Lecture 4

T-duality

T-duality transforms the metric g and 2-form B of a background to another metric $\widehat{g}$ and 2-form $\widehat{B}$ according to the Buscher rules. However, we can picture this transformation in terms of topological quantities rather than geometric quantities. We call this approach as topological T-duality.

Let us consider a S^1 bundle E over a base manifold M

$$\pi : E \to M.$$

The total space E is a fiber bundle which is locally homeomorphic to $S^1 \times M$ but globally can be different. A 3-form field H is defined locally as the exterior derivative of the 2-form field B

$$H = dB$$

and defines a cohomology class in $H^3(E, \mathbb{Z})$. When the local exactness is true globally, H corresponds to a trivial class in cohomology. The topological properties of the T-duality map do not depend on the details of the B field, but only on the class H.

On the other hand, S^1 bundles correspond to principal $U(1)$ bundles over the base manifold M and they are classified by the second cohomology group $H^2(M, \mathbb{Z})$. The element in $H^2(M, \mathbb{Z})$ characterising the bundle $\pi : E \to M$ is realized by the first Chern class c_1. It can be computed by calculating the curvature of a principal $U(1)$ connection 1-form A corresponding to the 2-form

$$F = dA.$$

Now, we have the following topological data: A principal $U(1)$ bundle $\pi : E \to M$, together with a pair of cohomology classes (F, H). The class $F \in H^2(M, \mathbb{Z})$ is the first Chern class and determines the isomorphism classes of the bundle, The class $H \in H^3(E, \mathbb{Z})$ is the cohomology class of the curvature of the B field. We will see that T-duality intermixes F and H. If we consider the T-dual bundle

$$\widehat{\pi} : \widehat{E} \to M.$$

There are also corresponding 3-form field $\widehat{H} \in H^3(E, \mathbb{Z})$ and 2-form Chern class $\widehat{F} \in H^2(M, \mathbb{Z})$. The meaning of T-duality is relating the couples (F, H) and $(\widehat{F}, \widehat{H})$ to each other. Remember that a 3-form H on E can have three components on M and no component on S^1 such as

$$H = H_{abc} e^a \wedge e^b \wedge e^c = H_3$$

where a, b, c are coordinate indices on M and take values from $1, ..., n$, or two components on M and one component on S^1 such as

$$H = H_{ab} e^a \wedge e^b \wedge e^\theta = H_2 \wedge e^\theta$$

where θ is the coordinate index on S^1. We denote these legs on M as H_3 and H_2, respectively and we have

$$H = H_3 + H_2 \wedge e^\theta.$$

Since, the 2-form F is defined on M, its both legs are on M and it and only has F_2 component

$$F_2 = F_{ab} e^a \wedge e^b.$$

Now, T-duality transforms the fields

$$\begin{aligned} H &= H_3 + H_2 \wedge e^\theta \\ F &= F_2 \end{aligned}$$

to the fields on the bundle $\widehat{\pi} : \widehat{E} \to M$ as

$$\begin{aligned} \widehat{H} &= H_3 + F_2 \wedge e^\theta \\ \widehat{F} &= H_2 \end{aligned}$$

So, it interchanges H_2 and F_2 to each other. Then, the Chern numbers of T-dual bundles are related as

$$\begin{aligned} F &= i_{X_\theta} \widehat{H} \\ \widehat{F} &= i_{X_\theta} H. \end{aligned}$$

Let us back to the T-duality picture that transforms the metric g and 2-form B of a background to another metric $\widehat{g}$ and 2-form $\widehat{B}$ according to the Buscher rules. T-duality is applied according to a Killing vector V satisfying

$$\begin{aligned} L_V g &= 0 \\ L_V H &= 0 \end{aligned}$$

where $H = dB$ which can also be written in terms of B' in the same cohomology class $B' = B + d\zeta$ with a 1-form ζ. So, T-duality can be written in terms of a Killing vector V and a 1-form ζ. We can construct a generalized vector $\mathcal{V} = V + \zeta$

to define the T-duality operation in the framework of generalized geometry. In fact, one can check that $\mathcal{V}$ corresponds to a generalized Killing vector satisfying

$$\mathbb{L}_{\mathcal{V}}\mathcal{G} = 0$$

and we can normalize $\mathcal{V}$ as

$$< \mathcal{V}, \mathcal{V} >= 1.$$

We define the T-duality operation with respect to $\mathcal{V}$ as $T_{\mathcal{V}} \in O(n,n)$

$$T_{\mathcal{V}} = I - 2\mathcal{V}\mathcal{V}^T\eta$$

where η is the bilinear form $<,>$ and can be written as the matrix

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

To see this, let us consider for the generalized vector $\mathcal{X} = X + \xi$ the following

$$\begin{aligned} \eta(\mathcal{X},\mathcal{X}) &= \mathcal{X}^T\eta\mathcal{X} \\ &= \begin{pmatrix} X & \xi \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} X \\ \xi \end{pmatrix} \\ &= \begin{pmatrix} X & \xi \end{pmatrix}\frac{1}{2}\begin{pmatrix} \xi \\ X \end{pmatrix} \\ &= \frac{1}{2}(i_X\xi + i_X\xi) \\ &= i_X\xi. \end{aligned}$$

The action of $T_{\mathcal{V}}$ on a generalized vector $\mathcal{X} = X + \xi$ gives

$$\begin{aligned} T_{\mathcal{V}}\mathcal{X} &= \mathcal{X} - 2\mathcal{V}\mathcal{V}^T\eta\mathcal{V} \\ &= \begin{pmatrix} X \\ \xi \end{pmatrix} - 2\begin{pmatrix} V \\ \zeta \end{pmatrix}\begin{pmatrix} V & \zeta \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} X \\ \xi \end{pmatrix} \\ &= \begin{pmatrix} X \\ \xi \end{pmatrix} - \begin{pmatrix} V \\ \zeta \end{pmatrix}\begin{pmatrix} V & \zeta \end{pmatrix}\begin{pmatrix} \xi \\ X \end{pmatrix} \\ &= \begin{pmatrix} X \\ \xi \end{pmatrix} - (i_V\xi + i_X\zeta)\begin{pmatrix} V \\ \zeta \end{pmatrix} \\ &= \begin{pmatrix} X - (i_V\xi + i_X\zeta)V \\ \xi - (i_V\xi + i_X\zeta)\zeta \end{pmatrix}. \end{aligned}$$

So, we can write $T_{\mathcal{V}}\mathcal{X} = \mathcal{X} - 2 < \mathcal{X}, \mathcal{V} > \mathcal{V}$. The condition $< \mathcal{V}, \mathcal{V} >= 1$ implies that $< T_{\mathcal{V}}\mathcal{X}, T_{\mathcal{V}}\mathcal{X} >=< \mathcal{X}, \mathcal{X} >$ and so $T_{\mathcal{V}} \in O(n,n)$ and $T_{\mathcal{V}}^2 = 1$. $T_{\mathcal{V}}$ transforms the generalized metric $\mathcal{G}$ to the T-dual generalized metric as

$$\begin{aligned} \widetilde{\mathcal{G}} &= T_{\mathcal{V}}^T\mathcal{G}T_{\mathcal{V}} \\ \widetilde{\mathcal{G}}(\mathcal{X},\mathcal{X}) &= \mathcal{G}(T_{\mathcal{V}}\mathcal{X}, T_{\mathcal{V}}\mathcal{X}). \end{aligned}$$

The action of $T_{\mathcal{V}}$ on generalized spinors is given by

$$\begin{aligned}\widetilde{\Phi} &= T_{\mathcal{V}}\Phi \\ &= i_V\Phi + \zeta\wedge\Phi\end{aligned}$$

where Φ is a generalized spinor and $\widetilde{\Phi}$ its T-dual.