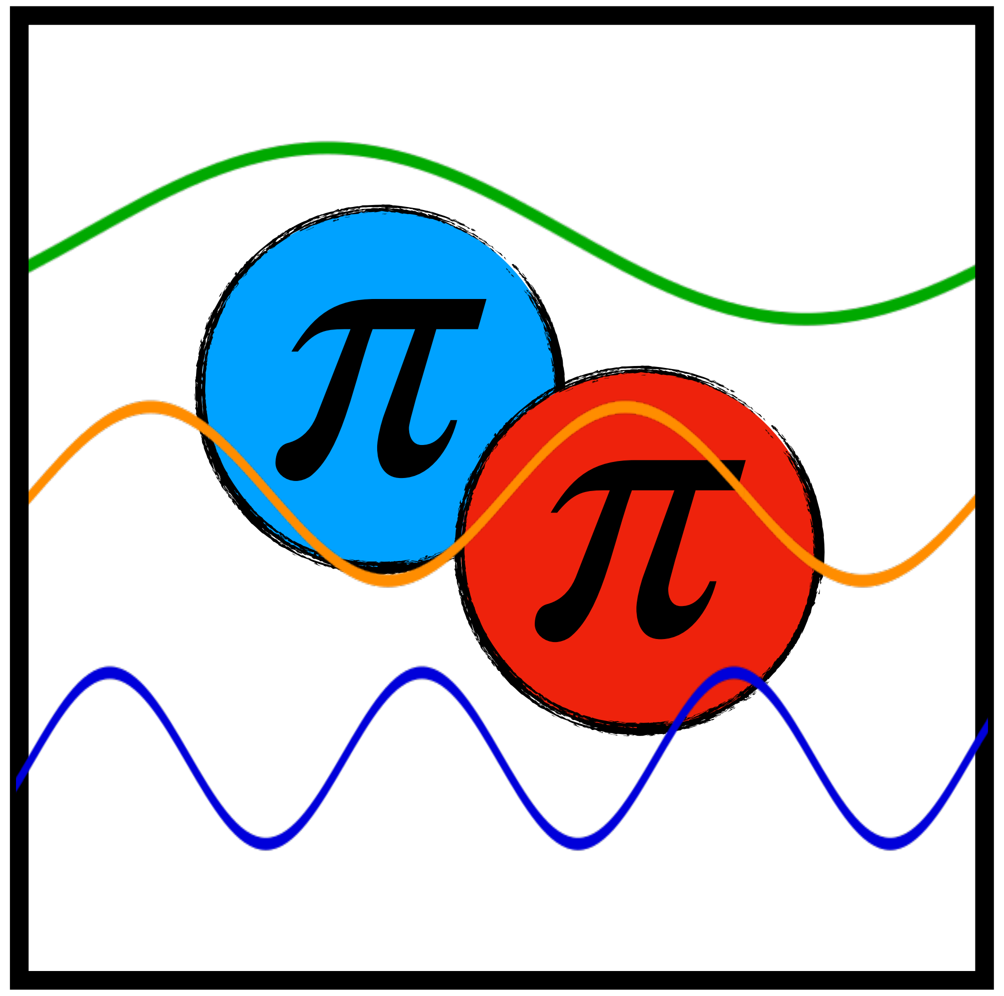
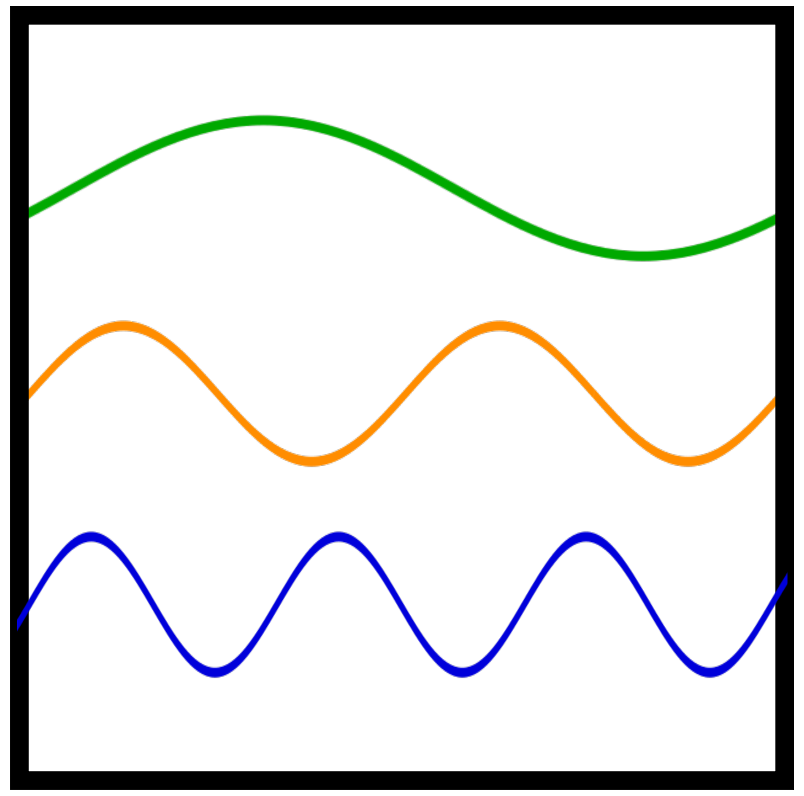


Fernando Romero-López, Uni Bern

Finite-volume systems

Free scalar particles in finite volume with periodic boundaries

$$\psi(\vec{x}) = \psi(\vec{x} + \vec{n}L)$$



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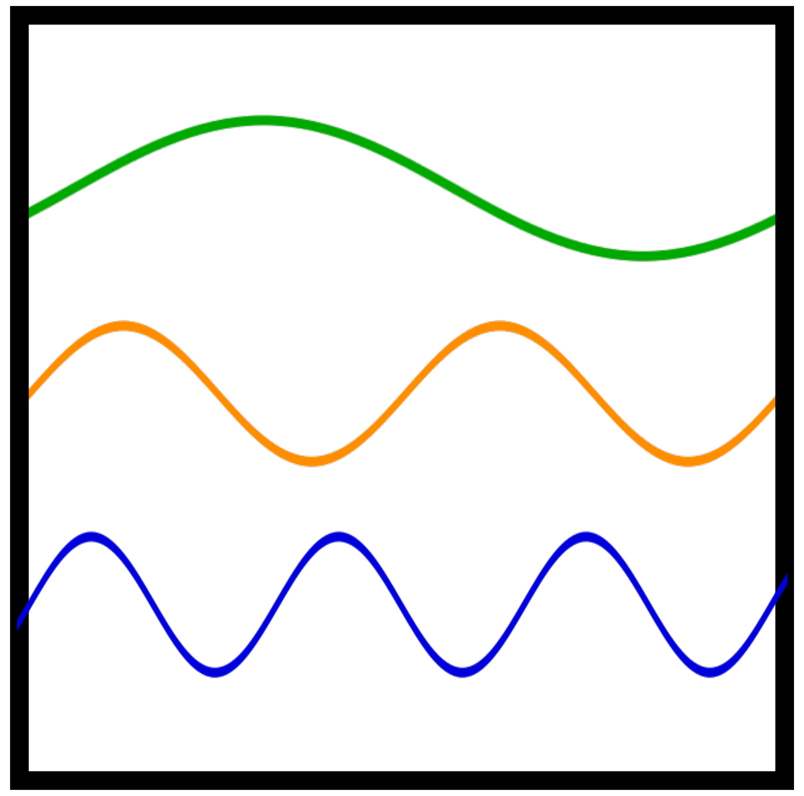

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Interactions change the spectrum:
it can be treated as a perturbation

Ground state to leading order

$$E_2 - 2m = \langle \phi(\vec{0})\phi(\vec{0}) \,|\, \mathbf{H_I} \,|\, \phi(\vec{0})\phi(\vec{0}) \rangle$$

$$\Delta E_2 = \frac{\mathcal{M}_2(E = 2m)}{8m^2L^3} + O(L^{-4})$$

[Huang, Yang, 1958]

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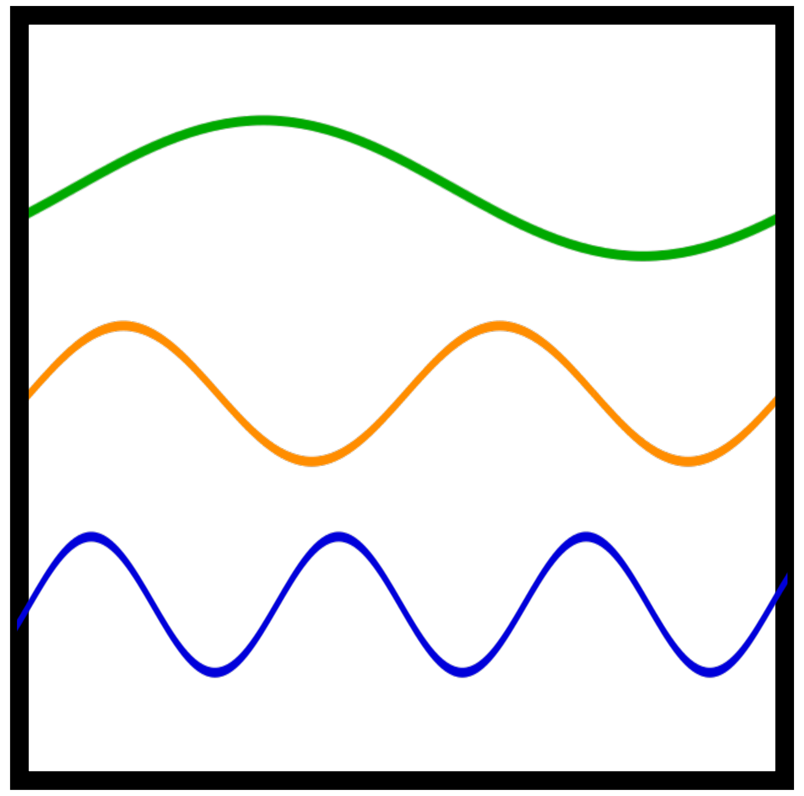

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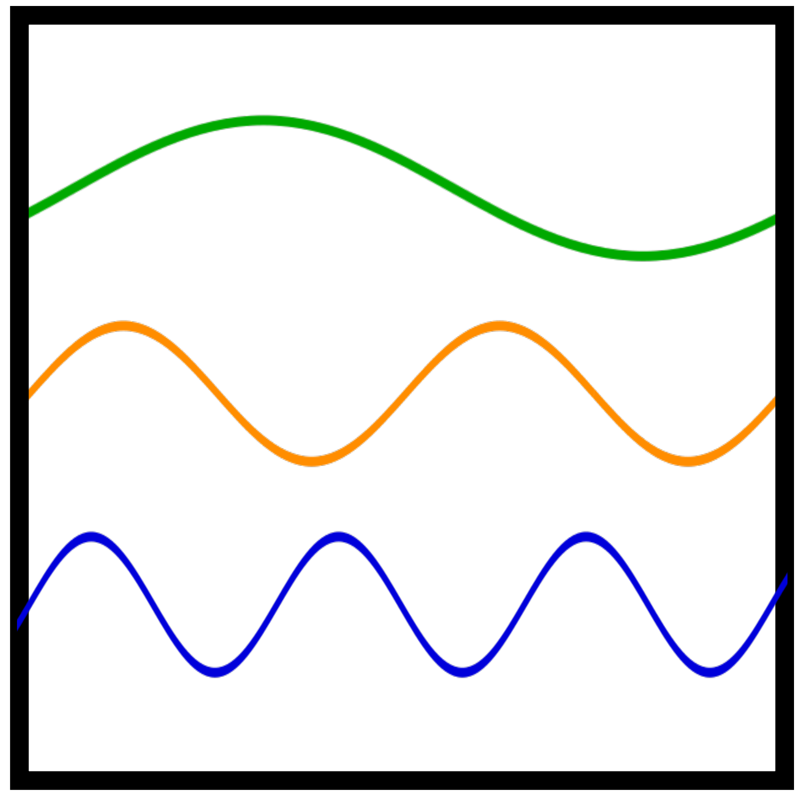

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In general a problem of
Quantum Field Theory
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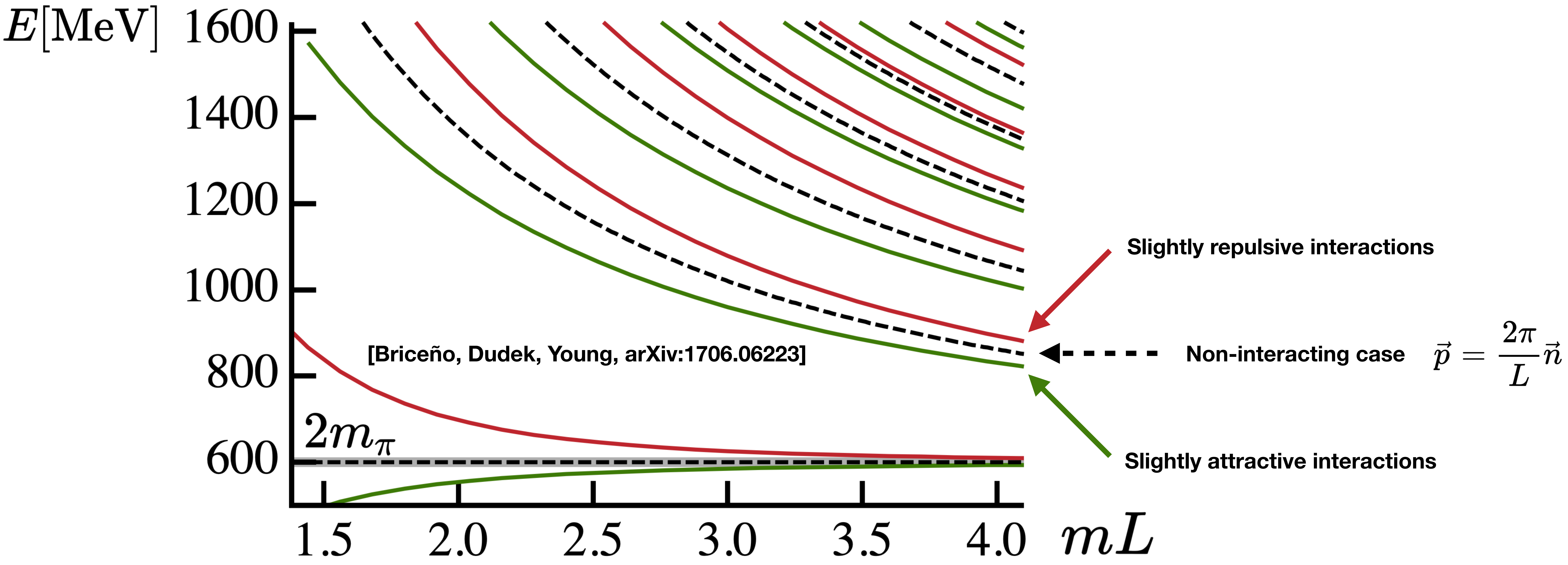
Key insight

Volume dependence of finite-volume energy states contains scattering information

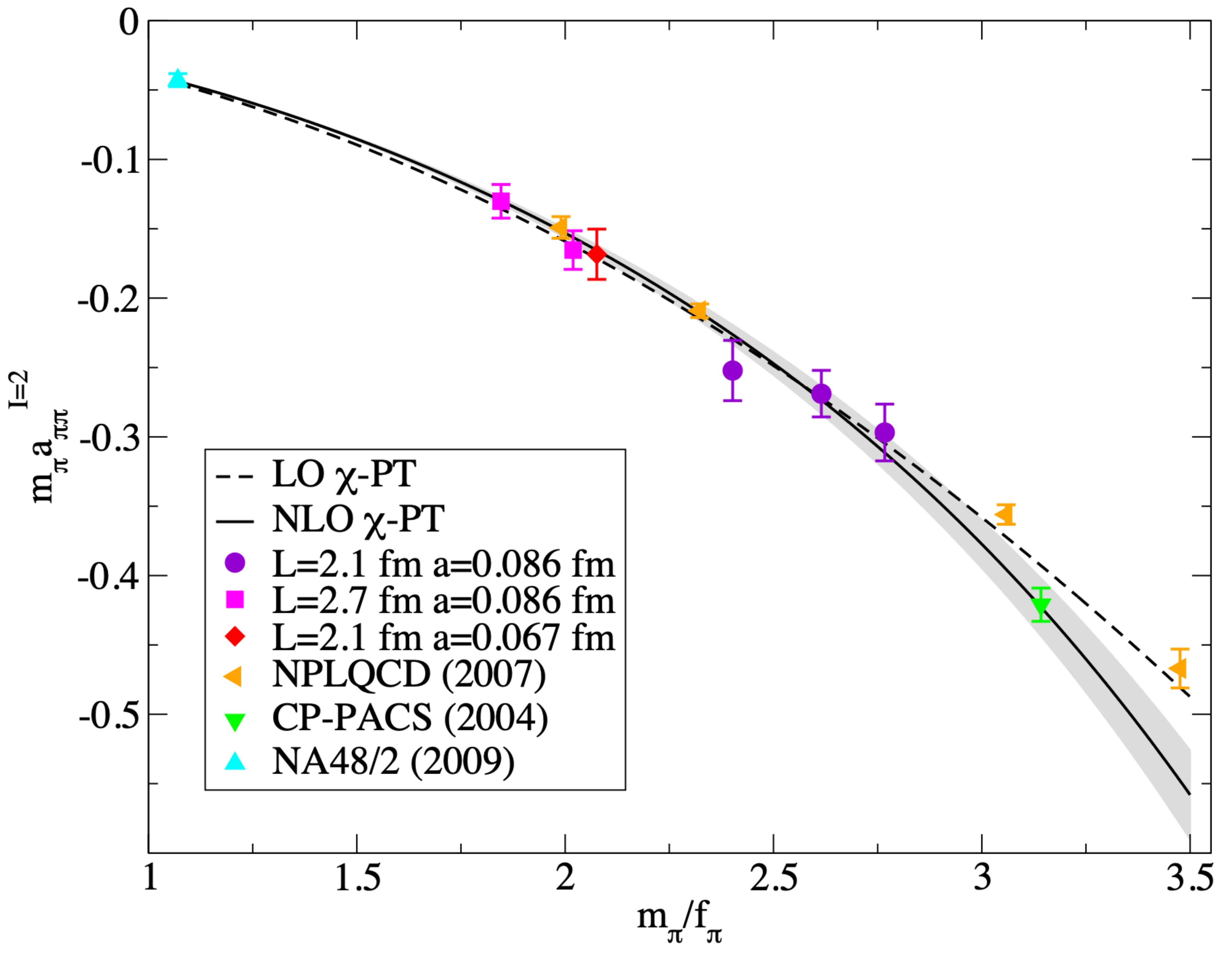


1/L expansions

- Perturbative expansions are useful for
  [Lüscher, 89']
  - Weakly interacting systems (no nearby bound states or resonances)
  - Just obtaining the scattering length

$$\Delta E_2 = E_2 - 2m = \frac{4\pi a_0}{mL^3}\left\{1 + c_1\left(\frac{a_0}{L}\right) + c_2\left(\frac{a_0}{L}\right)^2\right\} + \mathcal{O}(L^{-6})$$

$$(c_1 = 2.837,\ c_2 = 6.375)$$



Isospin-2 $\pi^+\pi^+$ scattering length
[Feng, Jansen, Renner, 0909.3255]

General approach

- In order to derive the full relation, consider the finite-volume correlator:

$$C_L(E, \overrightarrow{P}) = \int e^{iPx} \langle \mathcal{O}(x) | \mathcal{O}(0) \rangle =$$

[à la Kim, Sachrajda, Sharpe]

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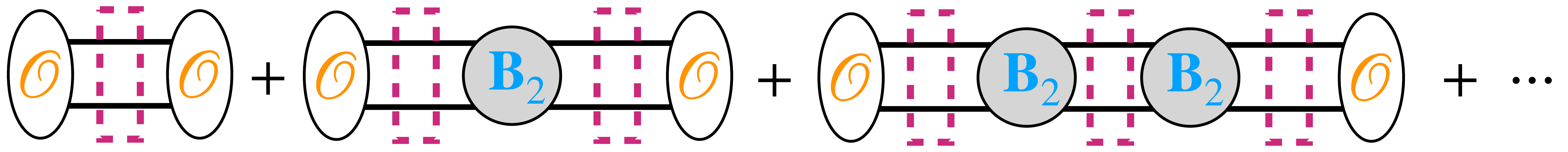

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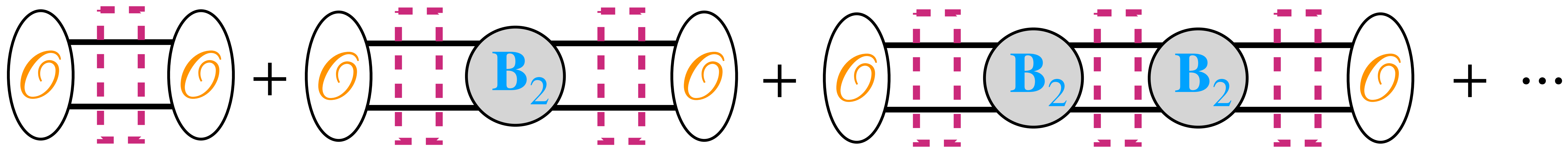

$\sum_{\vec{k}}$

Finite-volume sums

[à la Kim, Sachrajda, Sharpe]

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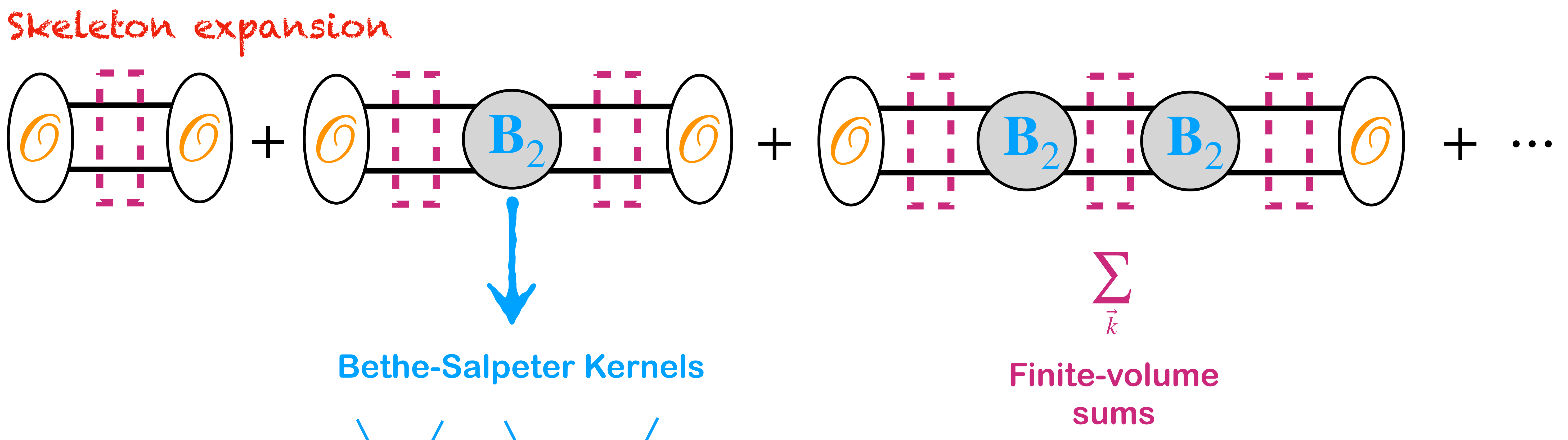
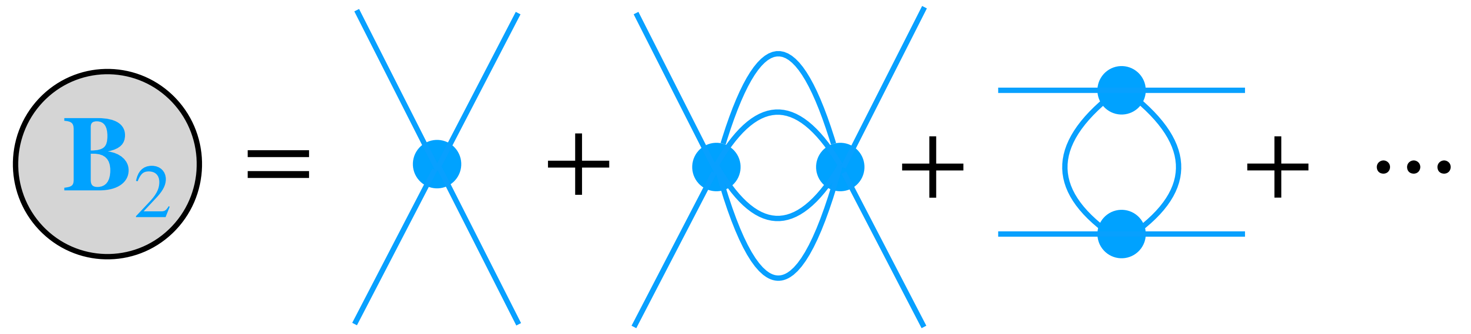


[à la Kim, Sachrajda, Sharpe]



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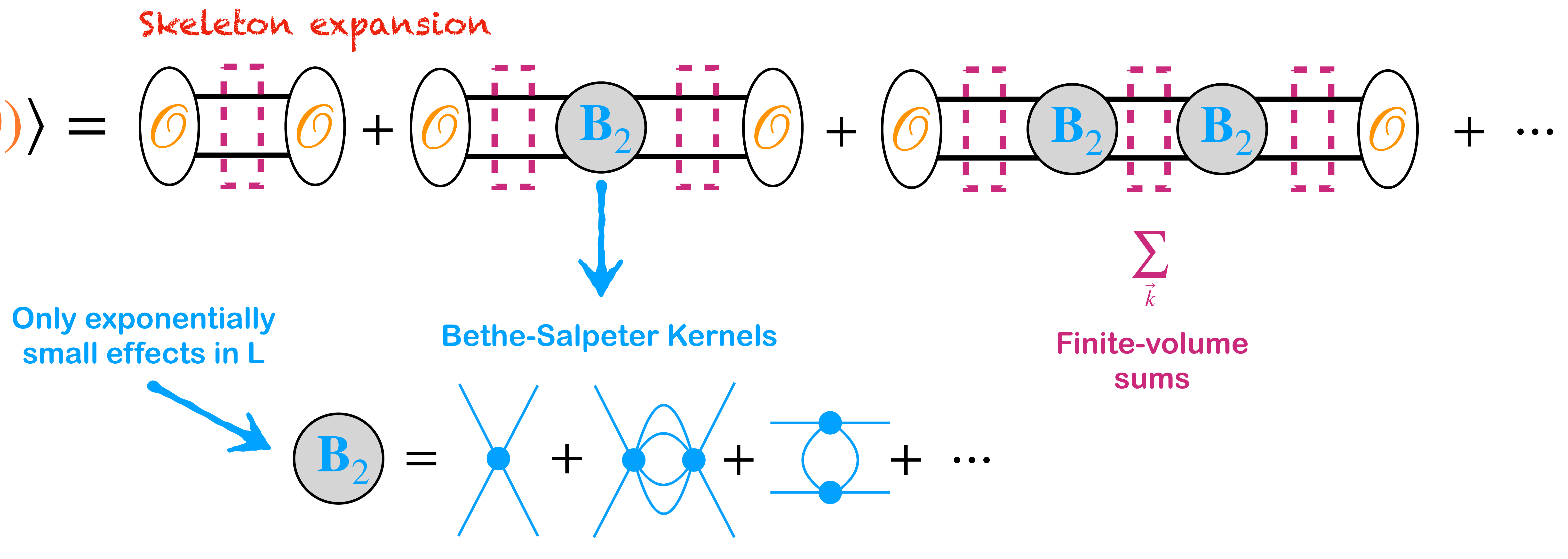

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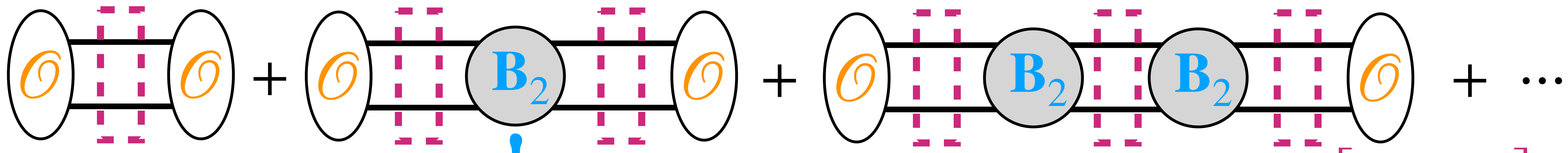

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Only exponentially small effects in L

Bethe-Salpeter Kernels

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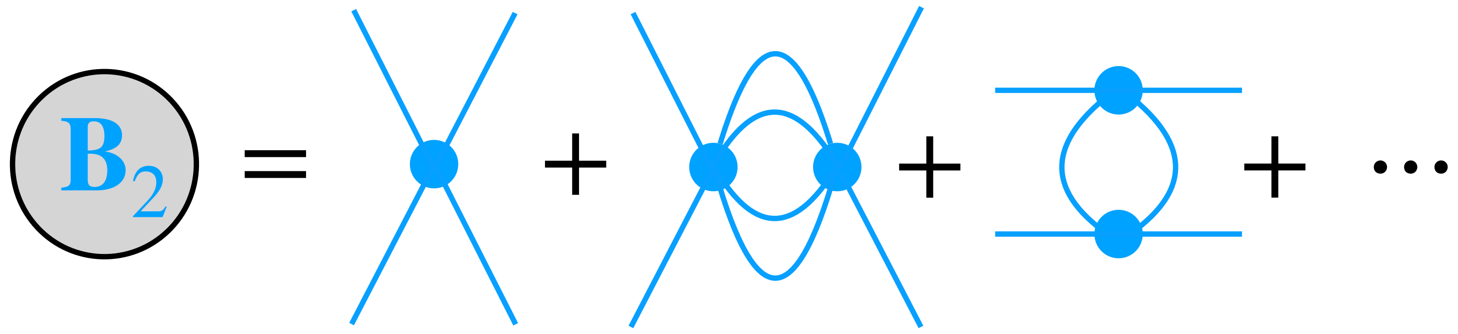

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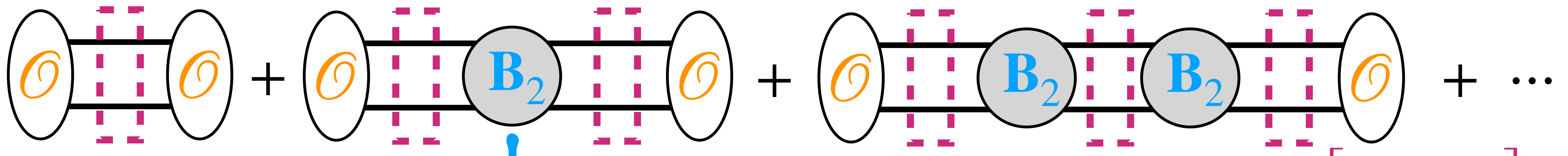

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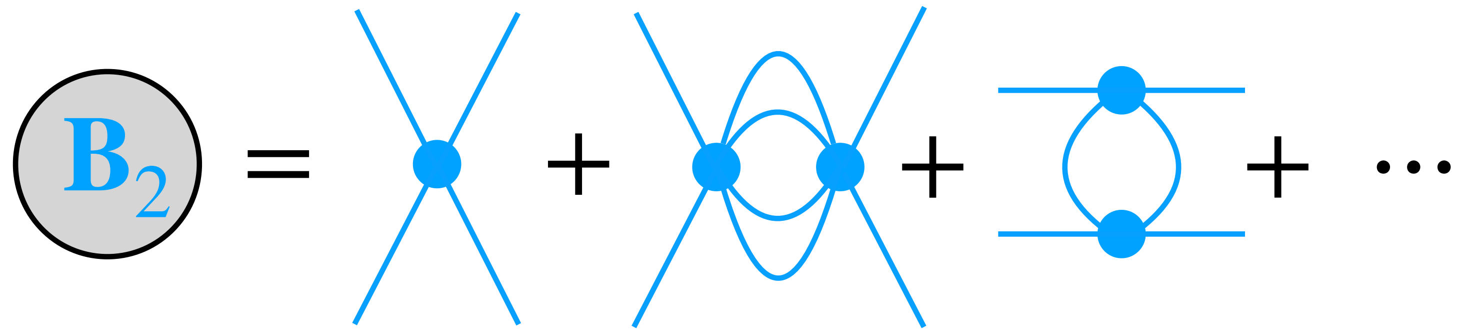

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2. Resumation of diagrams

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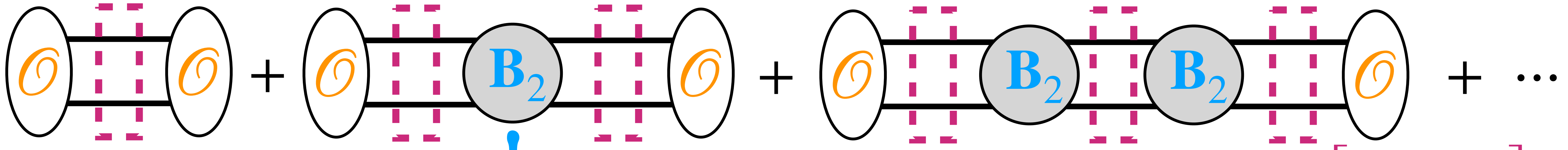

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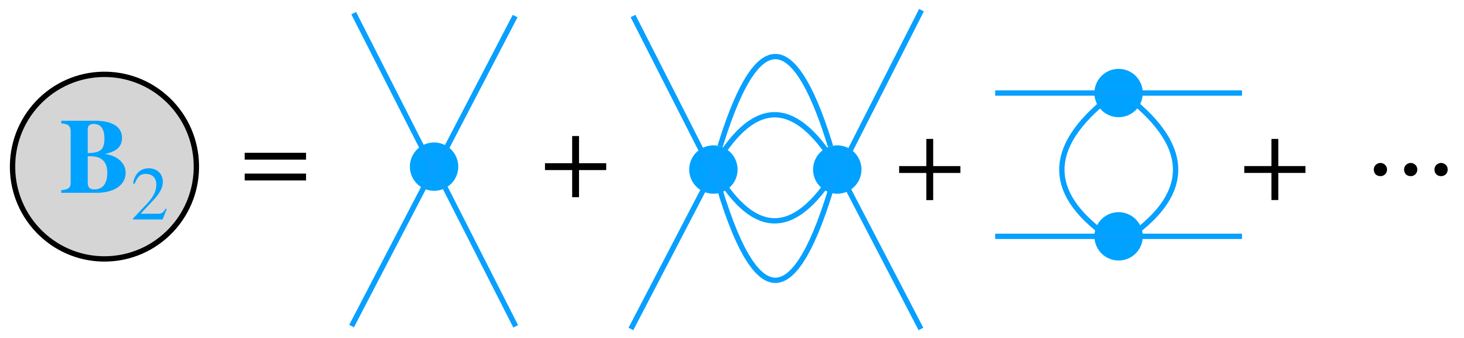

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Known kinematic function

$$\mathcal{M}_2^{-1} = \mathcal{K}_2^{-1} - i\sqrt{s - 2m^2}$$

General approach

$$C_L(E, \overrightarrow{P}) = \quad \text{some algebra ...} \quad = C_\infty(E, \overrightarrow{P}) + A^\dagger \frac{1}{\mathscr{K}_2 + F^{-1}} A + O(e^{-mL})$$

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Two-particle Quantization Condition

$$\det\left[\mathcal{K}_2(E_n) + F^{-1}(E_n, \overrightarrow{P}, L)\right] = 0$$

Scattering K-Matrix

Known kinematic function

"QC2"

! It holds below $E_{cm} < 4m$

The quantization condition

- Indirect connection between the spectrum and the two-particle scattering amplitude [Lüscher 89']

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$$\det_{\ell m}\left[\mathcal{K}_2(E_n) + F^{-1}(E_n, \vec{P}, L)\right]\Bigg|_{E=E_n} = 0$$

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"QC2"

Finite-volume information

$$F_{00}(q^2) \sim \left[\frac{1}{L^3}\sum_{\vec{k}} - \int\frac{d^3k}{(2\pi)^3}\right]\frac{1}{k^2-q^2}$$

- Fully general formalism (nondegenerate, spin, multi-channel):

[Rummukainen, Gottlieb, NPB 1995] [Kim, Sachrajda and Sharpe, NPB 2005]
[Bernard et al, JHEP 2008] [Davoudi, Savage, PRD 2011]
[Leskovec, Prevlosek, PRD 2012] [Göckeler et al, PRD 2012]
[Briceño, PRD 2014]

Example application

Two pions in s-wave

$$\mathcal{K}_2^{s-wave}(E_n) = \frac{-1}{F_{00}(E_n, \overrightarrow{P}, L)}$$

one energy level $\longrightarrow$ a phase shift point

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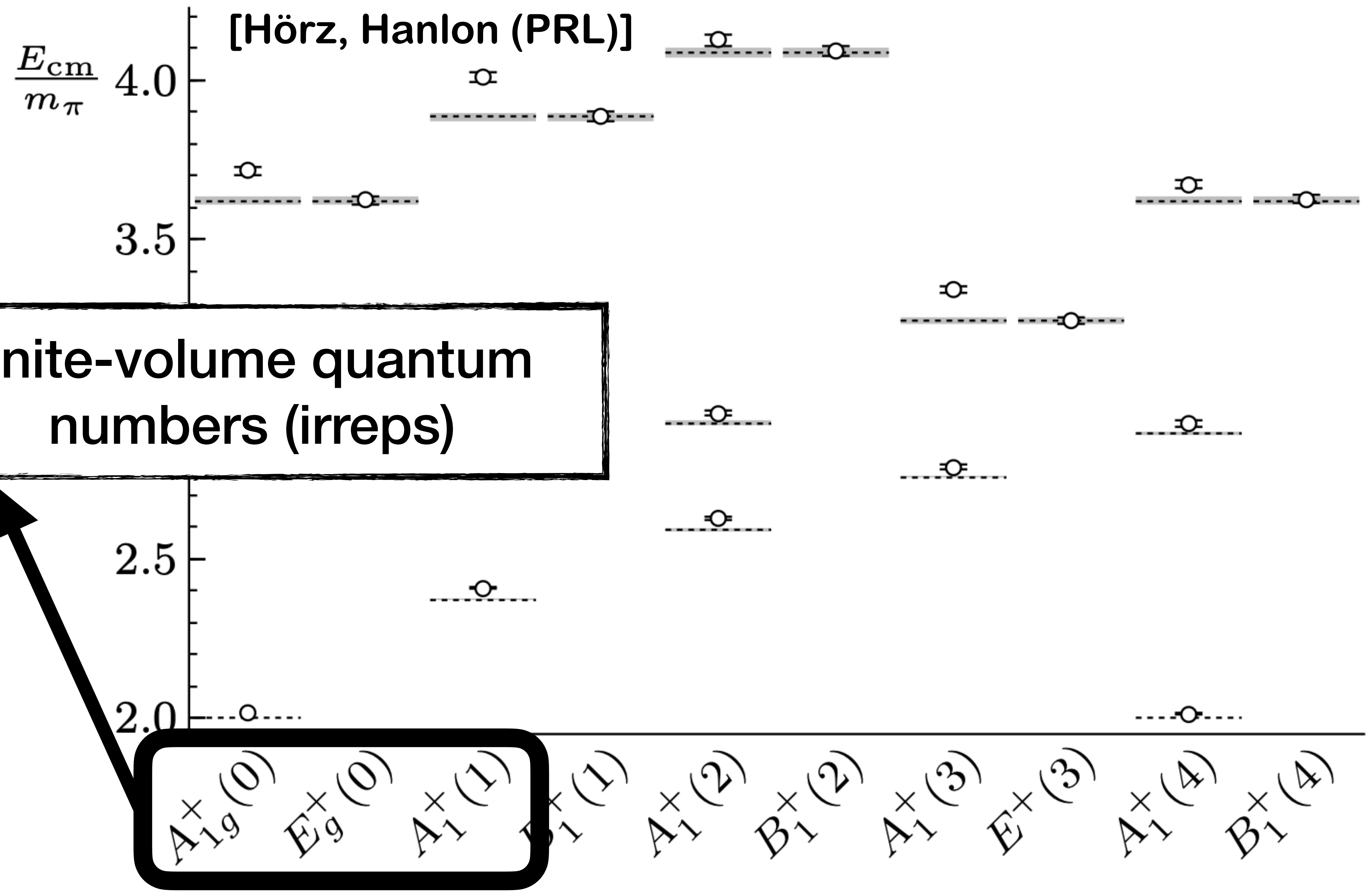
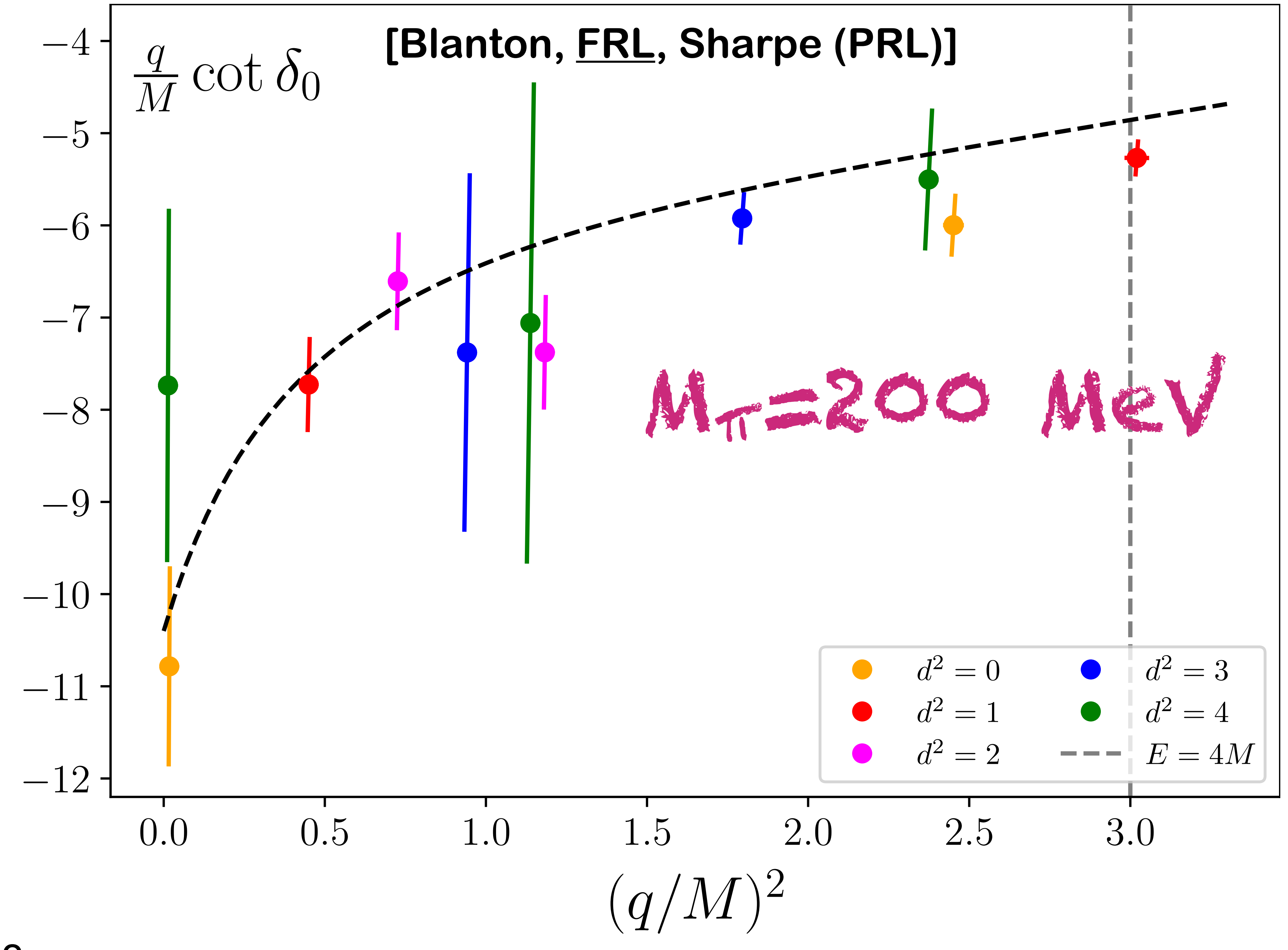


finite-volume quantum numbers (irreps)



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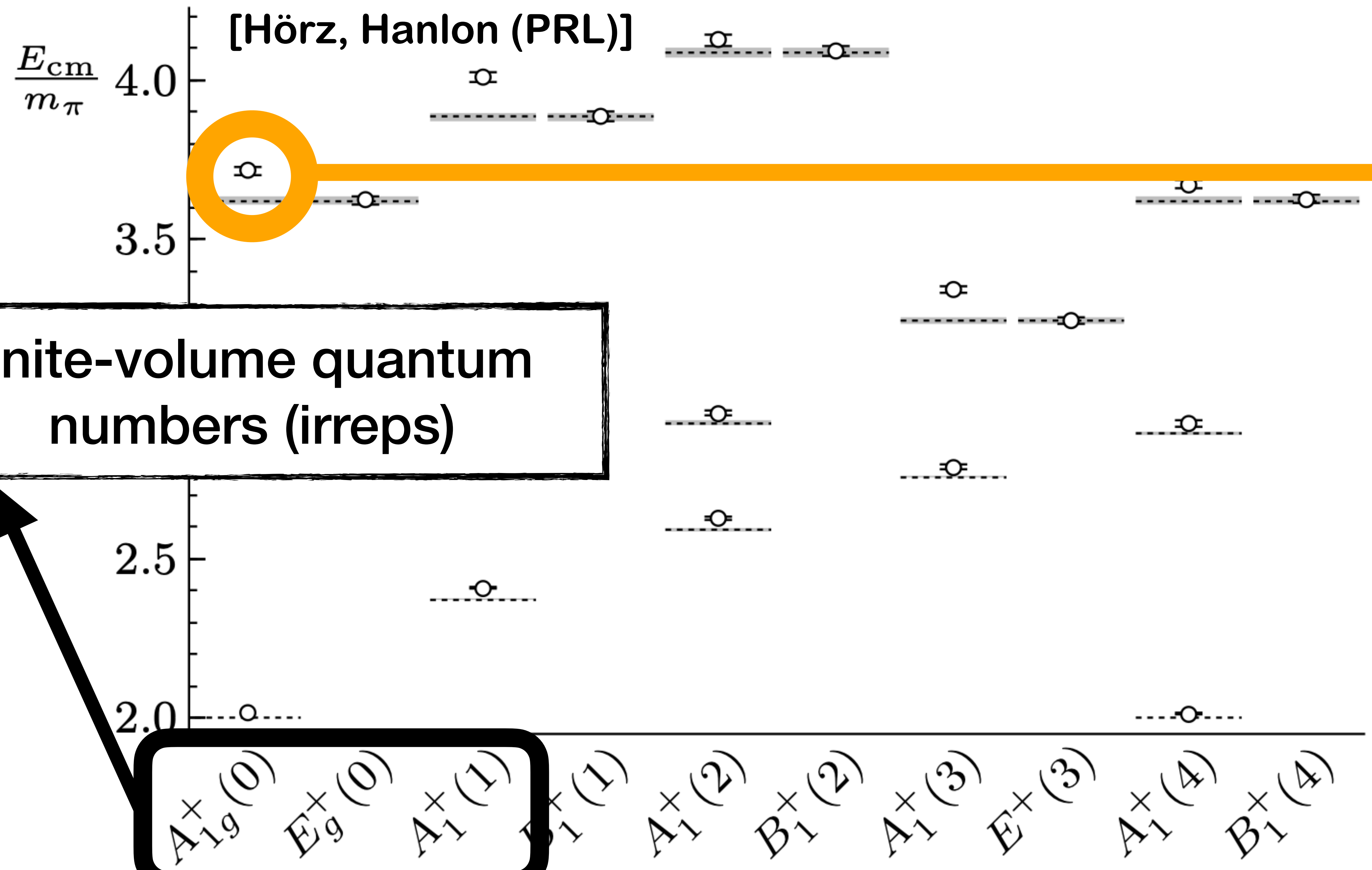

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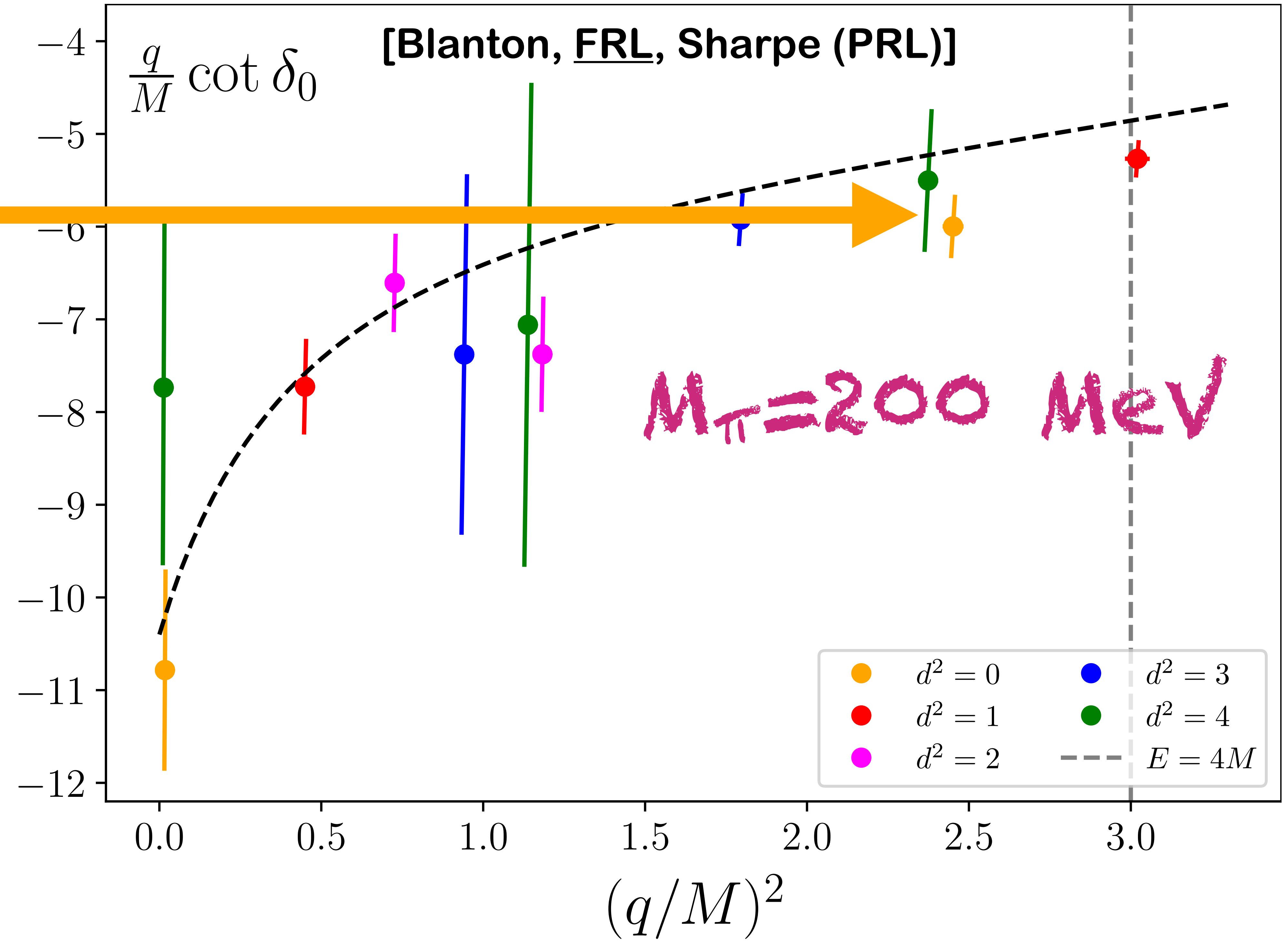

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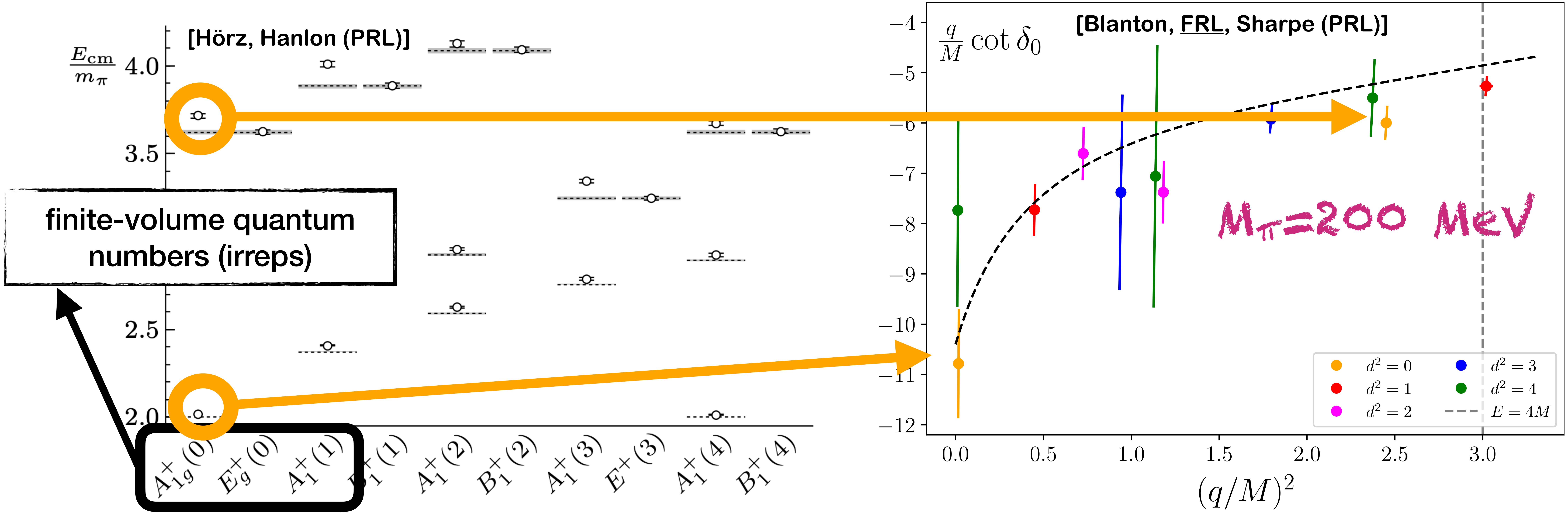

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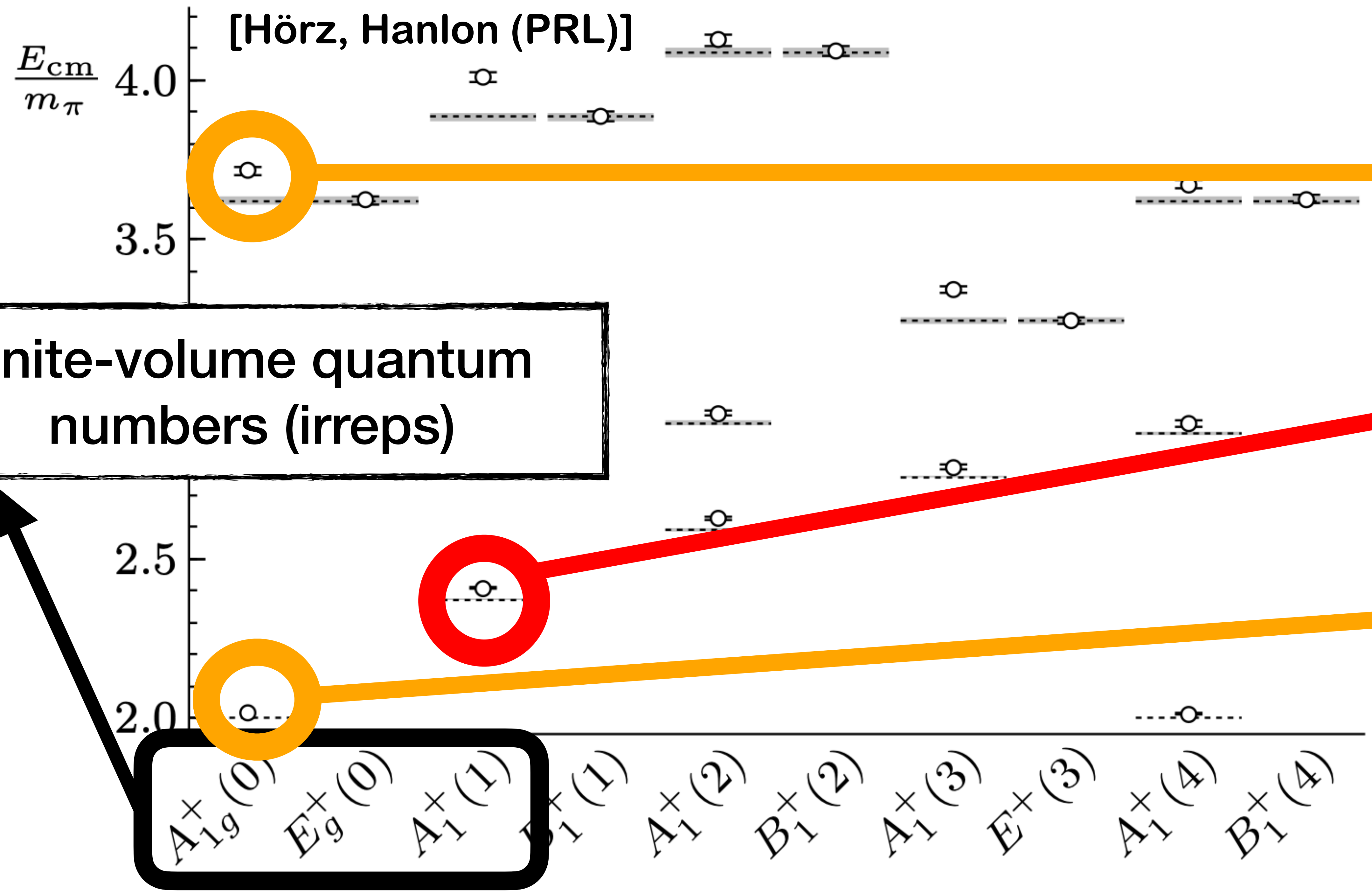

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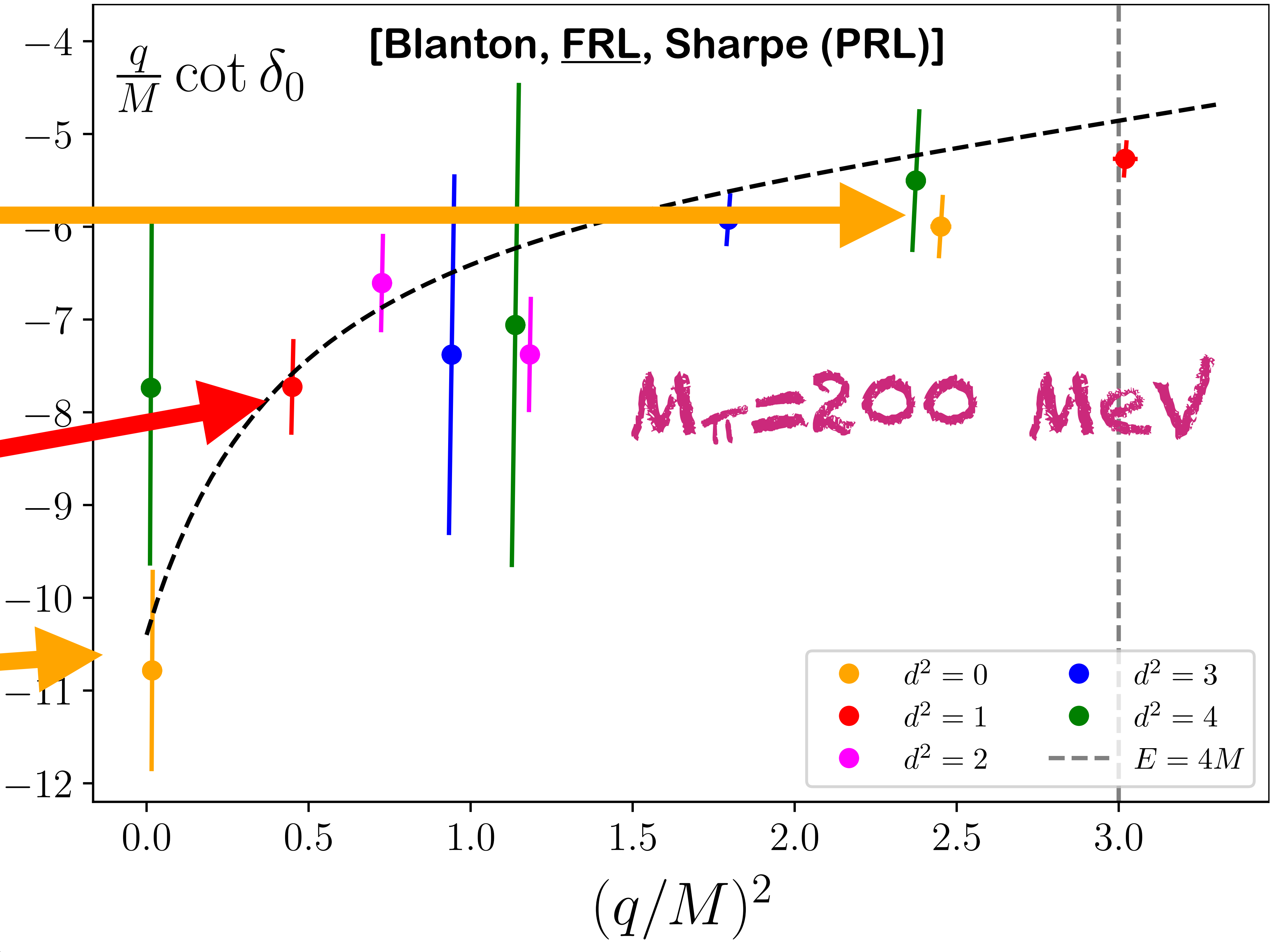

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Exercise

EXERCISE Code up the quantization condition and solve it for a system with:

$$mL = 4, \quad \frac{k}{m}\cot\delta_0 = -\frac{1}{ma_0}, \quad ma_0 = 0.1$$

You can use an easy implementation of the zeta function, and this form of the QC2:

$$\mathcal{Z}_0(k^2) = \lim_{\Lambda\to\infty} \sum_{\vec{n}\in\mathbb{Z}}^{|\vec{n}|<\Lambda} \frac{1}{\vec{n}^2 - (\frac{kL}{2\pi})^2} - 4\pi\Lambda \qquad k\cot\delta_0(k^2) = \frac{1}{\pi L}\mathcal{Z}_0(k^2, L)$$

What is the ground-state energy of the system in the frame $\vec{P} = 0$?

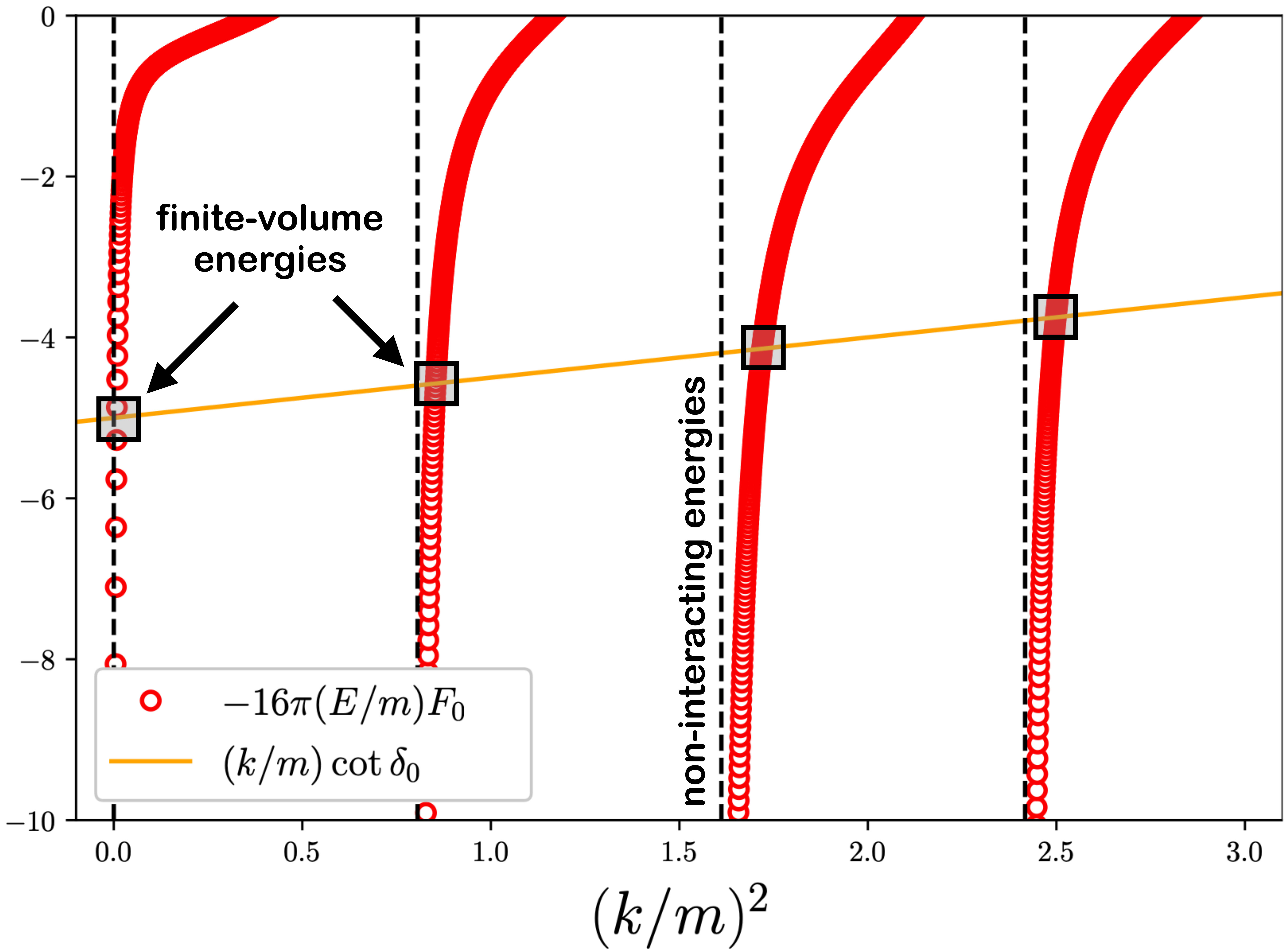
How does it compare to the 1/L expansion?

Solving the QC2

Leading-partial wave approximation:

QC2: $$\frac{1}{\mathcal{K}_2^{\ell=0}(k^2)} = -F_0(k^2, L)$$

$$\mathcal{K}_2^{\ell=0} = \frac{16\pi\sqrt{s}}{k\cot\delta_0}$$



Summary (3)

- Volume dependence of multi-hadron states contains information about scattering amplitude
- Energy shift to the non-interaction theory can be related to scattering parameters

$$\Delta E_2 = E_2 - 2m = \frac{4\pi a_0}{mL^3}\left\{1 + c_1\left(\frac{a_0}{L}\right) + c_2\left(\frac{a_0}{L}\right)^2\right\} + \mathcal{O}(L^{-6})$$

- Two-body quantization condition relates the spectrum to amplitude in generic systems

$$k \cot \delta_0(k^2) = \frac{1}{\pi L}\mathcal{Z}_0(k^2, L)$$

Selection of recent results

The $\Lambda(1405)$

$$\begin{pmatrix} \pi\Sigma \to \pi\Sigma & \pi\Sigma \to Kp \\ Kp \to \pi\Sigma & Kp \to Kp \end{pmatrix}$$

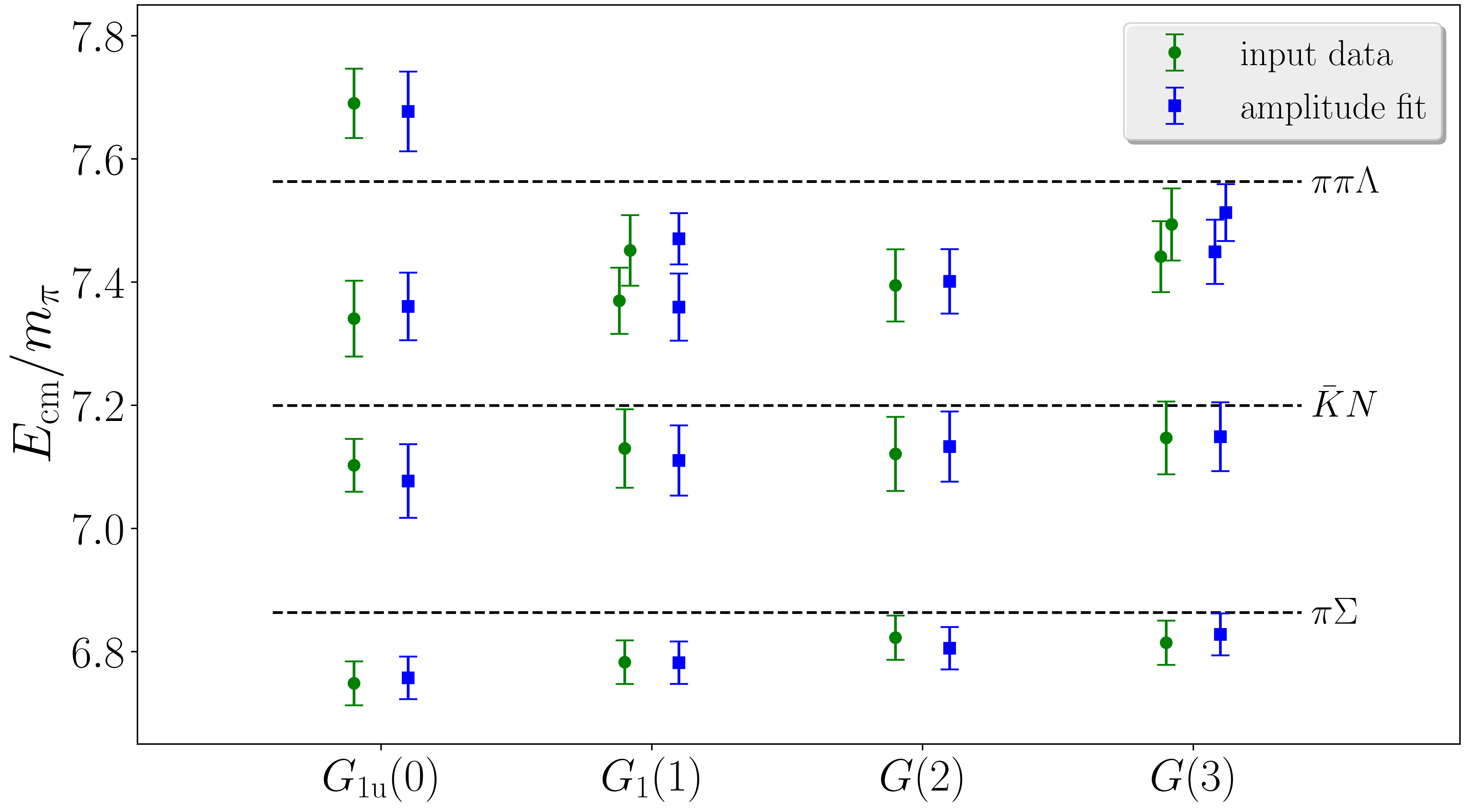
$$\det_{\ell m}\left[\begin{pmatrix} \widetilde{K}_{\pi\Sigma\to\pi\Sigma} & \widetilde{K}_{\pi\Sigma\to KN} \\ \widetilde{K}_{KN\to\pi\Sigma} & \widetilde{K}_{KN\to KN} \end{pmatrix} + \begin{pmatrix} F^{-1}_{\pi\Sigma}(E_n,\vec{P},L) & 0 \\ 0 & F^{-1}_{KN}(E_n,\vec{P},L) \end{pmatrix}\right] = 0$$

Multi-channel K-Matrix | Zeta function

$\boxed{\Lambda(1405)\ 1/2^-}$ $I(J^P) = 0(\frac{1}{2}^-)$ Status: ∗∗∗∗ 😀

$\boxed{\Lambda(1380)\ 1/2^-}$ $J^P = \frac{1}{2}^-$ Status: ∗∗ 😐

∗∗∗∗	Existence is certain.
∗∗∗	Existence is very likely.
∗∗	Evidence of existence is fair.
∗	Evidence of existence is poor.



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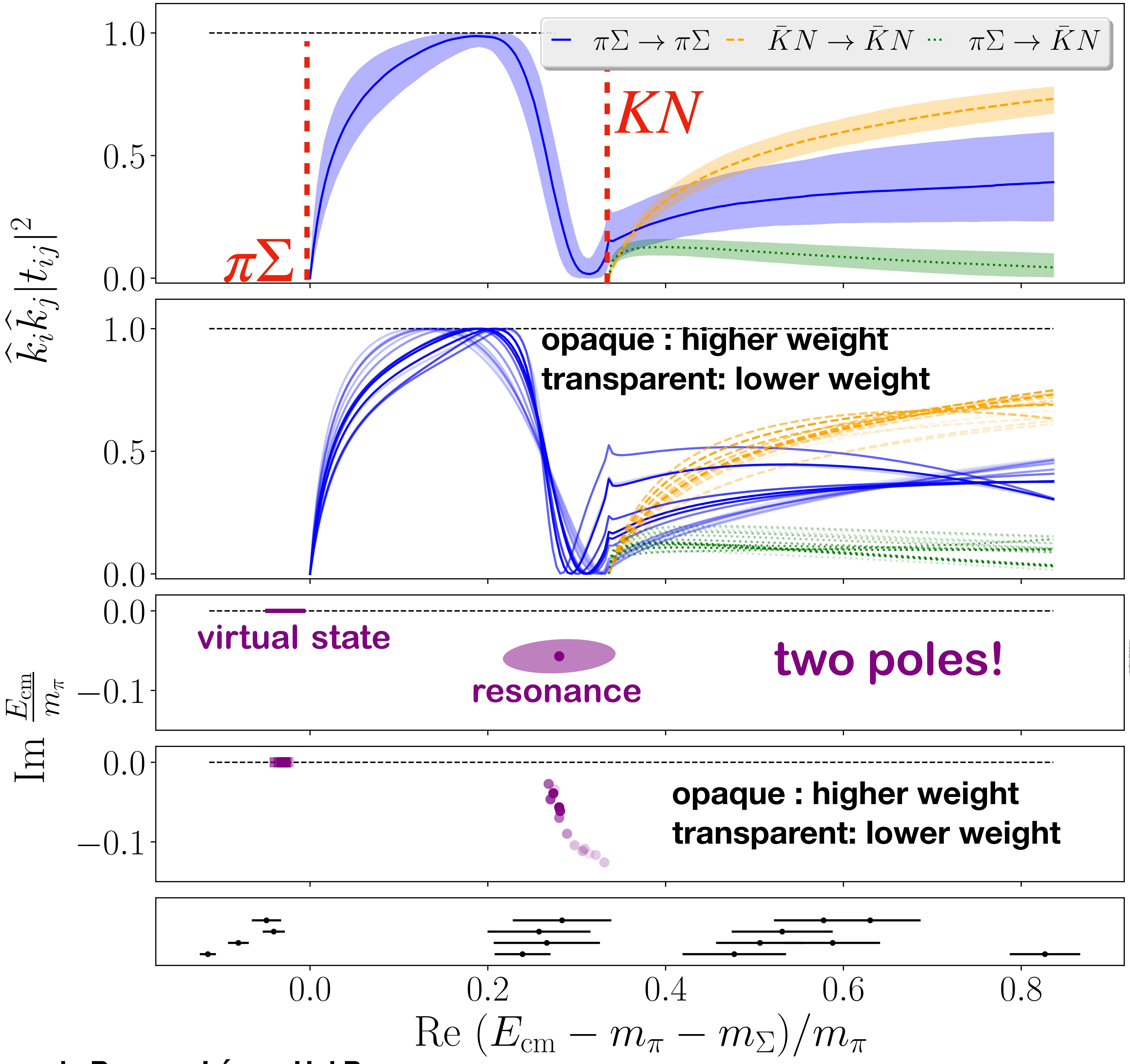
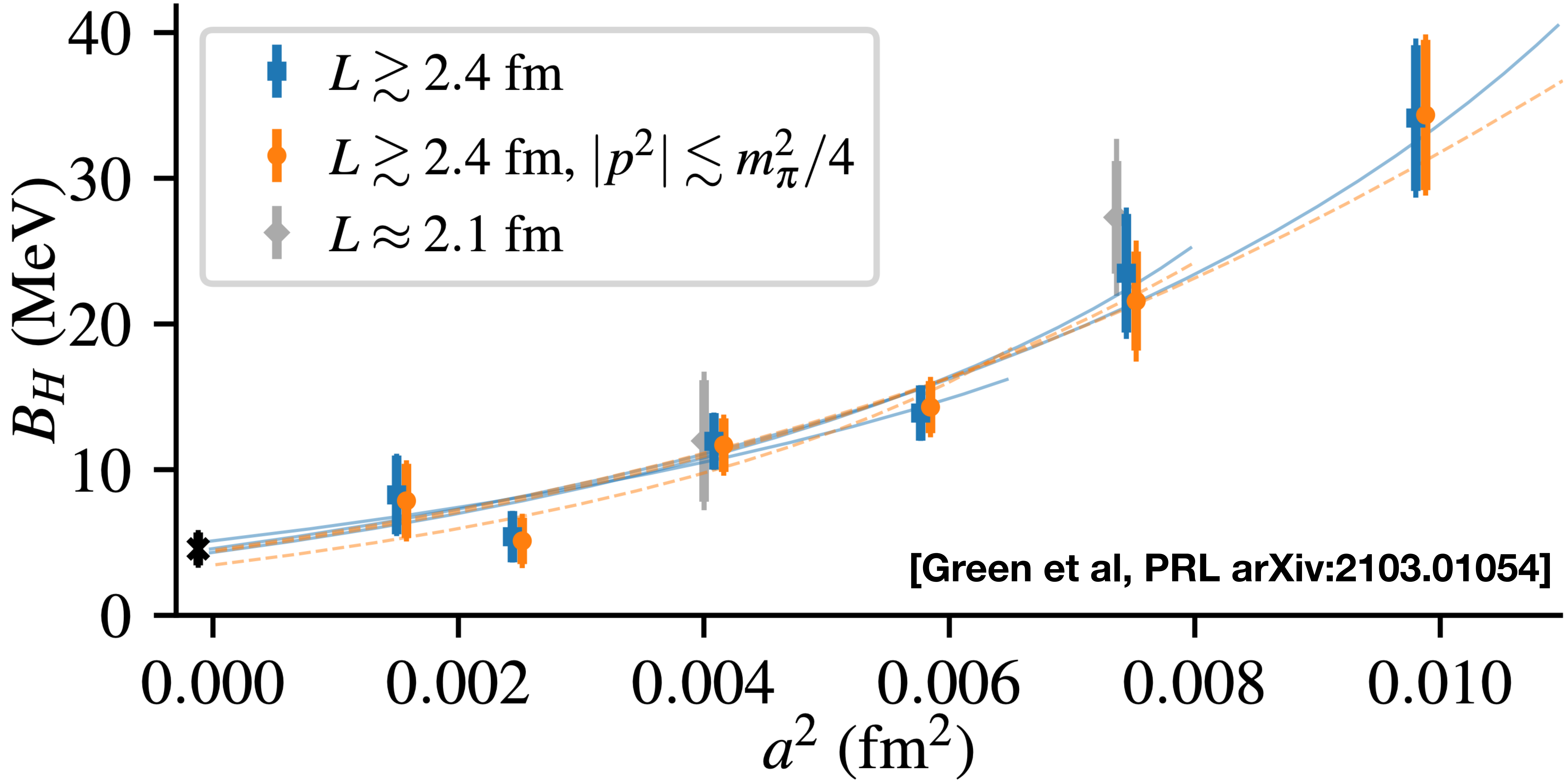


- Scattering amplitudes for "preferred" fit
  i.e. with lowest $\mathrm{AIC} = \chi^2 - 2\,\mathrm{dof}$
- Scattering amplitudes for different parametrizations
- Pole positions for "preferred" fit
- Pole positions for for different parametrization
  All find two poles!
- Lattice QCD energies used in fits

Continuum limit of H dibaryon

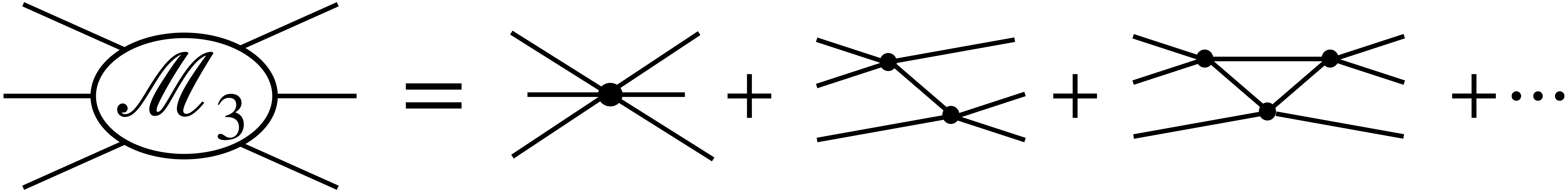
- H dibaryon is a ΛΛ bound state (uuddss quarks)



[Green et al, PRL arXiv:2103.01054]

Three-hadron amplitudes

Qualitatively more complicated than the two-particle case!



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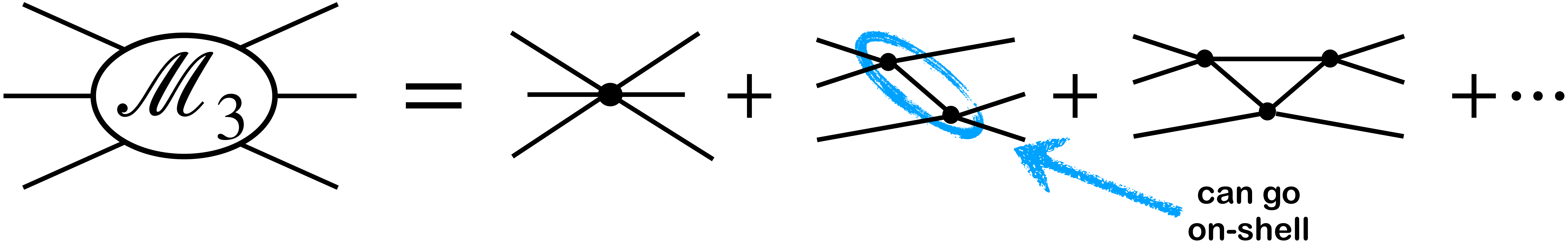

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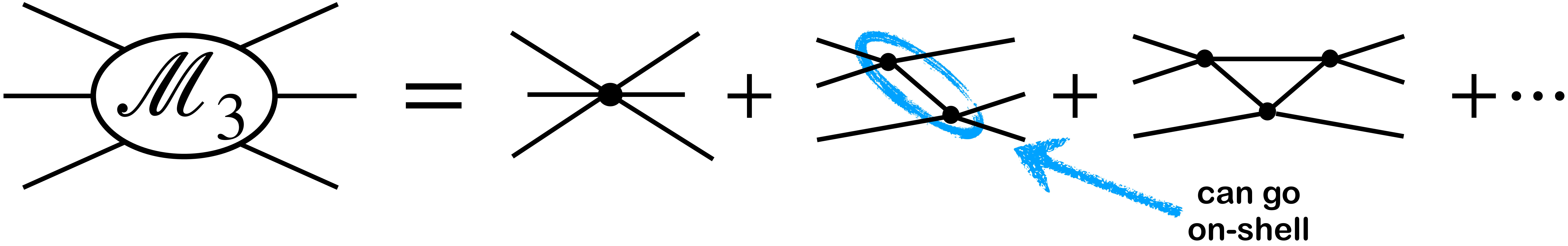
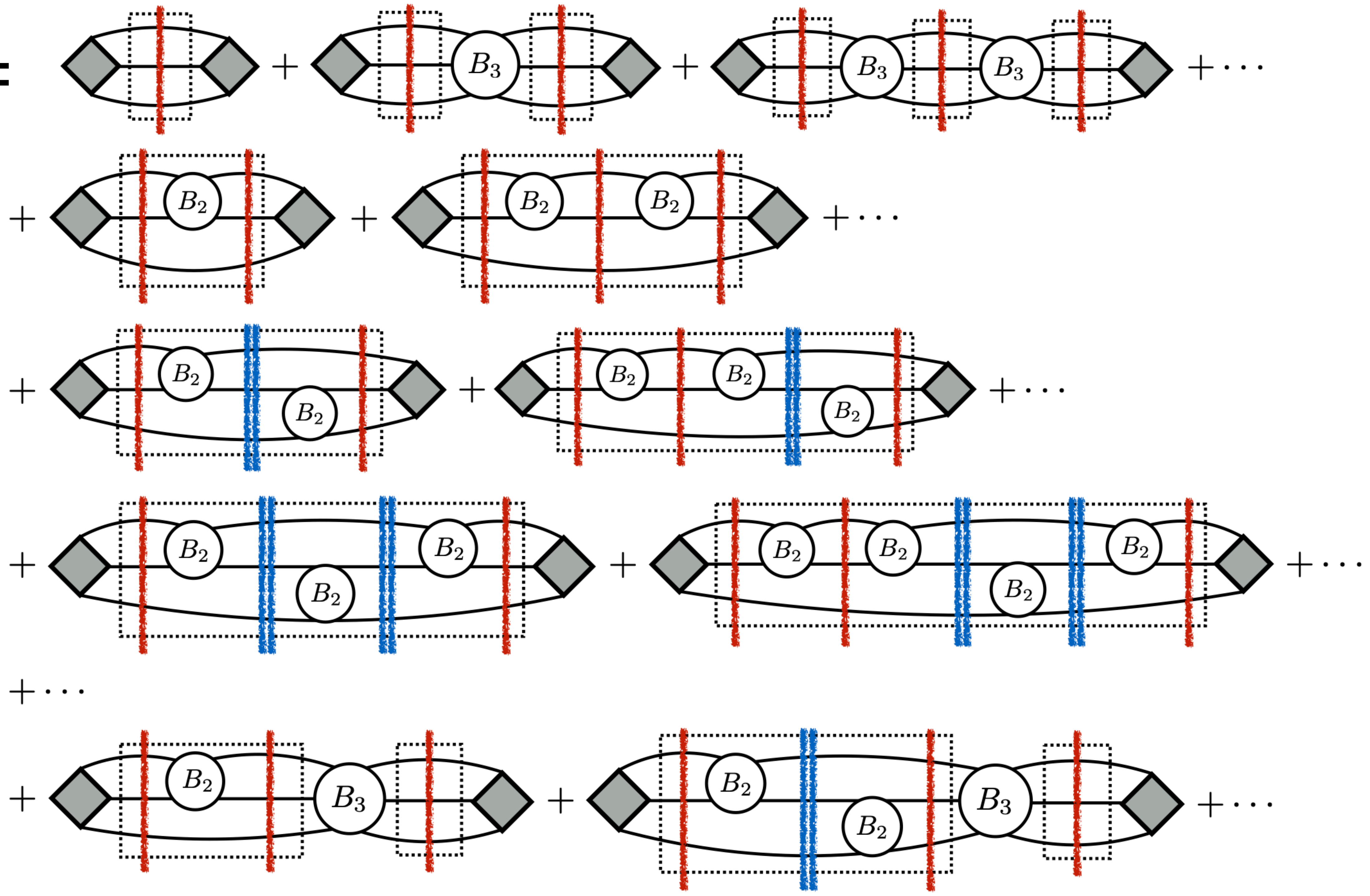


- Three-particle scattering amplitudes can be divergent for specific kinematics.
- They depend also on two-to-two interactions.
  - But any separation between “two-particle” and “three-particle” effects is not well-defined

Quantization Condition

Skeleton expansion

$C_L =$



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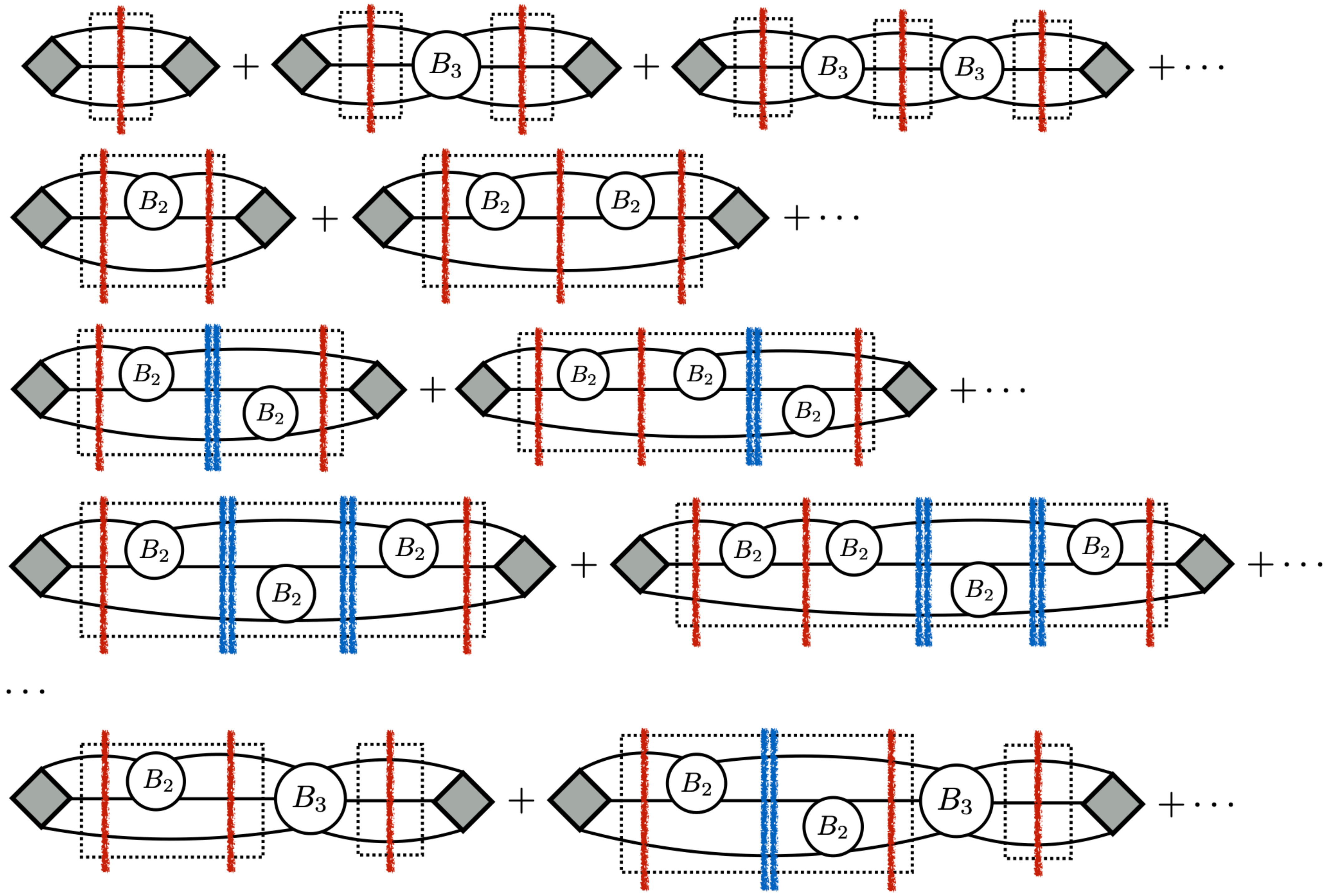

Separation of finite and infinite volume terms:

$$= C_\infty(P) + A_3 \frac{1}{\mathcal{K}_{df,3} + F_3^{-1}} A_3' + O(e^{-mL})$$

Easier derivation: Blanton, Sharpe [2007.16188]

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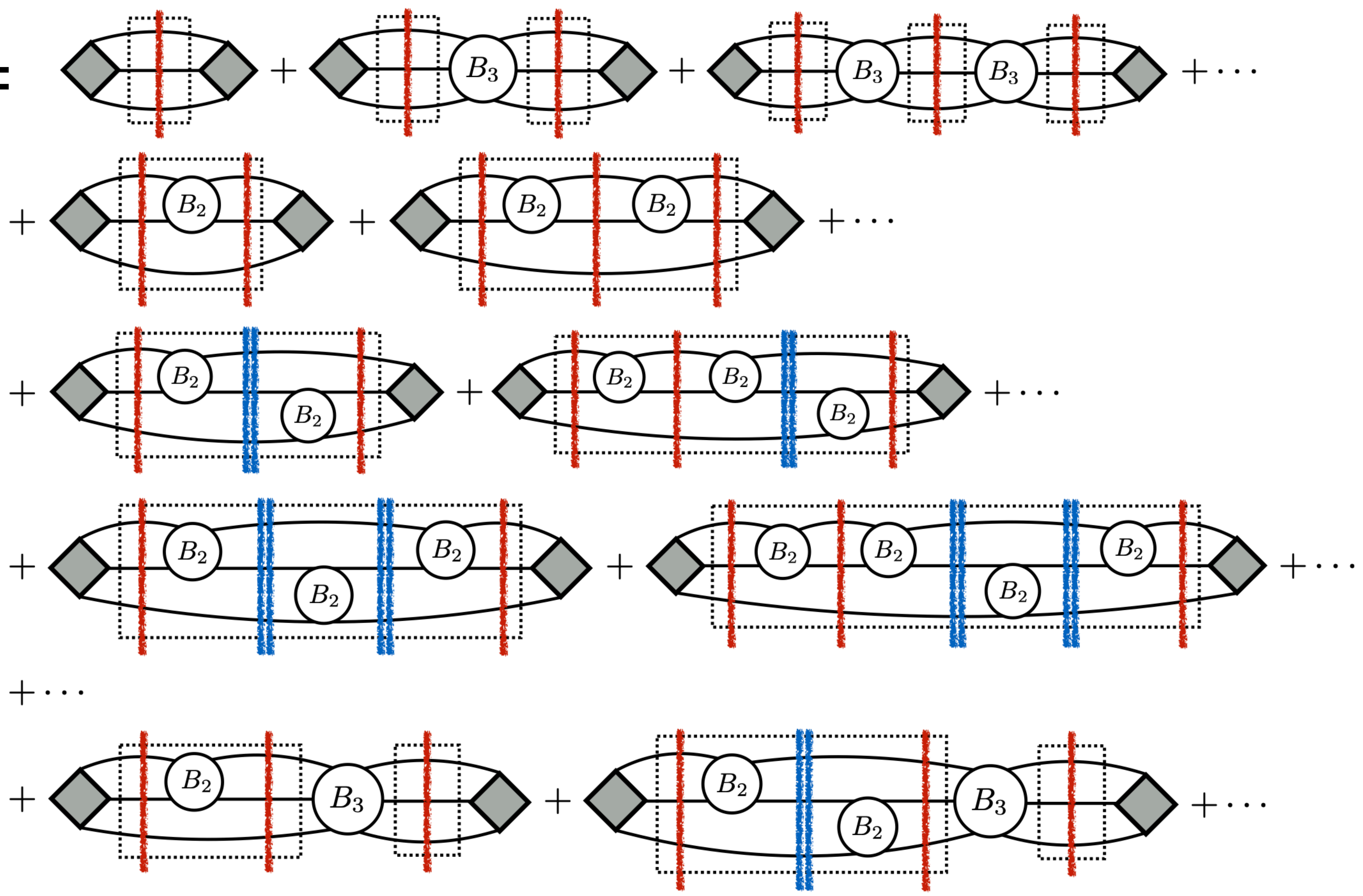

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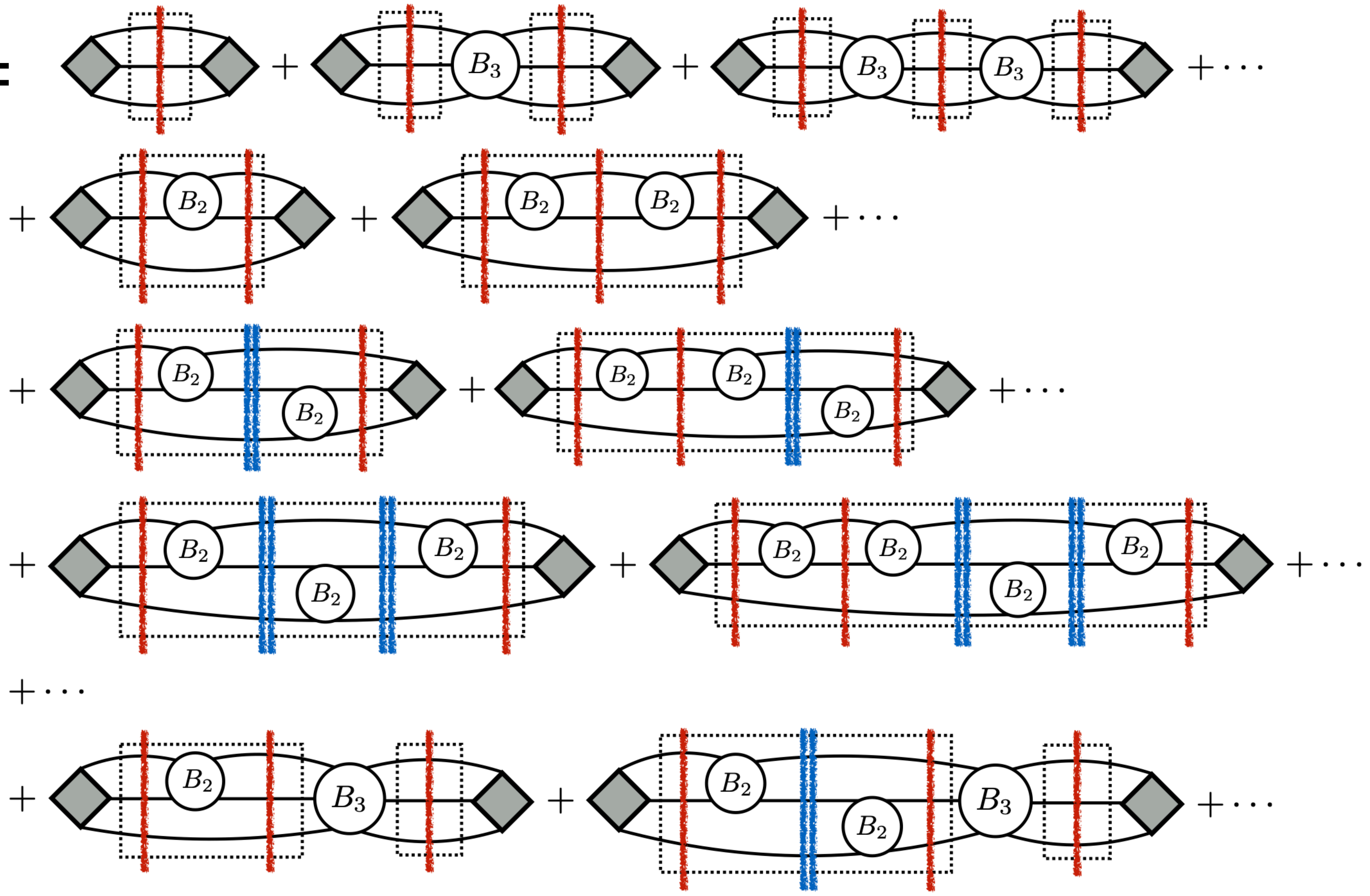

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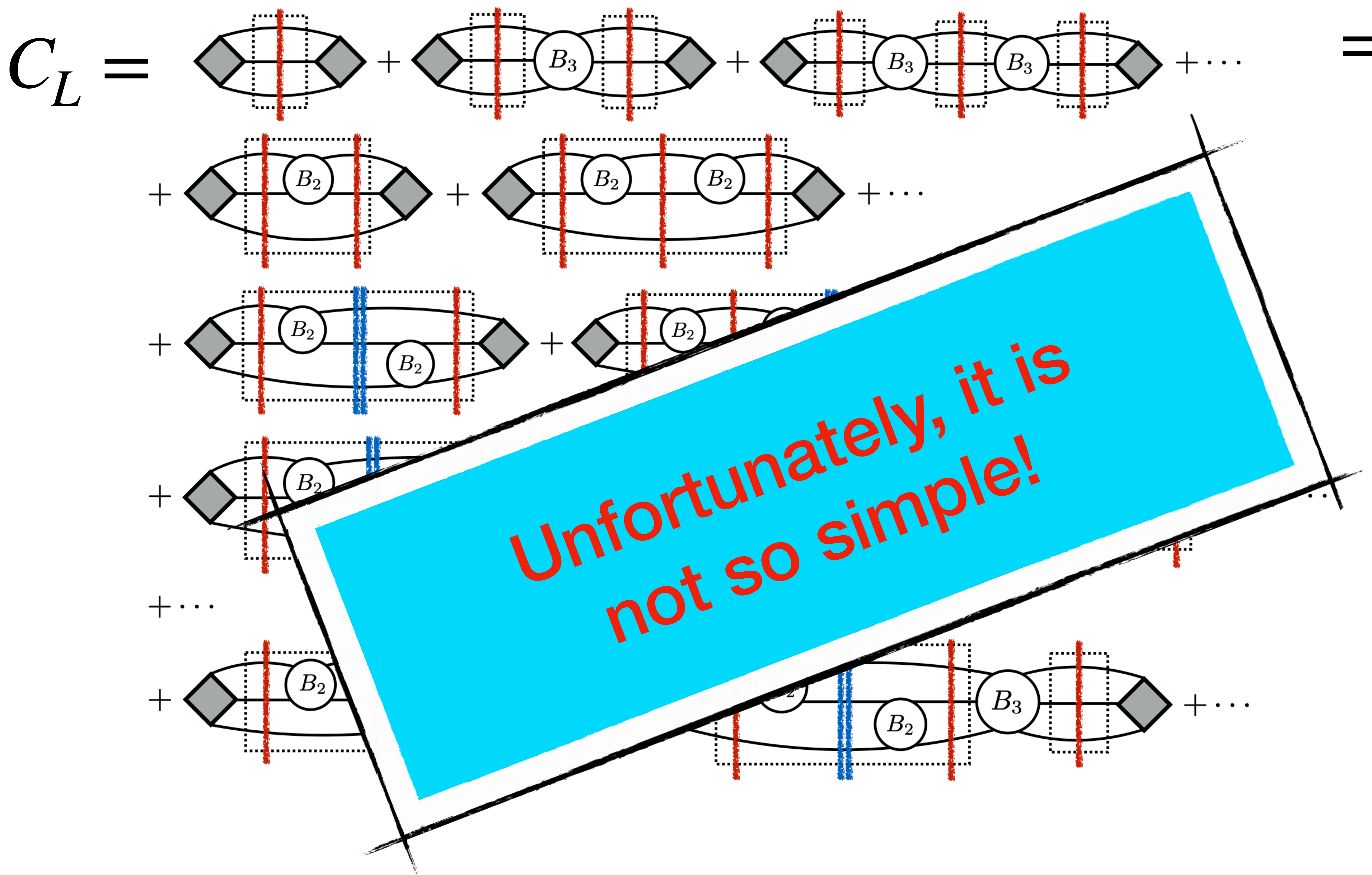

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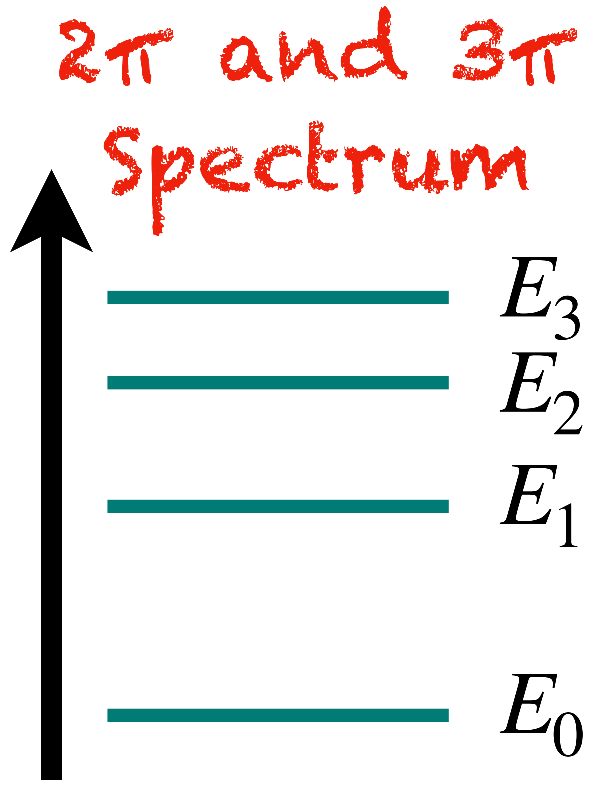

1. **Determine $\mathscr{K}_2$ and $\mathscr{K}_{df,3}$ from the two and three-pion spectrum**

Hansen, Sharpe [arXiv:1408.5933]

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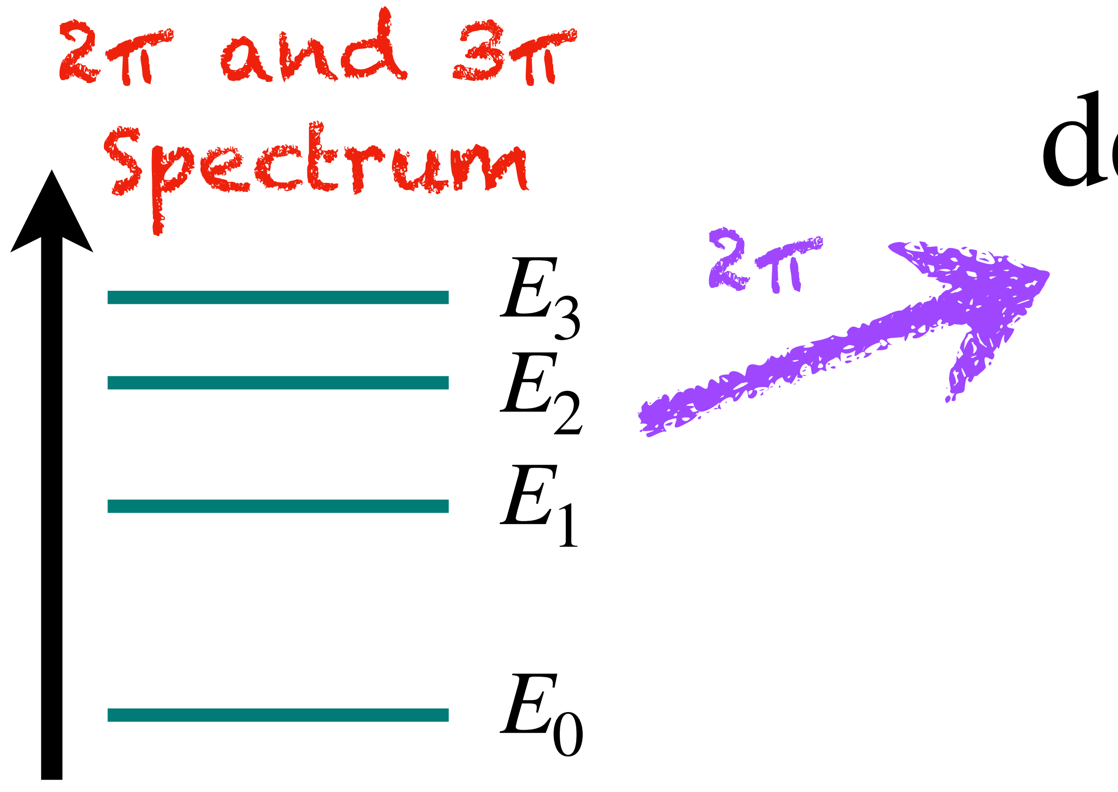

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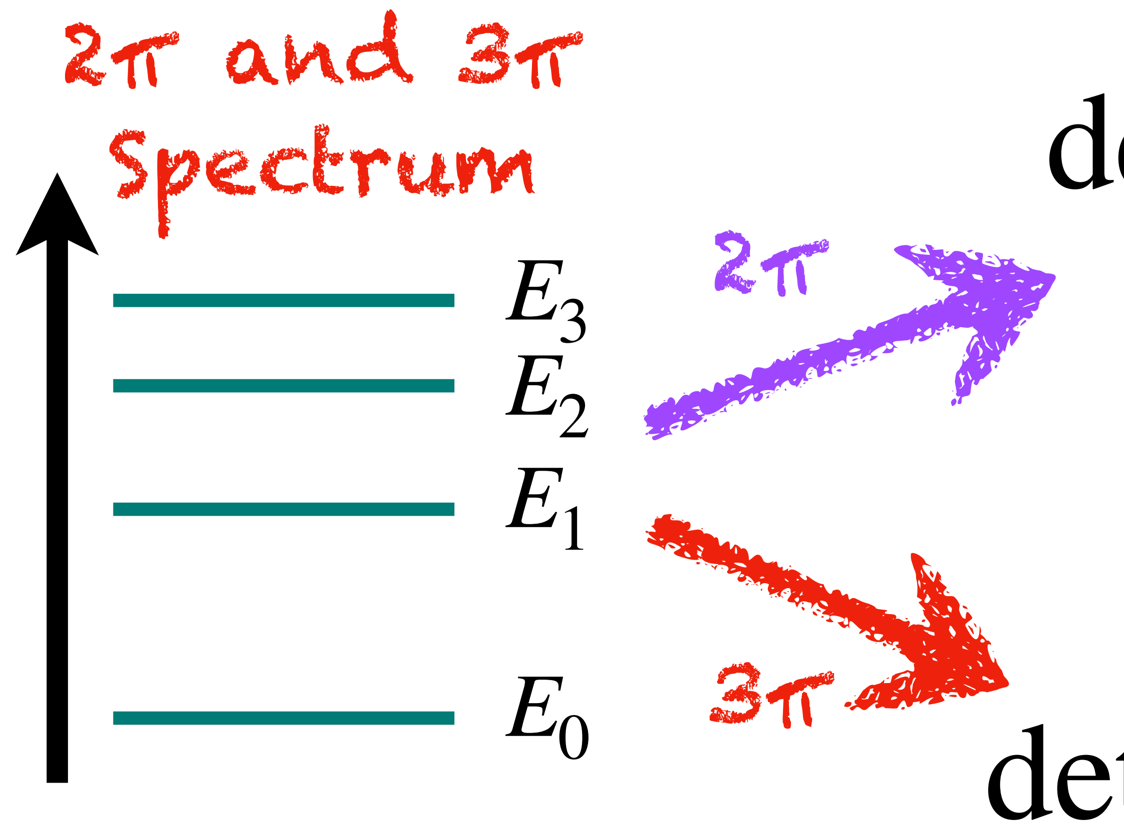

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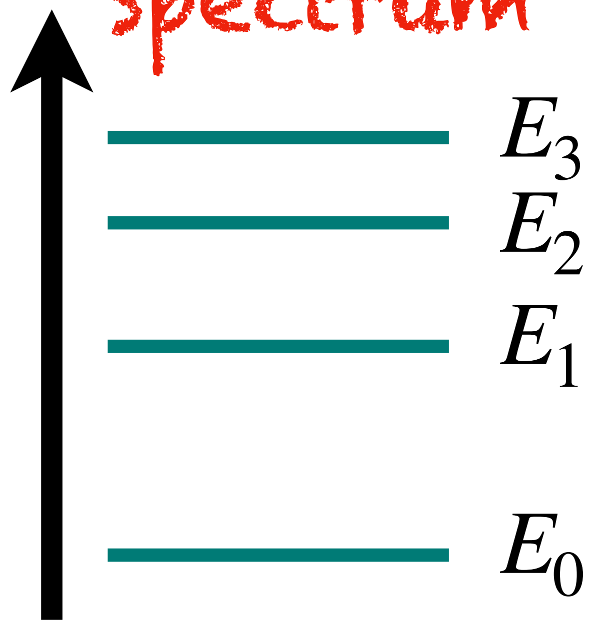
Three-hadron formalism

2π and 3π Spectrum



2π

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3π

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Hansen, Sharpe [arXiv:1408.5933]

2. **Solve integral equations to obtain The physical three-to-three amplitude**

Hansen, Sharpe [arXiv:1504.04248]

Physical 3->3 amplitude

$$\mathcal{K}_2, \mathcal{K}_{df,3} \xrightarrow[\text{Integral equations}]{} \mathcal{M}_3$$