Ek 1

Tensör & Dış Cebirler

(1)

V — $\{x_i\}$

$\dim_{\mathbb{R}} V = n < \infty$

V^* — $\{e^j\}$

$$T^r_s(V) = \otimes^r V^* \otimes^s V$$

$$\otimes ; \; T^r_s(V) \otimes T^p_q(V) \longrightarrow T^{r+p}_{s+q}(V)$$

$$a \quad , \quad b \longmapsto a \otimes b$$

$$\bigoplus_{r,s=0}^{\infty} T^r_s(V) := \mathrm{Tens}(V)$$

$\dim_{\mathbb{R}}$ ∞

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C$$

$$1 \in \mathrm{Tens}(V)$$

$$\mathbb{R}\{1\} \underset{v.s.}{\simeq} \mathbb{R}$$

$$\mathrm{Tens}(V) = \langle 1, V, V^* \rangle$$

$r = 0$, $\mathrm{Tens}_0(V)$

$s = 0$, $\mathrm{Ten}^0(V)$

bir Abel-sel grup

$a \in T^r(V) \Rightarrow \deg a = r \in \mathbb{Z}$

r. dereceden homojendir denir

$$\deg(a \otimes b) = \deg a + \deg b \; ; \; b \in T^s(V)$$

$$= r + s$$

2

involüter otomorfizm

$$\eta : \mathrm{Tens}^{\cdot}(V) \longrightarrow \mathrm{Tens}^{\cdot}(V)$$

$$a \in T^{r}(V) \Rightarrow \eta(a) = a^{\eta} := (-1)^{\deg a}\, a$$

$$= (-1)^{r}\, a$$

$$\eta^{2} = \mathrm{Id}_{\mathrm{Tens}^{\bullet}(V)} := 1$$

$$\eta(a \otimes b) = \eta(a) \otimes \eta(b)$$

$$\mathrm{Tens}^{\cdot}(V) = \mathrm{Tens}^{\cdot}_{+}(V)\ \hat{\oplus}\ \mathrm{Tens}^{\cdot}_{-}(V)$$

Abelsel grup

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_2 = \{[0], [1]\}$$

$$i \longmapsto \phi(i) = [i]$$

daha kaba bir derecelendirme

$$\xi : \mathrm{Tens}^{\cdot}(V) \longrightarrow \mathrm{Tens}^{\cdot}(V)$$

$$(\beta^{1} \otimes \beta^{2} \otimes \cdots \otimes \beta^{r})^{\xi} = \beta^{r} \otimes \cdots \otimes \beta^{2} \otimes \beta^{1}$$

$\in T^{r}_{dec.}(V)$ $\qquad \beta^{i} \in V^{*}$

$$(A \otimes B)^{\xi} = B^{\xi} \otimes A^{\xi}$$

açıkça $\xi^{2} = 1$

(3) **iç türev**

$X \in V$ için ve $A, B \in \mathrm{Tens}^{\bullet}(V)$

$$i_X(A \otimes B) = (i_X A) \otimes B + \eta(A) \otimes i_X B$$

i_X, $\mathrm{Tens}^{\bullet}(V)$'nin $\mathbb{R}$-çizgisel bir dönüşümüdür ve η otomorfizmine göre de bir (cebirsel) anti-türevdir.

$$i_X \beta = \beta(X) \quad ; \quad \beta \in V^*$$

$$i_X \lambda = 0 \quad ; \quad \lambda \in \mathbb{R} \simeq \mathbb{R}\{1\}$$

yani $i_X 1 = 0$

O halde iç türev $\mathbb{Z}$-derecelendirmesine göre $\mathrm{Tens}^{\bullet}(V)$'nin -1. dereceden homojen bir $\mathbb{R}$-çizgisel gönderimidir.

$$[i_X, \eta]_+ = 0$$

$$[i_X, i_Y]_+ = 0 \quad ; (i_X)^2 = 0$$

indeksi 2 nilpotent bir işlemci

④

Tamamen anti-simetrik tensörlerin

dış cebri

$$\Lambda^r(V) \underset{v.u.}{\subseteq} T^r(V)$$

dim → n

dim → n^r

olma özelliği sıralamadan bağımsız olmayı doğurur

$\{X_i\}$; $1 \leq i \leq n$

$$a \in \Lambda^r(V) \xrightarrow{\text{dim}} \#\{a(X_{i_1}, \ldots, X_{i_r})\} = \binom{n}{r}$$

$$1 \leq i_j \leq n$$

$$1 \leq j \leq r$$

$$\begin{aligned} \text{Alt} : T^r(V) &\longrightarrow \Lambda^r(V) \\ a &\longmapsto \text{Alt}(a) \end{aligned}$$

$$\text{Alt}(a)(Y_1, \ldots, Y_r) = \frac{1}{r!} \sum_{\sigma \in S(r)} \varepsilon(\sigma)\, a(Y_{\sigma(1)}, \ldots, Y_{\sigma(r)})$$

$$\# S(r) = r!$$

$$(\text{Alt})^2 = \text{Alt}$$

$$\otimes : T^r(V) \times T^s(V) \to T^{r+s}(V)$$

$$\begin{aligned} \wedge : \Lambda^r(V) \times \Lambda^s(V) &\longrightarrow \Lambda^{r+s}(V) \\ a \;,\; b &\longmapsto a \wedge b = \text{Alt}(a \otimes b) \end{aligned}$$

5

$$a \wedge b = (-1)^{r.s}\, b \wedge a$$

$$\Lambda^{\cdot}(V) = \bigoplus_{r=0}^{n} \Lambda^{r}(V)$$

V üzerindeki kovaryant dış cebir (çarpım $\wedge$)

$\binom{n}{r}$

$$\dim_{\mathbb{R}} \Lambda^{\cdot}(V) = \sum_{r=0}^{n} \binom{n}{r} = 2^{n}$$

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

Dış cebir tensör cebrinden $\mathbb{Z}$- derecelendirmesini miras alır ve "Alt" 0. dereceden homojen bir gönderim olduğu için $Tens^{\cdot}(V)$'nin, η $\Lambda^{\cdot}(V)$'nin de bir otomorfizmidir :

$$\eta(a \wedge b) = \eta(a) \wedge \eta(b)$$

$$\Lambda^{\cdot}(V) = \Lambda^{\cdot}_{+}(V) \,\hat{\oplus}\, \Lambda^{\cdot}_{-}(V)$$

$[\,Alt, \xi\,] = 0 \Rightarrow \xi$, $\Lambda^{\cdot}(V)$'nin de bir involüsyonudur.

$$a^{\xi} = (-1)^{\lfloor r/2 \rfloor}\, a$$

6

$\omega \in \Lambda^p(V)$

$(i_X\ \omega)(X_2, \dots, X_p) = p\ \omega(X, X_2, \dots, X_p)$

$V^* = \mathbb{R}\{e^i\}$

$$\omega = \sum_{i_1 < i_2 < \dots < i_p} \omega_{i_1 \dots i_p}\ e^{i_1} \wedge \dots \wedge e^{i_p} = \sum_I \omega_I\ e^I$$

$$= \frac{1}{p!}\ \omega_{i_1 \dots i_p}\ e^{i_1} \wedge \dots \wedge e^{i_p}$$

$\Lambda^p(V) = \Lambda^p_{dec}(V)$, $n = 1, 2, 3$

$p = 0, 1$; $0, 1, 2$; $0, 1, 2, 3$

e^1 $e^1 \wedge e^2$ $e^1 \wedge e^2$

$e^2 \wedge e^3$

$e^3 \wedge e^1$

$e^1 \wedge e^2 + e^2 \wedge e^3 + e^3 \wedge e^1$

$e^1 \wedge (e^2 - e^3) + e^2 \wedge e^3$

$e^1 \wedge (e^2 - e^3) + (e^2 - e^3) \wedge e^3$

$$\underbrace{(e^2 - e^3)}_{:= \beta^1} \wedge \underbrace{(e^3 - e^1)}_{:= \beta^2}$$

7

$$* : \Lambda^p(V) \longrightarrow \Lambda^{n-p}(V)$$

$\{e^a\}$, g^* - ortonormal ko-çerçeve ise

$$*(e^{i_1} \wedge \cdots \wedge e^{i_p}) = \frac{1}{(n-p)!} \varepsilon^{i_1 \cdots i_p}{}_{i_{p+1} \cdots i_n} e^{i_{p+1}} \wedge \cdots \wedge e^{i_n}$$

$$** \Big|_{\Lambda^p} = (-1)^{p(n-1)} \operatorname{sign}(\det g)$$

dün

$F \in \Gamma \Lambda^2 M$, $n = 4$, $\eta = \begin{pmatrix} +1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & -1 \end{pmatrix}$

$$** F = (-1)^{2(2-1)} (-1) F$$

$\mathcal{L}_V g = 0$, $g(V,V) < 0$ $\Rightarrow$ g durağan

V global

$+$

$V \in \Gamma T^{\perp} \Sigma_\alpha$; $\forall \alpha$

$\{\Sigma_\alpha\}$ uzaysal hiperyüzeylerin bir ailesi

$$\bigsqcup_\alpha \Sigma_\alpha = M$$