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Lattice Practices

Solvers II – Preconditioning and Deflation

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The scope of this excercise is to explore and play around with some options of preconditioning. Again demos for each task can be found in the `octave` folder for this exercise. The questions given on this sheet are meant to be discussed with your fellow lattice practicioners while inspecting the demo.

Task 1 *Preconditioned Conjugate Gradients*
We consider the sytsem $Ax = b$ with A the discrete Laplacian (see yesterday's exercises).

1. The first and most simple preconditioning one can use is diagonal scaling (a.k.a. Jacobi) preconditioning. In here

$$S = D^{-1}, \quad D = \mathrm{diag}(A).$$

   - Why does this preconditioning idea fail miserably (It does not help at all!)? (*Hint:* Inspect the diagonal of A.)

2. Preconditioning by $SSOR$ (symmetric successive over-relaxation)

$$\begin{aligned} x^{(k+1/2)} &= x^{(k)} + (\tfrac{1}{\omega}D + L)^{-1} r^{(k)} \\ x^{(k+1)} &= x^{(k+1/2)} + (\tfrac{1}{\omega}D + U)^{-1} r^{(k)} \end{aligned}$$

   reduces the condition number significantly. You can modify the over-relaxation parameter $\omega \in (0, 2)$ and look at the impact on preconditioning efficiency.

3. Compare the spectrum of the SSOR preconditioned matrix with the one you obtained in task 1 of yesterday's exercise.

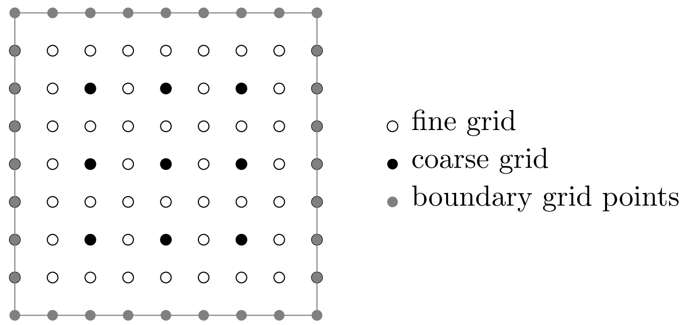
4. **Bonus***: Consider the situation, where the spectrum of A (hermitian positive definite) has the following structure. All the eigenvalues but one of A are contained in an interval $[a,b]$, the remaining eigenvalue is located at $c \gg b$ (or $0 < c \ll a$). Hence the condition number κ is given by
$$\kappa = \frac{c}{a} \quad (\text{or } \kappa = \frac{b}{c}).$$
Why do expect the CG method to converge much faster than predicted by the convergence theory? (*Hint:* Think about the interpretation of CG as approximating A^{-1} on the spectrum of A by a polynomial!)
   - Can you come up with a simple linear system $Ax = b$ to test the situation? (*Hint:* Prescribe the eigenvalues!)
   - Especially when using diverging preconditioners situations like the one described can occur, why? Assume that the preconditioner only diverges on a small subspace of eigenmodes.

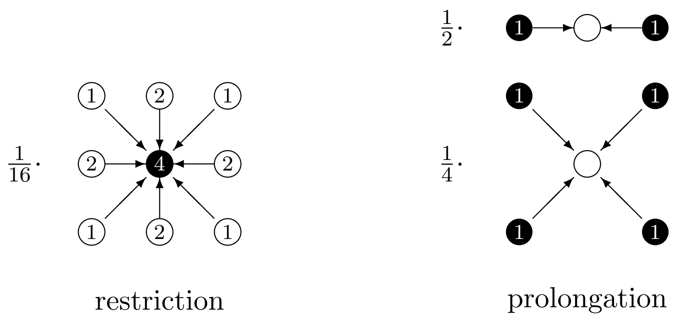
Task 2 *Preconditioned GMRES*
In order to show properties of the GMRES iteration we consider an example from Lattice QCD. The system matrix A is given by the Wilson discretization of the Dirac equation on a 4^4 lattice at $\beta = 6$ with an additive mass shift. The system matrix is non-hermitian with its eigenvalues in the right half-plane.

1. The first preconditioner to try for this problem is a domain decomposition approach with 2^4 blocks (including all 12 variables on each lattice site).
2. Next we use the odd-even preconditioner. In here we solve $S_c x_e = \tilde{b}_e$ by BiCGstab with a fixed accuracy. The method does not converge for an accuracy of 10^{-1}. What happened? What is the cure?

Task 3 *Multigrid*
For the discrete Laplacian, the symbolic stencil notation above describes a restriction operator R and a prolongation operator P.

1. Explore multigrid:
   - Run the multigrid method for $N = 7, 15, 31, 63$ and 127. For this, you need to run two lines in `Octave`:
     `>> b = rand(N);`
     `>> [x,iter] = multi_grid(b,ν);`
   - How does the number of iterations scale with N?





- Compare with CG. Run this with:

```
>> brs = reshape(b,N*N,1);
>> A = laplace(N);
>> [x, ~, ~, iter, ~, ~] = pcg(A,brs,1.0e-10,1000);
```

2. **Bonus*** Use multigrid as a preconditioner to GMRES:
   - How much do you gain as compared to “stand-alone” multigrid?
   - Why do we use GMRES and not CG, here?

3. **Bonus*** Explain why the multigrid idea is more difficult to apply to the gauge Laplacian (and to the Wilson-Dirac system).