Lecture 2

Spinors and Complex Structures

Generalized tangent bundle E has a natural action on the exterior bundle ΛM. For $\mathcal{X} = X + \xi \in \Gamma E$ and $\omega \in \Lambda M$, it is given by

$$(X + \xi).\omega = i_X \omega + \xi \wedge \omega. \tag{2.1}$$

Then, it can easily be seen that

$$\begin{aligned} (X+\xi)^2.\omega &= i_X(i_X\omega + \xi \wedge \omega) + \xi \wedge (i_X\omega + \xi \wedge \omega) \\ &= (i_X\xi)\omega \\ &= < \mathcal{X}, \mathcal{X} > \omega \end{aligned}$$

which defines a Clifford algebra structure on E. For $\mathcal{X}, \mathcal{Y} \in \Gamma E$, we have

$$\mathcal{X}.\mathcal{Y} + \mathcal{Y}.\mathcal{X} = 2 < \mathcal{X}, \mathcal{Y} > .$$

So, the Clifford algebra has a natural representation on $S = \Lambda M$ which is the generalized spinor space. Since for signature (n, n), the volume element z satisfies $z^2 = 1$, we have

$$S = S^+ \oplus S^-$$

which is equivalent to the decomposition

$$\Lambda M = \Lambda^{even} M \oplus \Lambda^{odd} M.$$

So, the positive helicity generalized spinors on E correspond to even degree differential forms on M and negative helicity generalized spinors on E correspond to odd degree differential forms on M. We can define an inner product on the space of generalized spinors. The spinor inner product of two generalized spinors ϕ and ψ is written as

$$(\phi, \psi) = (\phi^{\xi\eta} \wedge \psi)_n$$

which corresponds to the Mukai pairing of forms. Here $(\ )_n$ denotes the projection to the n-form part and ξ and η are inner automorphisms of the exterior algebra where η acts on a p-form ω as $\omega^\eta = (-1)^p \omega$ and ξ acts as $\omega^\xi = (-1)^{\lfloor \frac{p}{2} \rfloor}$ with $\xi\eta$ is the composition of both automorphisms. This inner product is symmetric or skew-symmetric depending on the dimension of M

$$(\phi, \psi) = (-1)^{n(n-1)/2} (\psi, \phi).$$

So, we have for $n = 0, 1 (\mathrm{mod}\ 4)$ symmetric and for $n = 2, 3 (\mathrm{mod}\ 4)$ skew inner products. Since, only the n-form part remains in the spinor inner product, one can see that, for $n = 0 (\mathrm{mod}\ 2)$, inner products of elements

$$S^+ \times S^- \text{ and } S^- \times S^+ \text{ are zero}$$

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For example, for $n = 4$, the inner product is symmetric and the even spinors are orthogonal to the odd spinors.

Let ϕ be any nonzero generalized spinor. Then, we define its null space $L_\phi \subset TM \oplus T^*M$ as follows

$$L_\phi = \{\mathcal{X} \in TM \oplus T^*M : \mathcal{X}.\phi = 0\}.$$

The key property of null spaces is that they are isotropic. If $\mathcal{X}, \mathcal{Y} \in L_\phi$, then

$$2 < \mathcal{X}, \mathcal{Y} > \phi = (\mathcal{X}.\mathcal{Y} + \mathcal{Y}.\mathcal{X}).\phi = 0$$

implying that $< \mathcal{X}, \mathcal{Y} >= 0$ for all $\mathcal{X}, \mathcal{Y} \in L_\phi$ which is the definition of the isotropic subspace. A generalized spinor ϕ is called a pure spinor when L_ϕ is maximally isotropic, i.e., has dimension n.

Example: Let $1 \in \Lambda M$ be the unit spinor. Then the null space is

$$\{X + \xi \in TM \oplus T^*M : (i_X + \xi \wedge) 1 = 0\} = TM$$

and $TM \subset TM \oplus T^*M$ is maximally isotropic. Hence 1 is a pure spinor. Similarly, let $z \in \Lambda M$ be the volume form on M. Then, its null space is

$$\{X + \xi \in TM \oplus T^*M : (i_X + \xi \wedge) z = 0\} = T^*M$$

which is maximally isotropic. Hence z is also a pure spinor.

Let us consider the tangent bundle TM of the manifold M. Suppose that we have a globally defined map

$$J : TM \to TM$$

which satisfies the property

$$J^2 = -I.$$

It follows that the action of J on TM has eigenvalues $+i$ and $-i$. J is called an almost complex structure. If J is integrable which corresponds to the following Nijenhuis tensor for vector fields $X, Y \in \Gamma TM$

$$N_J(X,Y) = J[JX,Y] + J[X,JY] - [JX,JY] + [X,Y]$$

vanishes

$$N_J(X,Y) = 0.$$

Then, J is called a complex structure.

On the other hand, if there is a globally defined non-degenerate 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$, then it is said that M admits a pre-symplectic structure. Moreover, if ω is integrable, namely it satisfies the condition

$$d\omega = 0$$

then ω is called a symplectic structure on M. In the presence of both a complex structure and a symplectic structure, a metric can be defined in terms of them as follows

$$g(X,Y) = \omega(X, JY).$$

In fact, in the presence of two of complex, symplectic and metric structures, the third can be written in terms of the other two.

One of the main properties of generalized geometry is that it unifies the complex and symplectic structures in the concept of generalized complex structure. A generalized almost complex structure is defined as a map

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$$

which satisfies

$$\mathcal{J}^2 = -I$$

and it also has the property

$$< \mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y} > = < \mathcal{X}, \mathcal{Y} > .$$

Associated to $\mathcal{J}$, there are two subbundles $L_{\mathcal{J}}, \overline{L}_{\mathcal{J}} \subset (TM \oplus T^*M) \otimes \mathbb{C}$ with fibers respectively $\pm i$ eigenspaces of the action of $\mathcal{J}$. $\mathcal{J}$ is called a generalized complex structure if $L_{\mathcal{J}}$ corresponds to a maximally isotropic subbundle. Namely for the sections of $L_{\mathcal{J}}$, $\mathcal{X}, \mathcal{Y} \in \Gamma L_{\mathcal{J}}$, they satisfy

$$< \mathcal{X}, \mathcal{X} > = 0$$

and

$$[\mathcal{X}, \mathcal{Y}]_C \in \Gamma L_{\mathcal{J}}.$$

We can obtain the ordinary complex and symplectic structures from the special cases of the generalized complex structure. For example, for the ordinary complex structure J, we can construct a generalized complex structure as follows

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix}$$

and from a symplectic structure ω, we can construct the following generalized complex structure

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}.$$

Hence, ordinary complex and symplectic structures are special cases for the generalized complex structure. In fact, generalized metric can be written as a combination of two generalized complex structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$ as

$$\mathcal{G} = -\mathcal{J}_J\mathcal{J}_\omega = -\begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix}\begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & J\omega^{-1} \\ J^T\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$