Generalized Geometry

Ümit Ertem

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Lecture 1

Basic Concepts

Let us consider an n-dimensional manifold M. We denote the tangent and cotangent bundles whose sections correspond to vector fields and 1-forms as TM and T^*M, respectively. The basic idea of generalized geometry is to combine vectors and 1-forms into a single object. So, we consider the direct sum of tangent and cotangent bundles $E = TM \oplus T^*M$ which is called the generalized tangent bundle. Sections of E are called generalized vectors and a generalized vector $\mathcal{X} \in \Gamma E$ can be written in terms of a vector field $X \in \Gamma TM$ and a 1-form $\xi \in \Gamma T^*M$ as

$$\mathcal{X} = X + \xi$$

or we can also denote it in the matrix form as $\mathcal{X} = \begin{pmatrix} X \\ \xi \end{pmatrix}$. On E, one can define a bilinear form $< , >$ which is written for generalized vectors $\mathcal{X} = X + \xi$ and $\mathcal{Y} = Y + \eta$ as follows

$$< \mathcal{X}, \mathcal{Y} >= \frac{1}{2}\big(i_X \eta + i_Y \xi\big) \tag{1.1}$$

where i_X denotes the interior derivative or contraction with respect to the vector field X. So, for a single generalized vector, we have $< \mathcal{X}, \mathcal{X} >= i_X \xi$. The symmetry group of this bilinear form is the orthogonal group

$$O(TM \oplus T^*M) = \{\mathbb{A} \in GL(TM \oplus T^*M)| < \mathbb{A}., \mathbb{A}. >=< ., . >\}$$

and this bilinear form has signature (n, n). We can see this as follows. For the basis vectors $\{X_a\} \in \Gamma TM$ and basis 1-forms $\{e^a\} \in \Gamma T^*M$ for $a = 1, ..., n$, we can define a basis for generalized vectors as

$$\mathcal{X}_A = X_a + \epsilon e_a$$

where $A = 1, ..., 2n$ and $\epsilon = \pm$. For $\epsilon = 1$, $< , >$ gives a $+$ sign and for $\epsilon = -1$, $< , >$ gives a $-$ sign. The elements of $O(n, n)$ are in the form of

$$\mathbb{Q} = \begin{pmatrix} \mathcal{A} & \beta \\ B & -\mathcal{A}^T \end{pmatrix}$$

where

$$\mathcal{A} : \Gamma TM \to \Gamma TM \qquad , \qquad \mathcal{A}^T : \Gamma T^*M \to \Gamma T^*M$$

$$\beta : \Gamma T^*M \to \Gamma TM \qquad , \qquad B : \Gamma TM \to \Gamma T^*M$$

and we can write $B \in \Lambda^2 T^*M$ as a 2-form and $\beta \in \Lambda^2 TM$ as a 2-vector. The invariance of $<,>$ under $O(n,n)$ gives shear transformations for generalized vectors.We have B- and β-transforms of $\mathcal{X}$ as follows

$$e^B(X+\xi) = X + \xi + i_X B \tag{1.2}$$

$$e^\beta(X+\xi) = X + i_\xi \beta + \xi. \tag{1.3}$$

A metric g defined on M can be seen as a map $g : TM \to T^*M$ which is invertible. So, we can define a generalized metric on E induced by g as follows

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{1.4}$$

So, the $\mathcal{G}$-dual of a generalized vector $\mathcal{X} = X + \xi$ can be written as

$$\widetilde{\mathcal{X}} = \mathcal{G}(\mathcal{X}) = \begin{pmatrix} \widetilde{X} \\ \widetilde{\xi} \end{pmatrix} \in \Gamma E^*$$

where $\widetilde{X}$ denotes the g-dual 1-form and $\widetilde{\xi}$ denotes the g-dual vector field and $E^* \simeq E$. In general, the B-transform of $\mathcal{G}$ can be written as

$$\mathcal{G}_B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \tag{1.5}$$

which is the generalized metric induced by g and B. Since $\mathcal{G}^2 = I$, ± 1 eigenspaces of $\mathcal{G}$ which are denoted by $V^\pm$ give a metric splitting of E

$$TM \oplus T^*M = V_+ \oplus V_-$$

corresponding to maximally positive/negative definite subbundles. The generalized metric $\mathcal{G}$ can be written in terms of the bilinear form restricted to $V_\pm$ as follows

$$\mathcal{G}(,) = < , >_+ - < , >_- . \tag{1.6}$$

We can also define an admissible metric for which the choices of $V_\pm$ correspond to

$$V_\pm = \left\{ X \pm \widetilde{X} : X \in \Gamma TM \right\}.$$

Similar to the Lie bracket $[\, , \,]$ of vector fields on ΓTM, we can also define a bracket operation on E called Courant bracket as $[\, , \,]_C : \Lambda^2 E \to \Gamma E$. For two generalized vectors $\mathcal{X} = X + \xi$ and $\mathcal{Y} = Y + \eta$, Courant bracket is defined as

$$[\mathcal{X}, \mathcal{Y}]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d\left(i_X \eta - i_Y \xi\right) \tag{1.7}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field X and d is the exterior derivative. If we define the anchor map $\pi : \Gamma E \to \Gamma TM$ as $\pi(\mathcal{X}) = X$, then the Courant bracket satisfies

$$\pi([\mathcal{X},\mathcal{Y}]_C) = [\pi(\mathcal{X}), \pi(\mathcal{Y})] \tag{1.8}$$

which can easily be seen from the definition. Although the Courant bracket is an antisymmetric bracket, it does not satisfy the Jacobi identity and hence does not correspond to a Lie bracket. However, with the definition of the Courant bracket, E admits a Courant algebroid structure. For $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma E$, a Courant algebroid $(E, [\,,\,]_C, <\,,\,>, \pi)$ satisfies the following properties

$$\pi([\mathcal{X},\mathcal{Y}]_C) = [\pi(\mathcal{X}), \pi(\mathcal{Y})]$$
$$[\mathcal{X}, f\mathcal{Y}]_C = f[\mathcal{X},\mathcal{Y}]_C + (\pi(X)f)\mathcal{Y} - <\mathcal{X},\mathcal{Y}> df$$
$$<df, dg> = 0$$
$$[\mathcal{X},[\mathcal{Y},\mathcal{Z}]_C]_C + [\mathcal{Y},[\mathcal{Z},\mathcal{X}]_C]_C + [\mathcal{Z},[\mathcal{X},\mathcal{Y}]_C]_C = dNij(\mathcal{X},\mathcal{Y},\mathcal{Z})$$

which is satisfed by the generalized tangent bundle $E = TM \oplus T^*M$. Here Nij denotes the Nijenhuis operator defined by

$$Nij(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \frac{1}{3}\left(<[\mathcal{X},\mathcal{Y}]_C, \mathcal{Z}> + <[\mathcal{Y},\mathcal{Z}]_C, \mathcal{X}> + <[\mathcal{Z},\mathcal{X}]_C, \mathcal{Y}>\right).$$

On E, one can define an isotropic splitting $s : \Gamma TM \to \Gamma E$ and this determines a closed 3-form H on M given by

$$H(X,Y,Z) = (i_Y i_X H)(Z) = <[s(X), s(Y)]_C, s(Z)> \tag{1.9}$$

for $X, Y, Z \in \Gamma TM$. The cohomology class of H classifies the Courant algebroids on M up to isomorphism. In the presence of non-zero H, one can modifiy the Courant bracket and define the twisted Courant bracket $[\,,\,]_H$ as follows

$$[\mathcal{X},\mathcal{Y}]_H = [\mathcal{X},\mathcal{Y}]_C - i_X i_Y H. \tag{1.10}$$

$[\,,\,]_H$ also defines a Courant algebroid structure for $dH = 0$. Another bracket operation on E which satisfies the Jacobi identity but not antisymmetric is the Dorfman bracket $[\,,\,]_D$ and it is defined as

$$[\mathcal{X},\mathcal{Y}]_D = [X,Y] + \mathcal{L}_X \eta - i_Y d\xi. \tag{1.11}$$

Courant bracket correspond to the antisymmetrization of the Dorfman bracket and it can also be written as

$$[\mathcal{X},\mathcal{Y}]_C = [\mathcal{X},\mathcal{Y}]_D - d <\mathcal{X},\mathcal{Y}>. \tag{1.12}$$

Dorfman bracket originates from the Lie derivative on differential forms and it is also considered as the generalized Lie derivative

$$\mathbb{L}_{\mathcal{X}}\mathcal{Y} = [\mathcal{X},\mathcal{Y}]_D$$

and it satisfies

$$\begin{aligned}\mathbb{L}_{\mathcal{X}}(f\mathcal{Y}) &= (\pi(\mathcal{X})f)\mathcal{Y} + f\mathbb{L}_{\mathcal{X}}\mathcal{Y} \\ [\mathbb{L}_{\mathcal{X}}, \mathbb{L}_{\mathcal{Y}}] &= \mathbb{L}_{[\mathcal{X},\mathcal{Y}]_D} \\ \mathbb{L}_{\mathcal{X}}(\mathcal{Y}\otimes\mathcal{Z}) &= \mathbb{L}_{\mathcal{X}}\mathcal{Y}\otimes\mathcal{Z} + \mathcal{Y}\otimes\mathbb{L}_{\mathcal{X}}\mathcal{Z}.\end{aligned} \tag{1.13}$$

We can define a generalized connection $\mathbb{D}$ on E which is compatible with the bilinear form $<,>$ corresponding to the linear operator

$$\mathbb{D} : \Gamma E \to \Gamma(E^* \otimes E)$$

satisfying the following identities

$$\mathbb{D}_{\mathcal{X}}(f\mathcal{Y}) = \pi(\mathcal{X})(f)\mathcal{Y} + f\mathbb{D}_{\mathcal{X}}\mathcal{Y} \tag{1.14}$$

$$\pi(\mathcal{X}) <\mathcal{Y},\mathcal{Z}> = <\mathbb{D}_{\mathcal{X}}\mathcal{Y},\mathcal{Z}> + <\mathcal{Y},\mathbb{D}_{\mathcal{X}}\mathcal{Z}> \tag{1.15}$$

where f is a function, $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Gamma E$ and we denote $\mathbb{D}_{\mathcal{X}}\mathcal{Y} = i_{\mathcal{X}}\mathbb{D}(\mathcal{Y})$. Here i is the contraction operator of elements in ΓE^* with elements in ΓE. Given a standard connection ∇ on TM, we can define the generalized connection $\mathbb{D}^{\nabla}$ induced by ∇ as

$$\mathbb{D}^{\nabla} = \nabla \oplus \nabla$$

corresponding to

$$\mathbb{D}^{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\pi(\mathcal{X})}\mathcal{Y}. \tag{1.16}$$

However, there is a freedom in choosing a connection as $\mathbb{D}' = \mathbb{D} + \alpha$ also defines a connection where $\alpha \in \Gamma(E^* \otimes o(E))$ and $o(E)$ denotes the bundle of skew-symmetric endomorphisms of E w.r.t. $<,>$. However, by considering relevant torsion-free, metric compatible and divergence-fixing conditions, we can define a unique connection on E [ref].