Lecture 3

Supergravity

Supergravity theories are supersymmetric generalizations of General Relativity in various dimensions. A supergravity action S consists of bosonic (ϕ_i) and fermionic (ψ_i) fields;

$$S = \text{bosonic } (\phi_i, \nabla\phi_i, ...) + \text{fermionic } (\psi_i, \nabla\psi_i, ...).$$

Bosonic part corresponds to gravitational and gauge field degrees of freedom (graviton (metric), p-form fields, etc.) and fermionic part consists of matter degrees of freedom (gravitino (spin-$\frac{3}{2}$ Rarita-Schwinger field), gaugino, etc). Supersymmetry transformations relate the bosonic and fermionic fields to each other. For a spinor parameter ϵ, supersymmetry transformations are in the following form

$$\begin{aligned} \delta_\epsilon \phi &= \bar{\epsilon}.\psi \\ \delta_\epsilon \psi &= (\nabla + f(\phi))\,\epsilon \end{aligned}$$

where δ_ϵ denotes the variation of the field. When we take the fermionic fields to be zero, we obtain the bosonic supergravity whose solutions give the consistent backgrounds of the theory. When $\psi_i \to 0$ we have $\delta_\epsilon \phi_i = 0$ automatically, and $\delta_\epsilon \psi = 0$ gives a constraint on the spinor parameter ϵ that is $(\nabla + f(\phi))\,\epsilon = 0$ which is called the supergravity Killing spinor equation. So, to obtain a consistent supergravity background, besides the field equations of the theory, we must also solve the supergravity Killing spinor equation. The low energy limits of string and M-theories are described by the bosonic sectors of ten- and eleven-dimensional supergravity theories. There are five different string theories in ten dimensions: type I, type IIA and IIB, and heterotic E8 $\times$ E8 and SO(32) theories. However, some dualities called T-duality, S-duality, and U-duality between strong coupling and weak coupling limits of these theories can be defined and these dualities can give rise to one unified M-theory in eleven dimensions.

For example the type II supergravity theories include the following bosonic fields

g	metric
B_2	2-form potential
ϕ	dilaton
C_1, C_3	RR-potentials for type IIA
C_0, C_2, C_4	RR-potentials for type IIB

For vanishing dilaton $\phi = 0$, the action of type II supergravity theories is given by

$$S_{II} = \frac{1}{2\kappa^2} \int R_{ab} \wedge *e^{ab} - \frac{1}{4\kappa^2} \int H \wedge H + \text{RR-field terms}$$

where $H = dB_2$ is the 3-form field strength and the Killing spinor equation for the common sector (without RR-sector) of type II theories is

$$\nabla_X \psi - \frac{1}{4} i_X H.\psi = 0.$$

So, the content of generalized geometry is relevant to describe the NS sector of type II supergravity theories. By considering the generalized tangent bundle $E = TM \oplus T^*M$, we can combine the non-zero NS fields g and B_2 into one geometrical object which is the generalized metric

$$\mathcal{G}_B = \begin{pmatrix} -g^{-1}B_2 & g^{-1} \\ g - B_2 g^{-1} B_2 & B_2 g^{-1} \end{pmatrix}.$$

We can write the 3-form field strength $H_3 = dB_2$ as the class of the Courant algebroid structure which appear in the twisted Courant bracket

$$[\mathcal{X}, \mathcal{Y}]_H = [\mathcal{X}, \mathcal{Y}]_C - i_X i_Y H_3.$$

In that way, we can write the field equations and supersymmetry conditions in terms of generalized geometry objects for type II supergravity theories. This was one of the motivations to construct generalized geometry for describing supergravity theories in a full geometrical framework. In fact, full content of type II supergravity theoires can also be described in terms of generalized geometry [ref].

For the case of 11-dimensional supergravity, the bosonic field content is simply described as

g	metric
A_3	3-form potential

and the action is given by

$$S_M = \frac{1}{12\kappa^2} \int \left(R_{ab} \wedge *e^{ab} - \frac{1}{2} F_4 \wedge *F_4 - \frac{1}{6} A_3 \wedge F_4 \wedge F_4 \right)$$

where $F_4 = dA_3$ is the 4-form field strength. Note that the first term is the gravity action, the second term is the Maxwell-like term and the third term is the Chern-Simons term. The field equations of the theory are given by

$$Ric(X,Y)*1 = \frac{1}{2}i_X F\wedge *i_Y F - \frac{1}{6}g(X,Y)F\wedge *F \tag{3.1}$$

$$d*F = \frac{1}{2}F\wedge F \tag{3.2}$$

where $X, Y \in TM$ and Ric is the Ricci tensor. The Killing spinor condition is given by

$$\nabla_X\psi + \frac{1}{24}\left(\widetilde{X}.F_4 - 3F_4.\widetilde{X}\right).\psi.$$

Moreover, we can rename the Hodge dual of the field strength as the 7-form field strength as $F_7 = *F_4$ and we can define a 6-form potential A_6 to write the 7-form field as

$$F_7 = dA_6 - \frac{1}{2}A_3\wedge dA_3.$$

However, the field content of 11-dimensional supergravity cannot fix into the framework of ordinary generalized geometry. To describe it in a full geometrical framework, we need to define a new kind of object.

The new object is called as exceptional generalized geometry and we define the the exceptional generalized tangent bundle as follows

$$E = TM\oplus\Lambda^2 M\oplus\Lambda^5 M.$$

Hence, we have the direct sum of tangent bundle, 2-form bundle and 5-form bundles. Then, we define an exceptional generalized vector $\mathcal{X}$ as a sum of a vector X, a 2-form α_2 and a 5-form α_5

$$\mathcal{X} = X + \alpha_2 + \alpha_5.$$

Remember that in the ordinary generalized geometry case, we add the cotangent bundle (1-form bundle) to the tangent bundle and we have a 2-form B-field transform of generalized vectors. In a similar sense, we can define 3-form B_3-field and 6-form B_6-field transforms of exceptional generalized vectors as follows

$$e^{B_3}(X+\alpha_2+\alpha_5) = X + \alpha_2 + i_X B_3 + \alpha_5 - B_3\wedge\alpha_2$$

$$e^{B_6}(X+\alpha_2+\alpha_5) = X + \alpha_2 + \alpha_5 - i_X B_6.$$

Generalization of the Lie bracket similar to the Courant bracket $[\, ,\,]_E$ in exceptional generalized geometry is defined as

$$\begin{aligned}[X+\alpha_2+\alpha_5, Y+\beta_2+\beta_5]_E &= [X,Y] + L_X\beta_2 - i_Y d\alpha_2\\ &\quad + L_X\beta_5 - i_Y d\alpha_5 + d\alpha_2\wedge\beta_2\end{aligned}$$

and can be twisted with 4-form F_4 and 7-form F_7 as

$$\begin{aligned}[X+\alpha_2+\alpha_5, Y+\beta_2+\beta_5]_{F_4,F_7} &= [X,Y] + L_X\beta_2 - i_Y d\alpha_2 + L_X\beta_5 - i_Y d\alpha_5\\ &\quad + d\alpha_2\wedge\beta_2 + i_X i_Y F_4 + i_X F_4\wedge\beta_2\\ &\quad + i_X i_Y F_7 - i_Y(F_4\wedge\alpha_2).\end{aligned}$$

Hence, the field content of 11-dimensional supergravity is in correspondence with exceptional generalized geometry and can be described in a full geometrical framework in terms of it. Namely, the field equations and Killing spinor equations can be written in terms of generalized objects.