From Permutaions To Scattering Amplitudes

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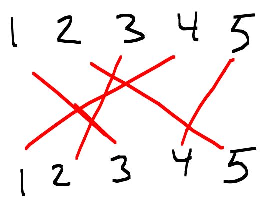
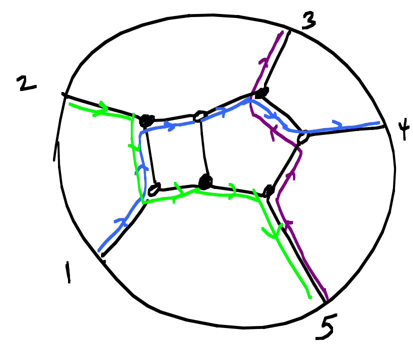
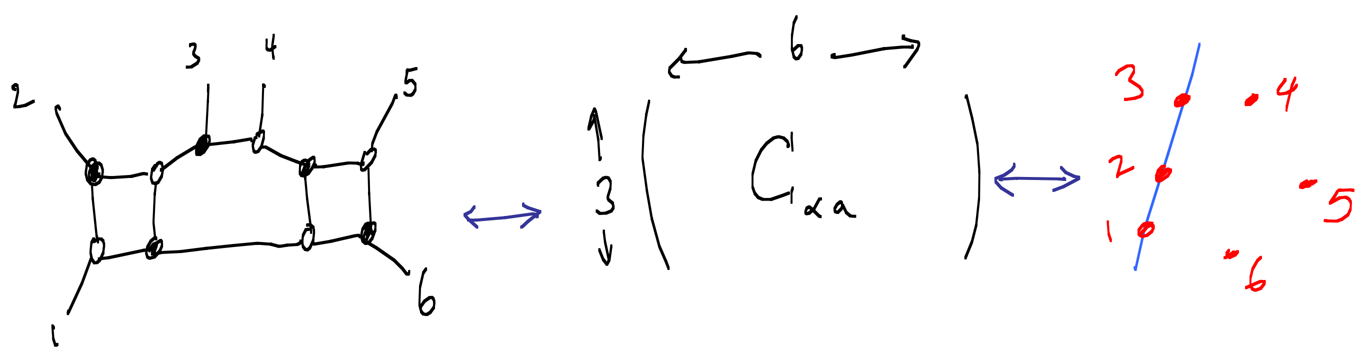
Outline

(0) Drawings for Permutations: a Vignette





(1) From On-Shell Diagrams → Grassmannian



(2) Planar $\mathcal{N}=4$ SYM:

* Complete classification/explicit formula/# of sol'n's for all on-shell diagrams + their relations

  Permutations $\leftrightarrow$ Conf. of Vectors $\leftrightarrow$ Positive Grassm.

* BCFW + Permutations

* Invariant top form $\leftrightarrow$ D.C.I,
  Yangian $\leftrightarrow$ "Positive" Diffs.

* All-loop integrand in the Grassmannian;
  d log form [$\rightsquigarrow$ "doing integrals"]

(3) Beyond Planar $\mathcal{N}=4$ SYM amps:

* ABJM
* Non-planar $\mathcal{N}=4$
* $\mathcal{N}=0$, planar/non-planar
* Gravity ...

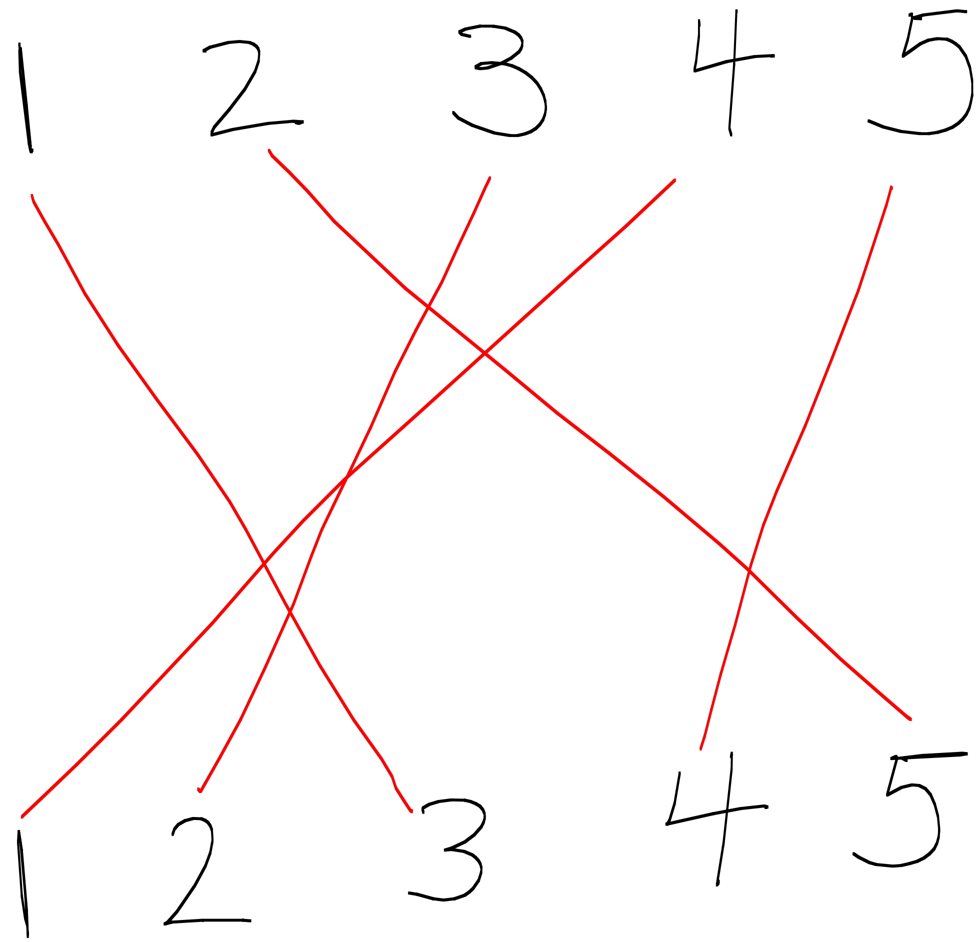
(4) Beyond Scattering Amplitudes

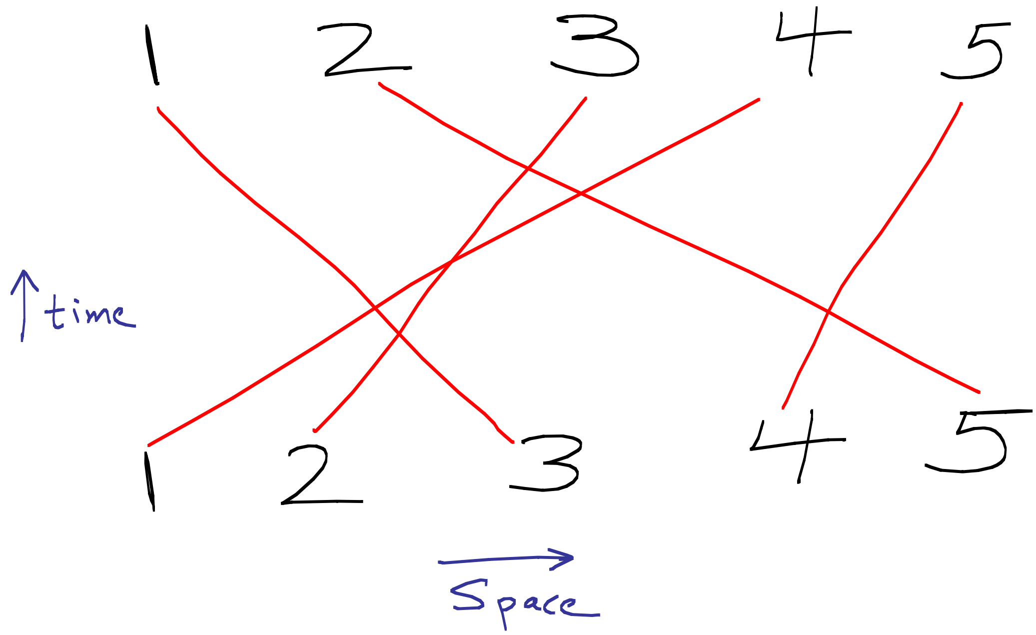
Drawings of Permutations

to

Pictures of Scattering

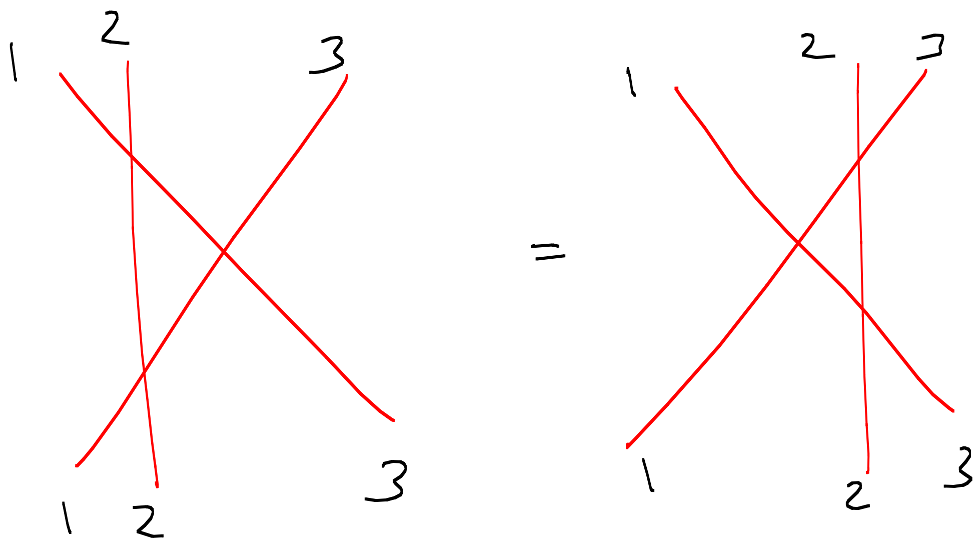
12345 → 35214


1 2 3 4 5
1 2 3 4 5



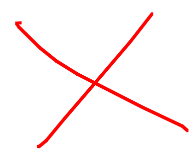
Breaking up permutation into product of adjacent transpositions [non-unique]

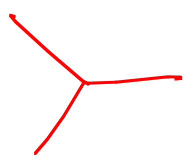
3 5 2 1 4
3 2 5 1 4
3 2 1 5 4
3 1 2 5 4
1 3 2 5 4
1 2 3 5 4
1 2 3 4 5

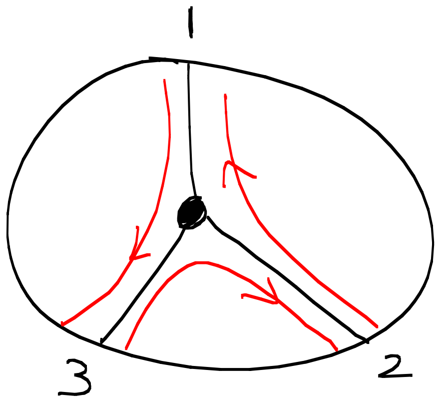


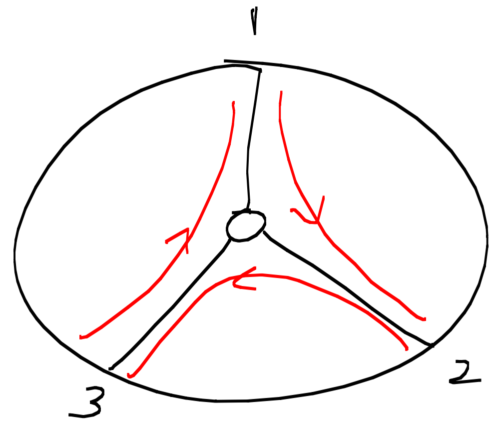
Yang-Baxter

Can't Apply to (3+1)-d

- No particle creation / destruction
- Fundamental interaction  ,

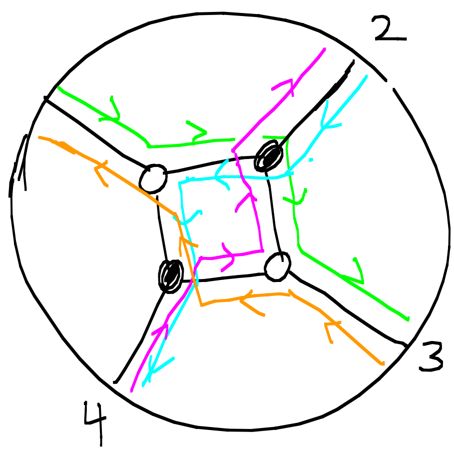
  not 





$1 \to 3$
$2 \to 1$
$3 \to 2$

$1 \to 2$
$2 \to 3$
$3 \to 1$



$1 \to 3$
$2 \to 4$
$3 \to 1$
$4 \to 2$

"Affine" or "Decorated" Permutation

$1 \to 3$
$2 \to 4$
$3 \to 1+4=5$
$4 \to 2+4=6$

$a \to p(a)$, $a+n \geq p(a) \geq a$

$p(a) \bmod n$ is a perm.

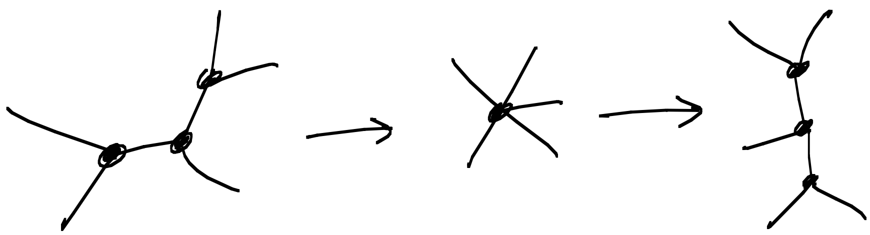
"K" = # that jump back $= \frac{1}{n}\sum_a (p(a)-a)$

So, a set of permutation for (n,K)

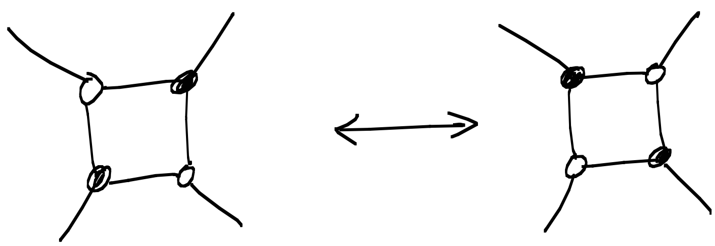
Enormous Redundancy

* First – can have infinitely many loops/faces! "Reduced" graphs → representing permutation with minimum # of faces.

* Even reduced graphs are far from unique, two "moves":

- 

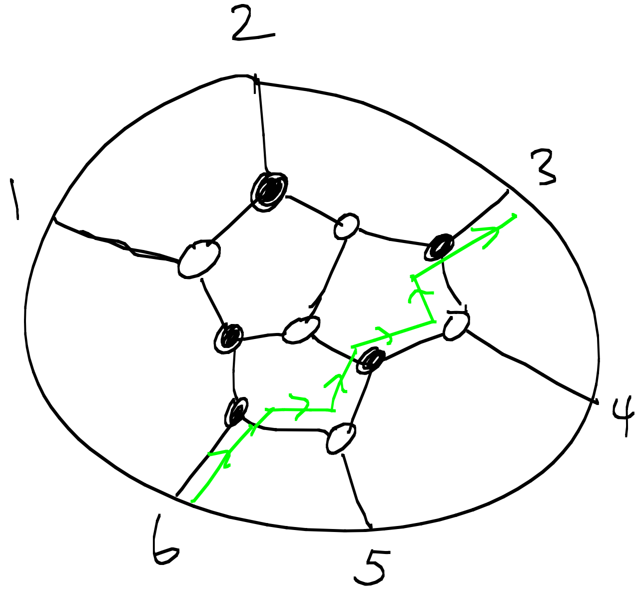
Merge B's, W's

- 

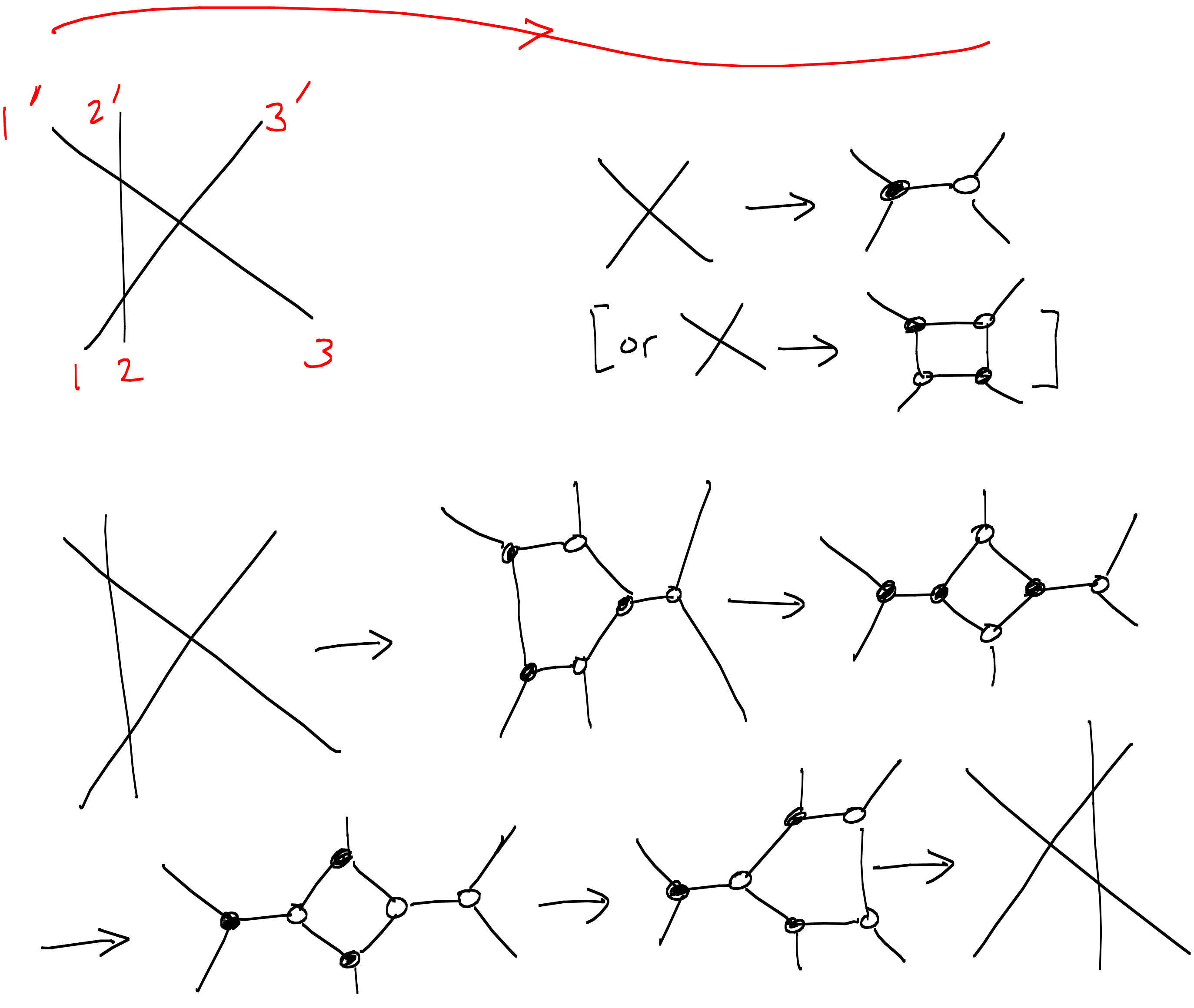
"Square Move"

* Every perm. can be represented like this

1 → 4
2 → 6
3 → 5
4 → 7
5 → 8
6 → 9



Old Perm. ⊂ New Perm.



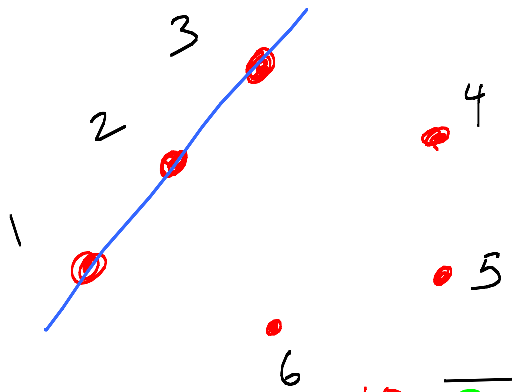
Yang-Baxter consequence of more "atomic" square move

Permutations $\longleftrightarrow$ Config. of Vectors

$\updownarrow$

Positive Grassmannian

Conf. of Vectors w/ Consecutive Lin. Dep

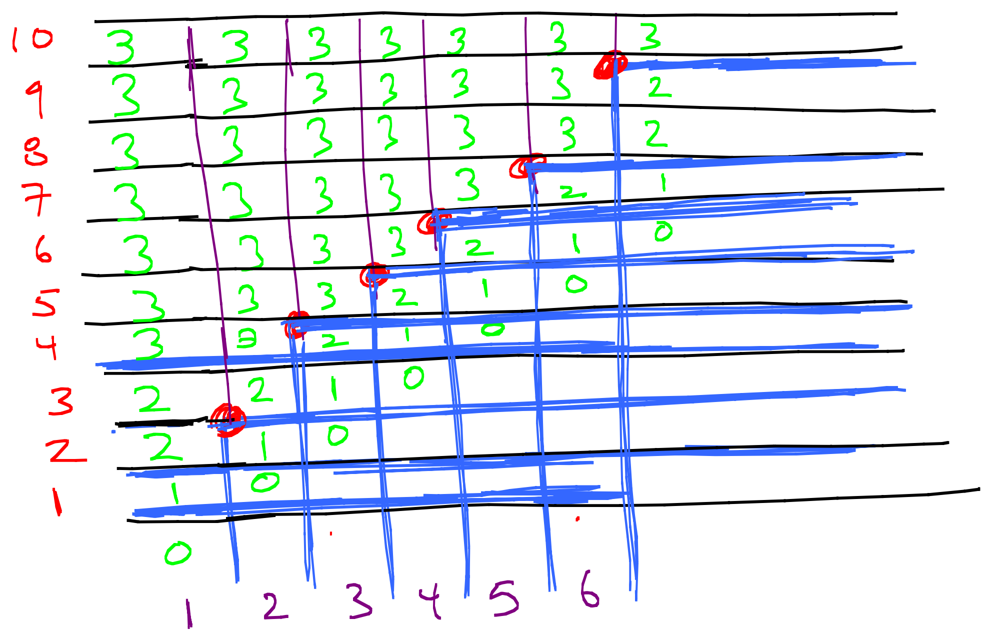


Fully Determined ⟷ By

$a \to p(a)$

$a \in \text{span}(a+1, \ldots, p(a))$

$1 \to 3$
$2 \to 5$
$3 \to 6$
$4 \to 7$
$5 \to 8$
$6 \to 10$



The Positive Grassmannian

$$C = \underbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}}_{\leftarrow n \rightarrow} \updownarrow k \qquad n \ \ k\text{-vectors.}$$

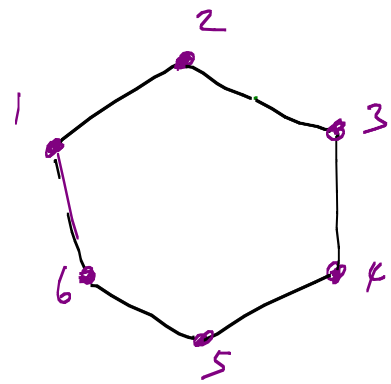
"Positive Part": $(c_{i_1} \cdots c_{i_k}) > 0$ for $i_k > \cdots > i_1$.

["All minors positive"].

Note: (twisted) cyclic structure

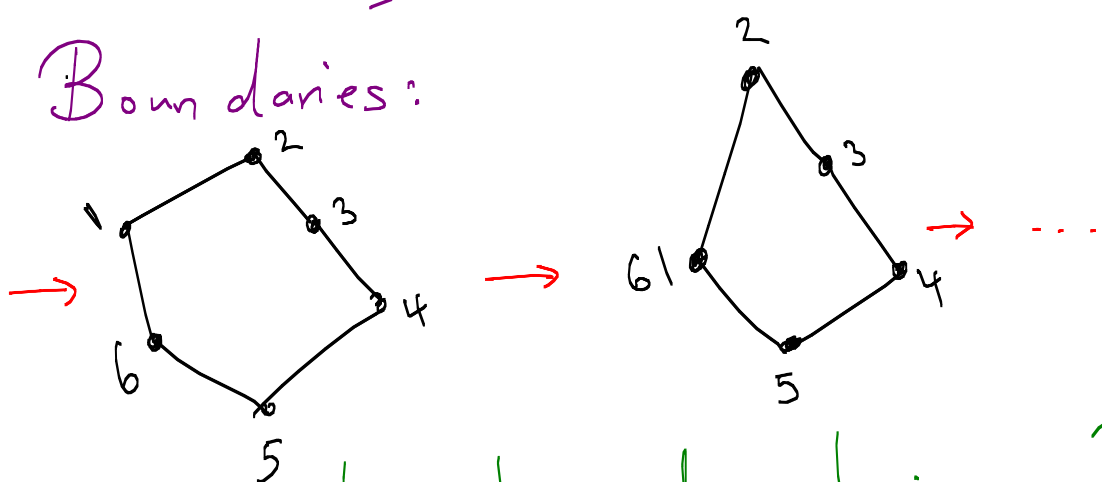
$c_1 \to c_2,\ c_2 \to c_3,\ \ldots,\ c_n \to (-1)^{k+1} c_1$

Positivity ⟷ Convexity



→ Convex Polygon

Boundaries:



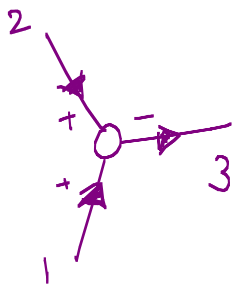
→ Consecutive linear dependencies → Permutations

From On-Shell Diagrams

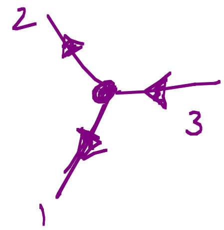
to

The Grassmannian

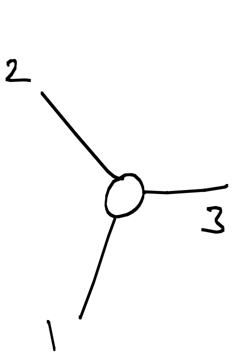
3-pt Vertices



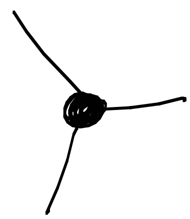
$$\frac{[12]^3}{[12][23]}\,\delta^4\Big(\sum_a p_a\Big)$$



$$\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}\,\delta^4\Big(\sum_a p_a\Big)$$

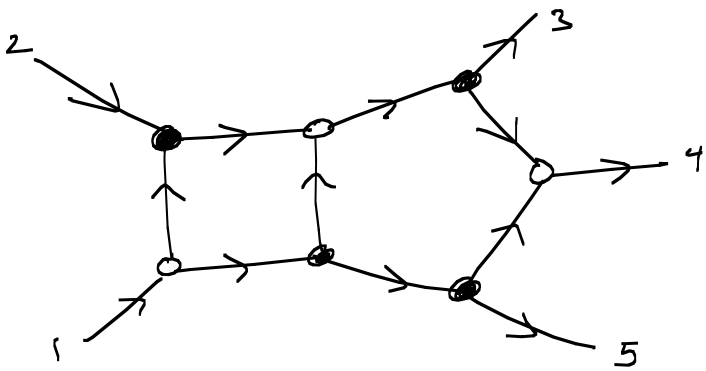


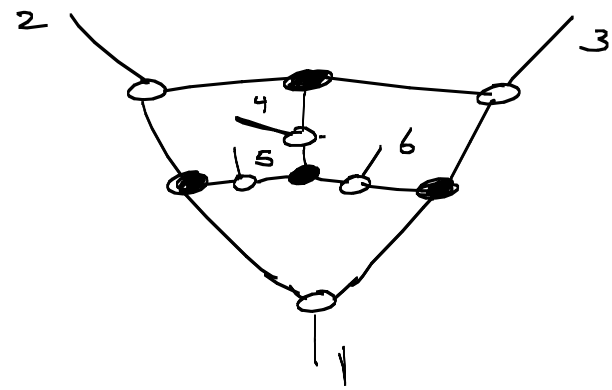
$$\frac{\delta^8\big(\sum \tilde{\lambda}_a \eta_a\big)}{[12][23][31]}\,\delta^4\Big(\sum_a p_a\Big)$$



$$\frac{\delta^8\big(\sum_a \lambda_a \tilde{\eta}_a\big)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,\delta^4\Big(\sum_a p_a\Big)$$

On-Shell Diagrams



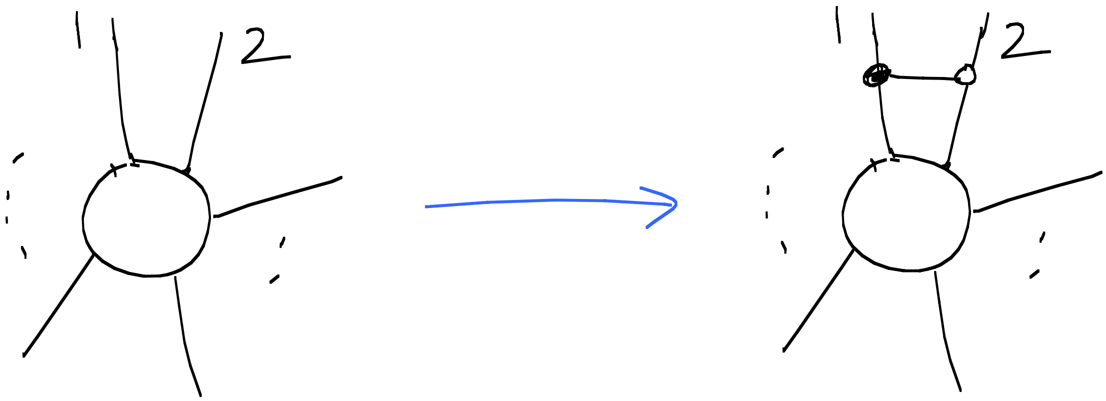


$\mathcal{N}=0$ $\mathcal{N}=4$

$$\left(\int \frac{d^2\lambda\, d^2\tilde{\lambda}}{GL(1)}, \int \frac{d^4 W}{GL(1)}\right) \qquad \left(\int \frac{d^2\lambda\, d^2\tilde{\lambda}\, d^4\tilde{\eta}}{GL(1)}, \int d^{4|4} \mathcal{W}\right)$$

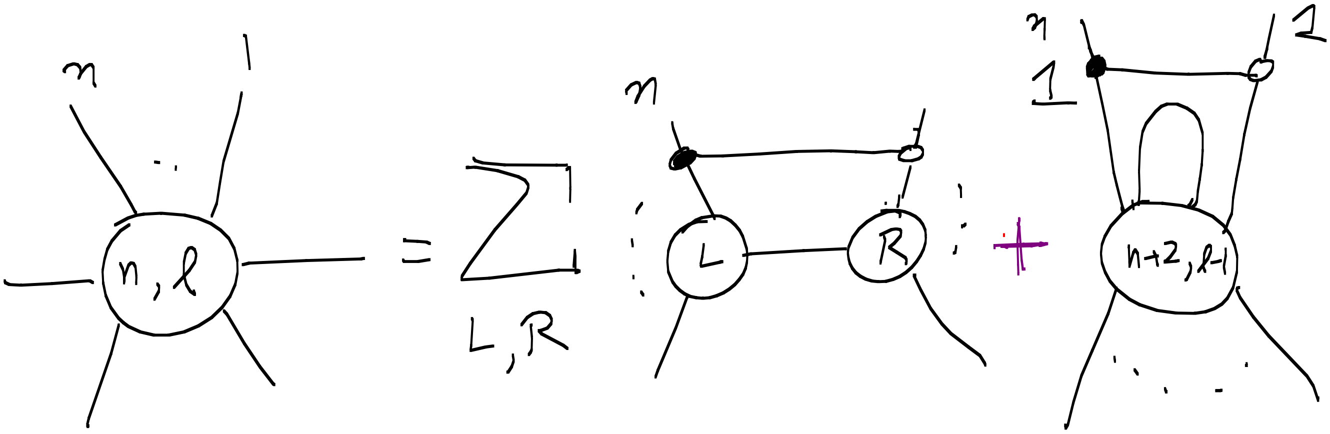
In general, a differential form

Ex: BCFW Deformation

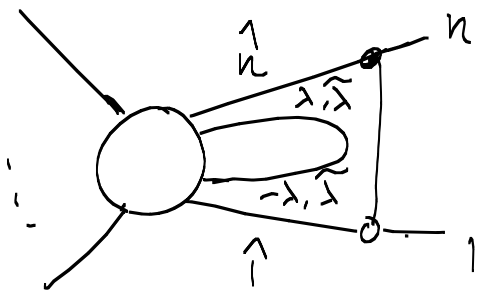


$$\int F[1,2,\ldots n] \longrightarrow \frac{d\tau}{\tau} \int F[\hat{1},\hat{2},\ldots,n]$$

All-Loop Recursion for Planar $\mathcal{N}=4$

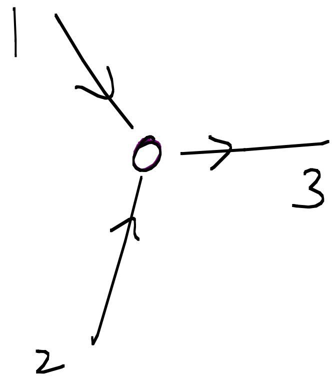


On-shell loops



$$\int \frac{d^2\lambda\, d^2\tilde\lambda}{GL(1)}\, \frac{d\tau}{\tau} \to \int d^4\ell$$

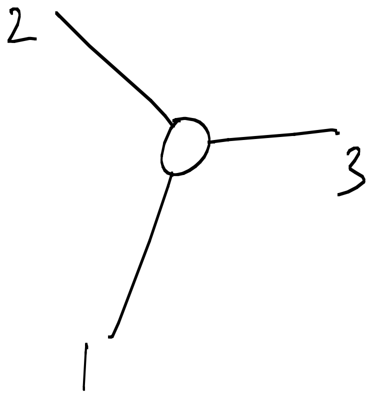
$$\ell = \lambda\tilde\lambda + \tau\lambda_1\tilde\lambda_2$$



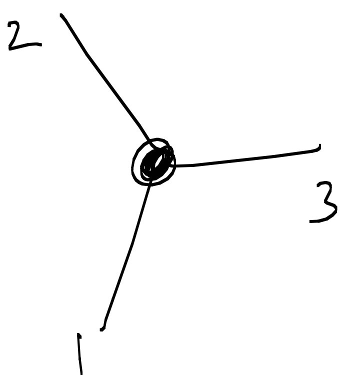
$$\frac{[12]^3}{[13][23]} \delta^4\left(\sum_a p_a\right)$$

$$\parallel$$

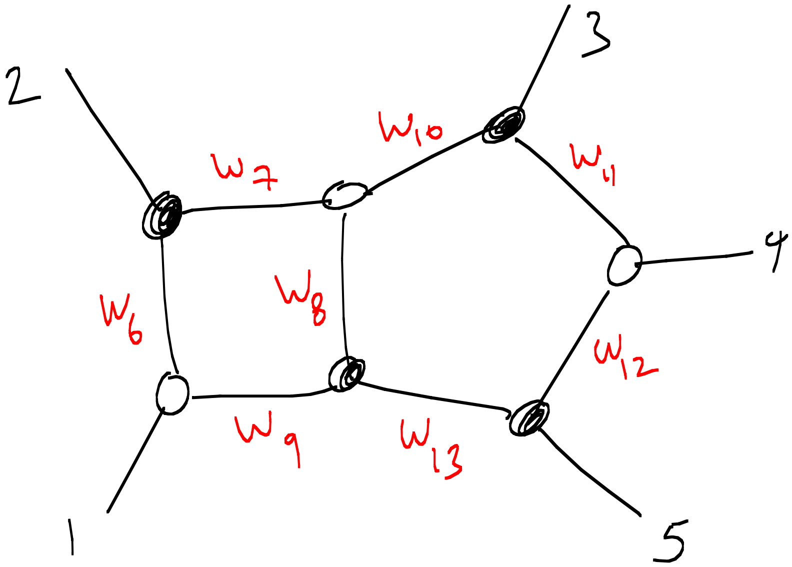
$$\int \frac{dt_1}{t_1} \frac{dt_2}{t_2} \quad \delta^2\left[\tilde{\lambda}_3 + t_2 \tilde{\lambda}_2 + t_1 \tilde{\lambda}_1\right] \delta^2\left[\lambda_3 - t_1 \lambda_1\right] \delta^2\left[\lambda_3 - t_2 \lambda_2\right]$$



$$= \int \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\Big/_{GL(1)}\, \delta^{4|4}(t_i N_i)$$



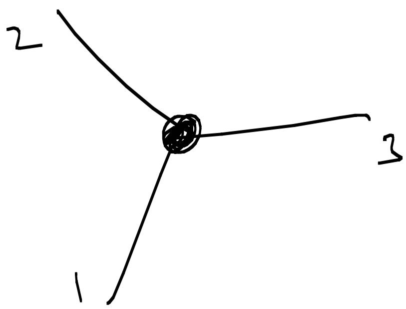
$$= \int d^2\vec{u}_1\, d^2\vec{u}_2\, d^2\vec{u}_3 \Big/_{GL(2)}\, \delta^{4|4}(\vec{u}_i N_i)$$

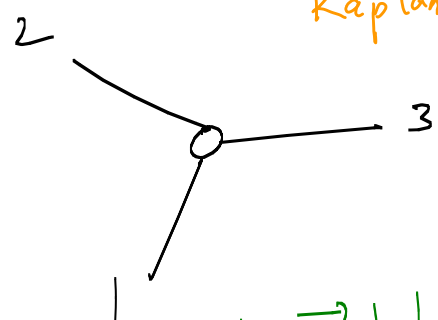


$$\int \frac{d^{4|4} W_{int}}{GL(1)'s},$$

trivial

{Mason + Skinner, Kaplan}





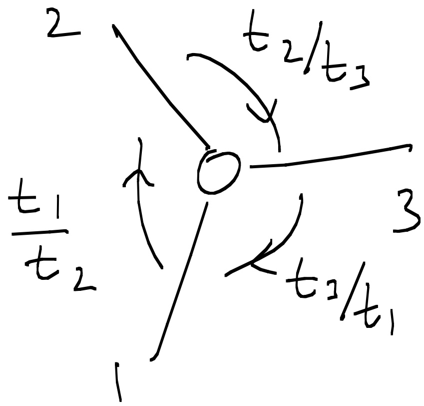
$$t_1 W_1 + t_2 W_2 + t_3 W_3 = 0$$

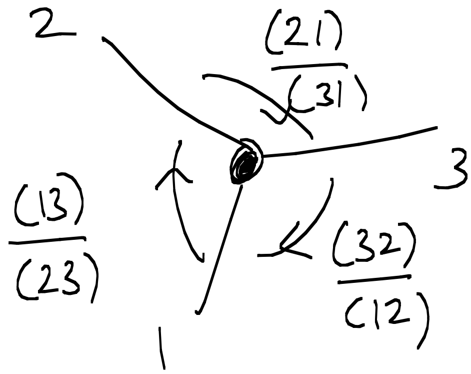
$$\vec{u}_1 W_1 + \vec{u}_2 W_2 + \vec{u}_3 W_3 = 0$$

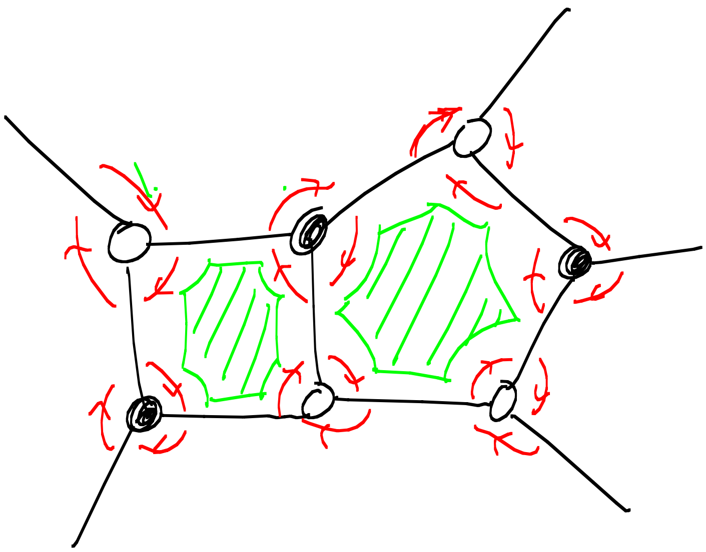
Now eliminate $W_{internal}$'s!

$$\rightarrow \prod_{\alpha=1}^{k} \delta^{4|4}\left[C_{\alpha a}[t's, \vec{u}'s] W_a \right]$$

Point in $G(k,n)$

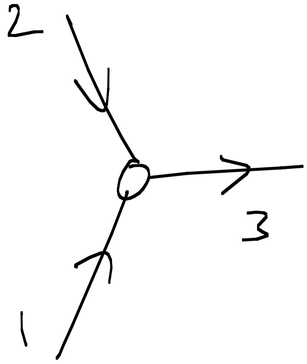




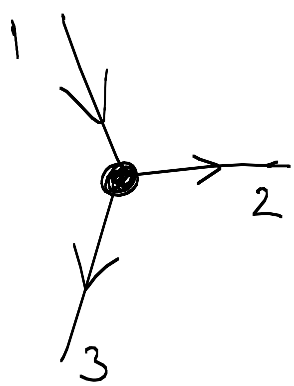


Variables associated with Faces

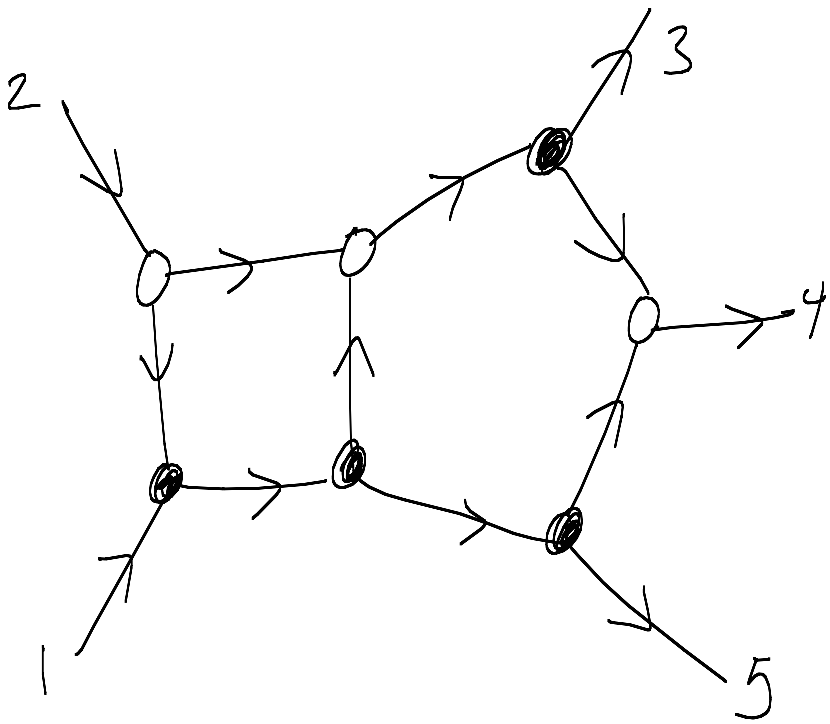
Convenient GFing + Explicit Co-ord.



$$x_3 W_3 = x_1 W_1 + x_2 W_2$$



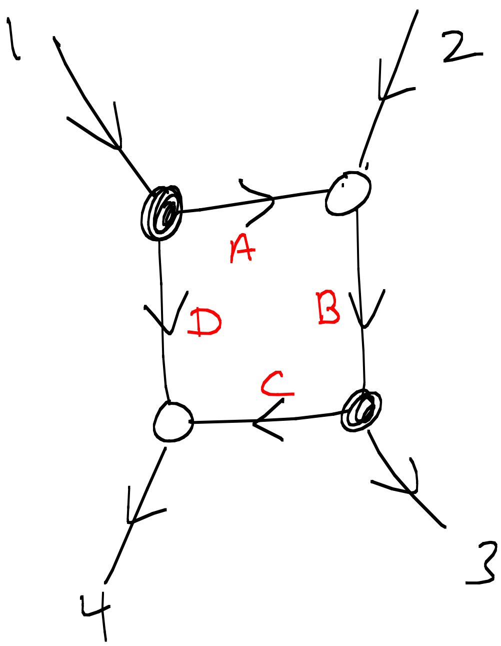
$$y_1 W_1 = y_2 W_2 = y_3 W_3$$



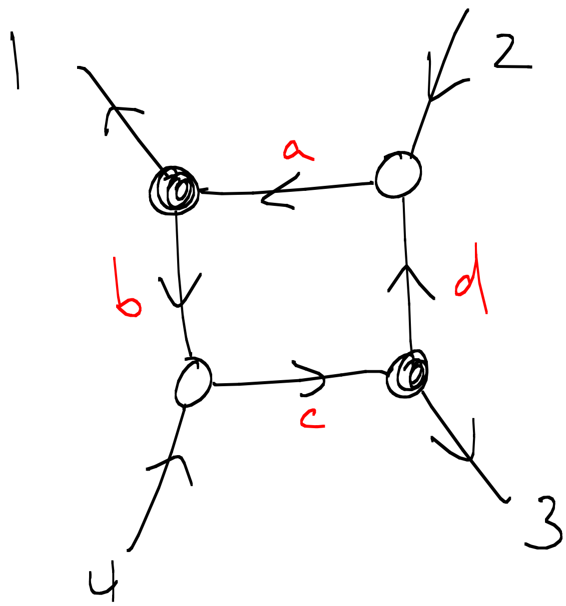
* Edge Var. $\leftrightarrow$ U(1) Gauge Field on Graph

* Face Var: Flux through faces

$$C_{iJ} = \sum_{\substack{\text{all paths}\\ i \to J}} \prod_{\substack{\text{edges}\\ \text{along path}}} e_\alpha$$



$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & AB & D+ABC \\ 0 & 1 & B & BC \end{bmatrix}$$



$$2 \to 1 \quad a + a(abcd) + a(abcd)^2 + \cdots = \frac{a}{1-abcd}$$

So, we have explicitly

$$\prod_{\text{edges}} \frac{de_\alpha}{e_\alpha} \Big/ GL(1)\text{'s}$$

or

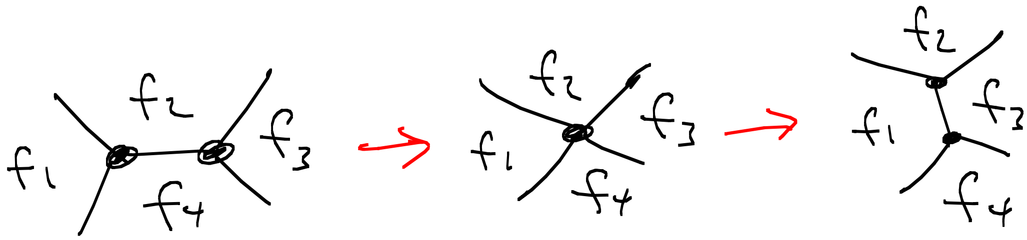
$$\prod_{\text{faces}} \frac{df_i}{f_i}$$

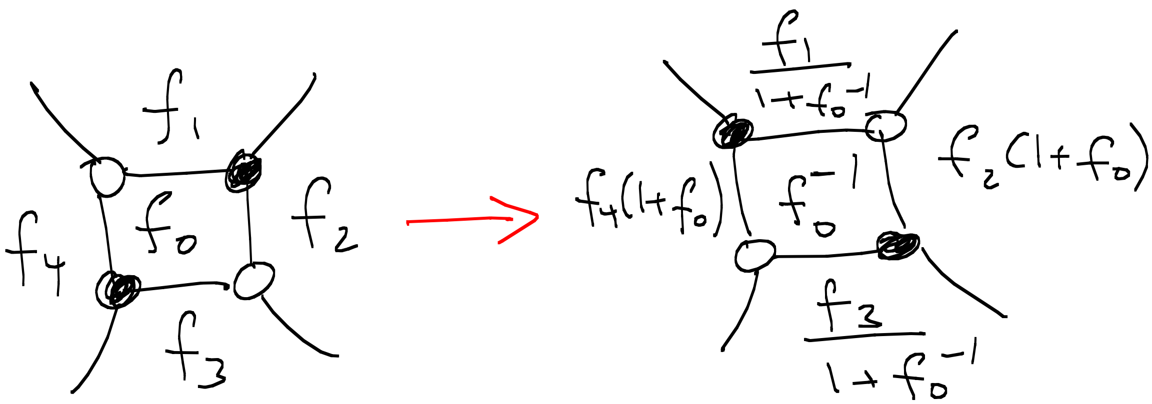
$$\prod_{\alpha=1} \delta^{4|4}\left[C_{\alpha a}[f] \, W_a\right]$$

The map from On-Shell-Diagrams $\longrightarrow$ Grassmannian is natural + Universal.

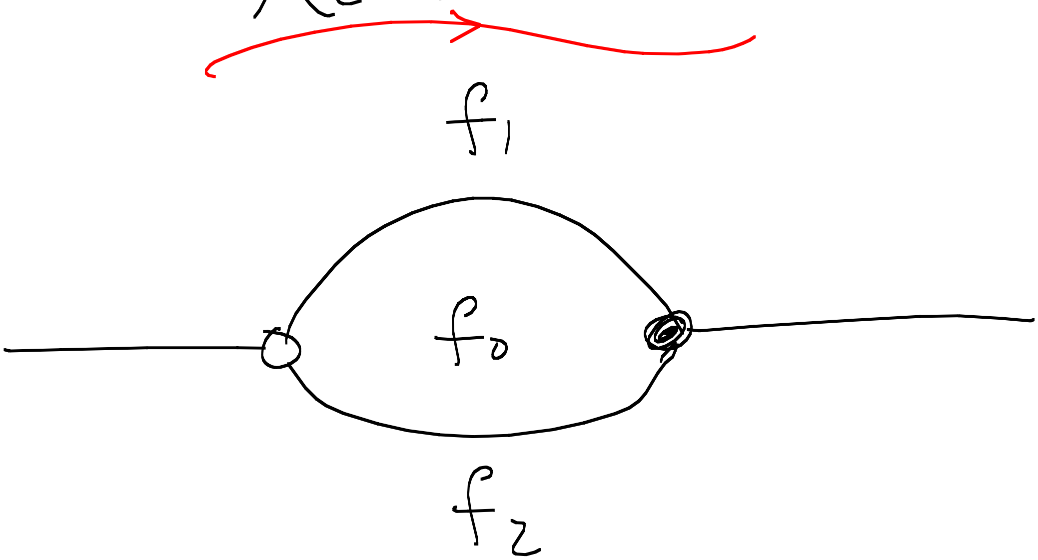
The measure depends on the theory....

"Moves"


f_1
f_2
f_3
f_4
f_2
f_1
f_3
f_4
f_2
f_1
f_3
f_4


f_1
f_4
f_0
f_2
f_3
$\frac{f_1}{1+f_0^{-1}}$
$f_4(1+f_0)$
f_0^{-1}
$f_2(1+f_0)$
$\frac{f_3}{1+f_0^{-1}}$

"Reduction"

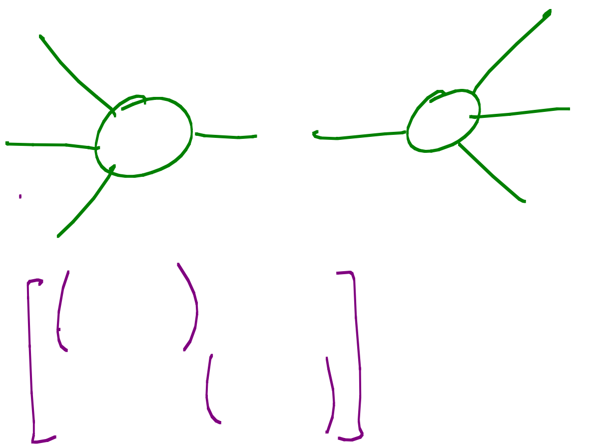


$$\downarrow$$

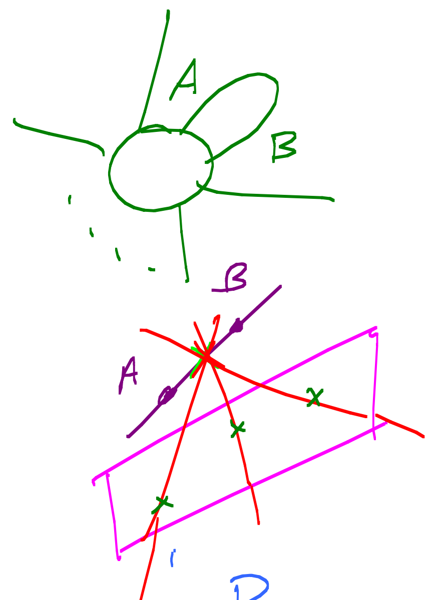
$$\frac{f_1(1+f_0)}{\dfrac{f_2}{1+f_0^{-1}}}$$

"Local" Picture

* Basic operations:



Direct Product



Proj

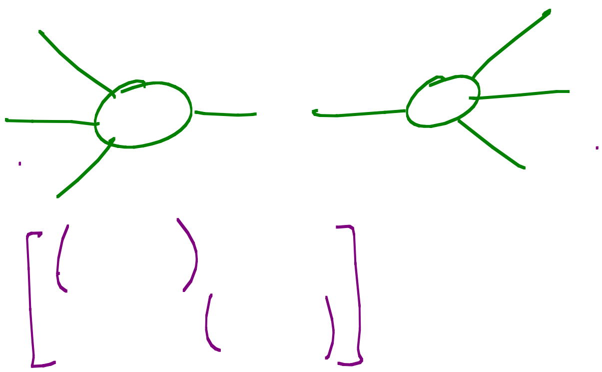
* Moves, Reductions Obvious

Planar $\mathcal{N}=4$ SYM

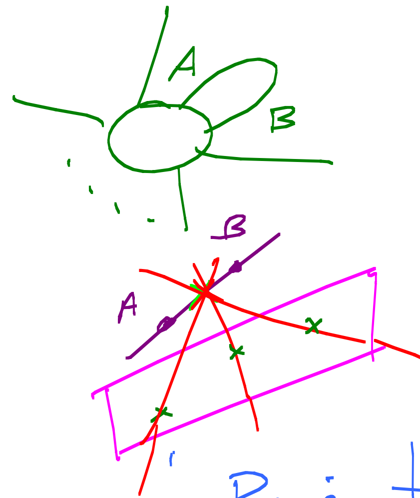
and

The Positive Grassmannian

* Basic operations:



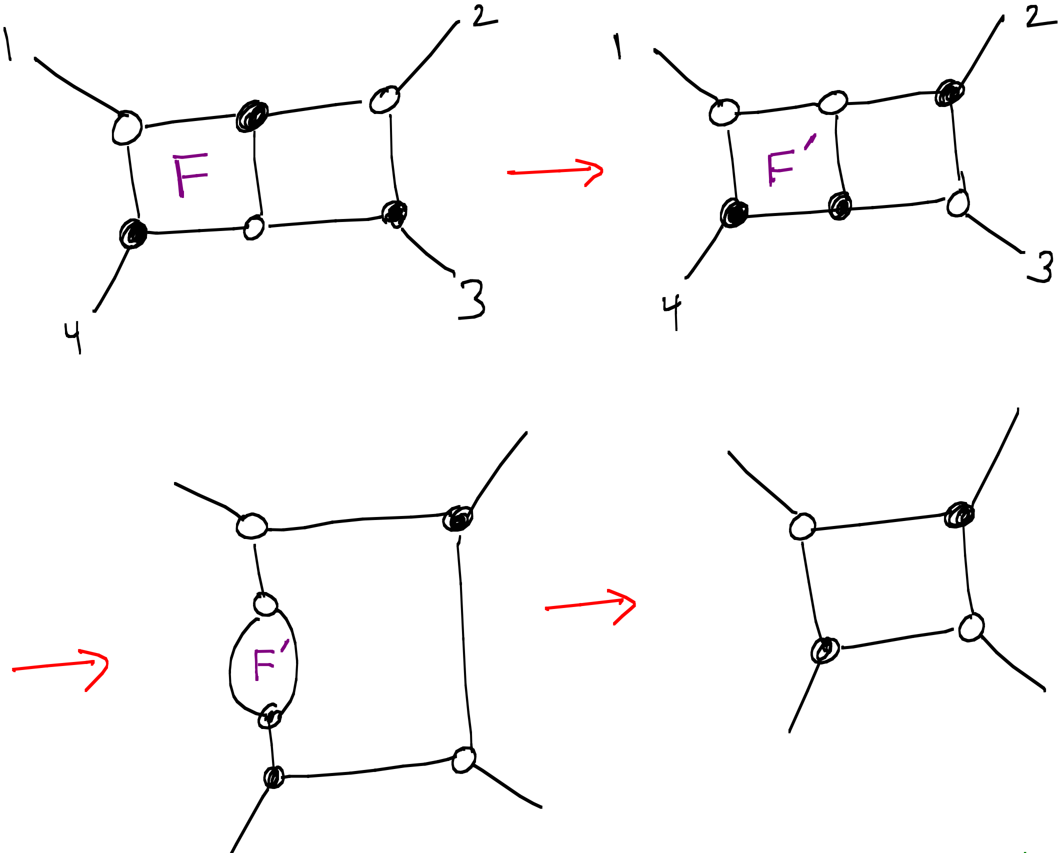
Direct Product



Projection

→ "Positive" Grassmannian Manifest for Planar Graphs from "local" picture

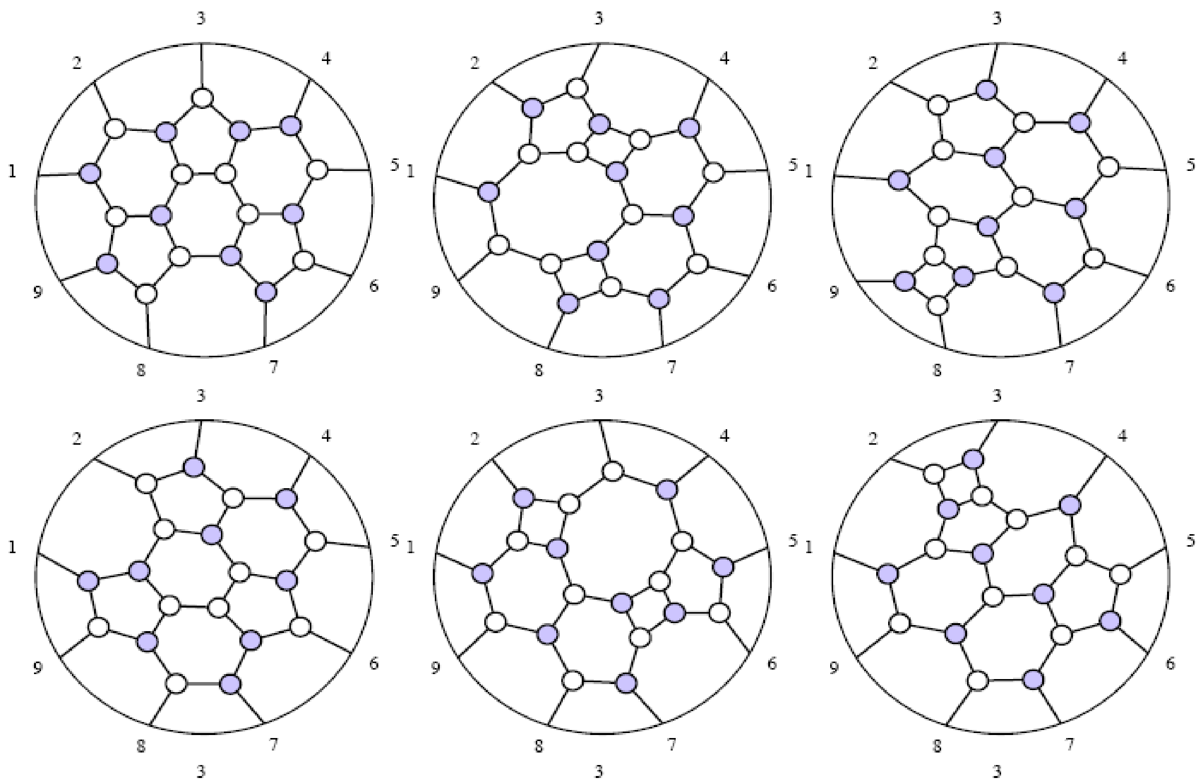
Reduction



$$\frac{dF}{F}\prod_a \frac{df_a}{f_a} \longrightarrow \left(\frac{dF'}{F'}\right)\prod_a \frac{df'_a}{f'_a}$$

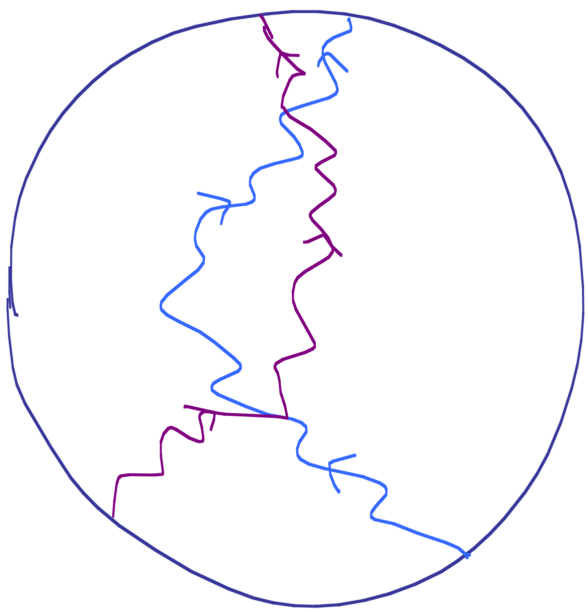
factors out

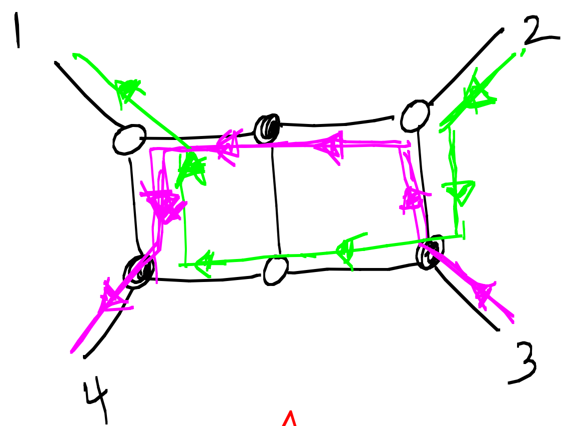
Invariants: Global Properties of L-R Paths!



Reduced? Same, different?

Reducible Graphs





"Bad double Crossing"

Once Reduced

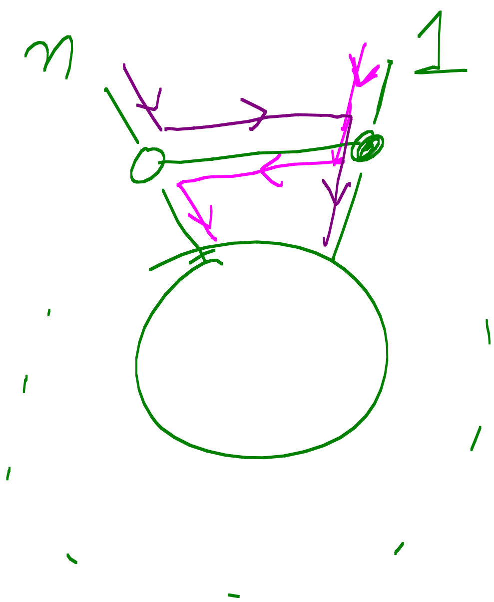
* All content is in L-R path permutation

↳ which gives lin dependencies of Grassmannian

↳ And Specifies cell of Positive Gr.

e

BCFW = Adjacent Transp.



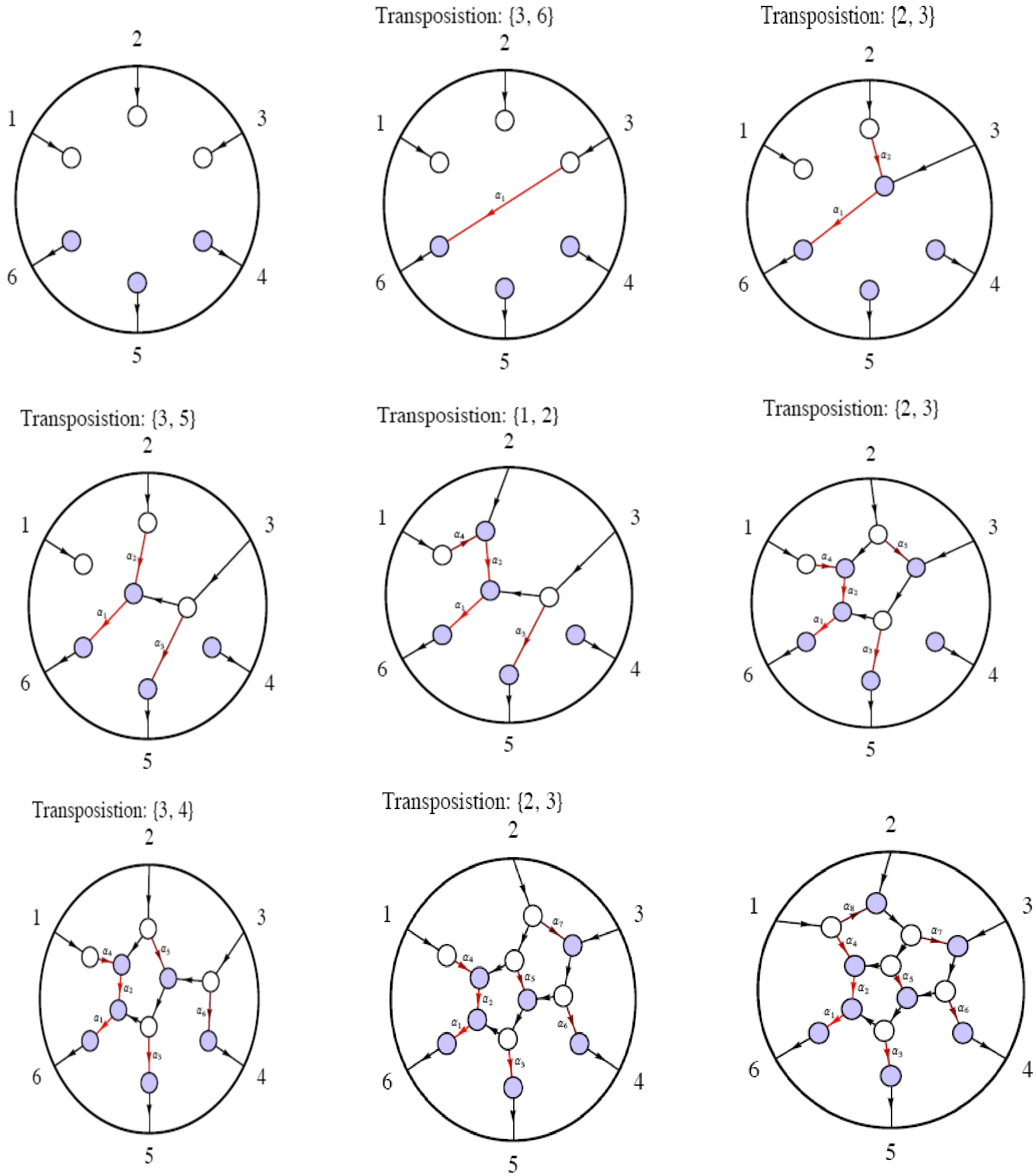
$n \longrightarrow p(1)$

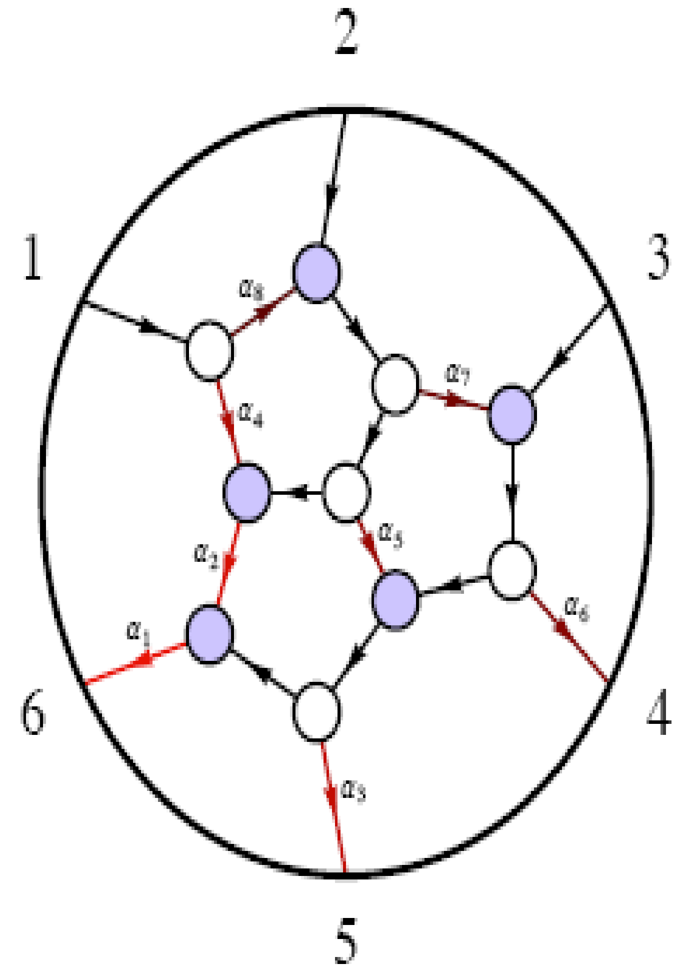
$1 \longrightarrow p(n)$

BCFW Perm. Decomp.

		(12)	(23)	(34)	(23)	(35)	(12)	(23)	(36)
1 →	4	6	6	6	6	6	7	7	7
2 →	6	4	5	5	7	7	6	8	8
3 →	5	5	4	7	5	8	8	6	9
4 →	7	7	7	4	4	4	4	4	4
5 →	8	8	8	8	8	5	5	5	5
6 →	9	9	9	9	9	9	9	9	6

Sequence of Adjacent Transpositions


Transposistion: {3, 6}
Transposistion: {2, 3}
Transposistion: {3, 5}
Transposistion: {1, 2}
Transposistion: {2, 3}
Transposistion: {3, 4}
Transposistion: {2, 3}



2
1
3
6
4
5
α_8
α_7
α_4
α_5
α_2
α_1
α_6
α_3

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[6 \to 6 + z_1 3]{(36)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z_1 \end{bmatrix}$$

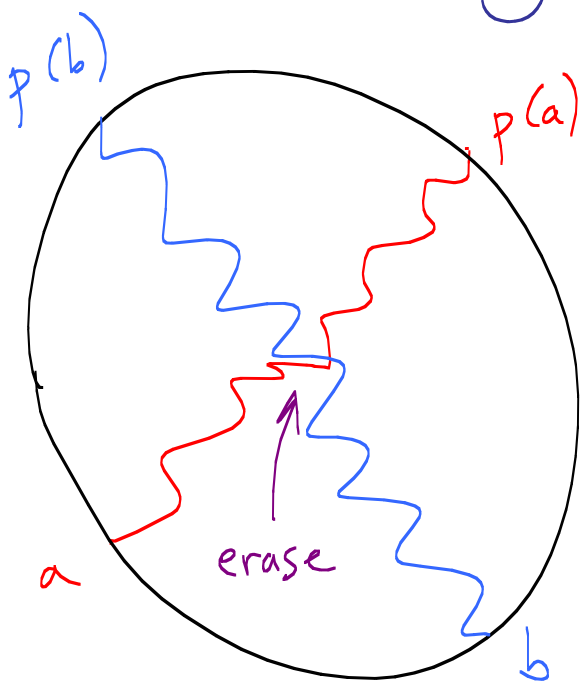
$$\xrightarrow[3 \to 3 + z_2 2]{(23)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z_1 \end{bmatrix} \quad \cdots\cdots$$

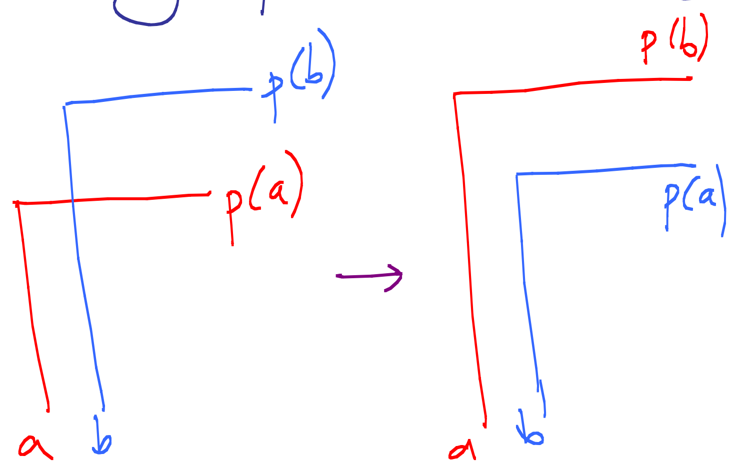
$$\longrightarrow \begin{bmatrix} 1 & z_4 + z_8 & z_4 z_5 + z_4 z_7 & z_4 z_5 z_6 & 0 & 0 \\ 0 & 1 & z_2 + z_5 + z_7 & (z_2 + z_5) z_6 & z_2 z_5 & 0 \\ 0 & 0 & 1 & z_6 & z_3 & z_1 \end{bmatrix}$$

(Manifestly Positive)

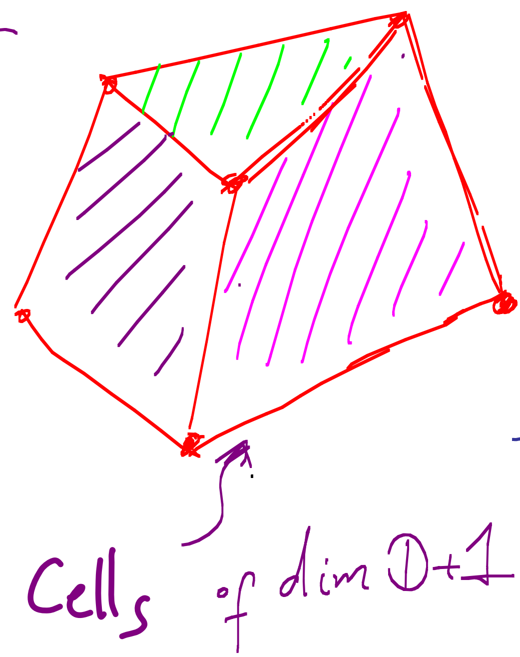
Singularities / Residues

Erase an edge [so graph still reducible!]

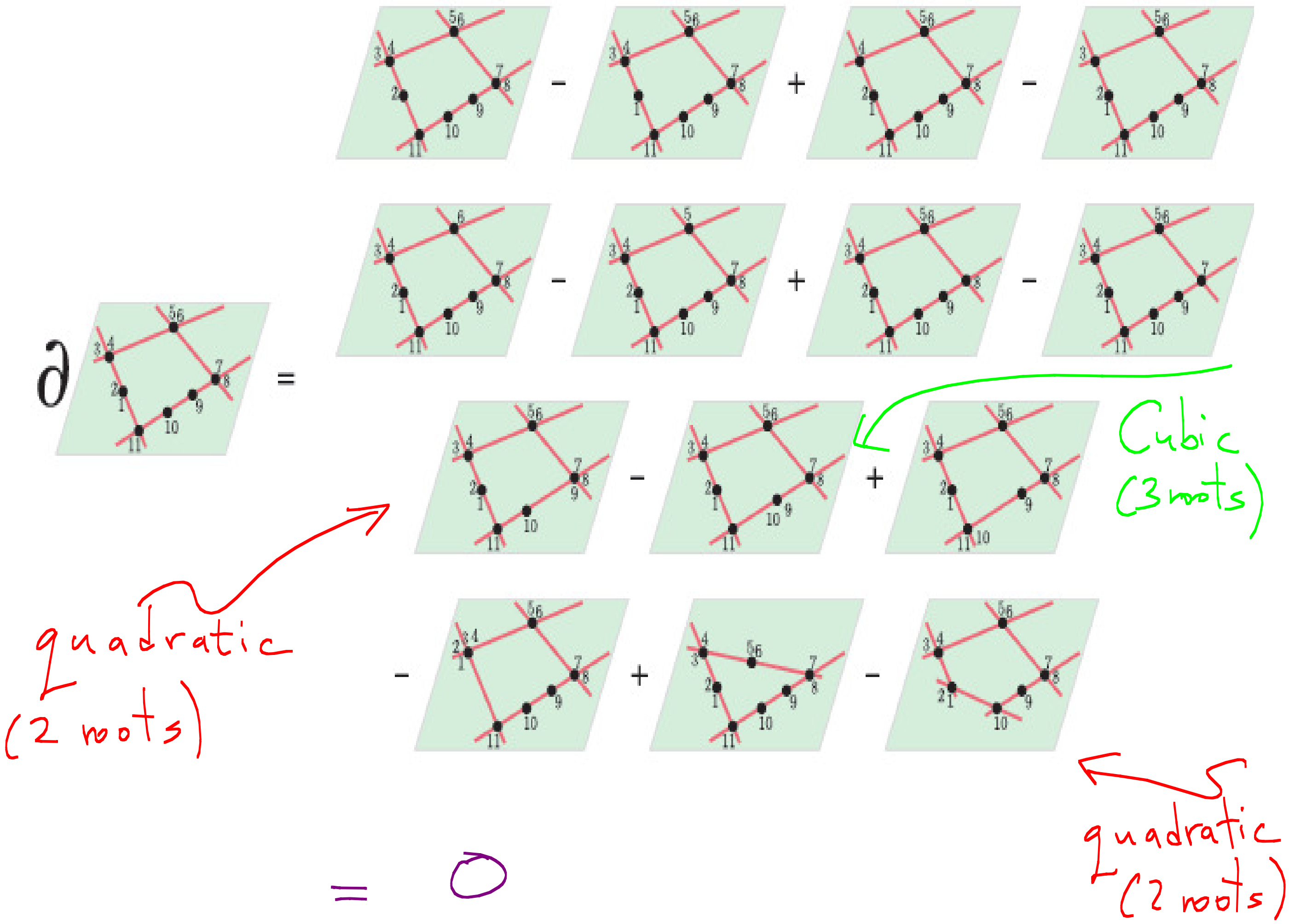


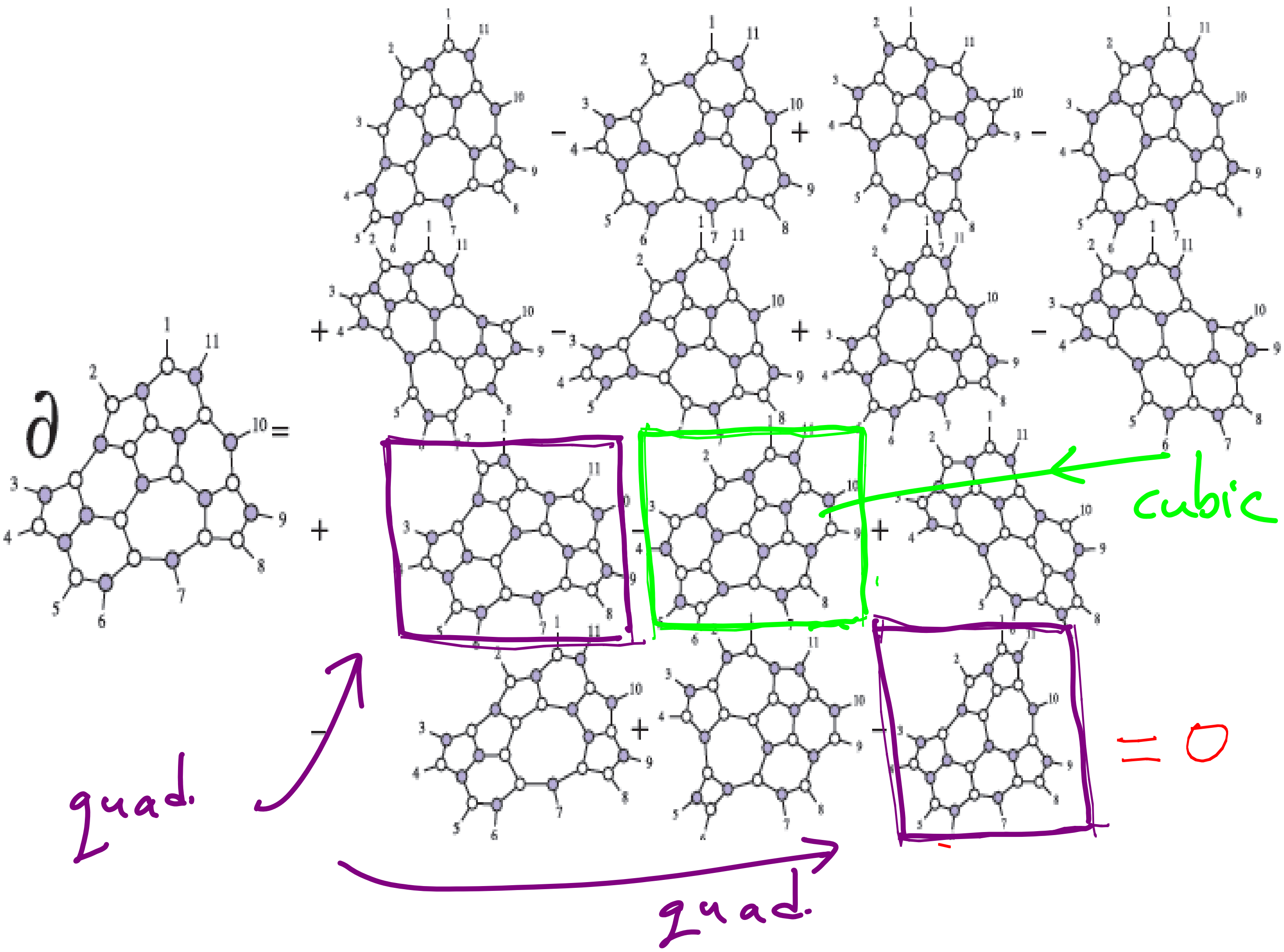


Relations

$\partial \Big[$  $\Big] = 0$

Ex: 14-term identity involving rationals, $\sqrt{\ }$'s, $\sqrt[3]{\ }$'s :




∂
cubic
quad.
quad.
= 0

Invariant Top Form

$$\frac{d^{k\times n} C}{\mathrm{vol}\, GL(k)\, (12\ldots k)\ldots(n1\ldots k-1)} = \frac{df_1}{f_1} \cdots \frac{df_{k(n-k)}}{f_{k(n-k)}}$$

→ Logarithmic singularities only on boundary of positive part!

Purpose in life : Makes

Dual. Conf. Inv. Manifest

At level of permutation

$a \rightarrow p(a)$

$a \rightarrow p(a) - 2$

$(k \rightarrow k-2)$

Original space

Dual Space

Yangian = Diffs on Positive Part

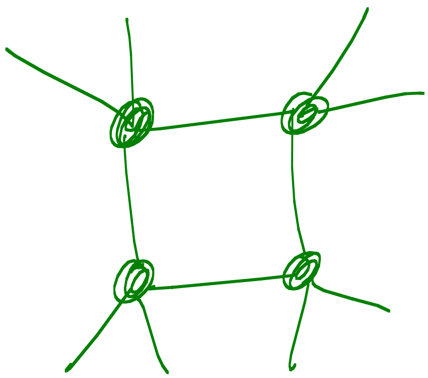
↑

Obvious symmetry of Geometry

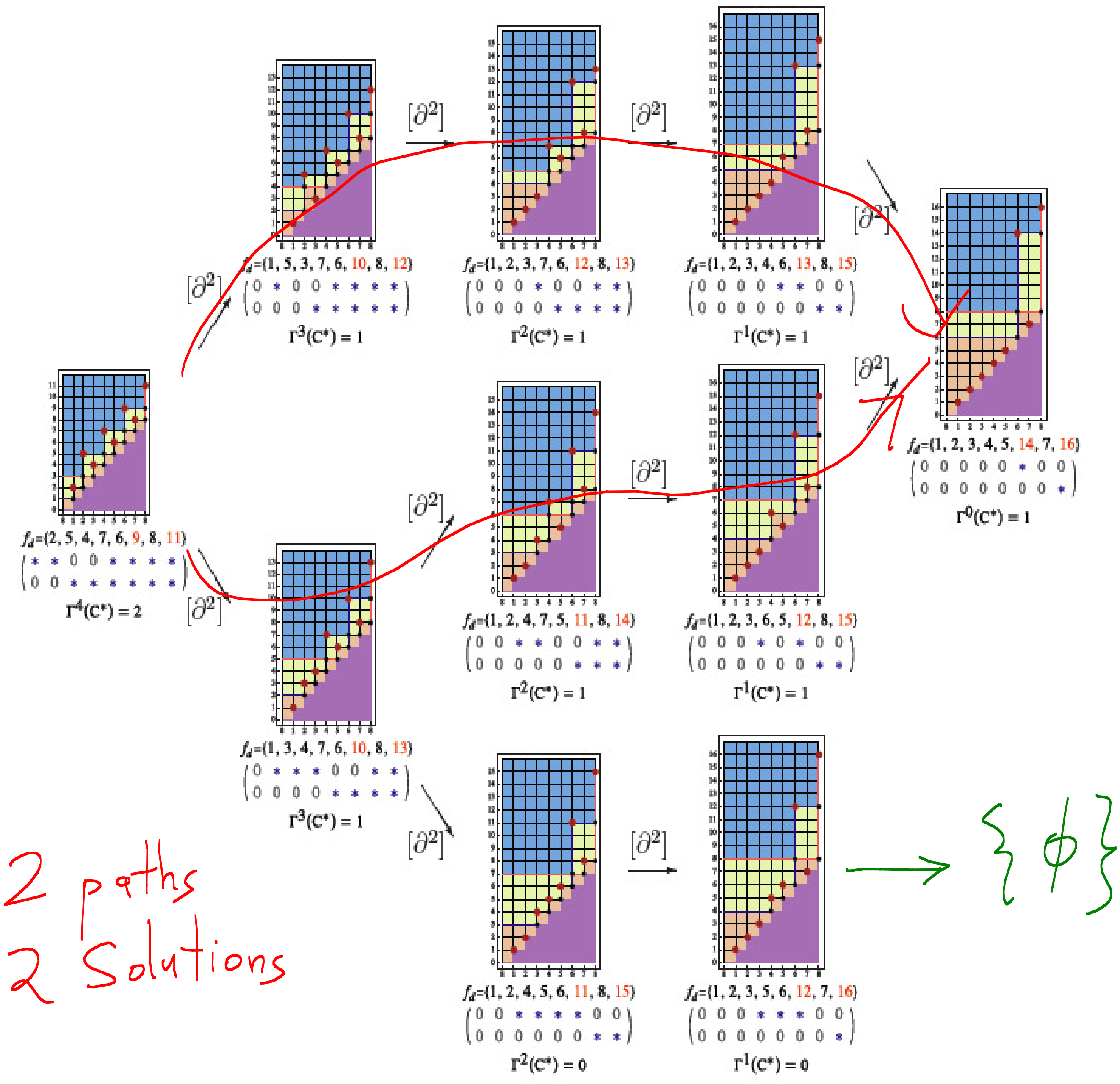
Beautiful action of BCFW vars.

Enumerative Geometry for Counting Sol'ns

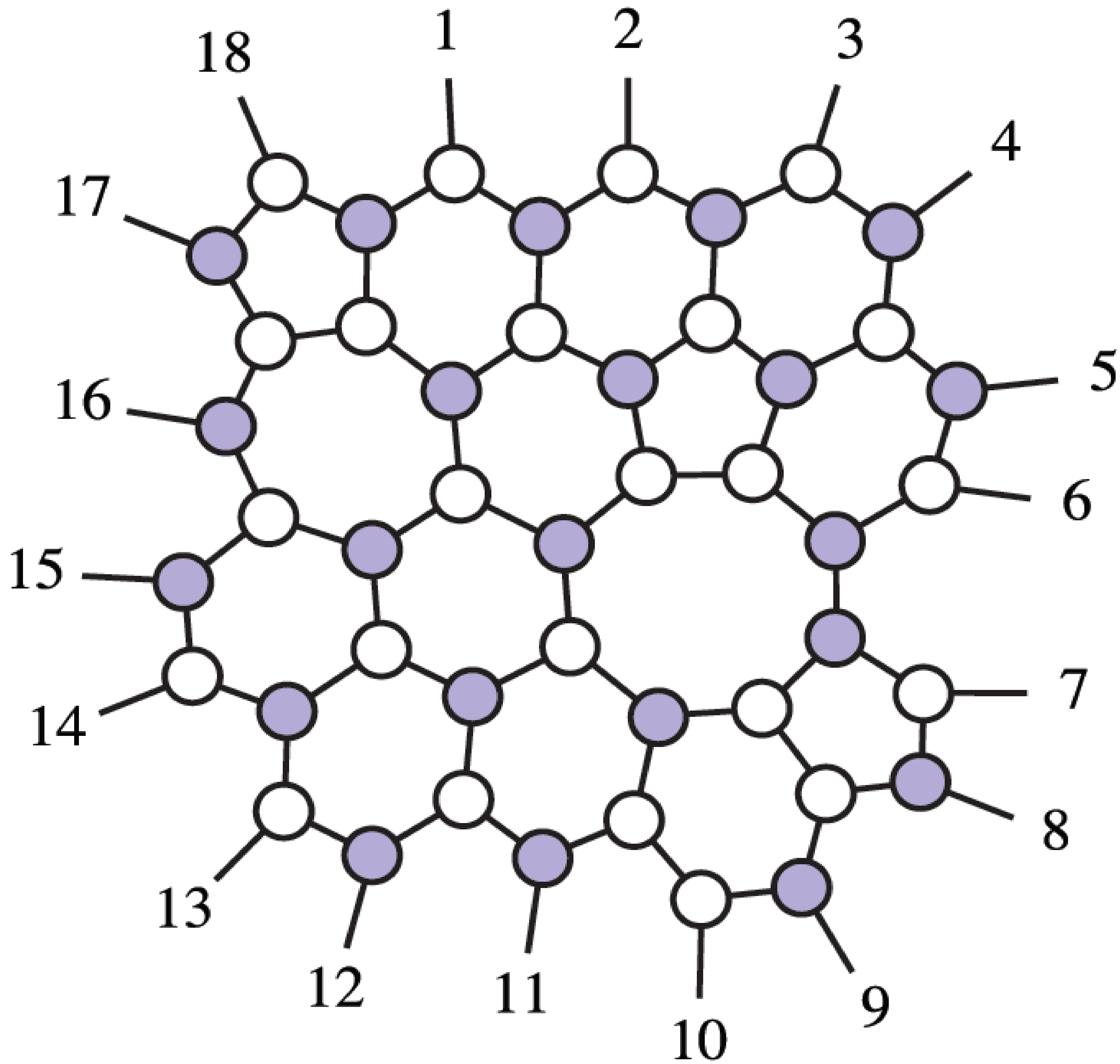
e.g



2 solutions
[some work!]



[∂2]
fd={2, 5, 4, 7, 6, 9, 8, 11}
Γ4(C*) = 2
fd={1, 5, 3, 7, 6, 10, 8, 12}
Γ3(C*) = 1
fd={1, 2, 3, 7, 6, 12, 8, 13}
Γ2(C*) = 1
fd={1, 2, 3, 4, 6, 13, 8, 15}
Γ1(C*) = 1
fd={1, 2, 3, 4, 5, 14, 7, 16}
Γ0(C*) = 1
fd={1, 3, 4, 7, 6, 10, 8, 13}
Γ3(C*) = 1
fd={1, 2, 4, 7, 5, 11, 8, 14}
Γ2(C*) = 1
fd={1, 2, 3, 6, 5, 12, 8, 15}
Γ1(C*) = 1
fd={1, 2, 4, 5, 6, 11, 8, 15}
Γ2(C*) = 0
fd={1, 2, 3, 5, 6, 12, 7, 16}
Γ1(C*) = 0
2 paths
2 Solutions
{ϕ}


18
1
2
3
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17
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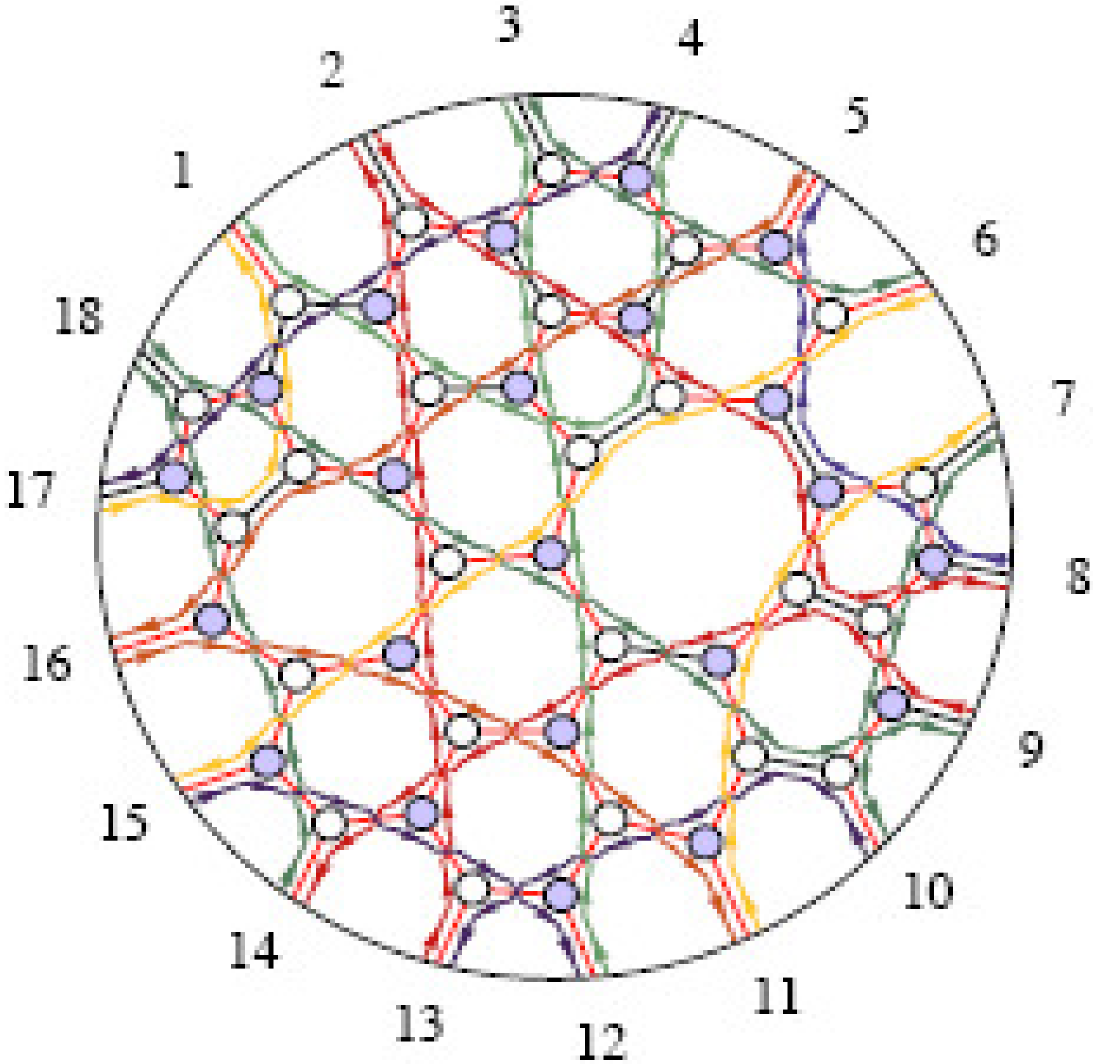
16
6
15
7
14
8
13
12
11
10
9



1
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12
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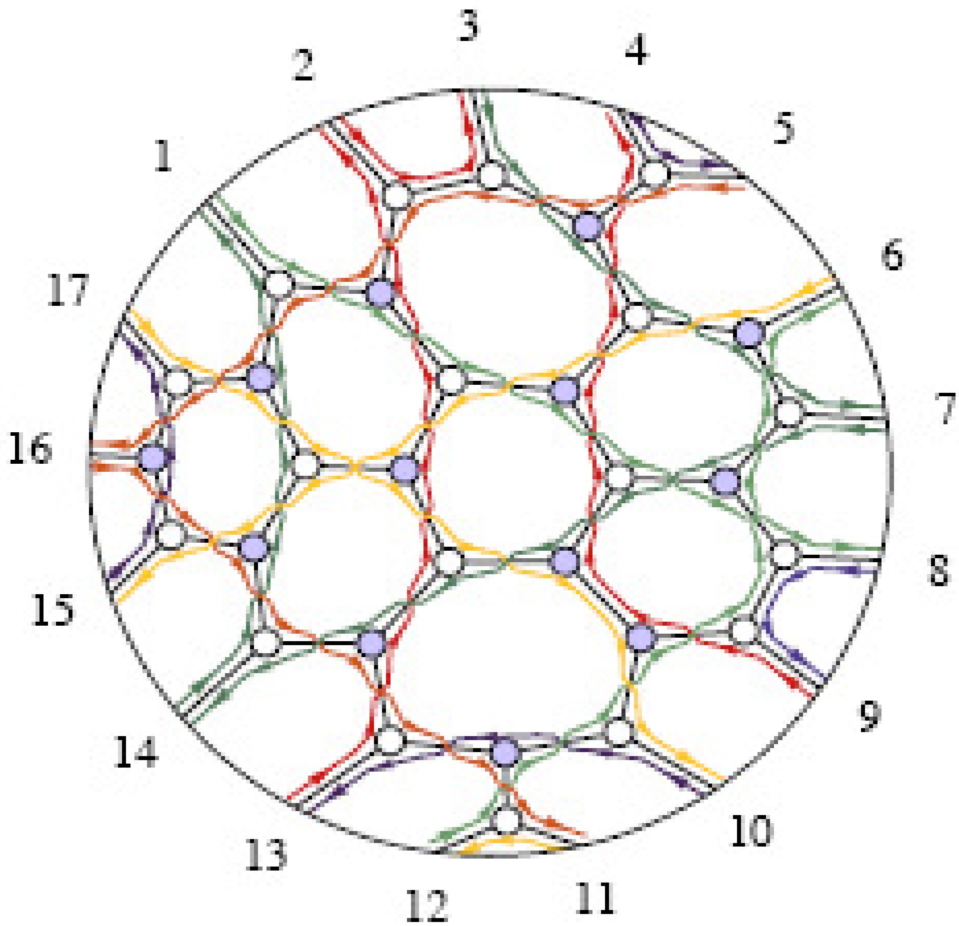
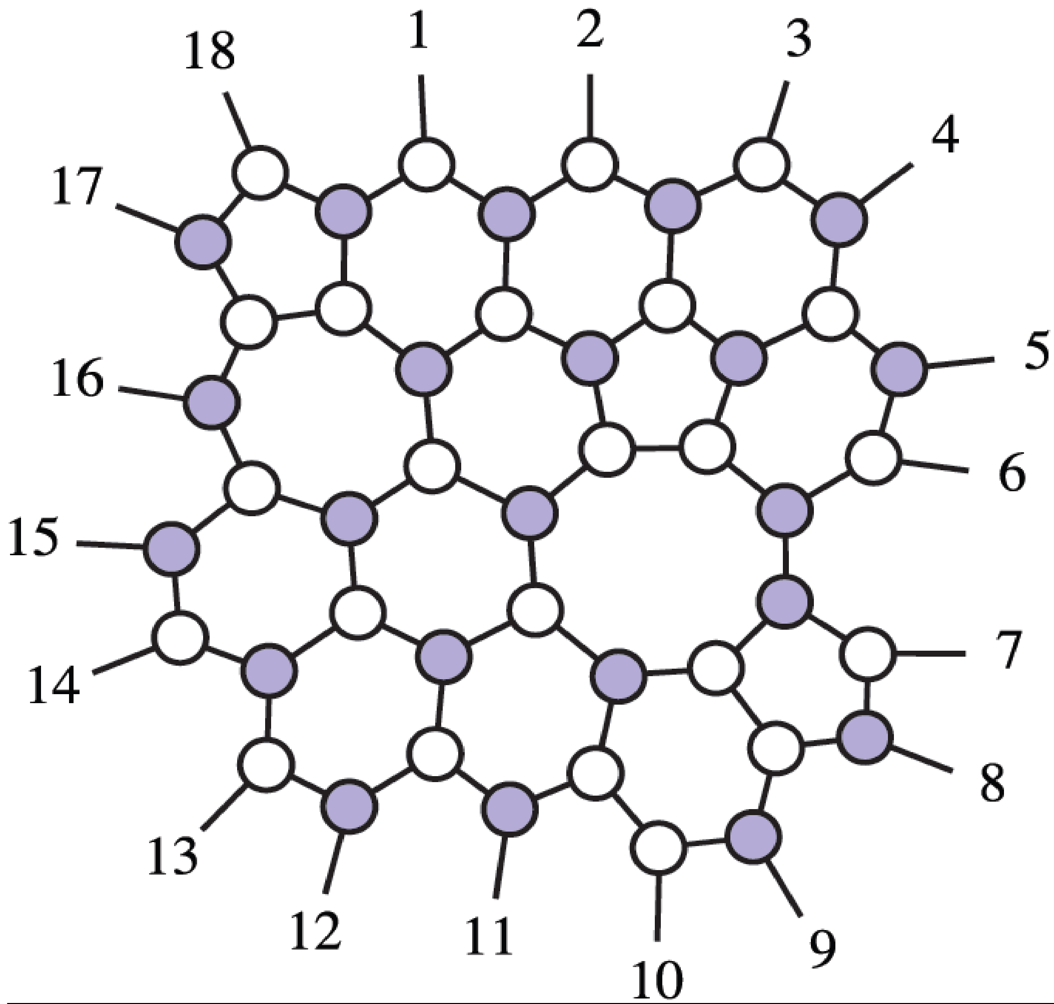
16
17
18



In Mom-Twistor Space

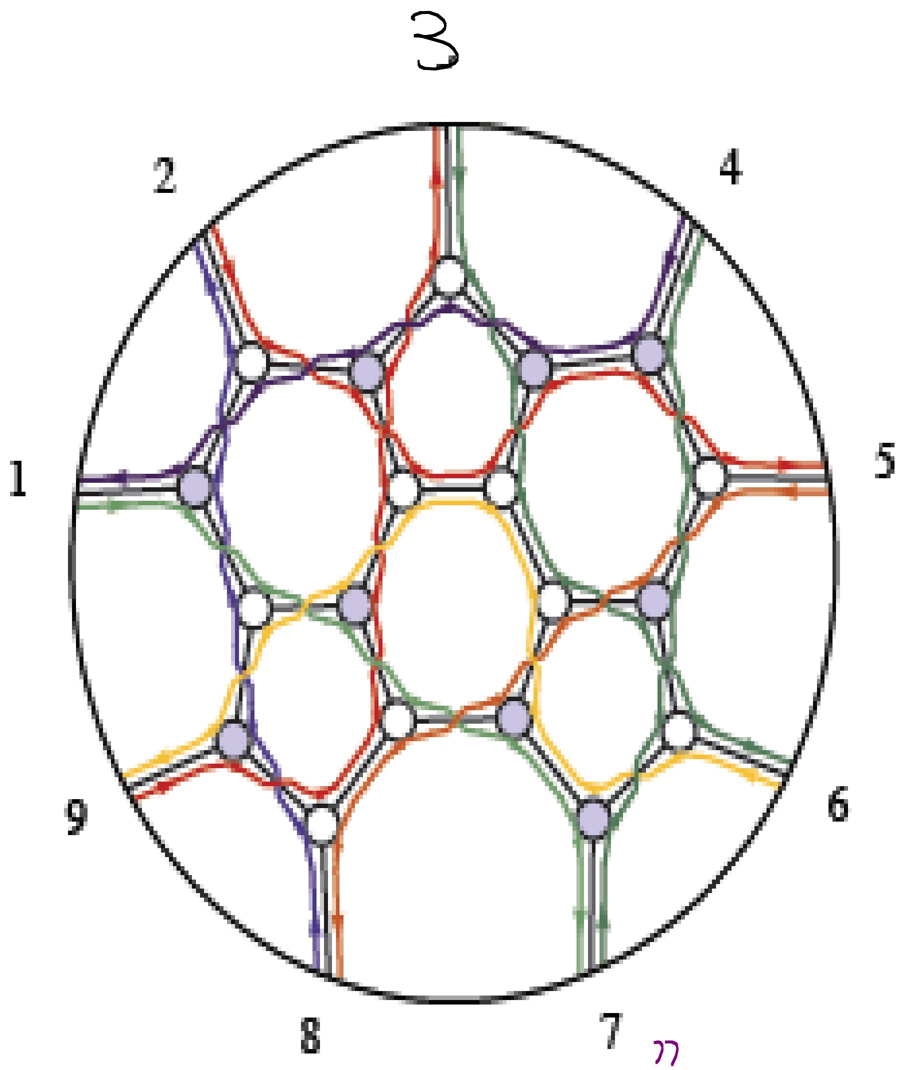


13 Solutions

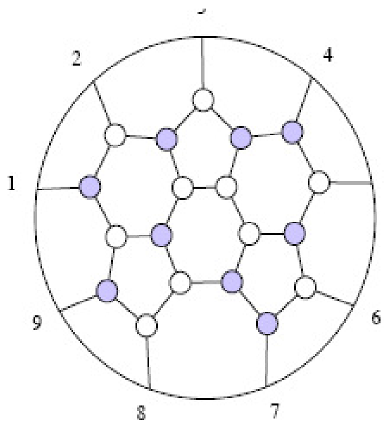
Easy to see, for LS, and generic external data:

No new objects for $n \geq 5(k-2)$

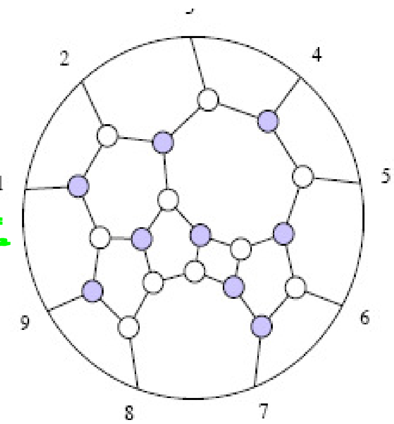
e.g. for $k=2$: BCFW terms, 4 m box and ....



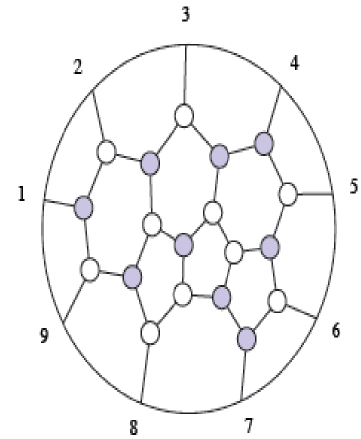
" Spurion "



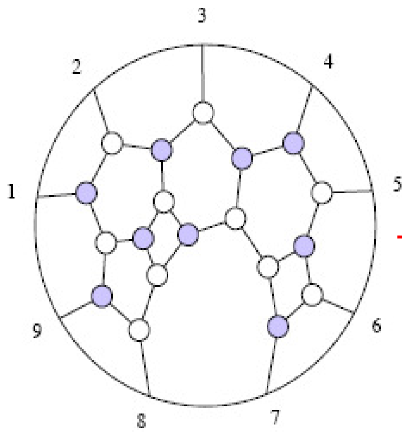
=



+



+



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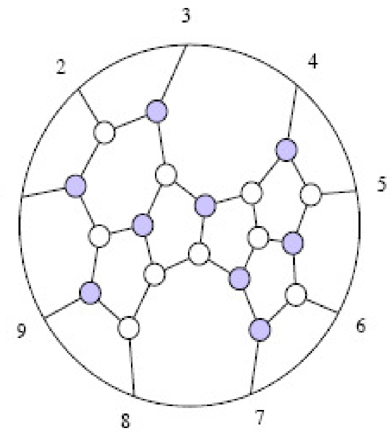
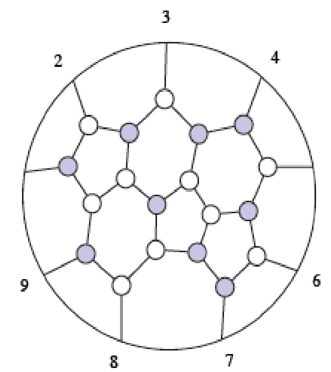
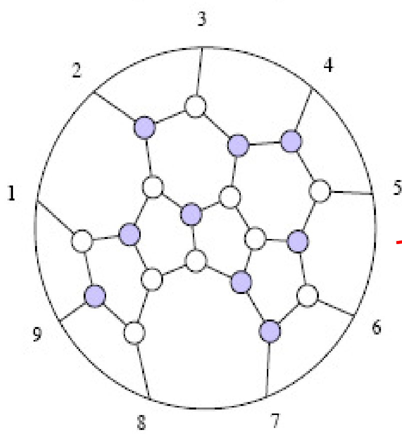


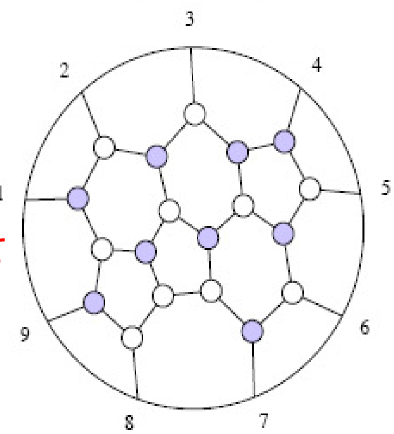
+



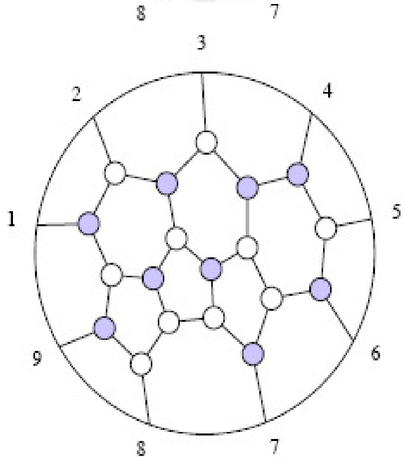
+

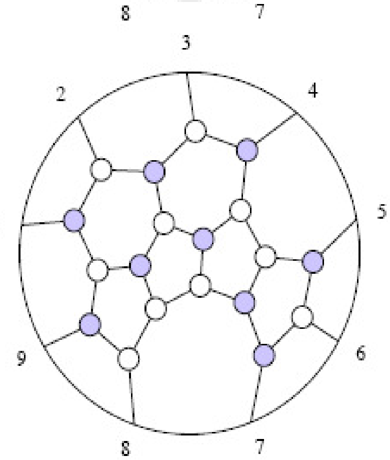


+

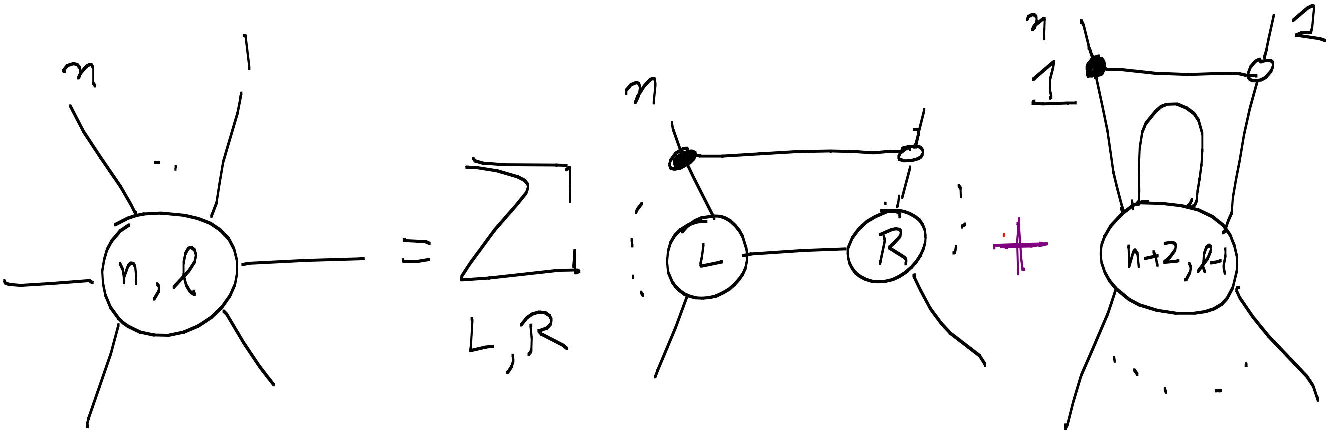


[All on RHS are BCFW terms]

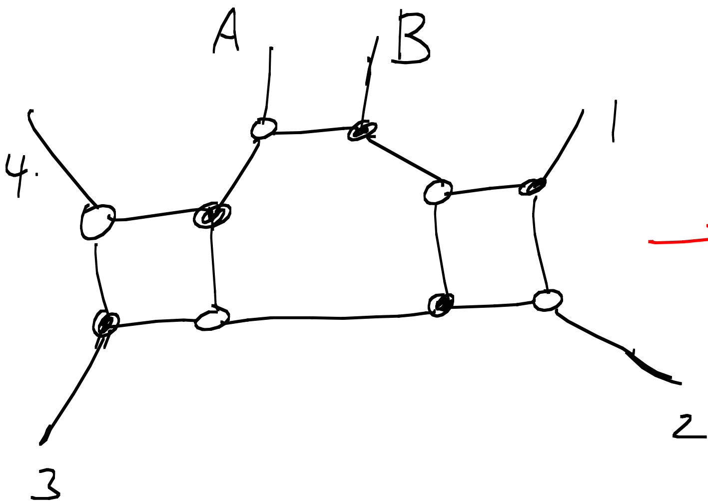


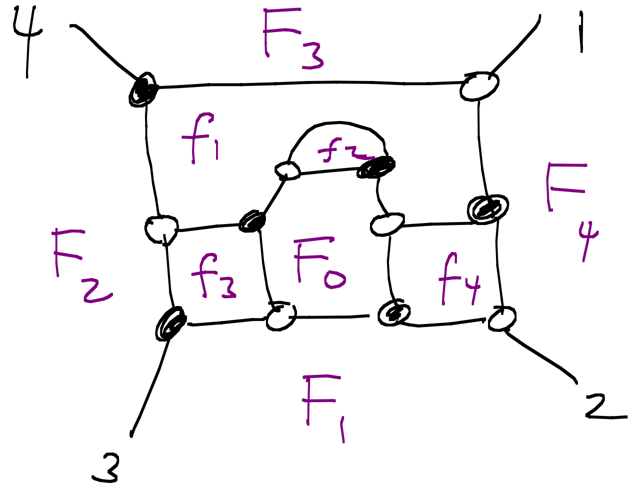


All-Loop Integrand in Grassm.

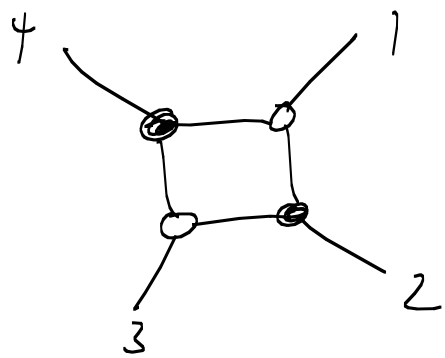


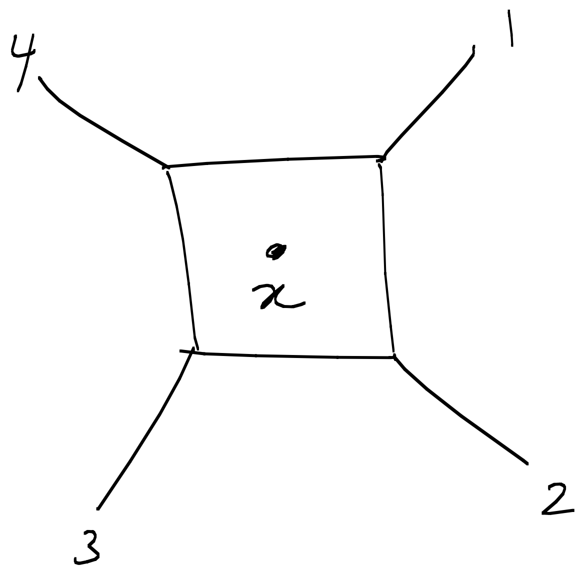
→ Explicitly, ∑'s of dlog's!





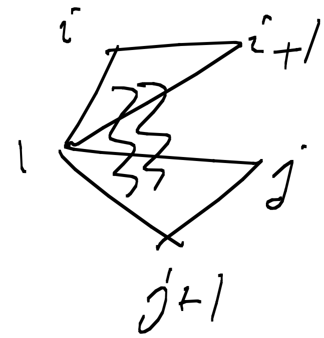
$$\underbrace{d\log f_1 \cdots d\log f_4}_{\text{Loop Integrand}}$$





$$\frac{d^4 x}{(x-x_1)^2 (x-x_2)^2 (x-x_3)^2 (x-x_4)^2}$$

$$= d\log \frac{(x-x^*_{13})^2}{(x-x_3)^2} \; d\log \frac{(x-x_2)^2}{(x-x_3)^2} \; d\log \frac{(x-x^*_{13})^2}{(x-x_1)^2} \; d\log \frac{(x-x_4)^2}{(x-x_1)^2}$$

$$MHV^{1-loop} = \sum_{i,j}$$ 

$$= \sum_{i} d\log\frac{\langle AB\,1\,i\rangle}{\langle AB\,i-1\,i\rangle}\, d\log\frac{\langle AB\,1\,i-1\rangle}{\langle AB\,i-1\,i+1\rangle}\, d\log\frac{\langle AB\,1\,i-\rangle}{\langle AB\,i+1\,1\rangle}\, d\log\frac{\langle AB\,i\,i+1\rangle}{\langle AB\,i+1\,1\rangle}$$

$$+ \sum_{j>i+1} d\log\frac{\langle AB\,1\,i\rangle}{\langle AB\,i\,i+1\rangle}\, d\log\frac{\langle AB\,1\,i+1\rangle}{\langle AB\,i\,i+1\rangle}\, d\log\frac{\langle AB\,1\,j\rangle}{\langle AB\,j\,j+1\rangle}\, d\log\frac{\langle AB\,1\,j+1\rangle}{\langle AB\,j\,j+1\rangle}$$

↑

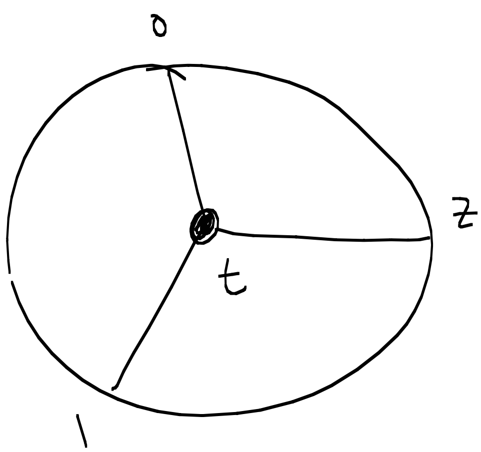
Finite

Non-trivial only because of **reality**

$$\bar{Z}_{A,B} \, Z_{A,B} = 0$$

Generalizing Goncharov's Theory of Polylogs

Goncharov Polylogs = Tree-level correlators of $(\partial\phi)^2 + \phi(\partial\phi)^2$ in $(1+1)$-d



$$\int_{\mathbb{CP}^1} \log|t| \; d\log|t-1| \; d\log|t-z|$$

$$= \int_{CP^1 \times CP^1} d\log|t'| \; d\log|t'-t| \; d\log|t-1| \; d\log|t-z|$$

In general

$$G^{(d+1)}(a_1, \dots, a_d) = \int \underbrace{d\log(\,)\; d\log(\,) \cdots d\log(\,)}_{2(d+1)}$$

Reality condition $\tilde{\lambda}\lambda^{*} = 0 \Rightarrow \lambda^{*} = \tilde{\lambda}$

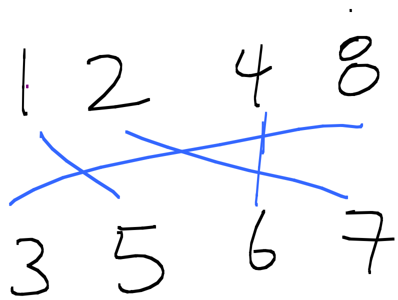
We have a generalization, with a more non-linear reality condition $\rightsquigarrow$ Good, because we must get much richer functions than Polylogs!

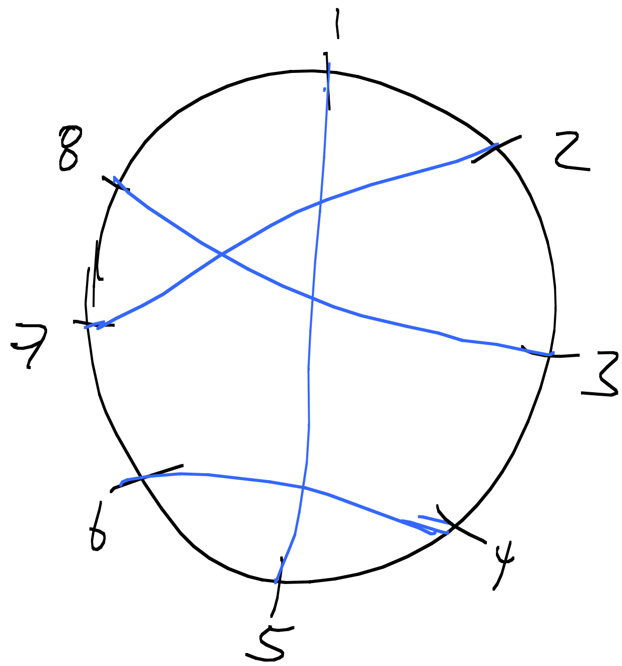
[Elliptic integrals + more ....]

Beyond

Planar $\mathcal{N} = 4$

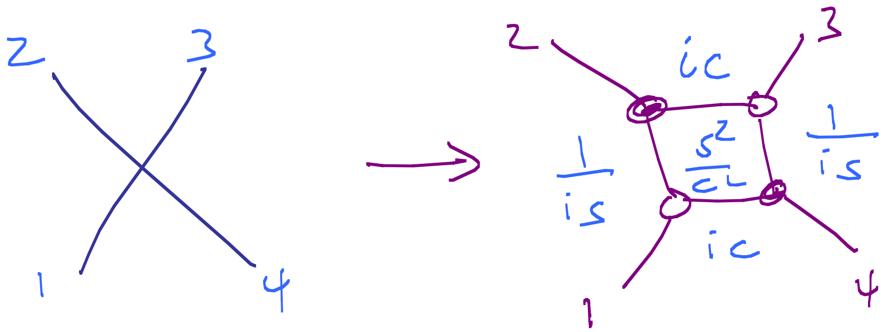
Back to Permutations





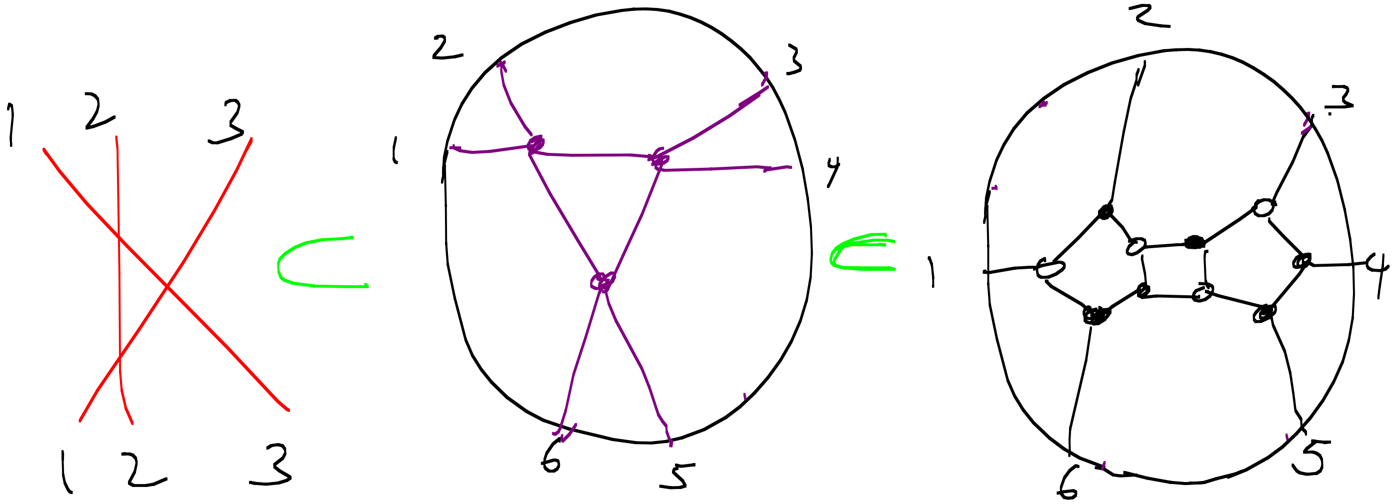
On-shell diagrams for ABJM

ABJM $\quad \mathcal{N}=4$ SYM



$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & ic & is \\ 0 & 1 & -is & ic \end{bmatrix}$$

"Orthogonal" Grassmannian

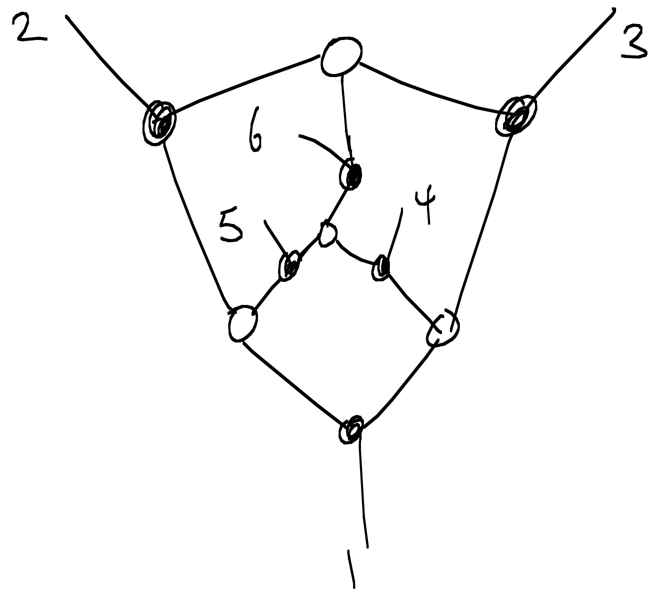


1+1-d ⊂ 2+1-d ⊂ 3+1-d

Y-B ⊂ ABJM ⊂ $\mathcal{N}=4$ SYM

On-shell diagrams for Non-planar amplitudes

* Solved for MHV:
  - Reduction
  - Computing reduced graphs



$$= \frac{\left(\langle 14\rangle\langle 36\rangle\langle 25\rangle - \langle 15\rangle\langle 26\rangle\langle 34\rangle\right)^2}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 36\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle}$$

= Sum of PT factors!

On-Shell Diagrams for $\mathcal{N}<4$

$$\left(\prod \frac{de_i}{e_i}\right) \mathcal{J}^{\mathcal{N}-4} \prod_{\alpha=1}^{K} \delta^{4|\mathcal{N}}\left[C_{\alpha a}[e] W_a\right]$$

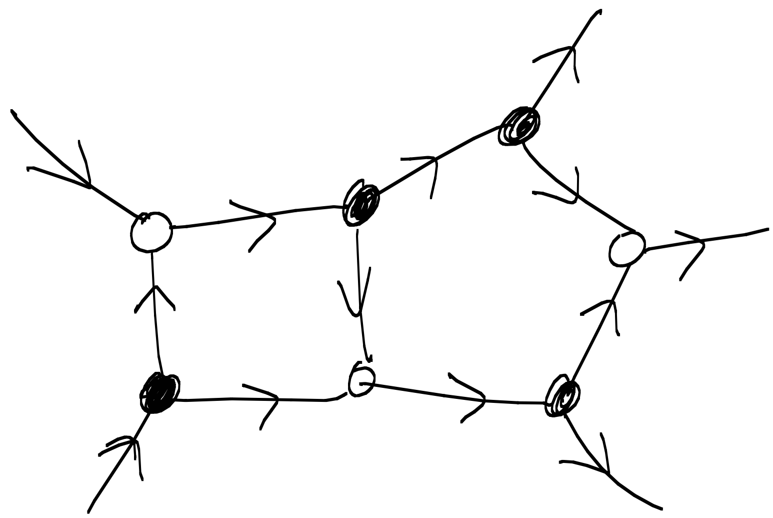
New

Universal

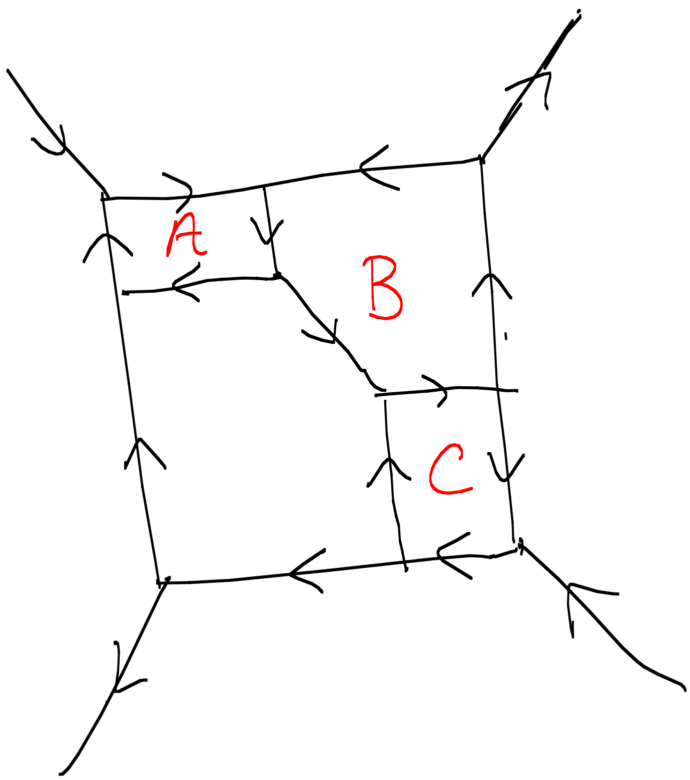
$$J = \det(\mathbb{1} - A)$$

$$A_{ij} = \begin{cases} 0 & \text{if no } i \to j \\ e_{ij} & \text{if } i \to j \end{cases}$$

"Spectrum" of Graph



$J = 1$ if no "orientation loops"



$$J = 1 + A + B + C + AC$$

* Some residues

$$\prod_i \frac{de_i}{e_i} \, \delta^4(C \cdot W) \, J^{\mathcal{N}-4}$$

residues here

multiplicative factor on $\mathcal{N}=4$ answer

* More interesting

$$\prod \frac{de_i}{e_i} \; J^{N-4} \qquad \delta^{4|N}(C \cdot W)$$

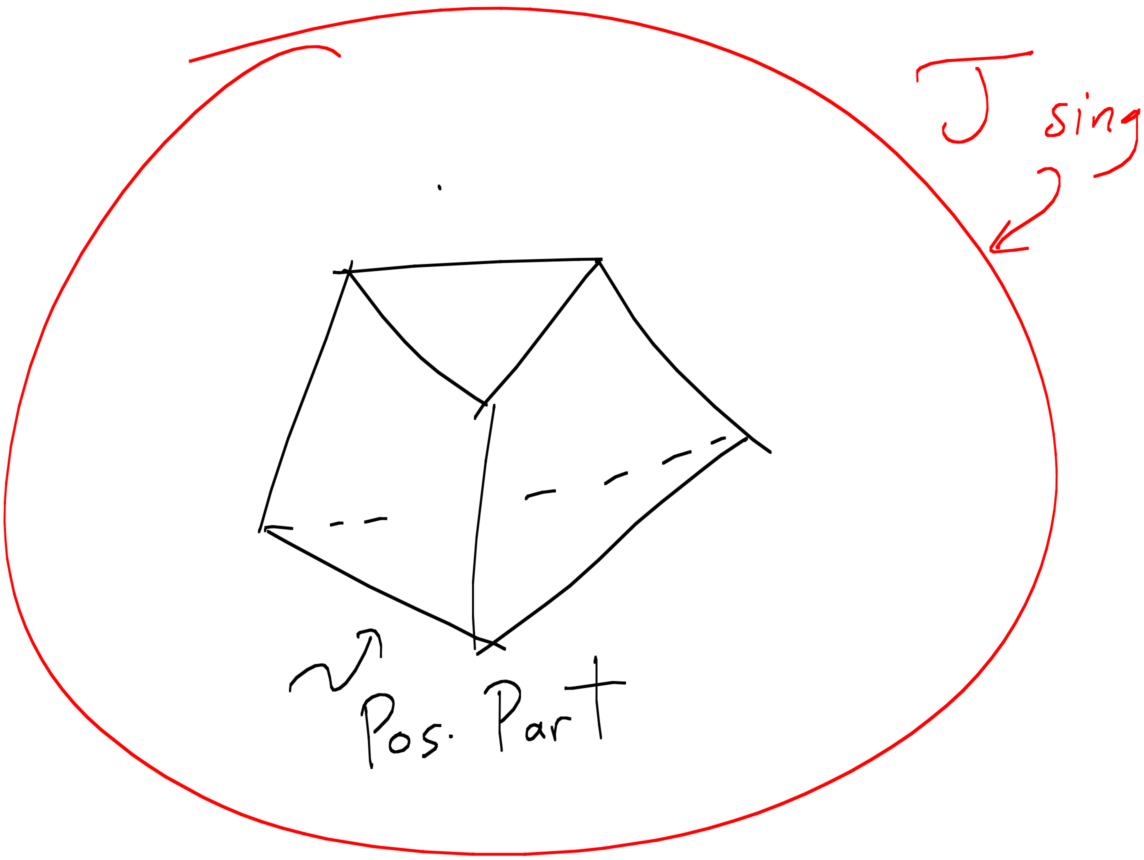
also use these singularities!

These correspond to "Poles at ∞"

* Triangles ....
* BCFW residues @ ∞ for:
  "bad shifts"

  ⋮

Remarkably



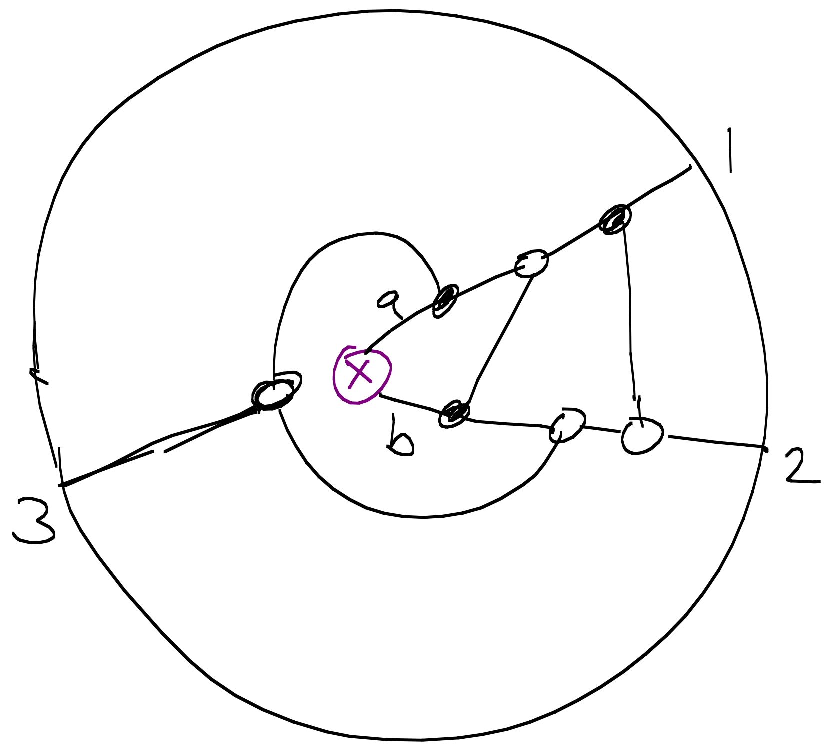
Locus of $\mathcal{J} = 0$ "UV"

Positive Grassmannian "IR"

Naturally Paired

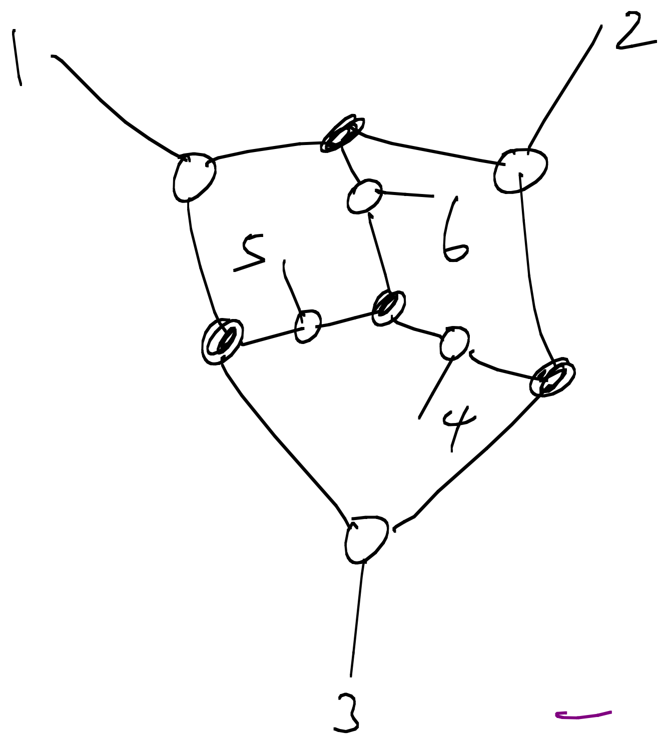
Beyond

Scattering Amplitudes



On-shell diagram for form factors

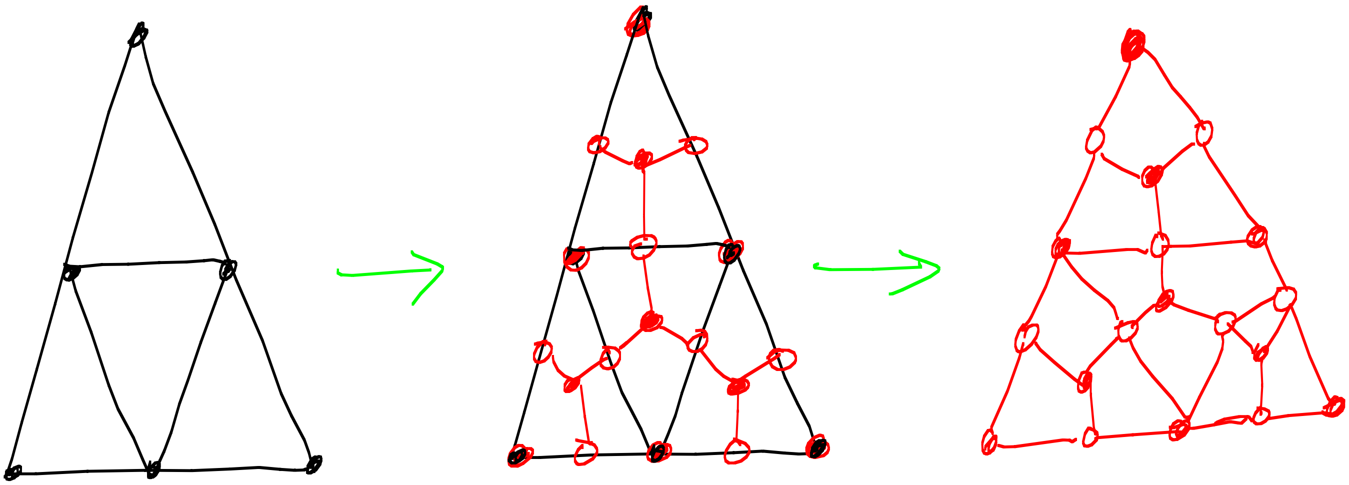
[Correlators ...] {Huang, Xie, ... in progress}

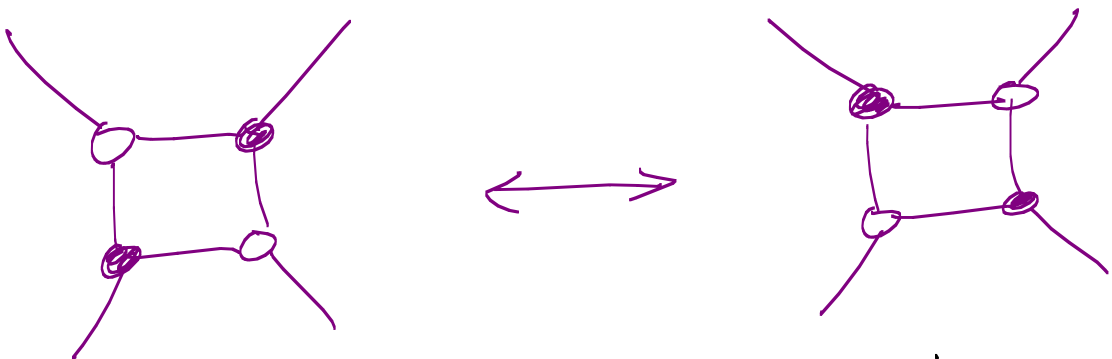


Depending on how external legs are connected

– O.S. diagram for Amp, Form Factors, Corr. @ diff orders in $1/N_c$

* Amazingly — exactly the same mathematical structure shows up in wall-crossing for Gaiotto $\mathcal{N}=2$ theories, and amps @ strong coupling





Square Move Seiberg Duality

Open Questions

* Complete On-Shell/Grassmannian connection:

Non-planar $\mathcal{N}=4$ [Beyond MHV]

$\mathcal{N}=0$, Gravity, ...

* Can we use the on-shell information to systematically construct non-planar amps/form factors/correlators { is there any useful notion of "integrand"? }

* Can we use

$$\int \underbrace{d\log(\ )\, d\log(\ ) \cdots d\log(\ )}_{4L}$$

↓

Transcendental Functions

(Well beyond Polylogs/Symbol...)

Most Fascinating (to me) Questions

(1) We have seen enormous amounts of magic associated with the still mysterious (2,0) theory in 6D.

* The remarkable mathematical structures we are seeing are clearly intimately tied to the (2,0) theory → S-duality (in wall-crossing) and T-duality (in scattering amps)

* Can we see these structures in action directly in 6D?