Induced Gravitational Waves After Ultra Slow-Roll Inflation.

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HELMHOLTZ

DESY.

Why Inflation?

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$$d\tau = dt/a(t)$$

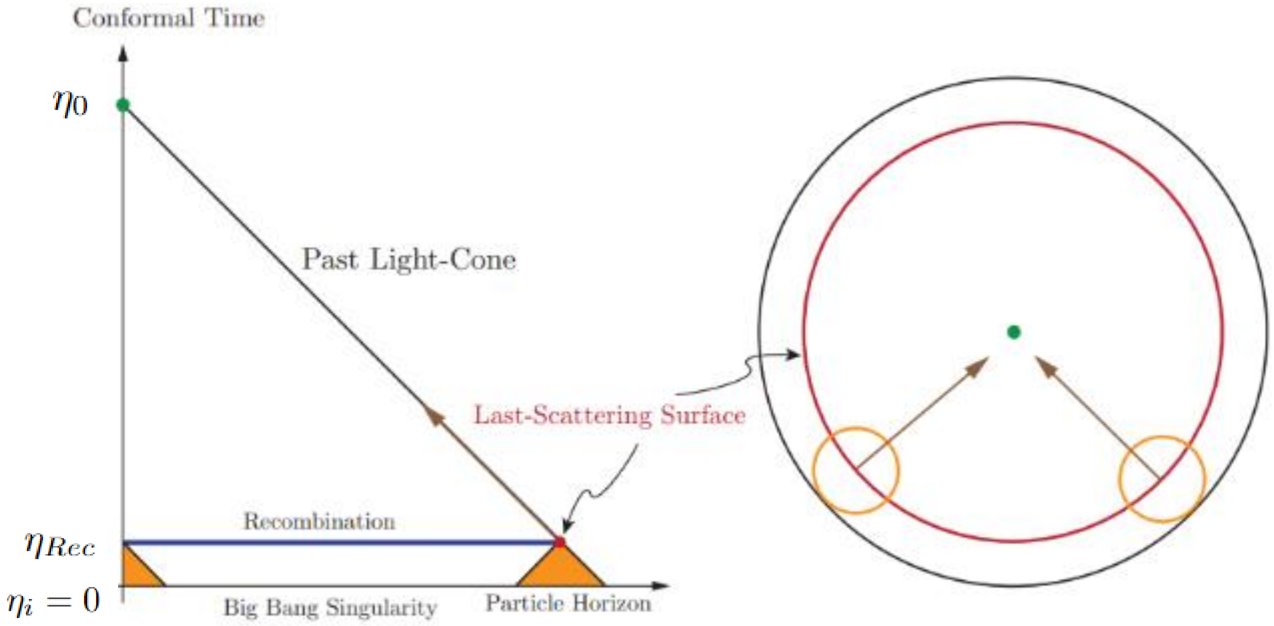
$$ds^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 d\Omega^2\right]$$

$$\Delta r_{max} = \eta - \eta_i = \int_0^t \frac{dt'}{a(t')}.$$

$$H^2 = \frac{\rho}{3M_p^2} \qquad \dot{H} + H^2 = -\frac{\rho + 3p}{6M_p^2}$$

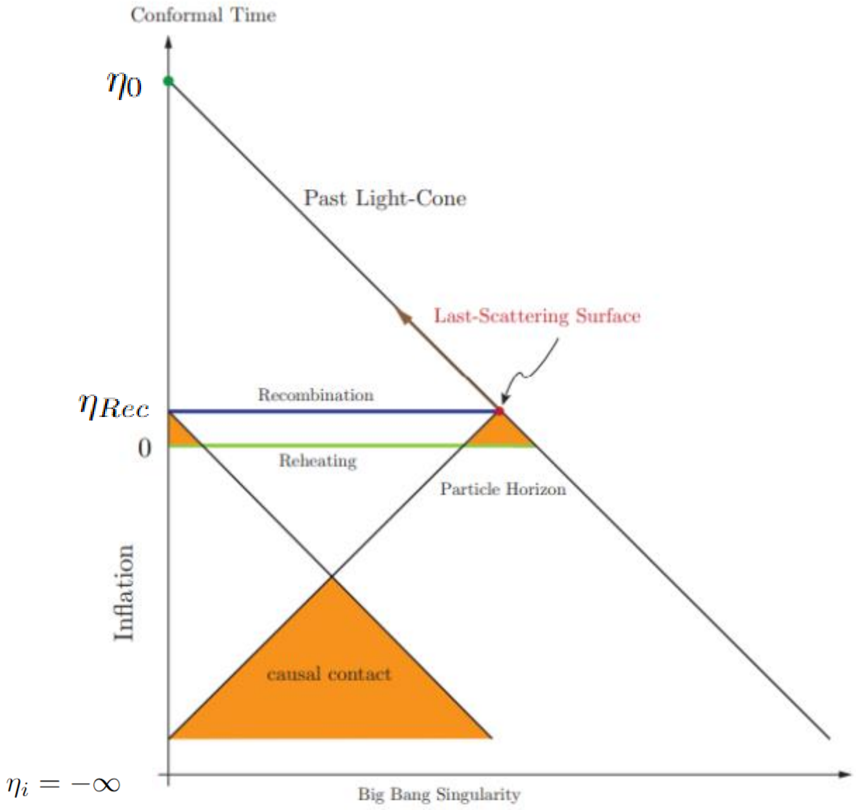
$$\omega = \frac{p}{\rho}$$

$$1 + 3\omega > 0 \implies \eta_i = 0$$



Daniel Baumann. Tasi lectures on inflation, 2012.

How can we have Inflation?



Daniel Baumann. Tasi lectures on inflation, 2012.

Slow-roll conditions.

$$\frac{d}{dt}[aH]^{-1} < 0 \implies \epsilon = \frac{-\dot{H}}{H^2} < 1 \iff 1 + 3\omega < 0$$

$$\eta = \frac{-\dot{\epsilon}}{2H\epsilon}, \qquad |\eta| < 1$$

Single-field inflaton weakly coupled to gravity.

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} - \partial_\mu\phi\partial^\mu\phi - V(\phi)\right]$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \qquad H = \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + V\right]$$

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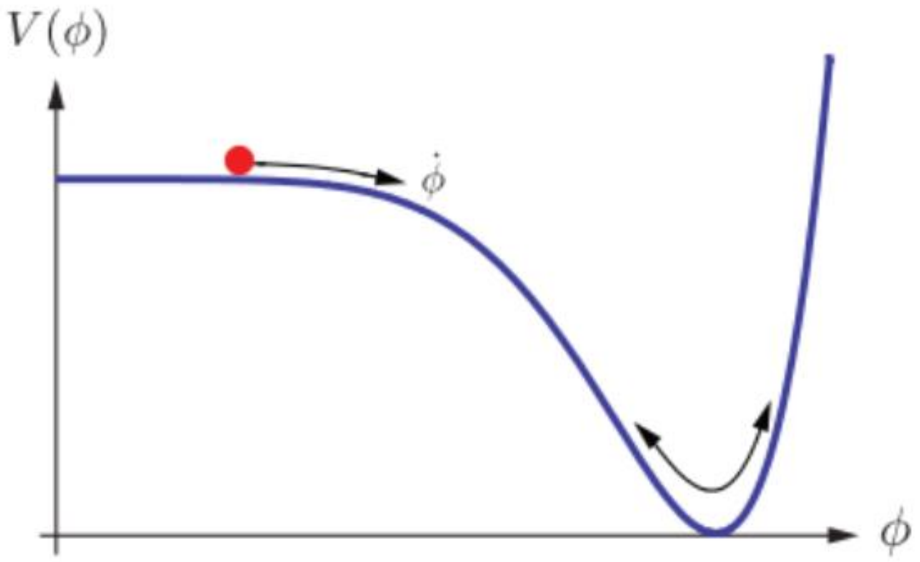

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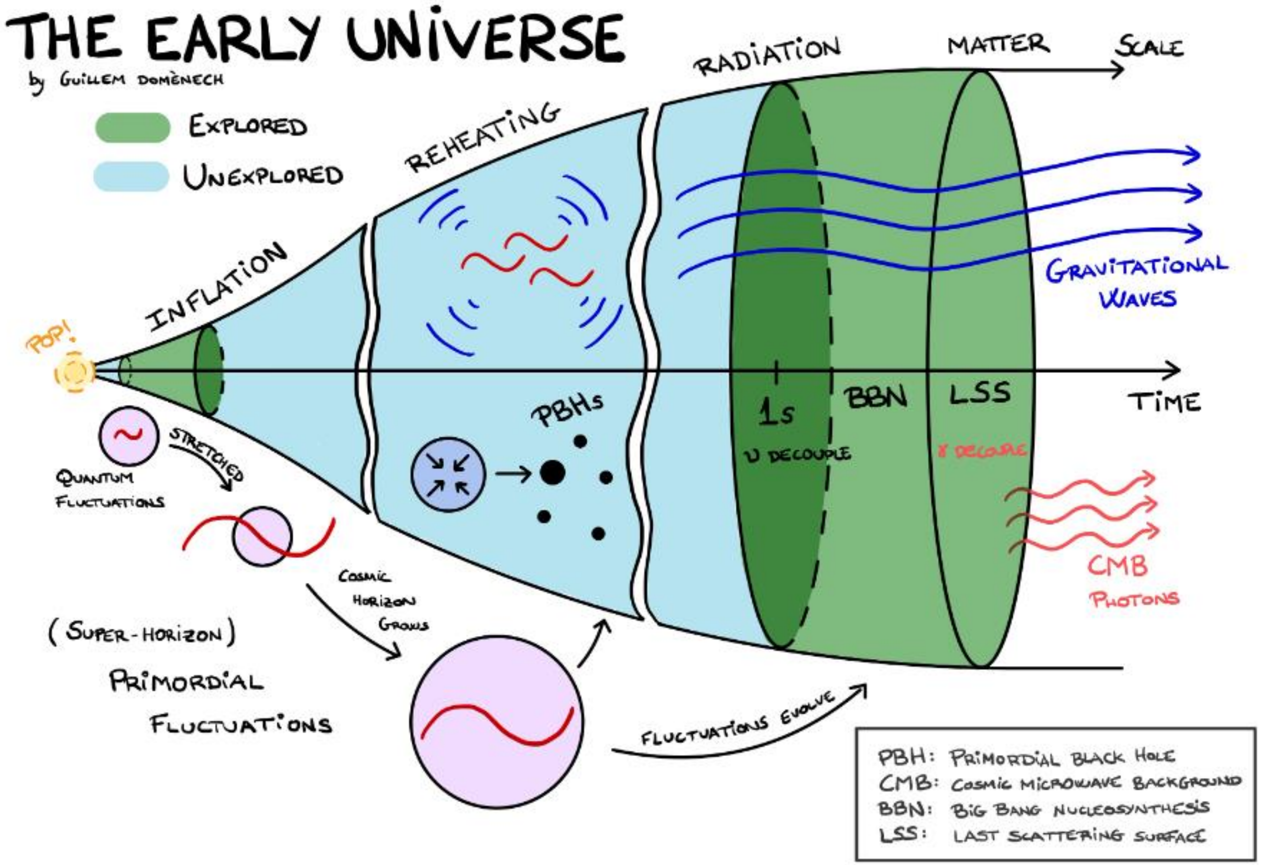
Why Ultra Slow-Roll Inflation?



https://domenechcosmo.netlify.app/post/induced-gws/

Enhancement of the power spectrum after inflation on a certain length scale.

Production of Primordial Black Holes as a consequence of density perturbations

Gravitational wave two-point function.

Perturbed metric.

$$ds^2 = a^2(\eta)\left[-(1+2\Phi)d\eta^2 + \left[(1-2\psi)\delta_{ij} + h_{ij}\right]dx^i dx^j\right]$$

Second order expansion of Einstein Eqs. in Fourier space.

$$h_{\boldsymbol{k}}^{s\,\prime\prime} + 2\mathcal{H}h_{\boldsymbol{k}}^{s\,\prime} + k^2 h_{\boldsymbol{k}}^s = S_{\boldsymbol{k}}^s(\eta)$$

s=(+, ×)

Projector

Source term

$$S_{\boldsymbol{k}}^s = \int \frac{d^3p}{(2\pi)^3}\left[p_i e_{ij}^s(\boldsymbol{k})p_j\right]\left[8\psi_{\boldsymbol{p}}\psi_{\boldsymbol{k}-\boldsymbol{p}} + \frac{16}{3(1+\omega)}\left(\psi_{\boldsymbol{p}} + \frac{\psi_{\boldsymbol{p}}'}{\mathcal{H}}\right)\left(\psi_{\boldsymbol{k}-\boldsymbol{p}} + \frac{\psi_{\boldsymbol{k}-\boldsymbol{p}}'}{\mathcal{H}}\right)\right]$$

Green's function method.

$$h_{\boldsymbol{k}}^s(\eta) = \int_0^\infty d\eta' g_{\boldsymbol{k}}(\eta,\eta')S_{\boldsymbol{k}}^s(\eta').$$

Comoving curvature perturbation and Transfer function.

$$\psi_{\boldsymbol{k}}(\eta) = T(\eta,k)\mathcal{R}_{\boldsymbol{k}}, \qquad \psi_{\boldsymbol{k}}(0) = \frac{3+3\omega}{5+3\omega}\mathcal{R}_{\boldsymbol{k}}$$

$$\psi_{\boldsymbol{k}}'' + 3(1+\omega)\mathcal{H}\psi_{\boldsymbol{k}}' + \omega k^2\psi_{\boldsymbol{k}} = 0$$

Gravitational wave two-point function.

GW compact solution.

Transfer function term.

$$h^s_{\boldsymbol{k}}(\eta) = \frac{1}{k^2}\int \frac{d^3p}{(2\pi)^3}\left[p_i p_j e^s_{ij}(\boldsymbol{k})\right]\mathcal{R}_{\boldsymbol{p}}\mathcal{R}_{\boldsymbol{k}-\boldsymbol{p}} I_k(\eta, p, |\boldsymbol{k}-\boldsymbol{p}|)$$

Quantization.

$$h^s_{\boldsymbol{k}} \to \hat{h}^s_{\boldsymbol{k}}$$

Power spectrum definition.

$$\langle \hat{h}^s_{\boldsymbol{k}} \hat{h}^t_{\boldsymbol{p}}\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_h(k)(2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{p})\delta^{st}$$

GW density.

$$\rho_{\rm GW}(\tau, \boldsymbol{x}) = \frac{M_p^2}{16a^2}\overline{\nabla_k h_{ij}(\boldsymbol{x})\nabla_k h_{ij}(\boldsymbol{x})}$$

GW density parameter.

$$\frac{\langle \rho_{\rm GW}(\tau, \boldsymbol{x})\rangle}{\rho_c(\tau)} = \int d\log q\left[\frac{1}{24}\frac{q^2}{\mathcal{H}^2}\overline{\mathcal{P}_h(q)}\right] \equiv \int d\log q\,\Omega_{\rm GW}(q,\tau)$$

$$\Omega_{\rm GW}(q,\tau) = \boxed{\frac{\Omega_\gamma(T_0)}{24}\frac{g_\star(T)}{g_\star(T_0)}\left[\frac{g_{\star s}(T_0)}{g_{\star s}(T)}\right]^{4/3}}\frac{q^2}{\mathcal{H}^2}\overline{\mathcal{P}_h(q)}$$

Entropy conservation.

Cosmological Feynman Diagrams.

To calculate the 2 point function of GW we need the 4 point function of R.

We can represent our fields with the following lines.

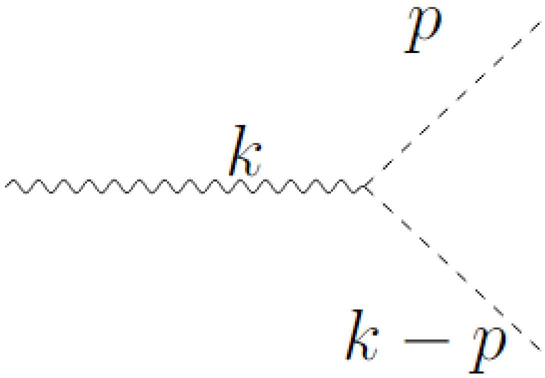
$\varphi \sim$

$\mathcal{R} \sim$

$h \sim$

Cosmological Feynman Diagrams.

Conversion between tensor perturbations h and comoving curvature perturbations $\mathcal{R}$.

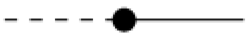


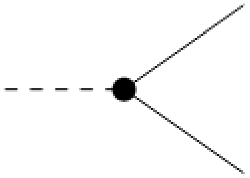
$$\sim \frac{1}{k^2} I_k(\eta, p, |\boldsymbol{k}-\boldsymbol{p}|)\,[\boldsymbol{p}\cdot\boldsymbol{e}^s(\boldsymbol{k})\cdot\boldsymbol{p}]$$

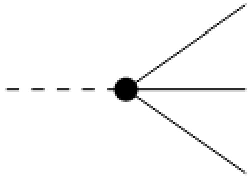
Cosmological Feynman Diagrams.

If we allow a non linear relation between R and φ we have non gaussianities.

$$\mathcal{R} = \varphi + f_{\rm NL}\varphi^2 + g_{\rm NL}\varphi^3 + \cdots$$

$1 \sim$ 

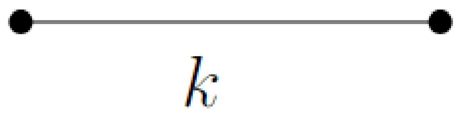
$f_{NL} \sim$ 

$g_{NL} \sim$ 

Cosmological Feynman Diagrams.

The φ propagator gives de dimensionless power spectrum of R.

$$|\varphi_k(t)|^2 = \Delta_k^2(t) = \mathcal{P}_{\mathcal{R}}(k)\frac{2\pi^2}{k^3} \quad \sim$$

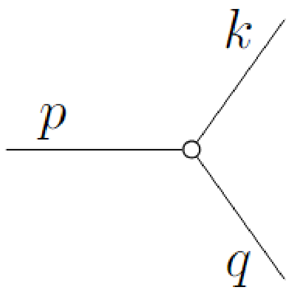


Cosmological Feynman Diagrams.

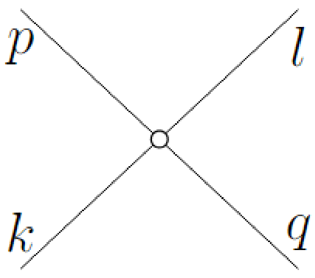
If we don’t assume φ is a Gaussian variable and admit self interactions we will have intrinsic non gaussianities.

In-In Formalism

$v_3 \sim$



$v_4 \sim$



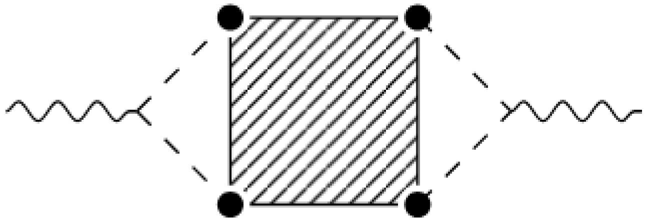
$$\frac{1}{3!}\int_{-\infty}^{t} dt'\, a^3(t')\, v_3(t')\, 2\,\mathrm{Im}[\varphi_p(t)\varphi_p^*(t')\varphi_q(t)\varphi_q^*(t')\varphi_k(t)\varphi_k^*(t')]$$

$$\frac{1}{4!}\int_{-\infty}^{t} dt'\, a^3(t')\, v_4(t')\, 2\,\mathrm{Im}[\varphi_p(t)\varphi_p^*(t')\varphi_q(t)\varphi_q^*(t')\varphi_k(t)\varphi_k^*(t')\varphi_l(t)\varphi_l^*(t')]$$

Two point function up to 2 loops.

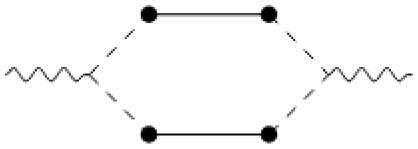
We have to consider all posible diagrams that fit inside the black box.

$$2\frac{2\pi^2}{k^3}\mathcal{P}_h(k) =$$



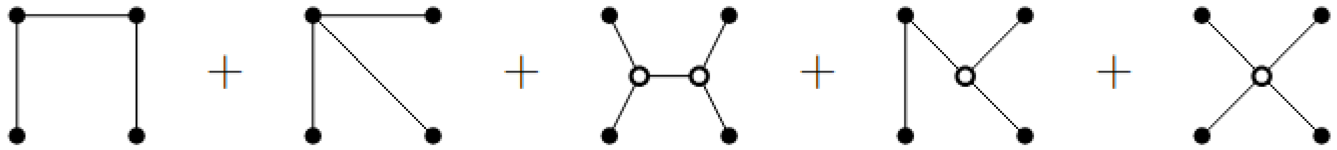
At leading order we have the Gaussian contribution.

$$2\frac{2\pi^2}{k^3}\mathcal{P}_h^G(k) =$$



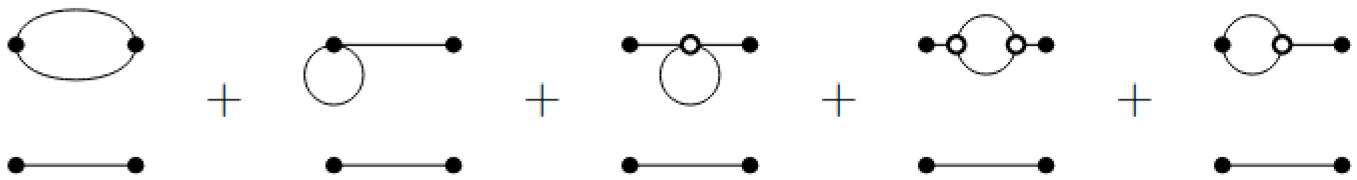
Two point function up to 2 loops.

We have 5 diagrams that violate helicity conservation.



Not connected or Tadpole diagrams don’t contribute

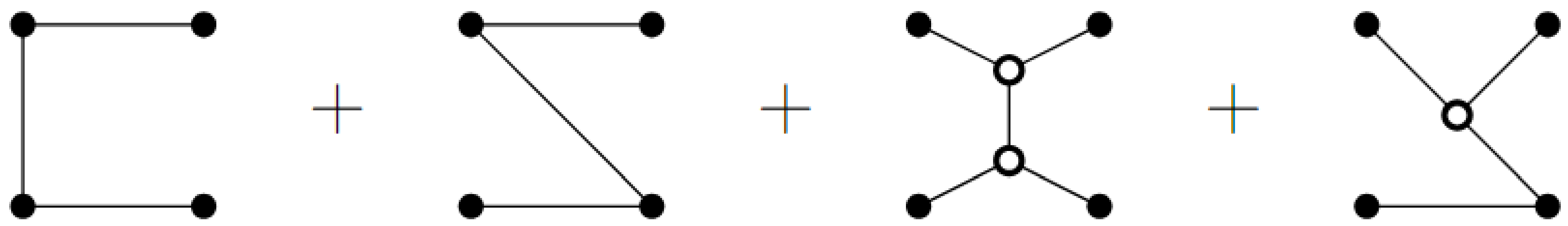
We also have 5 one-loop corrections to the propagator.



Their contributions have to be renormalized in some cases

Two point function up to 2 loops.

We have 4 diagrams that contribute (C,Z,X,M).



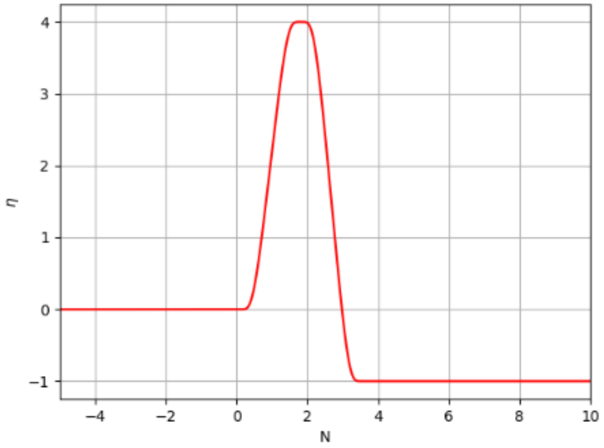
Slow-Roll parameter Model.

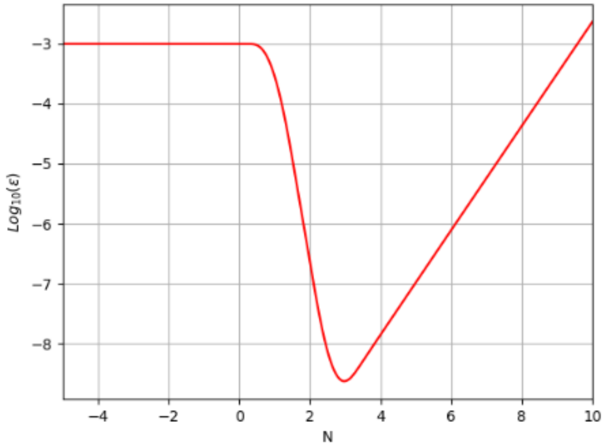
Smooth transition function between phases.

$$\Theta(N) \equiv \frac{\Theta_{\mathrm{H}}(N)e^{-\delta N/N}}{\Theta_{\mathrm{H}}(N)e^{-\delta N/N} + \Theta_{\mathrm{H}}(\delta N - N)e^{-\delta N/(\delta N - N)}}$$

Inflation model considered.

$$\eta(N) = \eta_{USR}[\Theta(N) - \Theta(N - \Delta N_{USR})] + \eta_{CR}\Theta(N - \Delta N_{USR})$$





Slow-Roll parameter Model.

Evolution of the field is given by Mukhanov-Sasaki equation.

$$\frac{d^2u_k}{dN^2} + (1-\eta)\frac{du_k}{dN} + \left(\frac{k^2}{\mathcal{H}^2} + (1-\eta)(\eta+\epsilon-2) + \frac{d\eta}{dN}\right)u_k = 0$$

$$u = z\varphi = \sqrt{2a^2\epsilon}\varphi$$

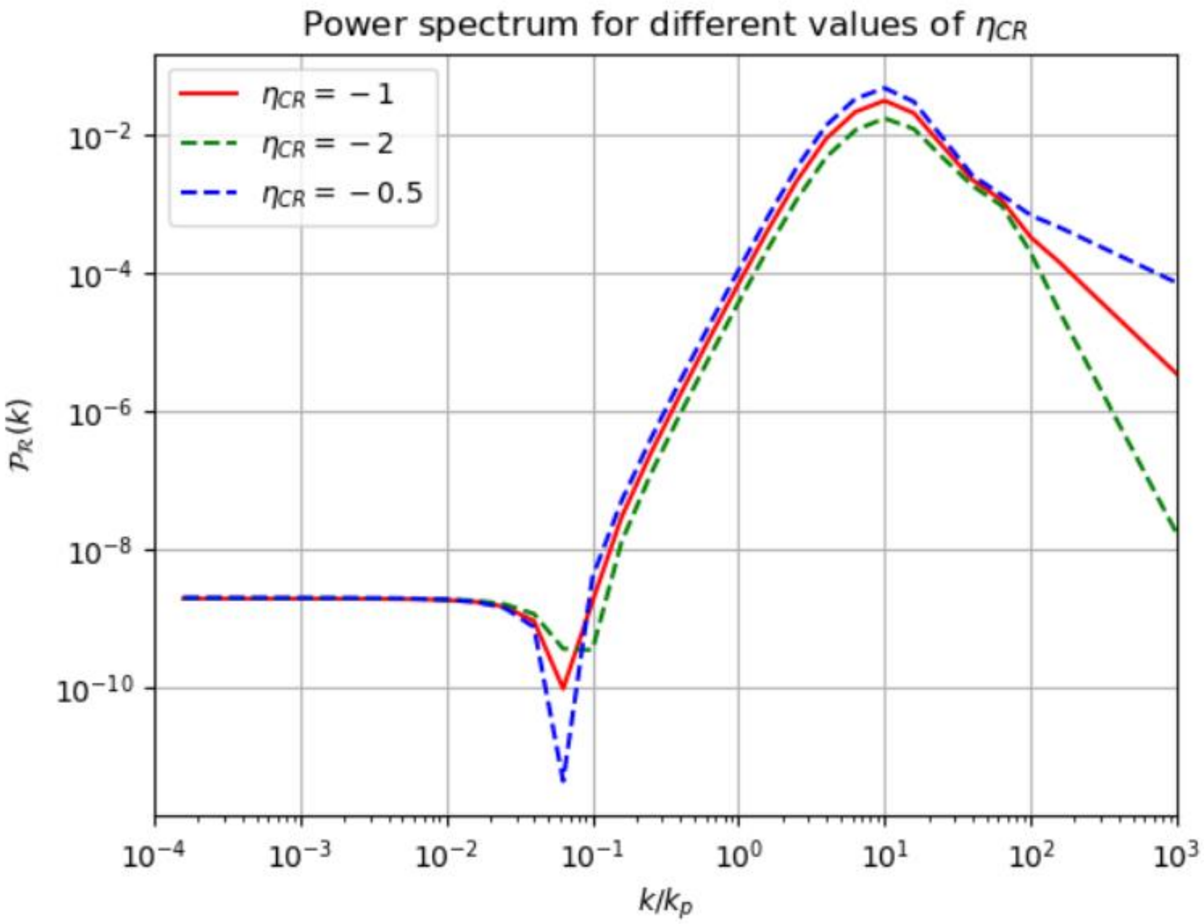
With Bunch-Davis Vacuum solution as Initial condition.

$$u_k(N_i) = \frac{1}{\sqrt{2k}}, \qquad u_k'(N_i) = -i\frac{k}{\sqrt{2}k_i}$$

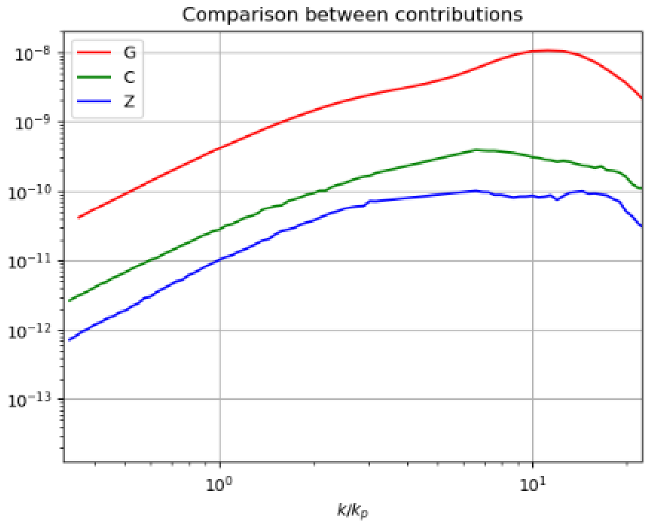
$$k_i = \mathcal{H}(N_i) << k, \text{ we choose } k_i = 10^{-3}k$$

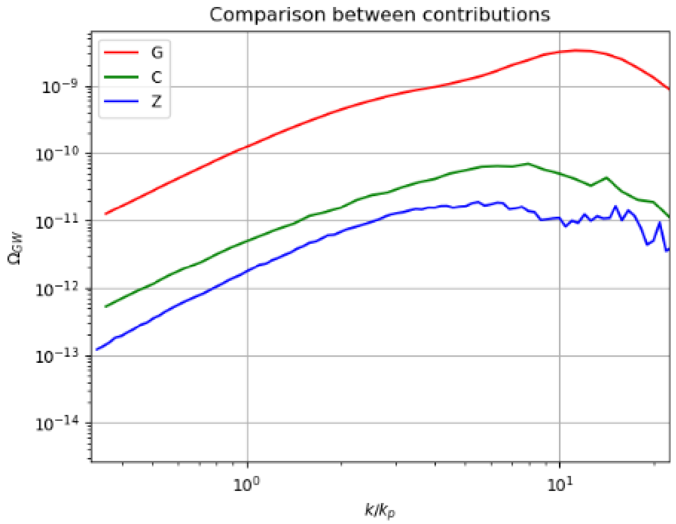
Slow-Roll parameter Model.

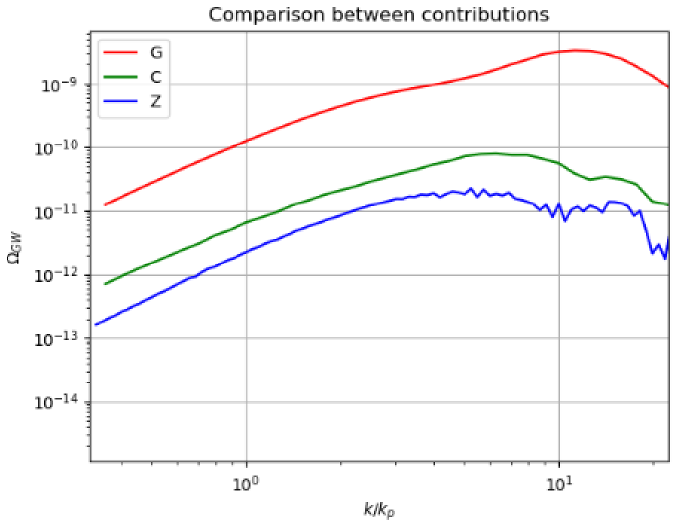
We calculate the power spectrum of R at late times.



Contributions of G,C and Z diagrams to the density parameter.







Conclusions

- Systematic method for two point functions with posiblity to calculate n-point functions if you are patient enough.
- Tested with some non Gaussian corrections to a specific slow-roll parameter model.

References

[1] Daniel Baumann. *Cosmology*. Cambridge University Press, 2022.

[2] Daniel Baumann. Tasi lectures on inflation, 2012.

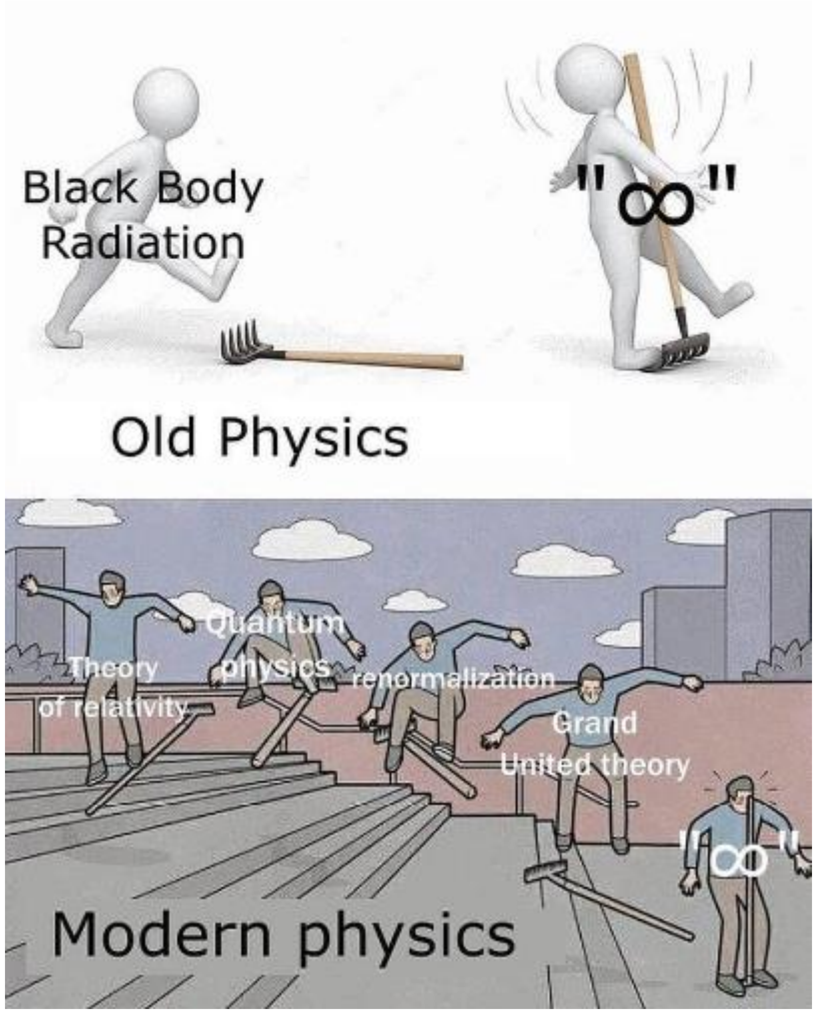
[3] J.R. Espinosa, D. Racco, and A. Riotto. A cosmological signature of the sm higgs instability: gravitational waves. *Journal of Cosmology and Astroparticle Physics*, 2018(09):012–012, September 2018.

[4] Jun-Peng Li, Sai Wang, Zhi-Chao Zhao, and Kazunori Kohri. Complete analysis of the background and anisotropies of scalar-induced gravitational waves: primordial non-gaussianity f nl and g nl considered. *Journal of Cosmology and Astroparticle Physics*, 2024(06):039, June 2024.

[5] Guillermo Ballesteros and Jesús Gambín Egea. One-loop power spectrum in ultra slow-roll inflation and implications for primordial black hole dark matter, 2024.

[6] Guillermo Ballesteros and Marco Taoso. Primordial black hole dark matter from single field inflation. *Physical Review D*, 97(2), January 2018.

Thank you for your atention



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