Recent Results from MicroBooNE

First results using the full dataset

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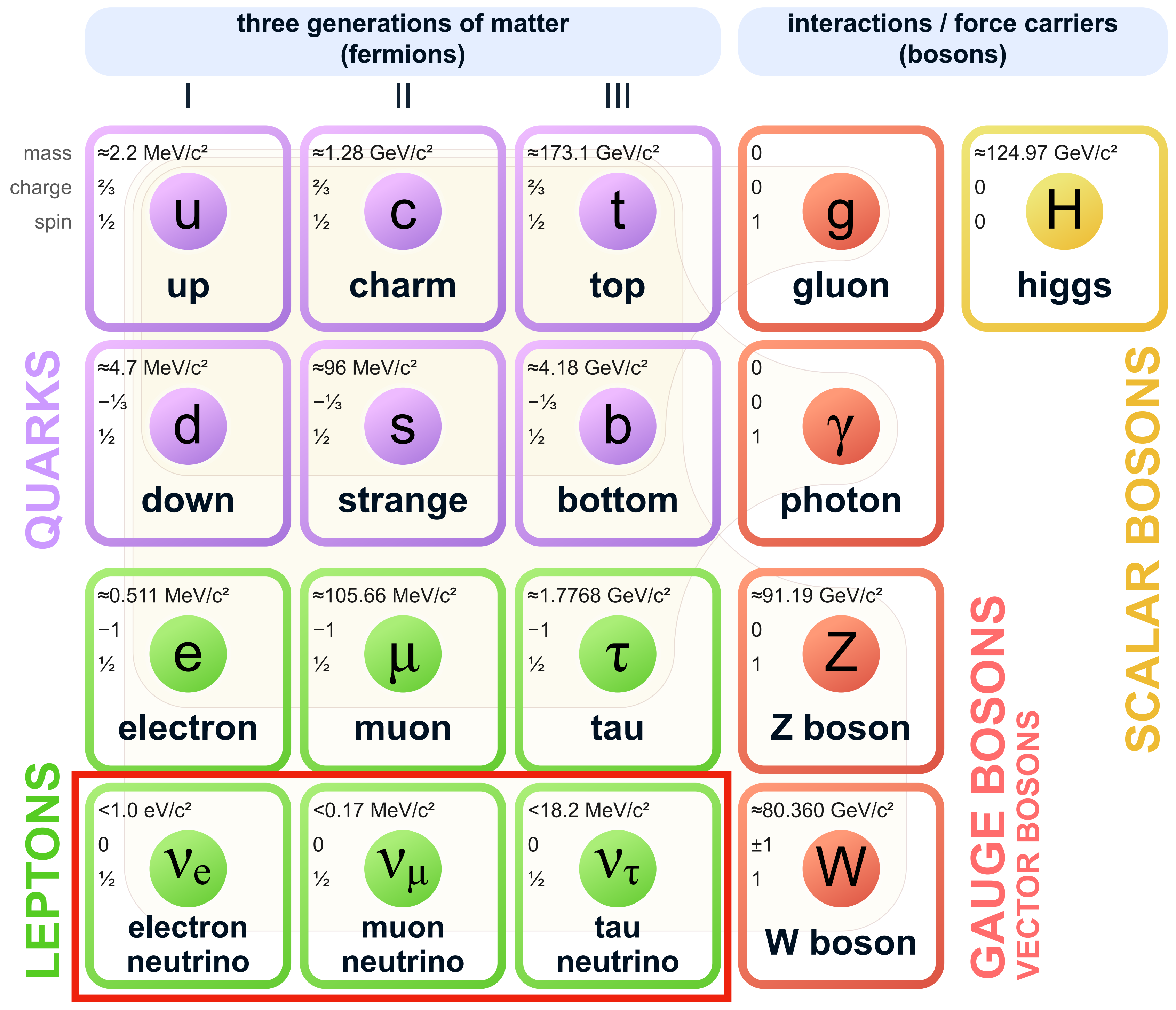
Introduction

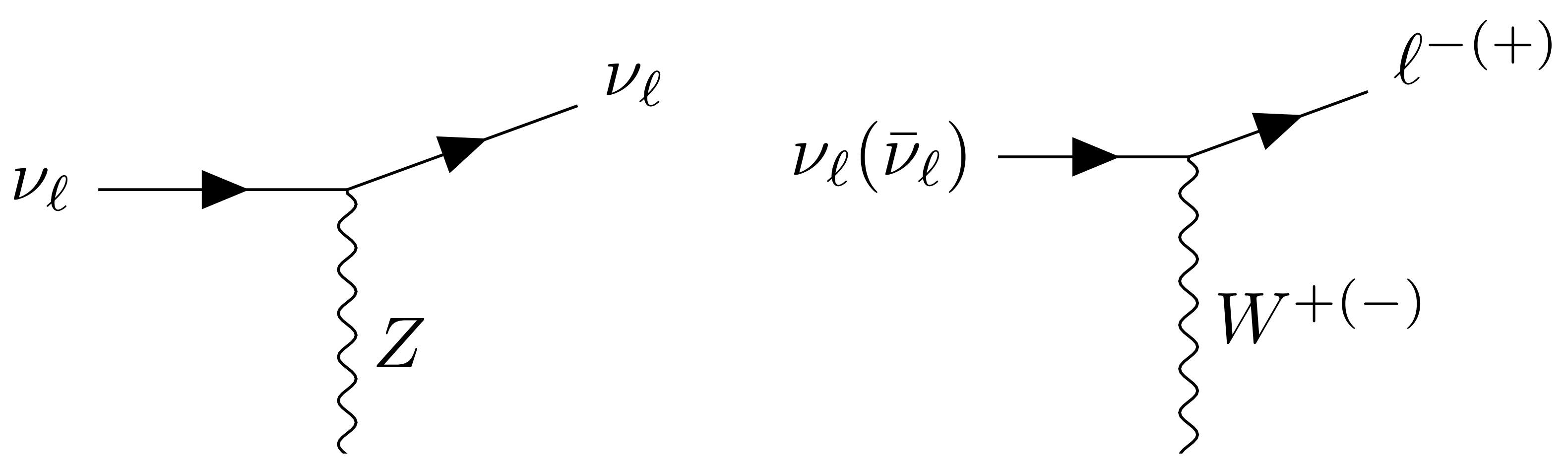
Neutrinos in the Standard Model

Standard Model of Elementary Particles



Credit: Wikimedia Commons

Weak Interactions



Neutral-current interactions

Charged-current interactions

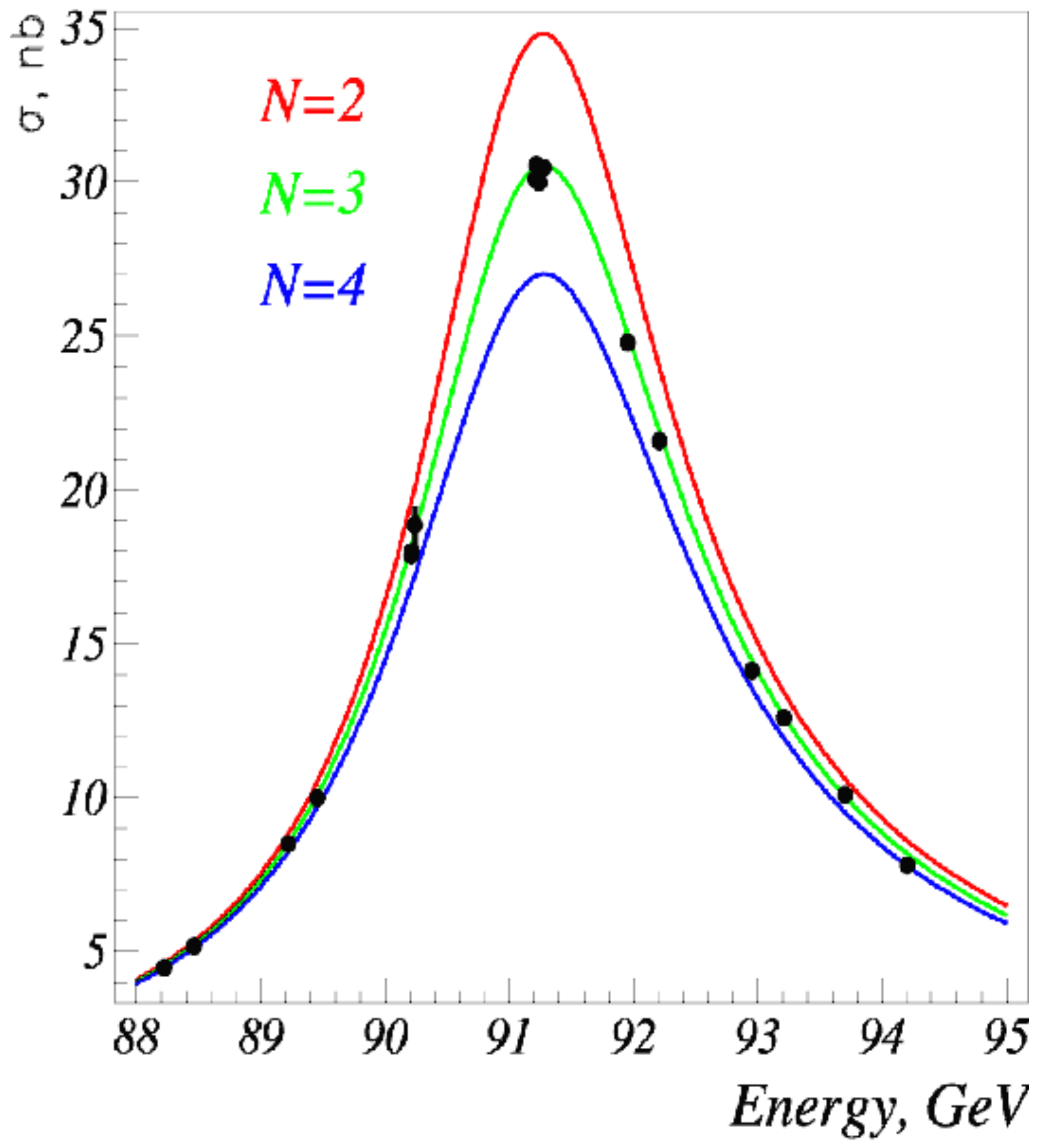
Neutrino Mixing

Mass eigenstates

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Flavor eigenstates

Number of Flavors



LEP measurement at the Z-resonance. *arXiv:hep-ex/0503050*

Neutrino Oscillations (in vacuum)

Mixing Angles

$R \equiv$ *real rotation matrix* $\qquad s_{ij} = \sin\theta_{ij}$

$\tilde{R} \equiv$ *complex rotation matrix* $\qquad c_{ij} = \cos\theta_{ij}$

$$U = \begin{pmatrix} U_{e1} U_{e2} U_{e3} \\ U_{\mu1} U_{\mu2} U_{\mu3} \\ U_{\tau1} U_{\tau2} U_{\tau3} \end{pmatrix} = R_{23}(\theta_{23})\tilde{R}_{13}(\theta_{13},\delta_{13})R_{12}(\theta_{12})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

"atmospheric" *"reactor"* *"solar"*

Transition probability

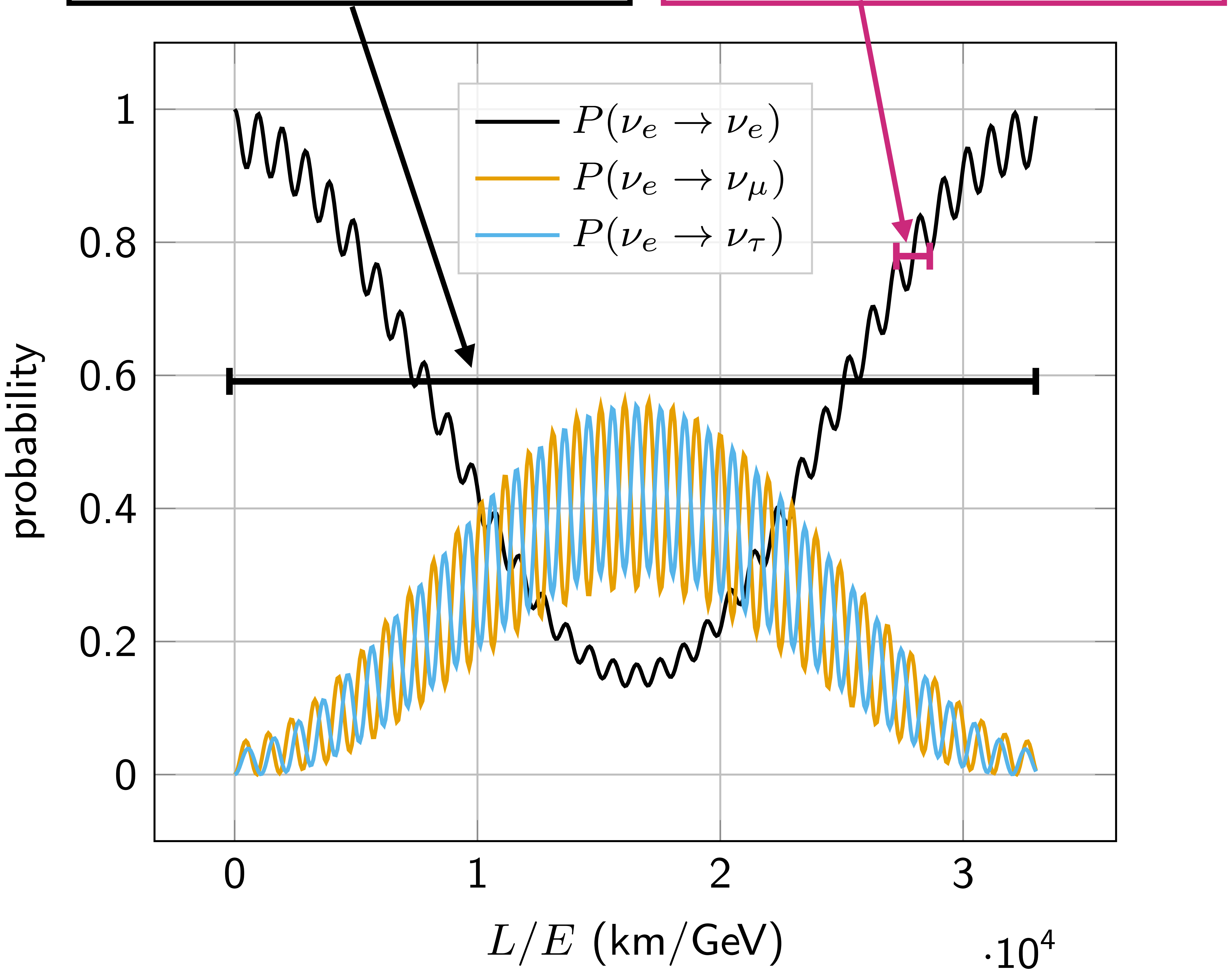
mass splitting: $\Delta m^2_{kj} = m^2_k - m^2_j$

$$P_{\nu_\alpha \to \nu_\beta}(L,E) = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i\frac{\Delta m^2_{kj} L}{2E}\right)$$

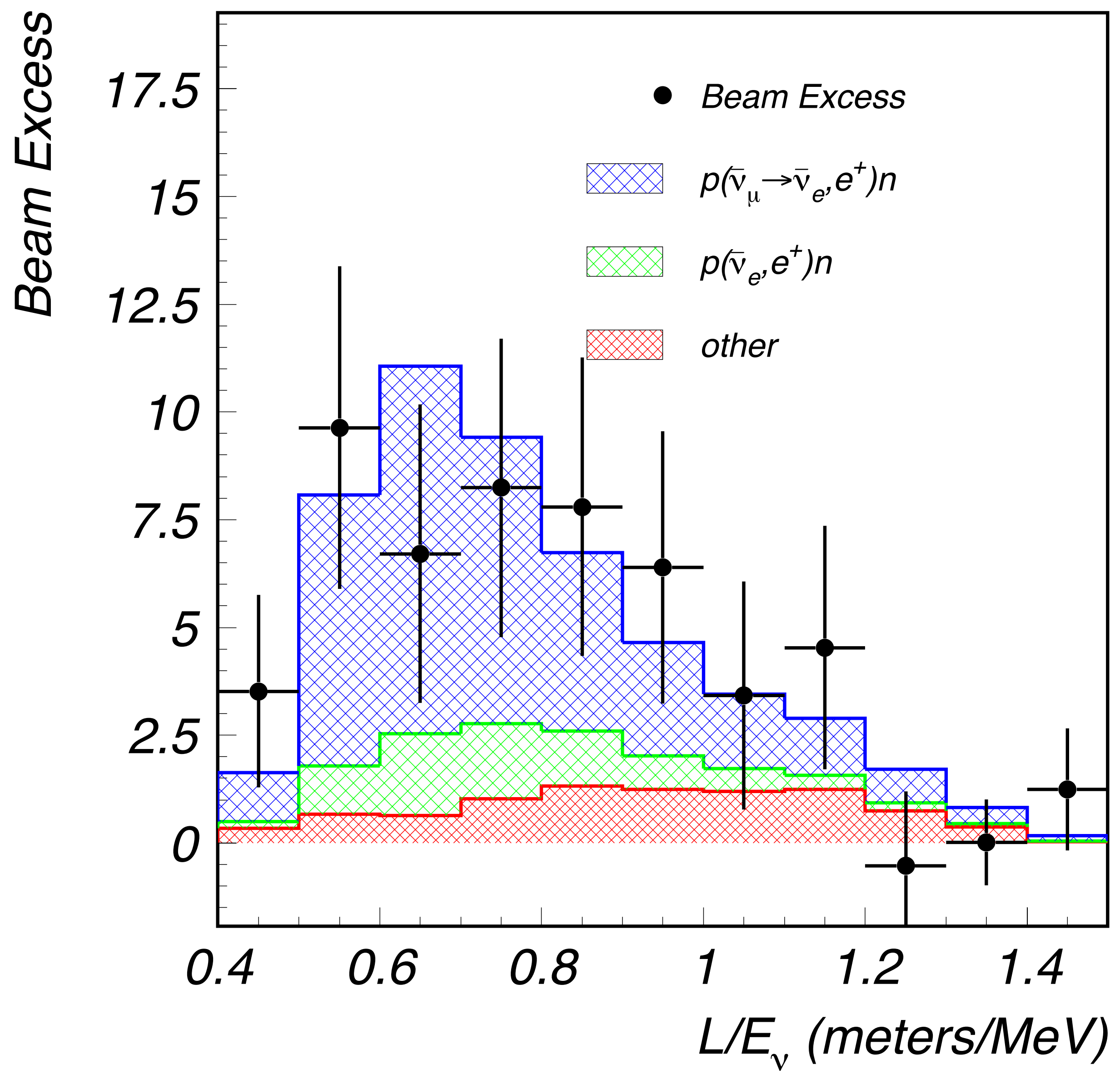
two-flavor approximation: $\approx \sin^2 2\vartheta \sin^2\left(1.267 \times \frac{L}{\text{km}} \frac{\text{GeV}}{E} \frac{\Delta m^2}{\text{eV}^2}\right)$

"solar" mass splitting $\quad \Delta m^2_{21} \approx 7.5 \times 10^{-5}\ \text{eV}^2$

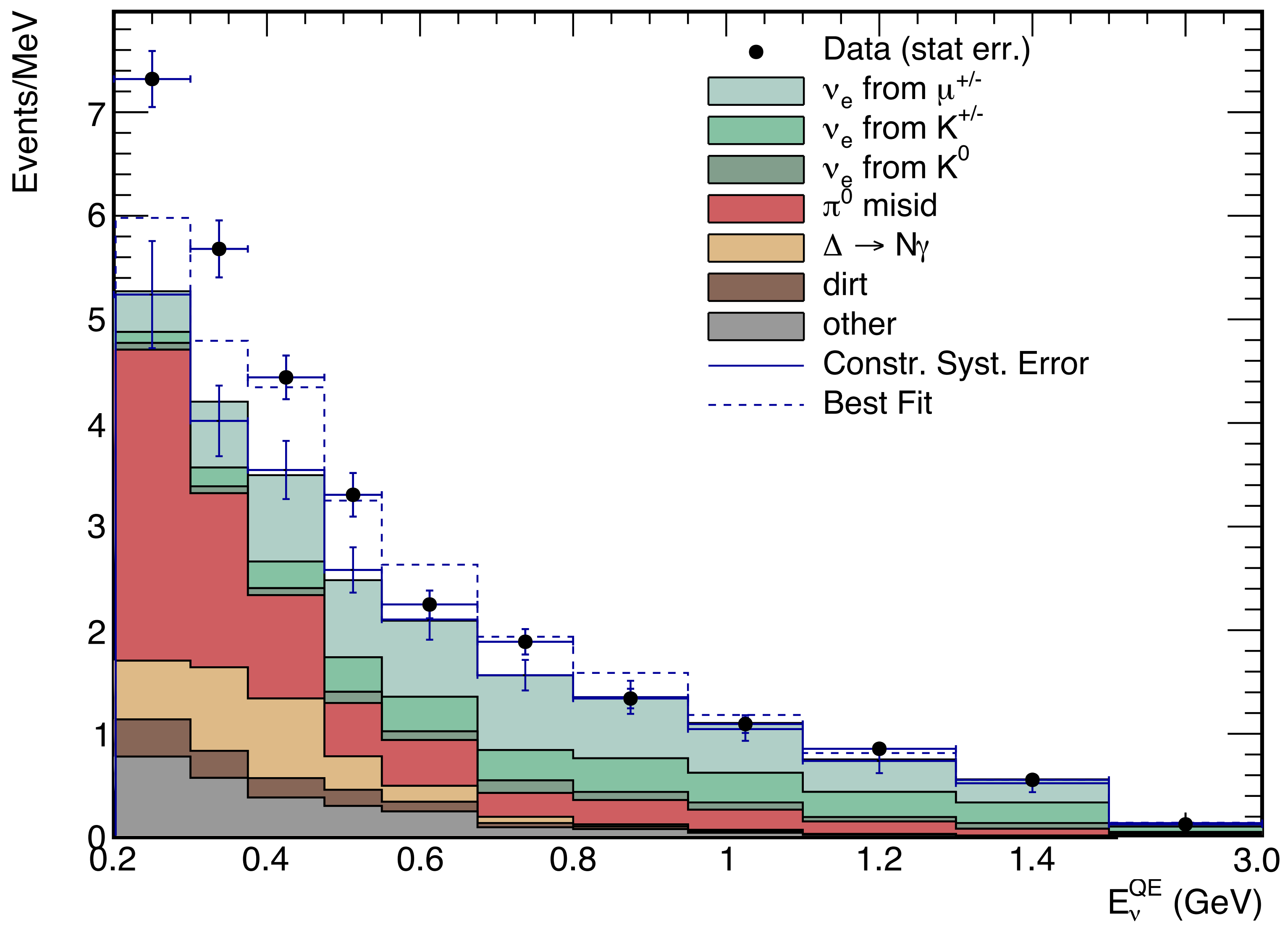
"atmospheric" mass splitting $\quad \Delta m^2_{31} \approx 2.5 \times 10^{-3}\ \text{eV}^2$



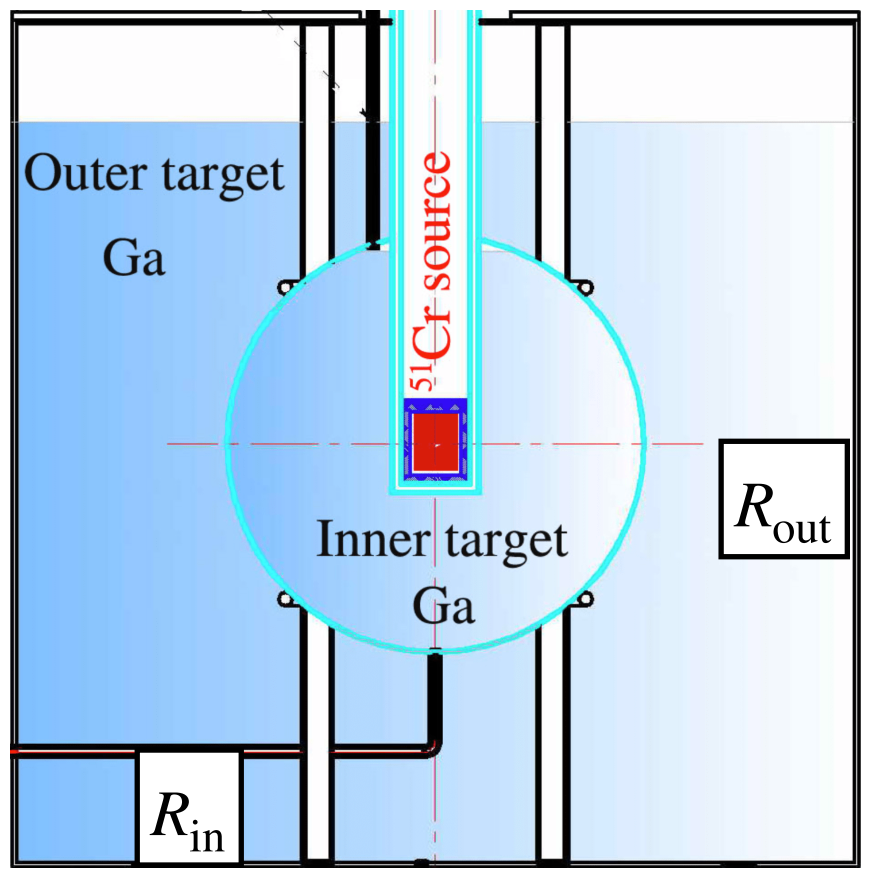
Experimental Anomalies



Phys.Rev.D 64 (2001) 112007



Phys.Rev.D 103 (2021) 5, 052002



Experimental setup of the BEST experiment.
Phys.Rev.Lett. 128 (2022) 23, 232501

- Excess in $\nu_\mu \to \nu_e$ channel
- 3.8σ significance
- $L/E \approx \dfrac{30\text{ m}}{30\text{ MeV}} \to \Delta m^2 \sim \mathcal{O}(1\text{ eV}^2)$

- Excess in $\nu_\mu \to \nu_e$ channel
- 4.8σ significance
- $L/E \approx \dfrac{541\text{ m}}{700\text{ MeV}} \to \Delta m^2 \sim \mathcal{O}(1\text{ eV}^2)$

Deficit in $\nu_e \to \nu_e$ channel:

$R_{in} = 0.79 \pm 0.05$

$R_{out} = 0.77 \pm 0.05$

Fourth mass eigenstate at $\Delta m^2 \sim \mathcal{O}(1\text{ eV}^2)$?

Sterile Neutrinos?

Adding a fourth mass eigenstate to the picture

- Know from Z-decay width that there are only 3 interacting flavors

  ➡ Additional mass eigenstate can only correspond to non-interacting "sterile" state in the flavor basis

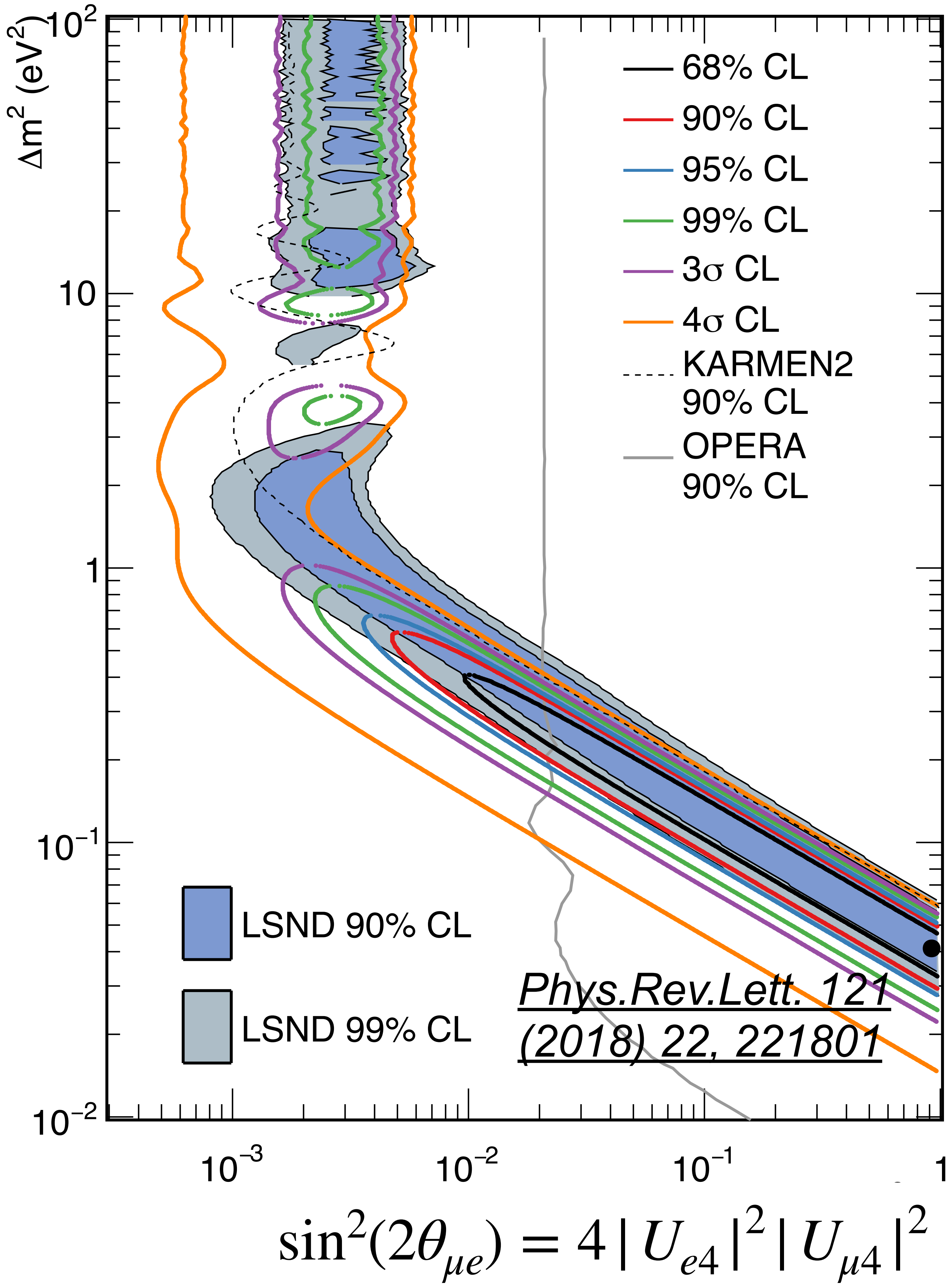
Simplest "3+1" model

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \nu_s \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}$$

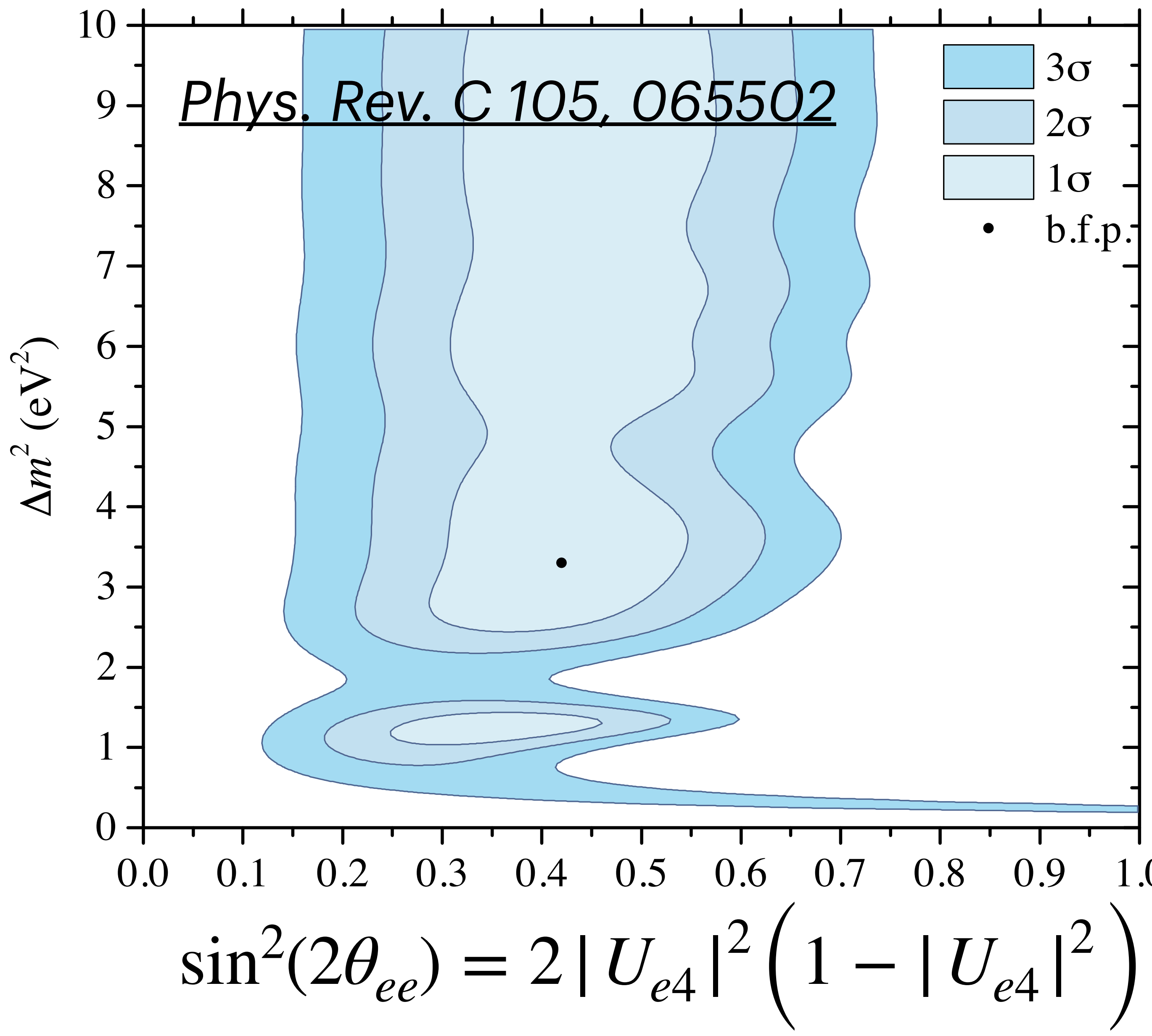
unobservable "sterile" state ν_s in flavor basis

Mass eigenstate with mass splitting Δm^2_{41}

MiniBooNE Anomaly



Gallium Anomaly



Sterile Neutrinos?

A conflicted experimental landscape

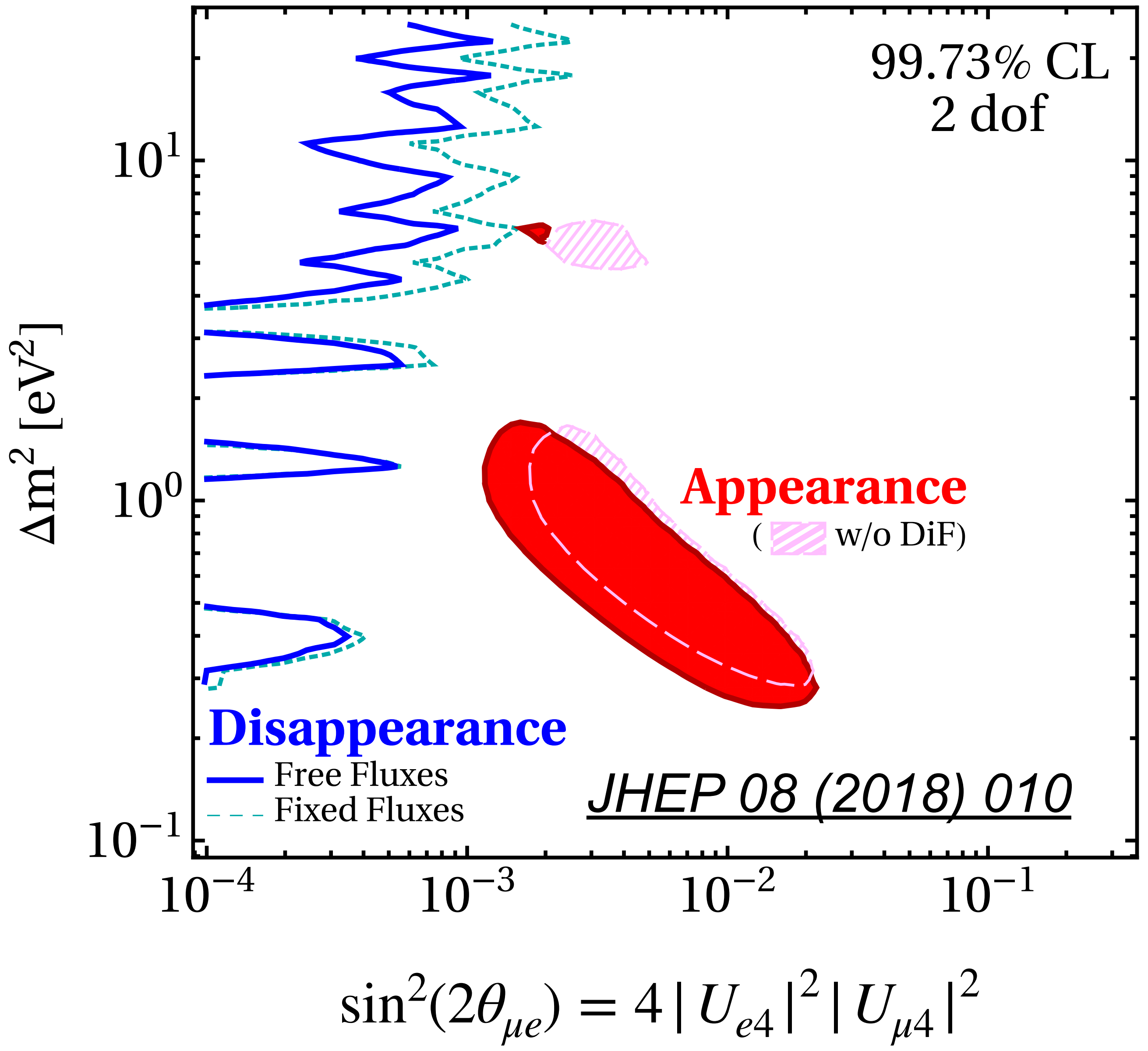
- Mixing parameters preferred by LSND and MiniBooNE can be constrained independently using *disappearance* measurements

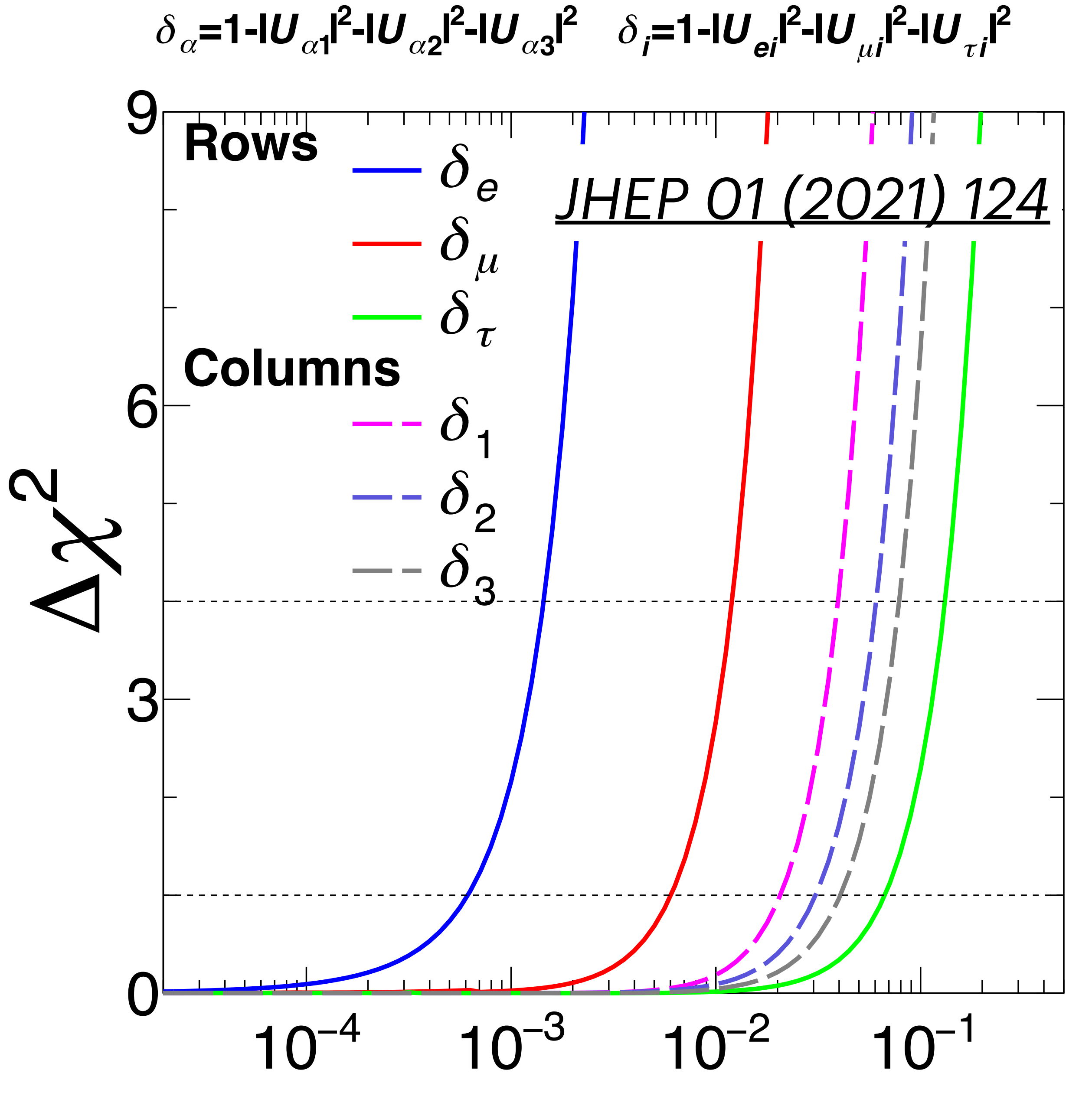
  ➡ Tension at 4.7σ between datasets

- Global unitarity constraints do not allow $|U_{e4}|^2$ high enough to be compatible

- Cosmological constraints from Planck collaboration (A&A 641, A6 (2020)):

$$\sum m_\nu < 0.12 \text{ eV}, 95\,\%$$



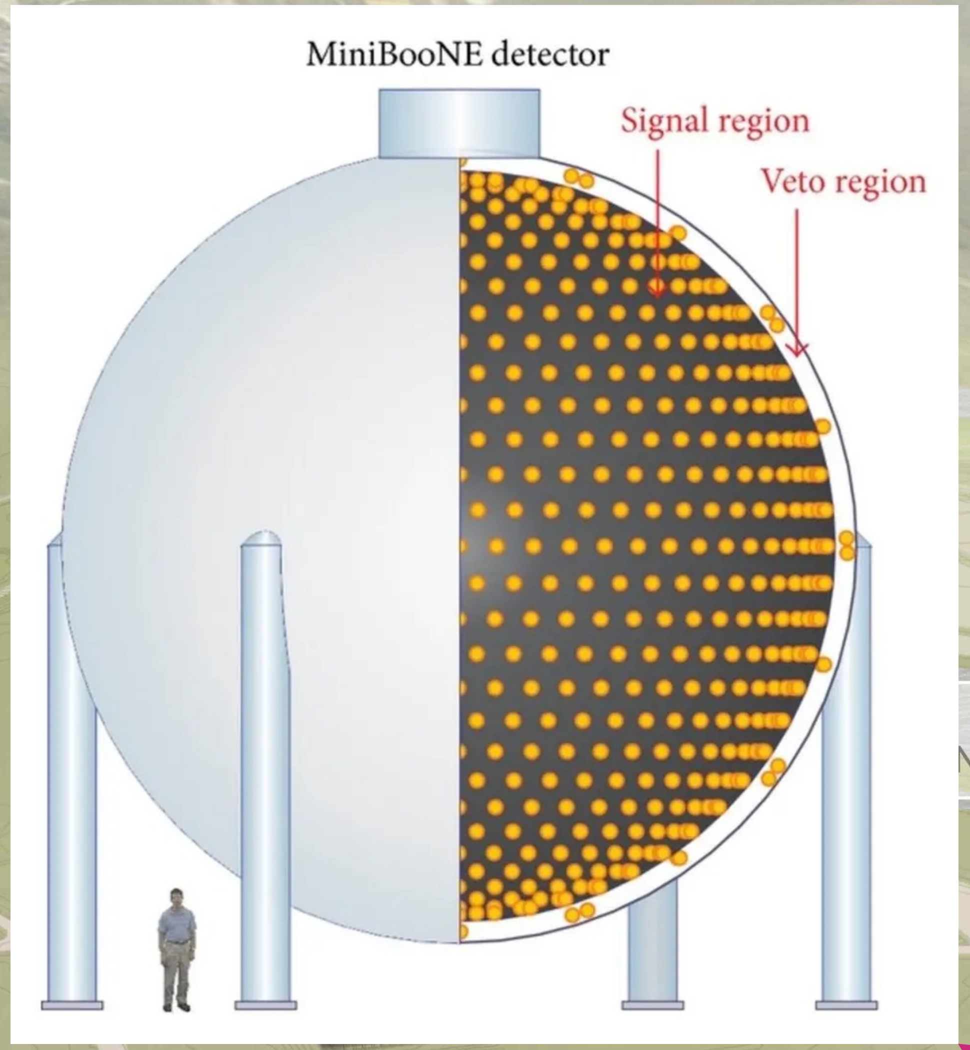


Constraints from unitarity of PMNS matrix.

channel	mixing angle definition	experiments
$\nu_\mu \to \nu_e$	$\sin^2(2\theta_{\mu e}) \equiv 4\lvert U_{\mu 4}\rvert^2\lvert U_{e4}\rvert^2$	LSND, MiniBooNE, OPERA, ...
$\nu_e \to \nu_e$	$\sin^2(2\theta_{ee}) \equiv 4\lvert U_{e4}\rvert^2(1-\lvert U_{e4}\rvert^2)$	Reactor, solar, Gallium, ...
$\nu_\mu \to \nu_\mu$	$\sin^2(2\theta_{\mu\mu}) \equiv 4\lvert U_{\mu 4}\rvert^2(1-\lvert U_{\mu 4}\rvert^2)$	MiniBooNE, MINOS, IceCube, ...

The MiniBooNE Experiment

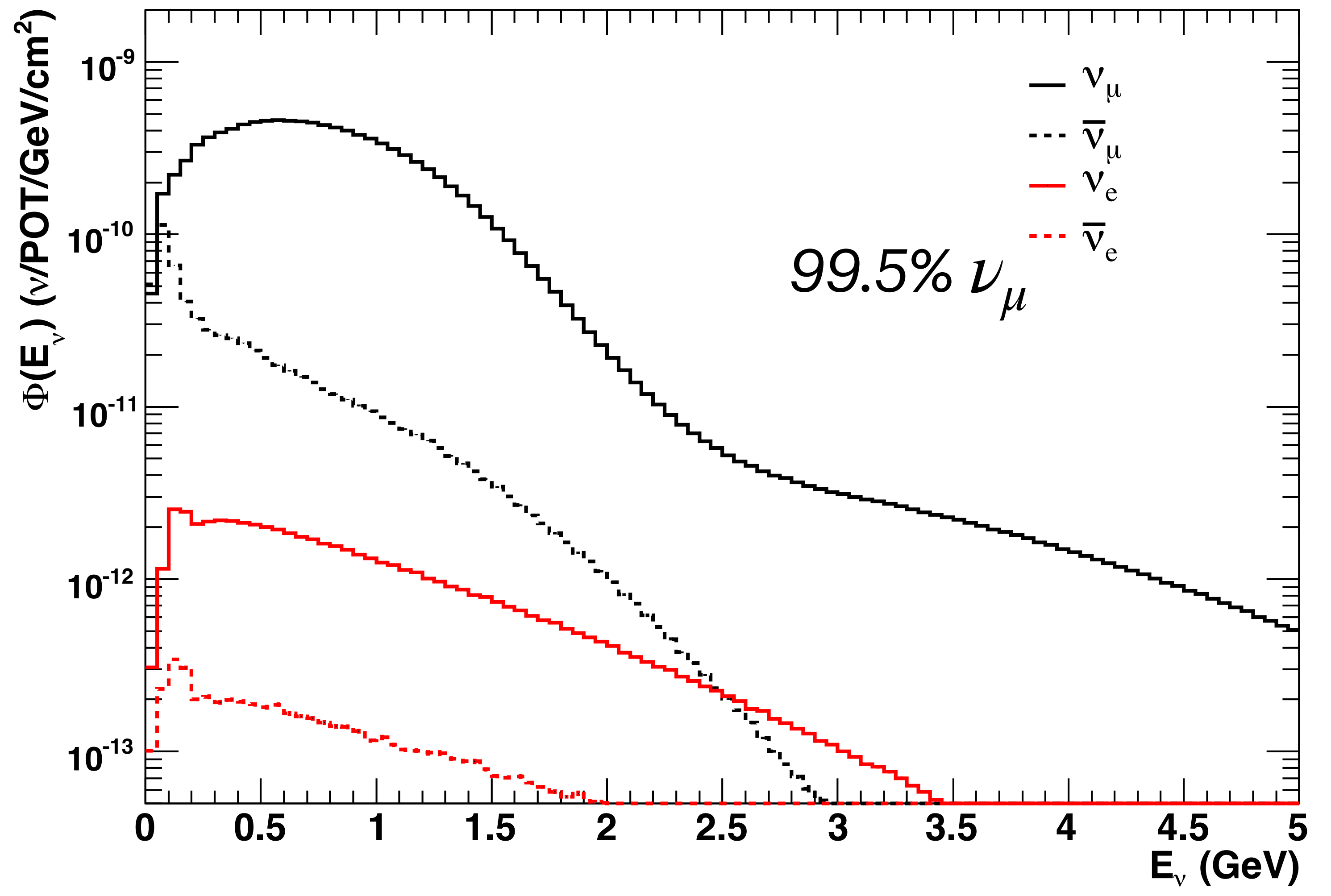
Located in the Booster Neutrino Beam at Fermilab



MiniBooNE:

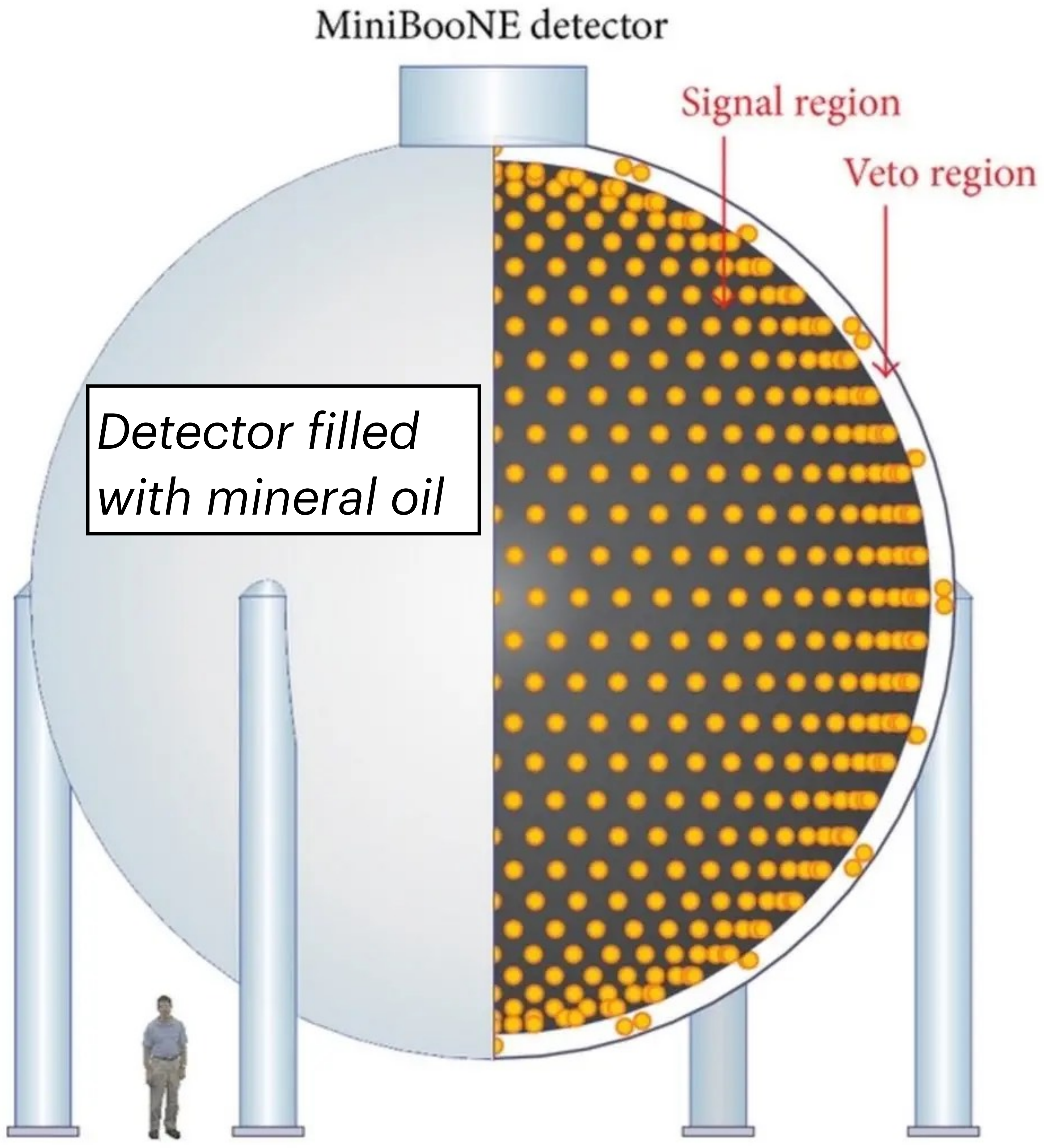
- *541 m baseline*
- *800 tons mineral oil*
- *Cherenkov effect*
- *Active 2002 — 2018*

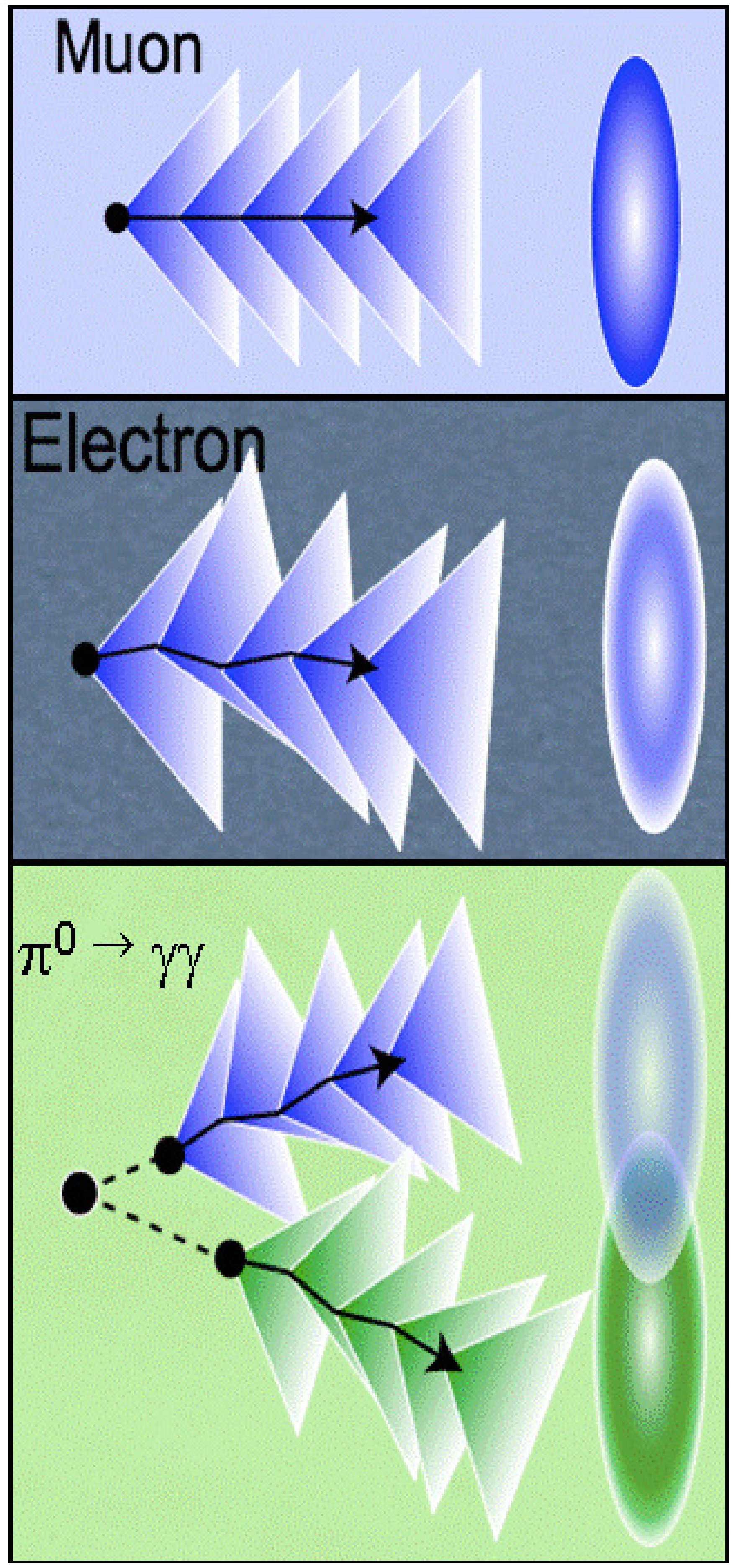
Neutrino Flux

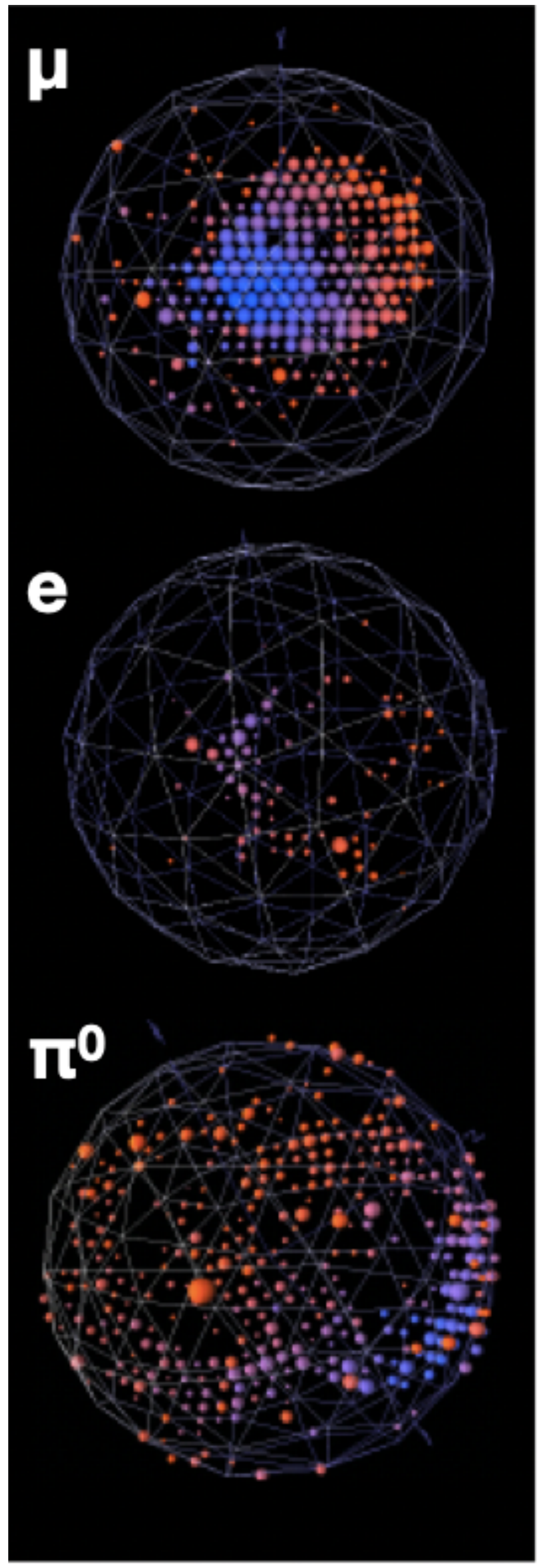


Total predicted flux at the MiniBooNE detector in neutrino mode. Phys.Rev.D 79 (2009) 072002

Event Signatures in MiniBooNE

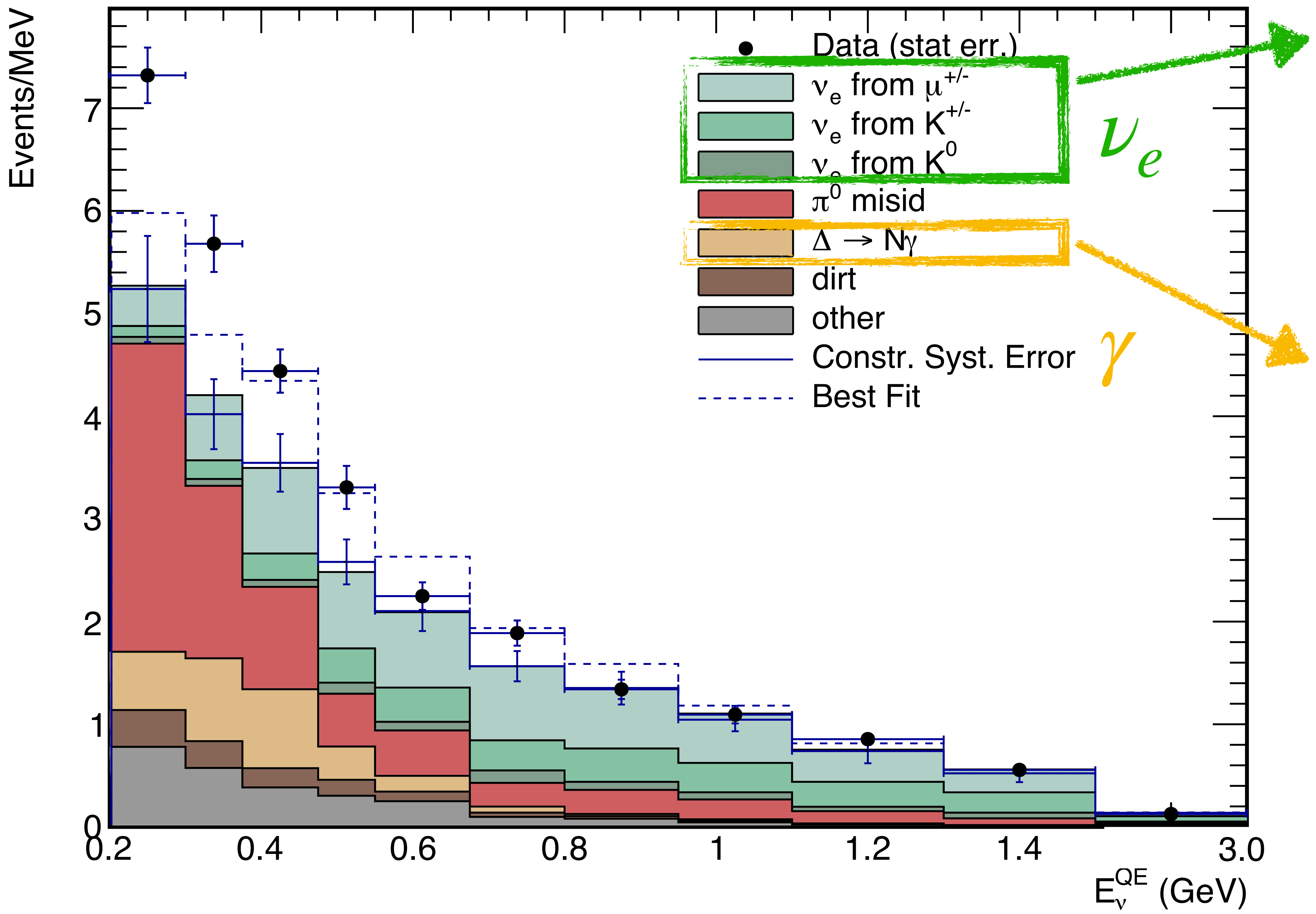




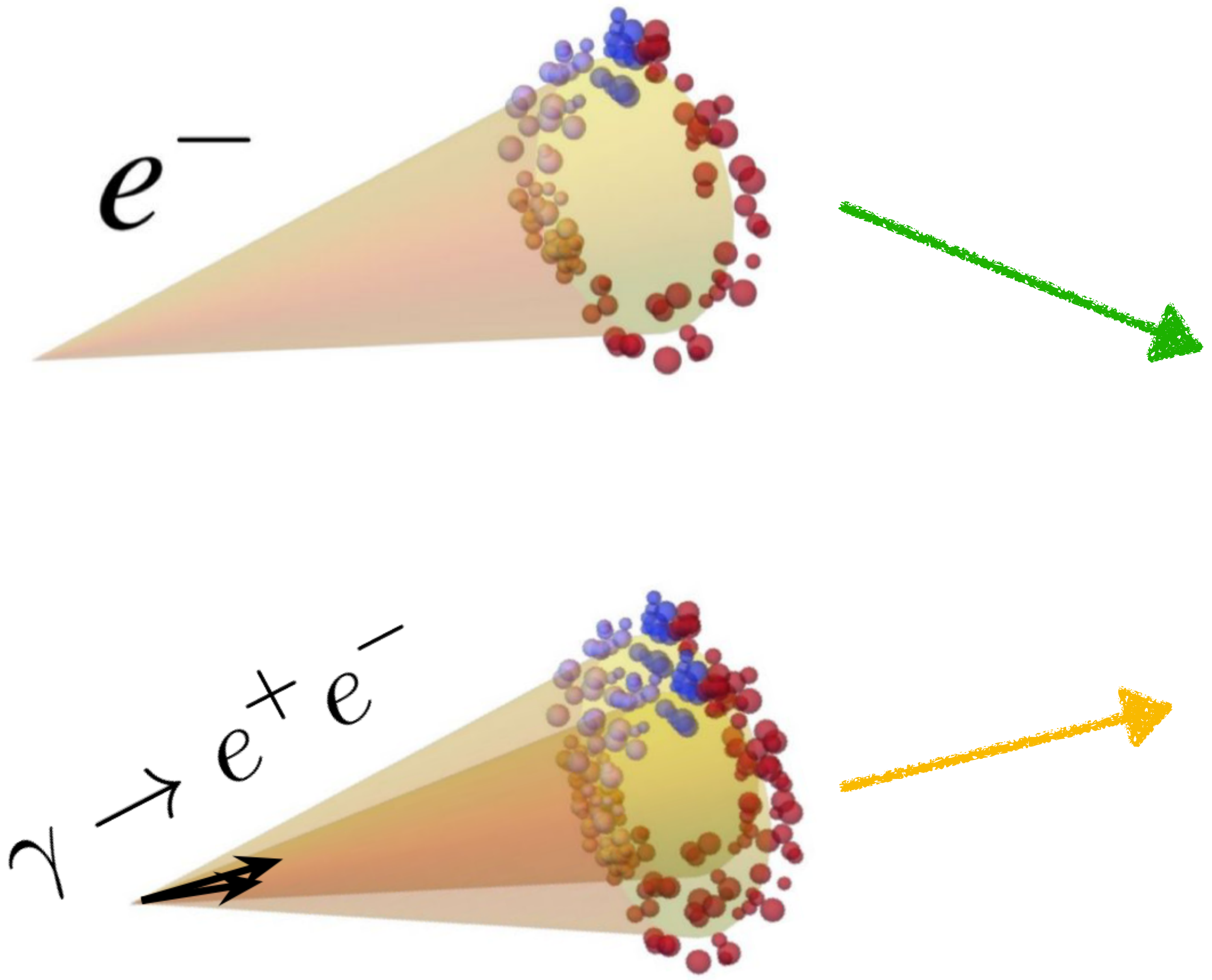


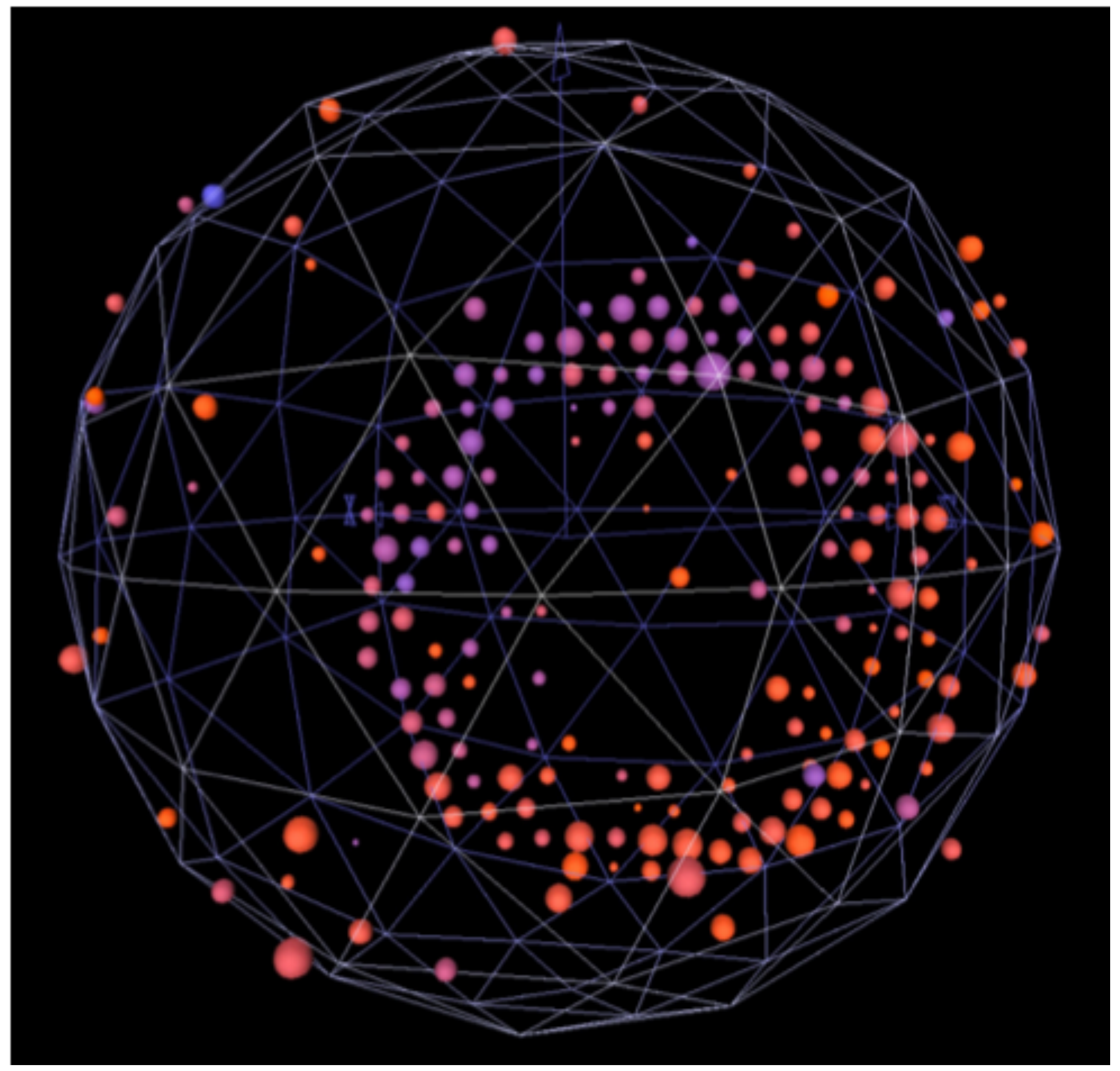
What Could The Excess Be?

Electron-neutrino-like signatures from different sources



Phys.Rev.D 103 (2021) 5, 052002



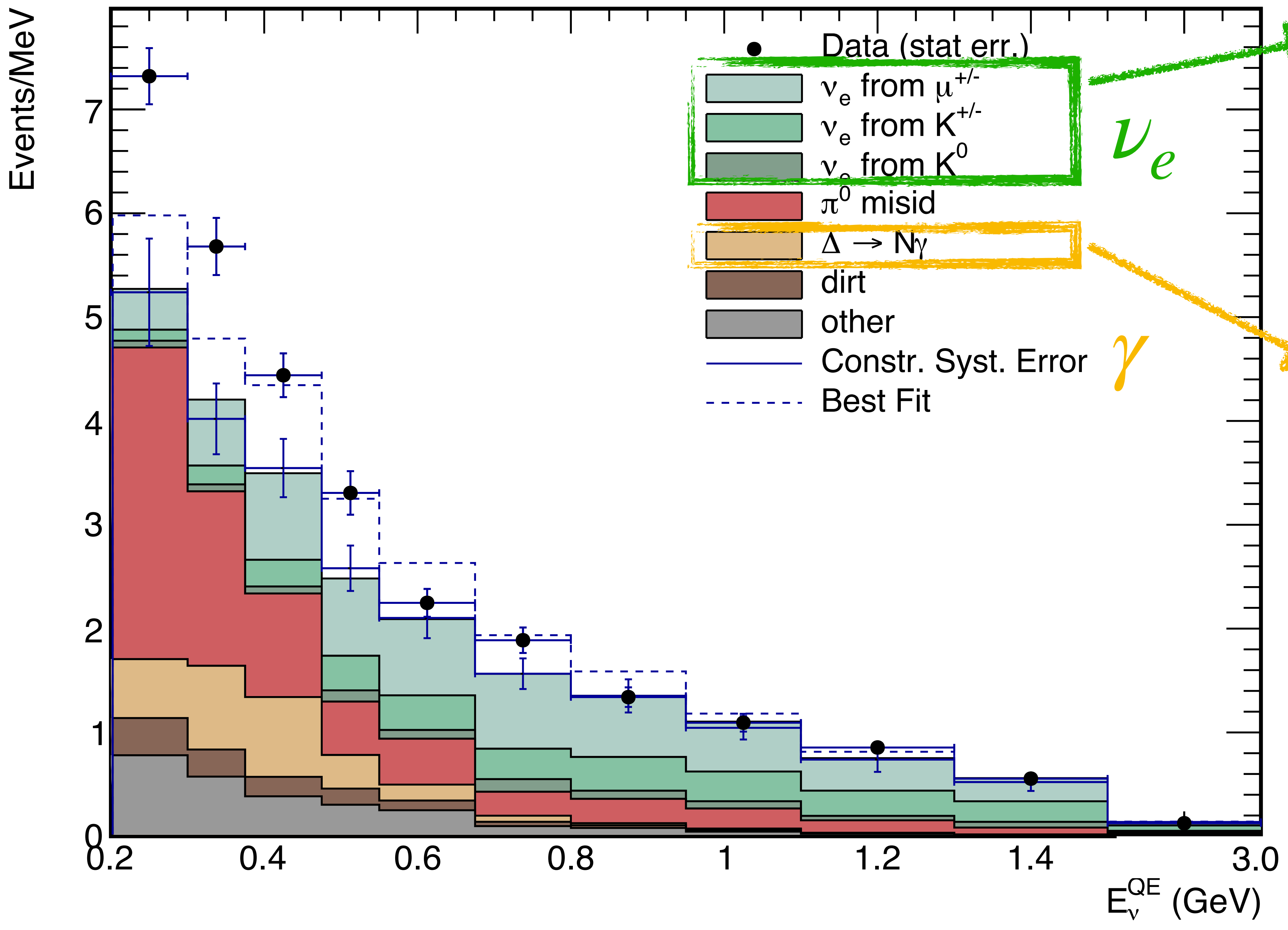


Electron neutrino signature in MiniBooNE

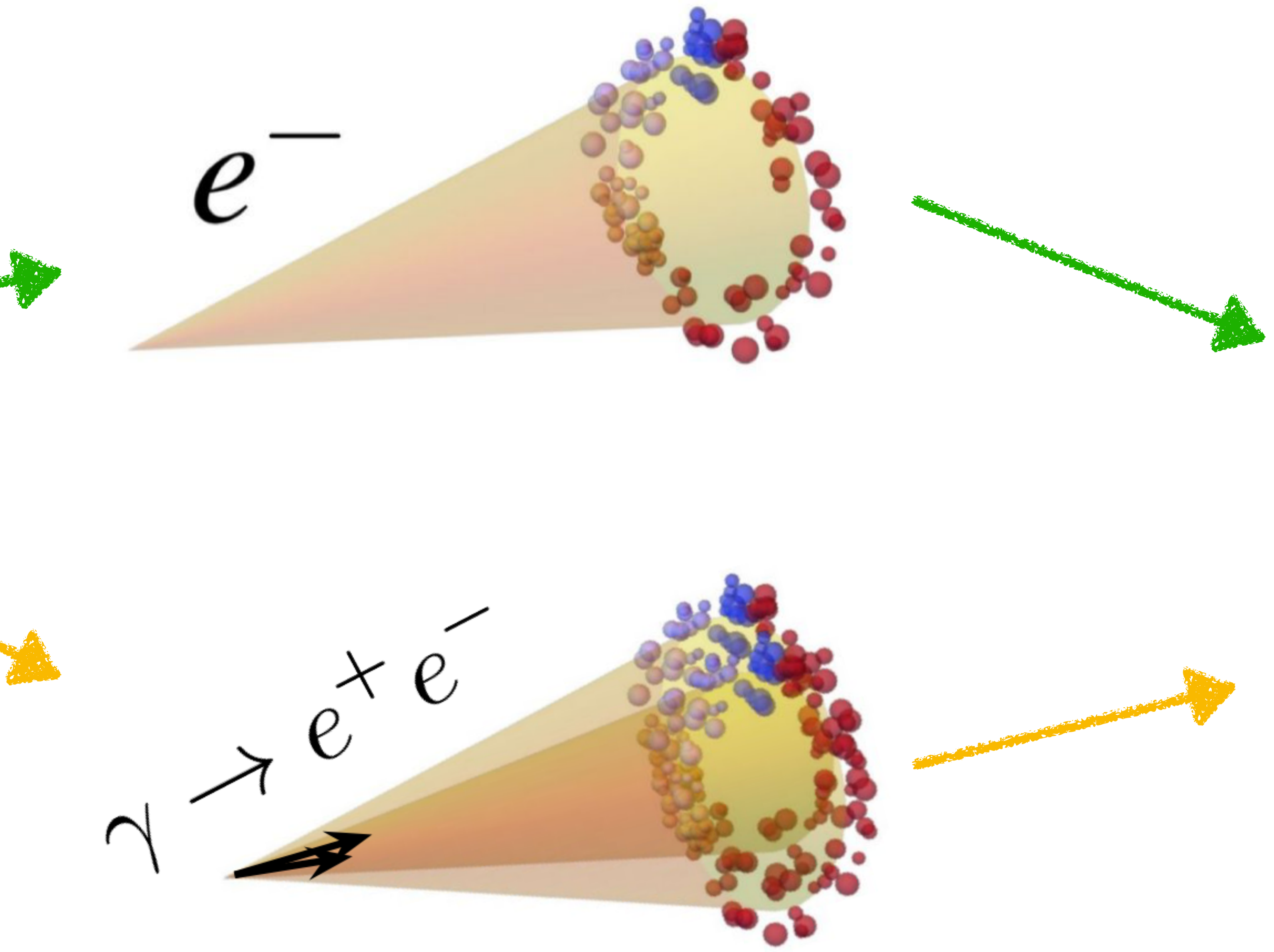
➡ Electron neutrinos and photons can both produce "single Cherenkov ring" signature in MiniBooNE!

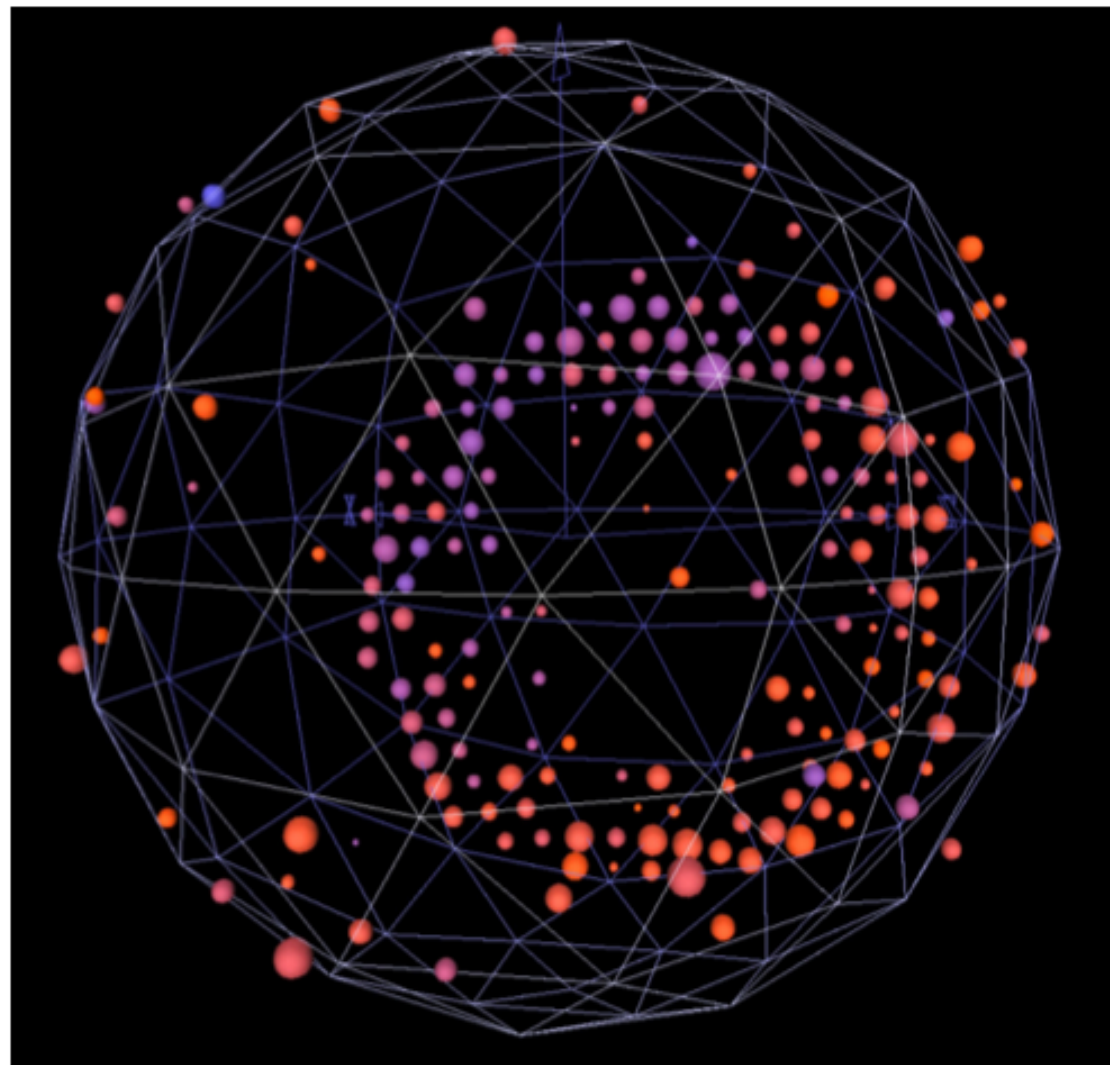
What Could The Excess Be?

Electron-neutrino-like signatures from different sources



Phys.Rev.D 103 (2021) 5, 052002





Electron neutrino signature in MiniBooNE

➡ Electron neutrinos and photons can both produce "single Cherenkov ring" signature in MiniBooNE!

MicroBooNE motivation: Determine the origin of the excess!

The MicroBooNE Experiment

Placed in the same neutrino beam as MiniBooNE

MiniBooNE:

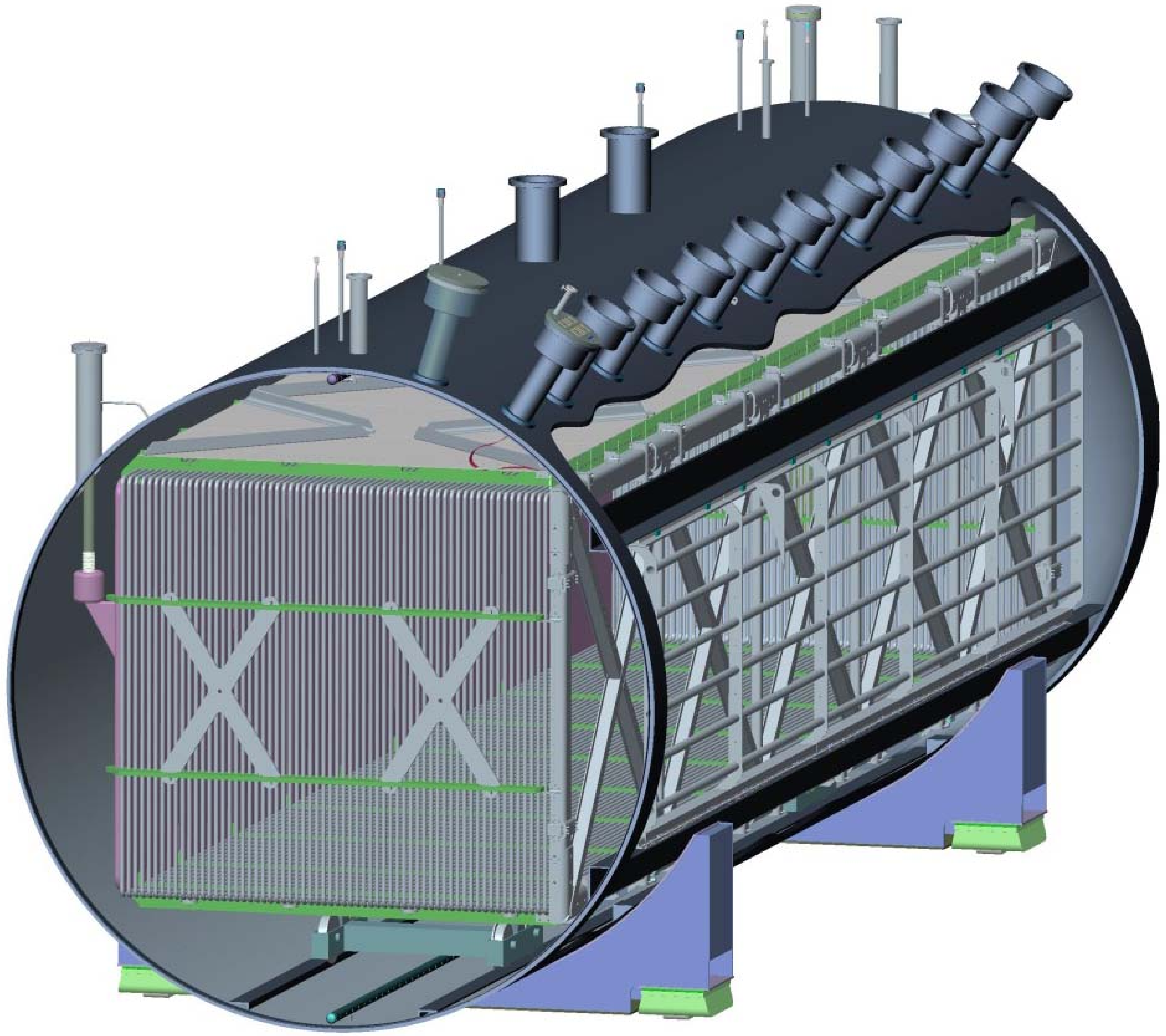
- *541 m baseline*
- *800 tons mineral oil*
- *Cherenkov effect*
- *Active 2002 — 2018*

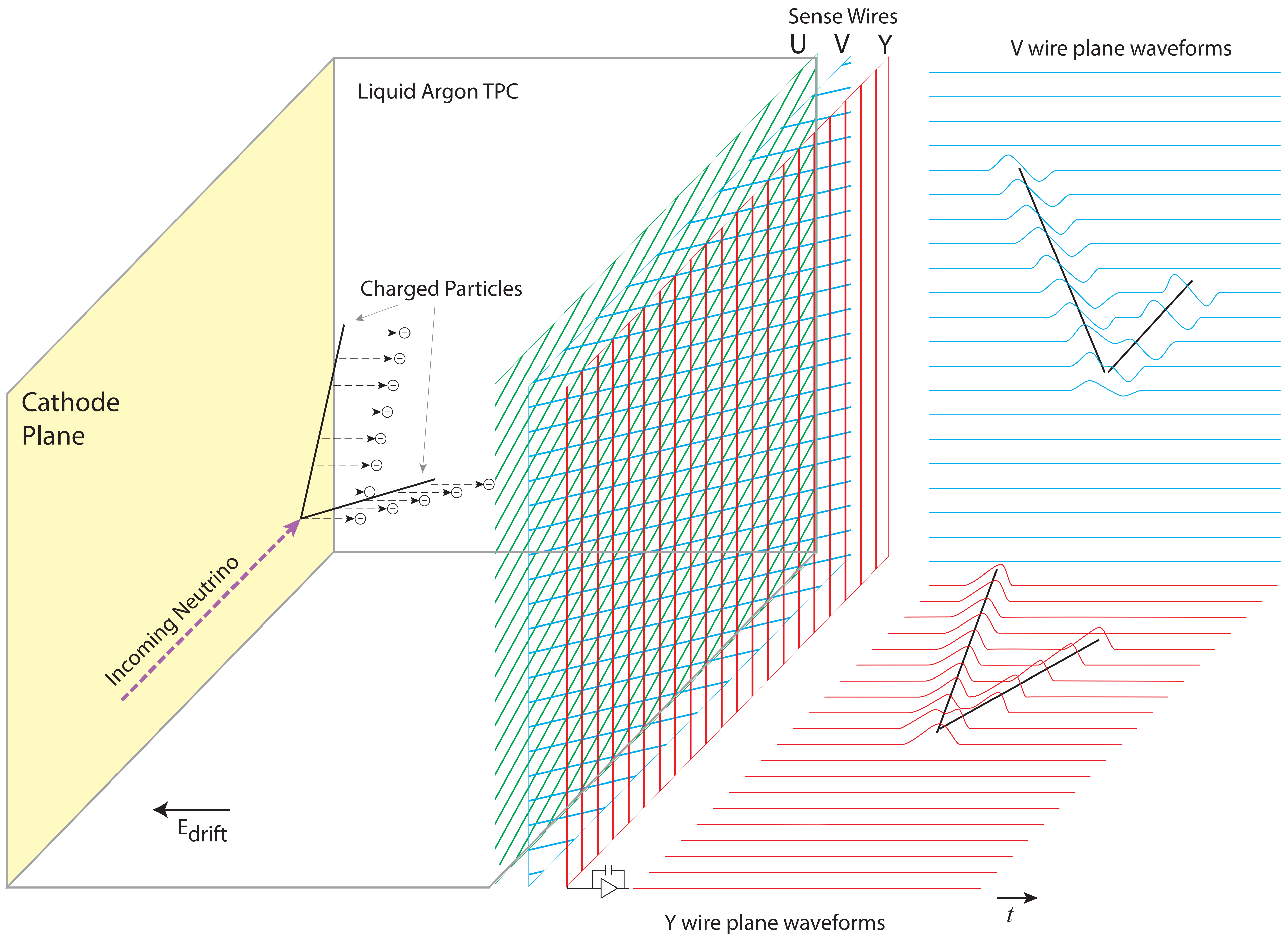
MicroBooNE:

- *470 m baseline*
- *90 tons of Argon*
- *Time projection chamber*
- *Active 2015 — 2020*

The MicroBooNE Experiment

A Liquid Argon Time Projection Chamber



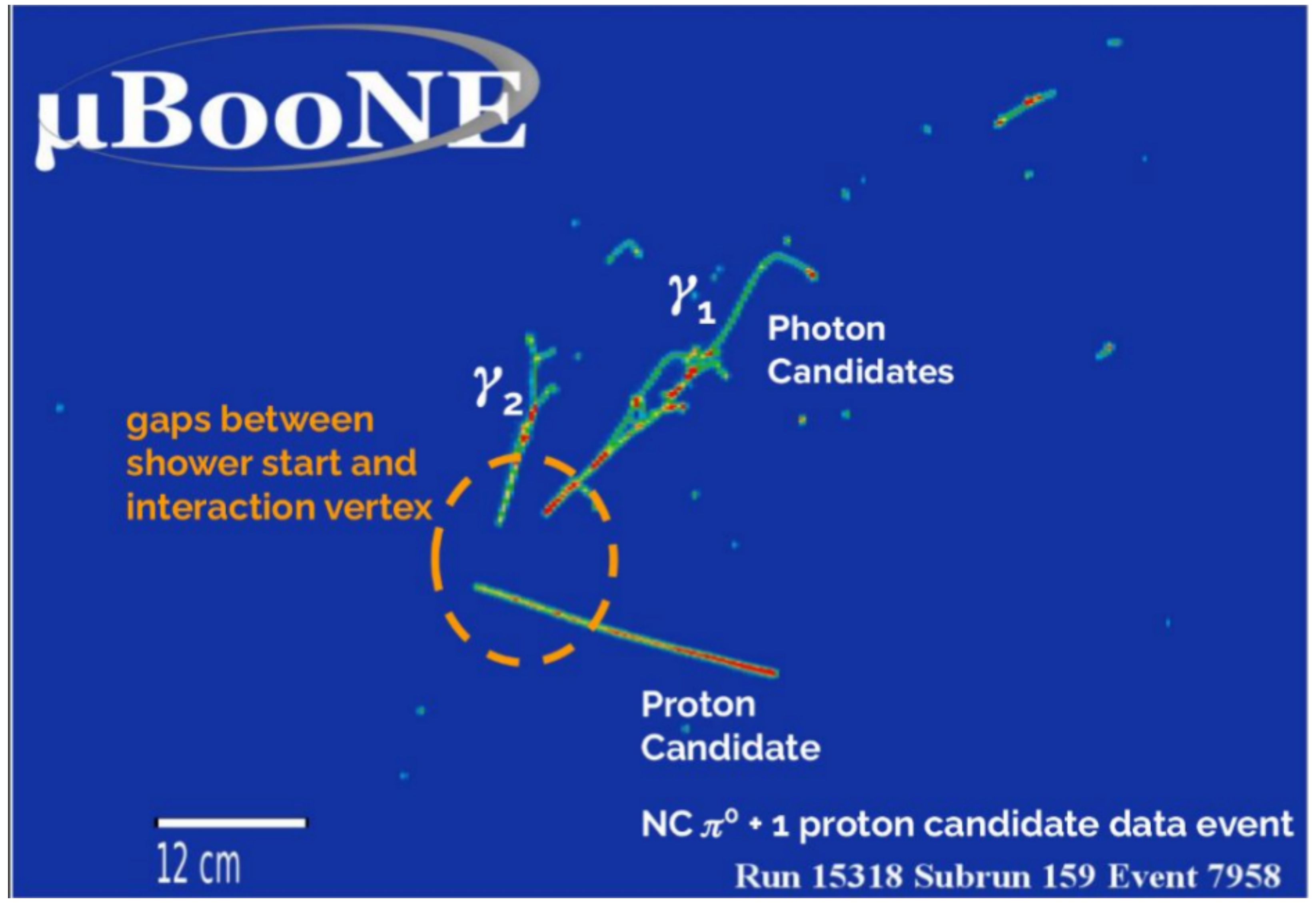


JINST 12 (2017) 02, P02017

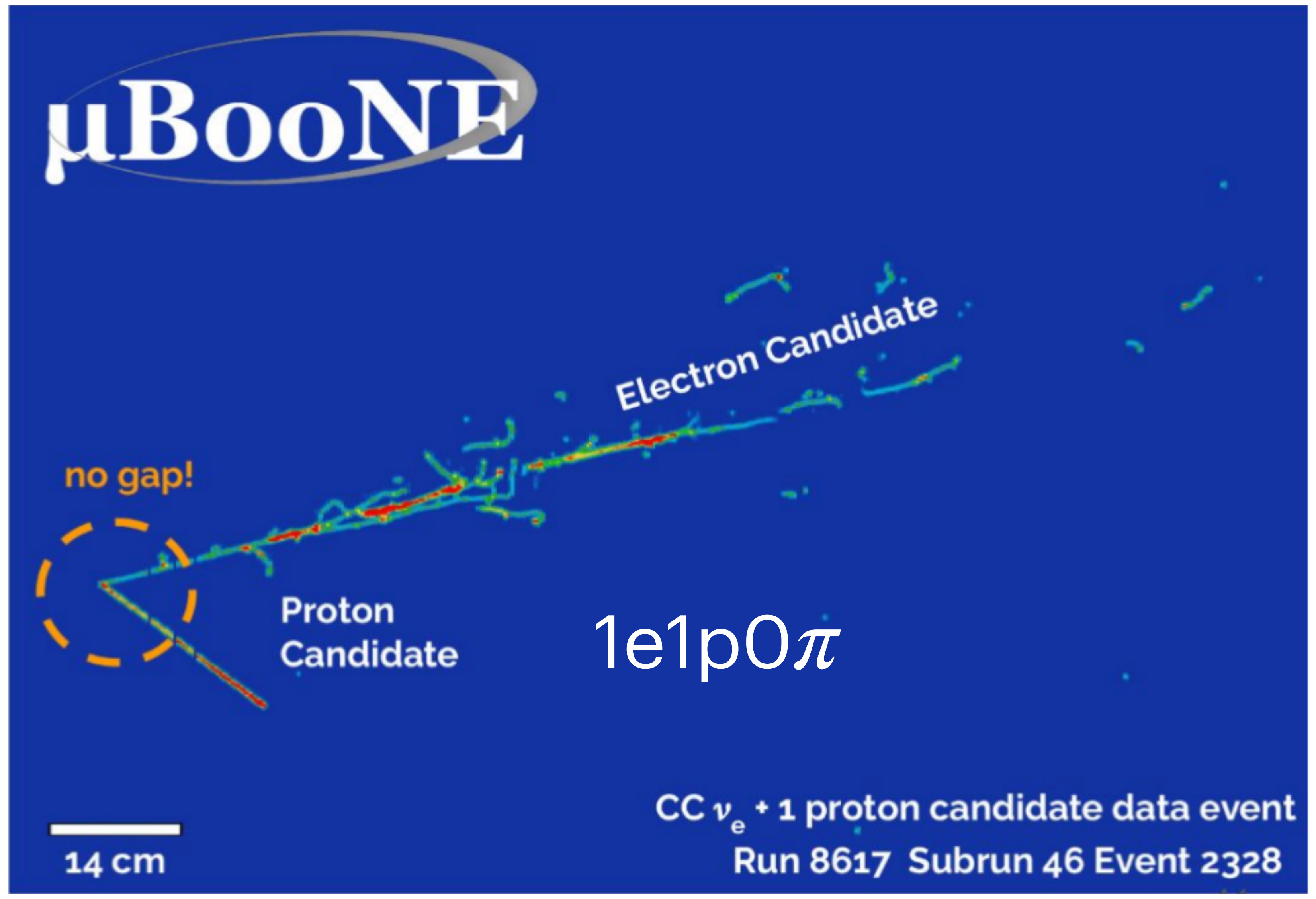
The MicroBooNE Experiment

Separation of event signatures

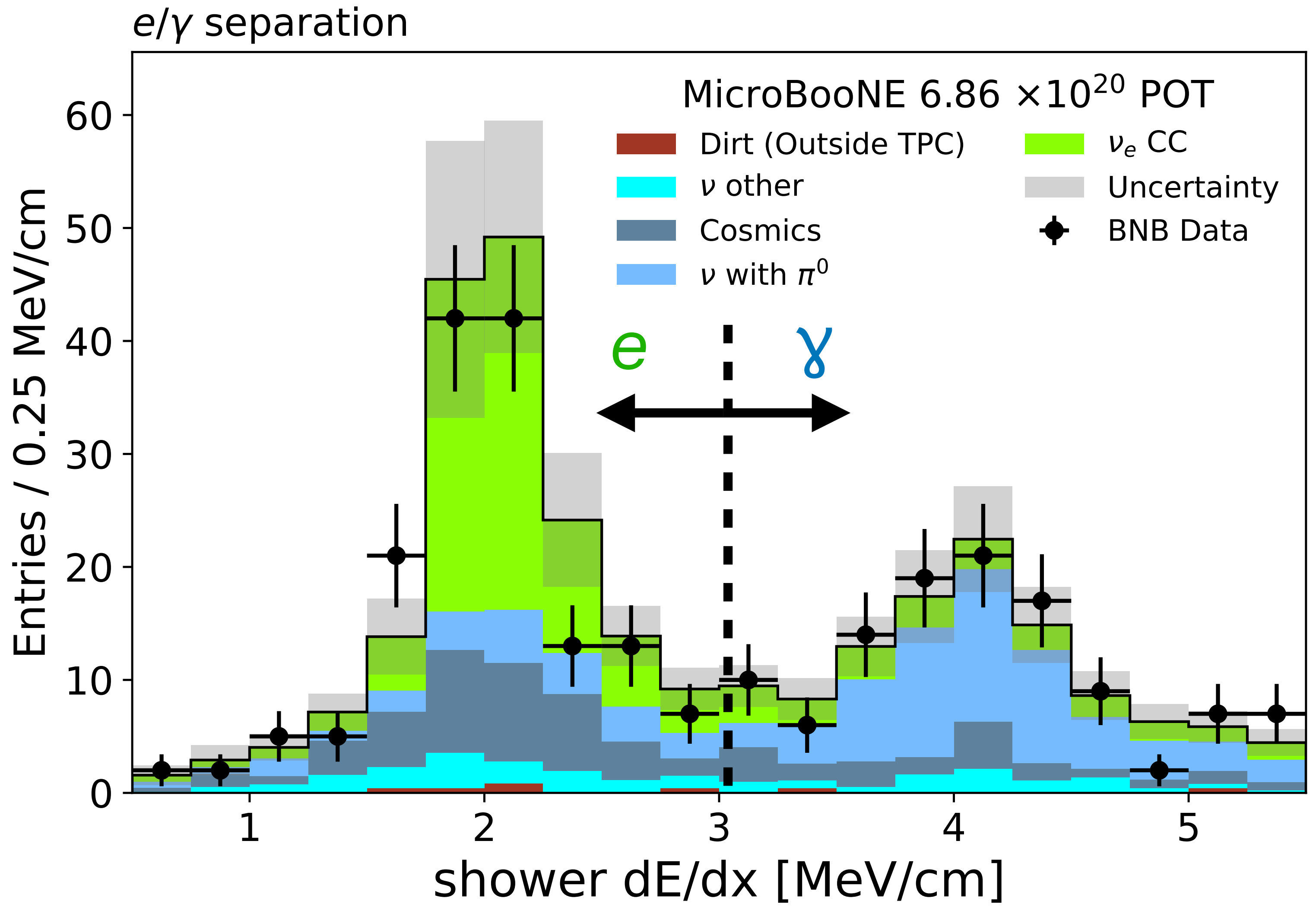
Photon signature



Electron neutrino signature



Energy resolution



Phys.Rev.D 105 (2022) 11, 112004

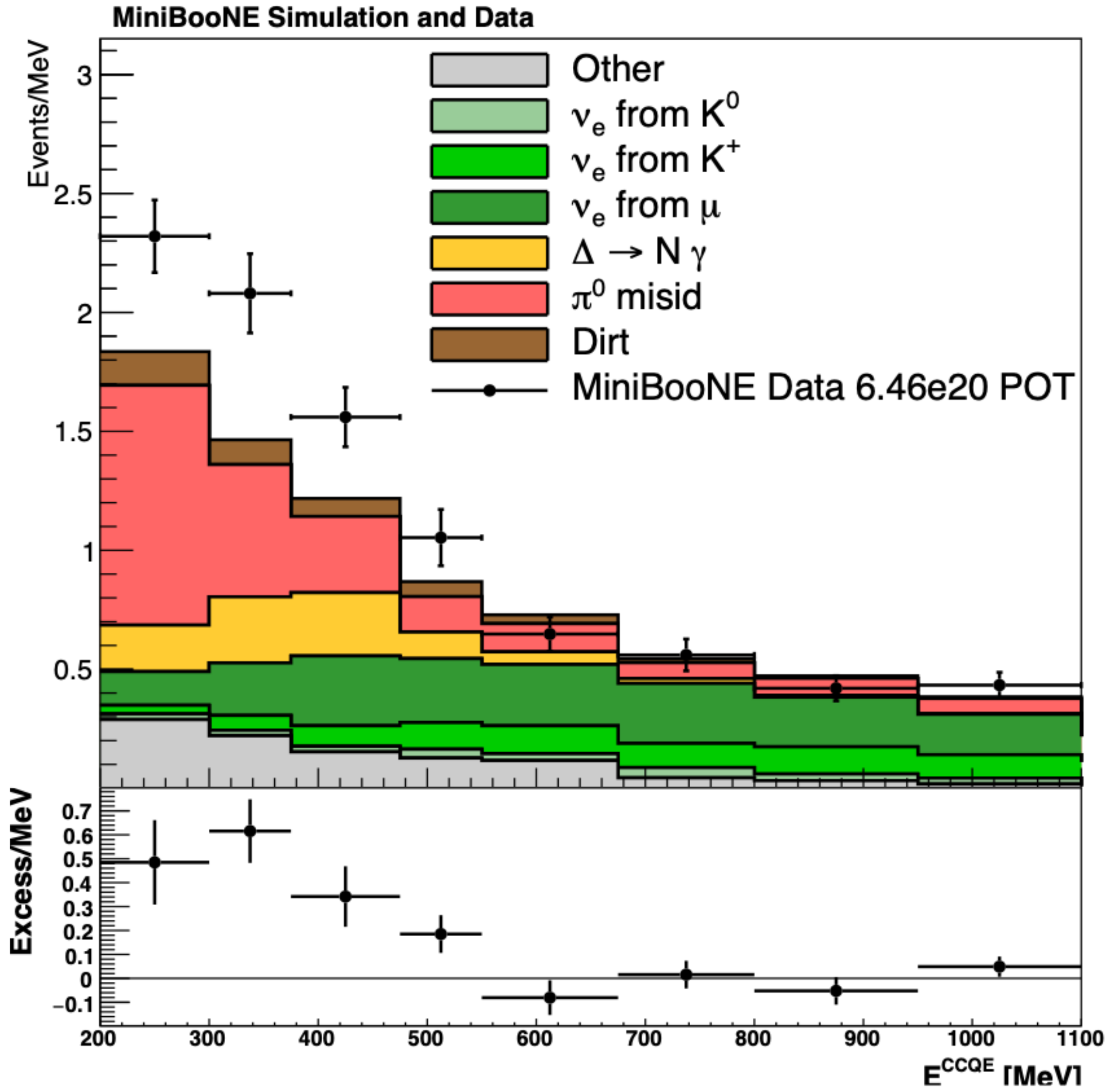
- MicroBooNE can clearly distinguish between photons from NC π^0 decay and CC ν_e
- Precise shower energy loss reconstruction aids in e/ɣ separation
- Can measure number and kinematics of protons (invisible to MiniBooNE)

First Low-Energy Excess Search

From *"Search for an Anomalous Excess of Charged Current Electron Neutrino Interactions Without Pions in the Final State with the MicroBooNE Experiment", Phys. Rev. D 105, 112004 — Published 13 June 2022*

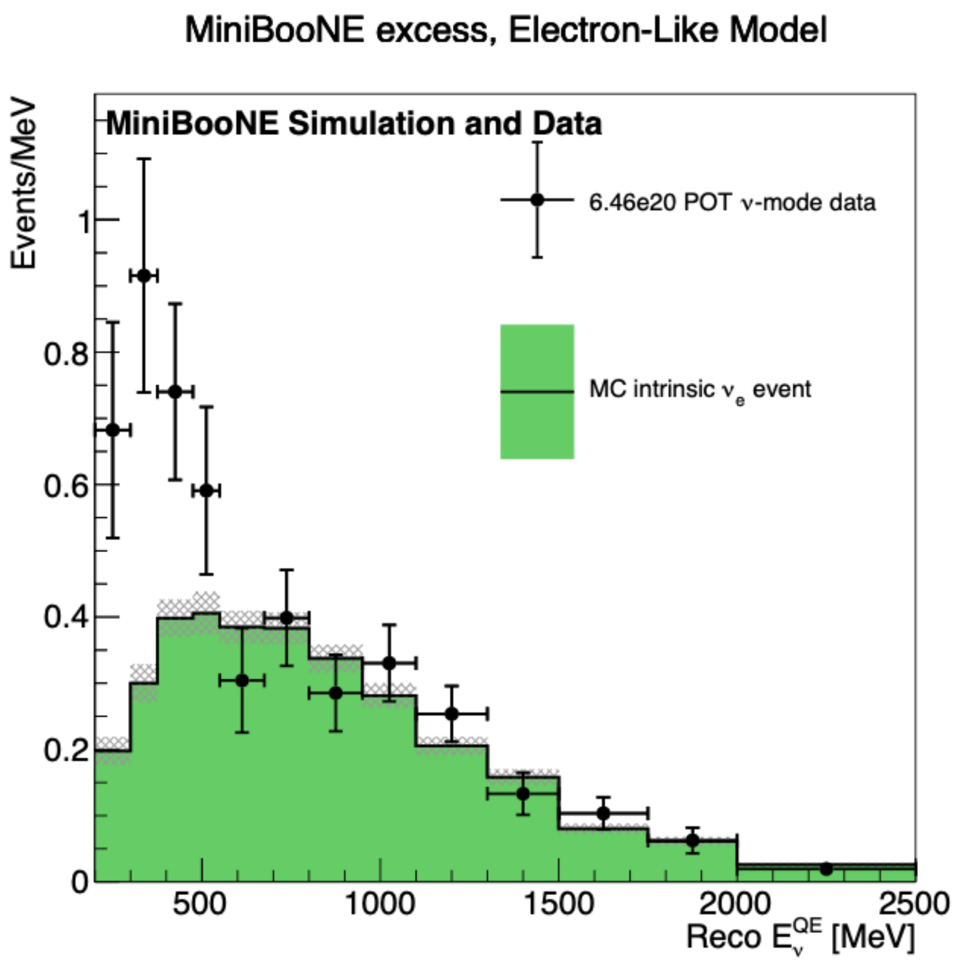
A Generic Signal Model

Testing the LEE in a physics-agnostic way



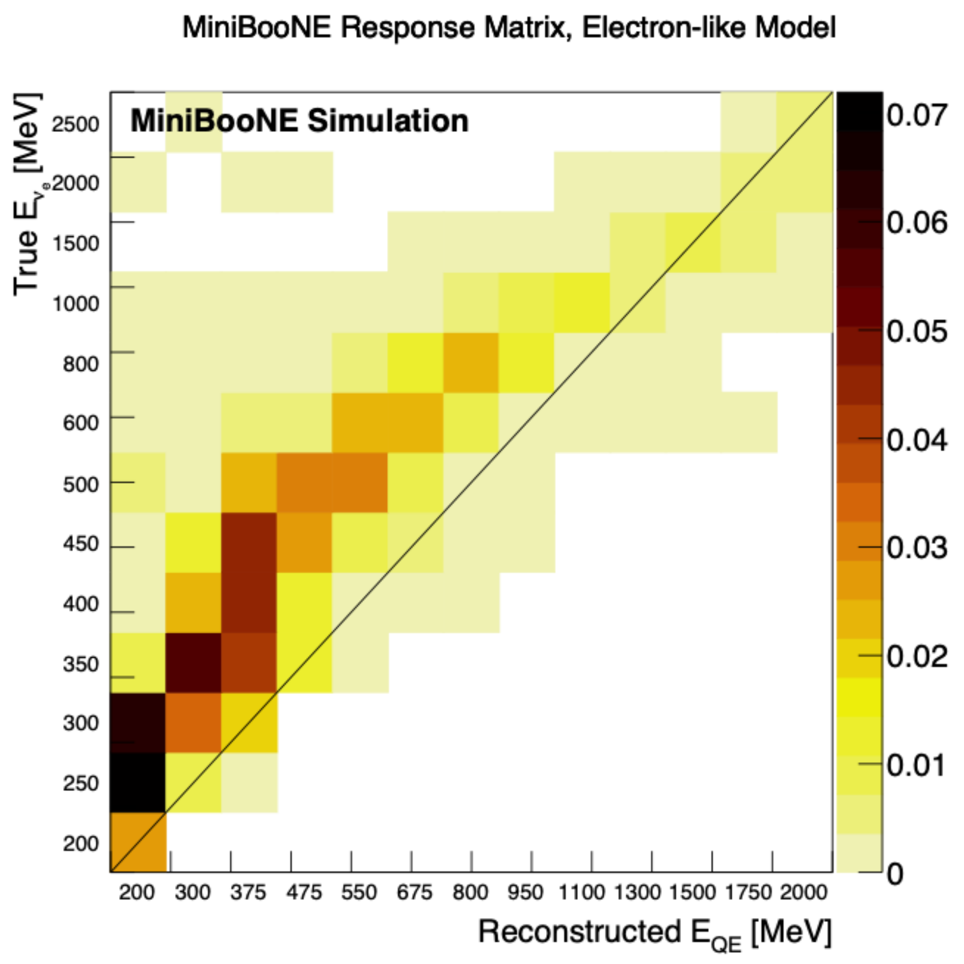
Distribution of electron-neutrino-like events in MiniBooNE as a function of reconstructed energy.
MICROBOONE-NOTE-1043-PUB

Subtract back-ground

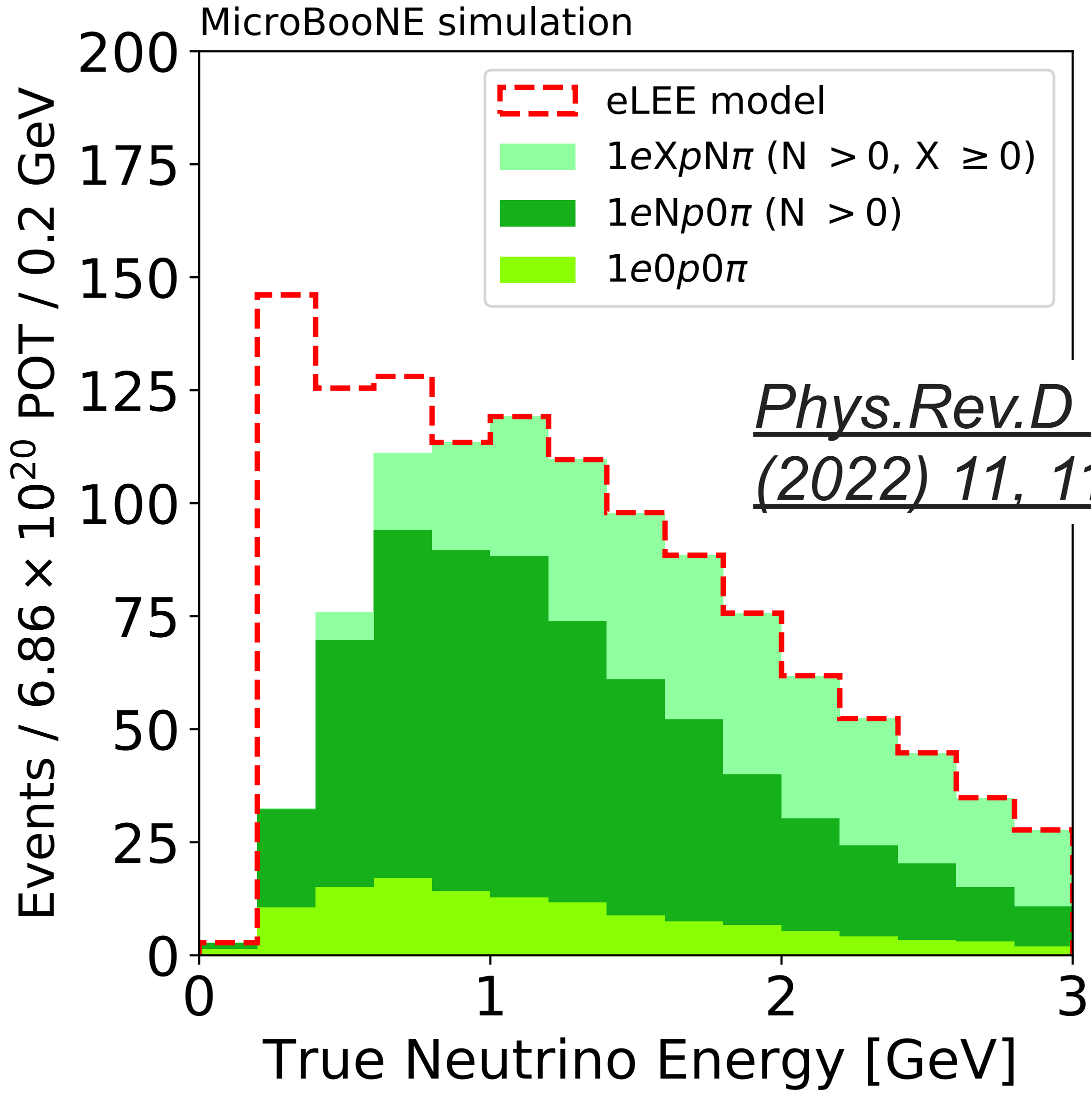


+

Response matrix from simulation



Unfold to distribution in ***true*** *energy*



Phys.Rev.D 105 (2022) 11, 112004

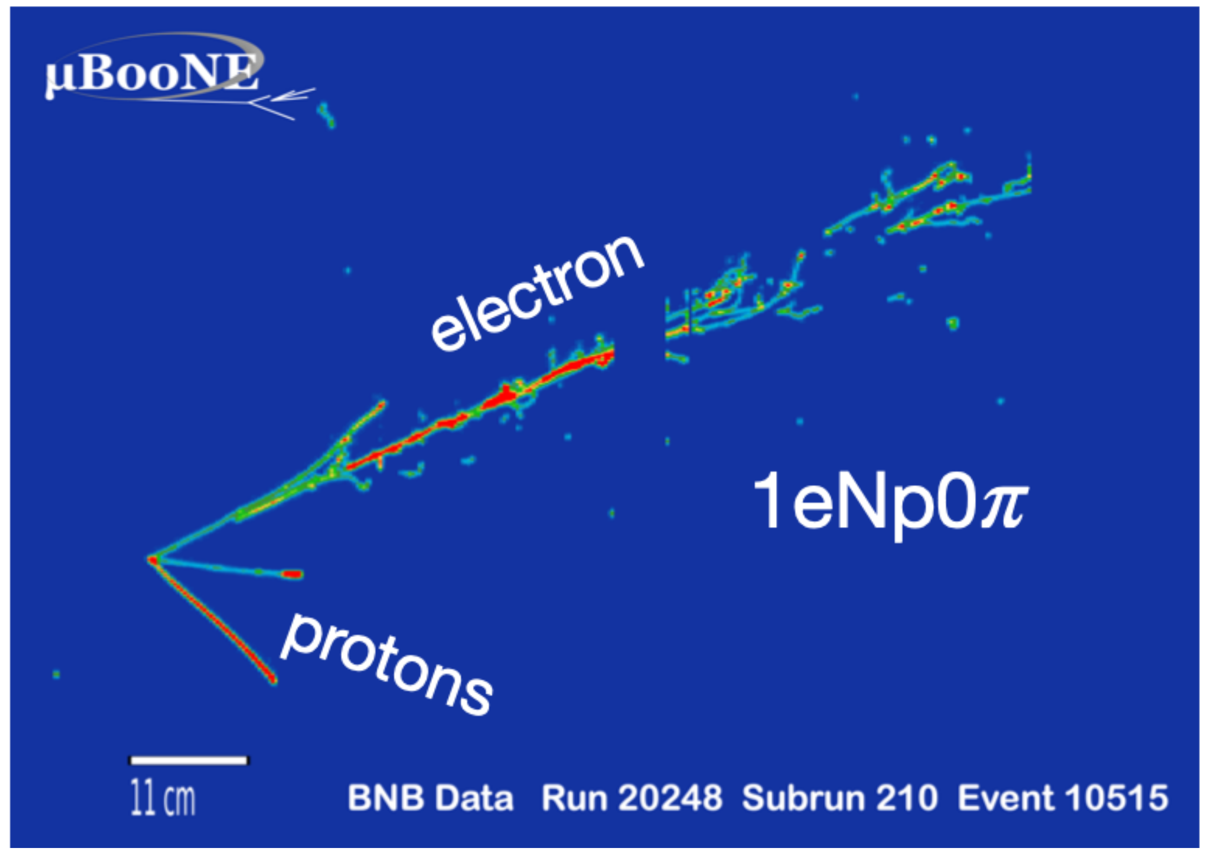
Assumptions made:

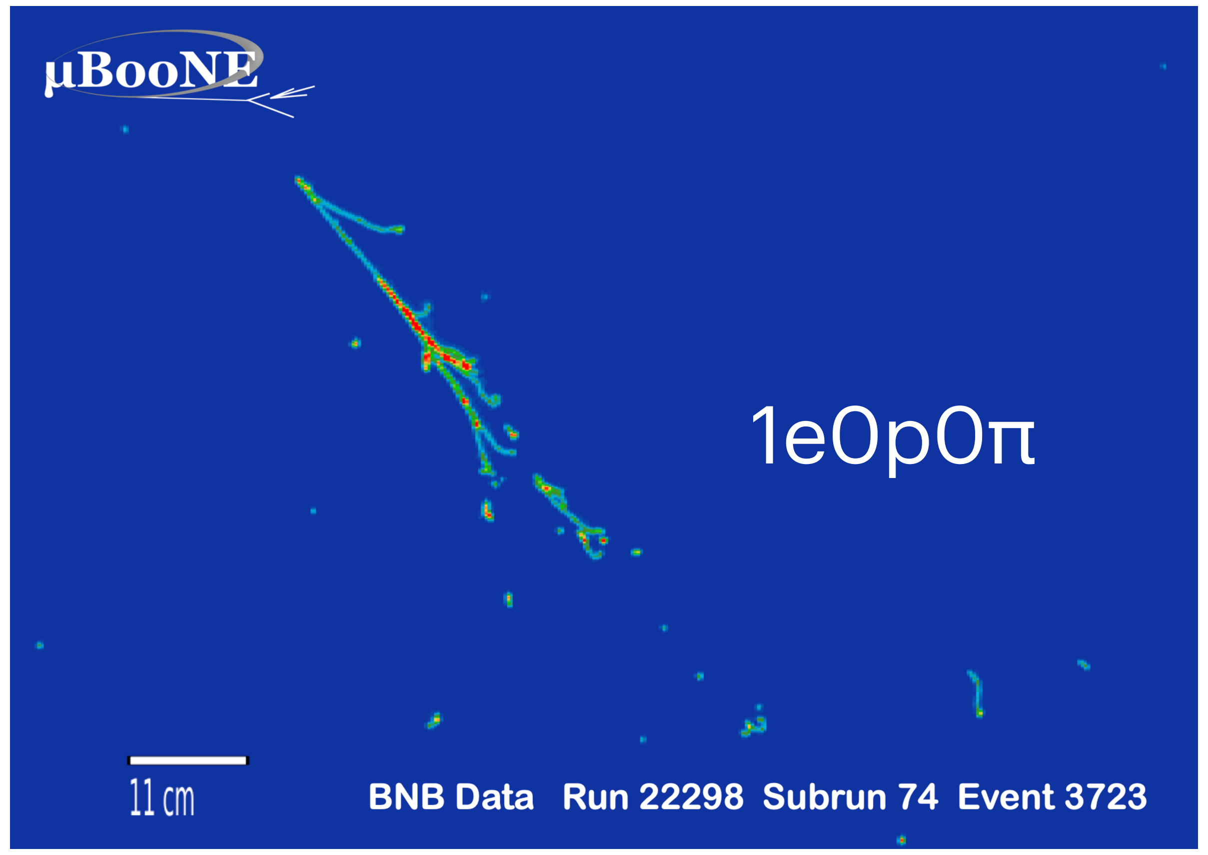
- LEE is due to ν_e flux modification
- Relationship between true and reconstructed energy is correctly simulated
  - ➡ Process for $N\gamma$ model analogous

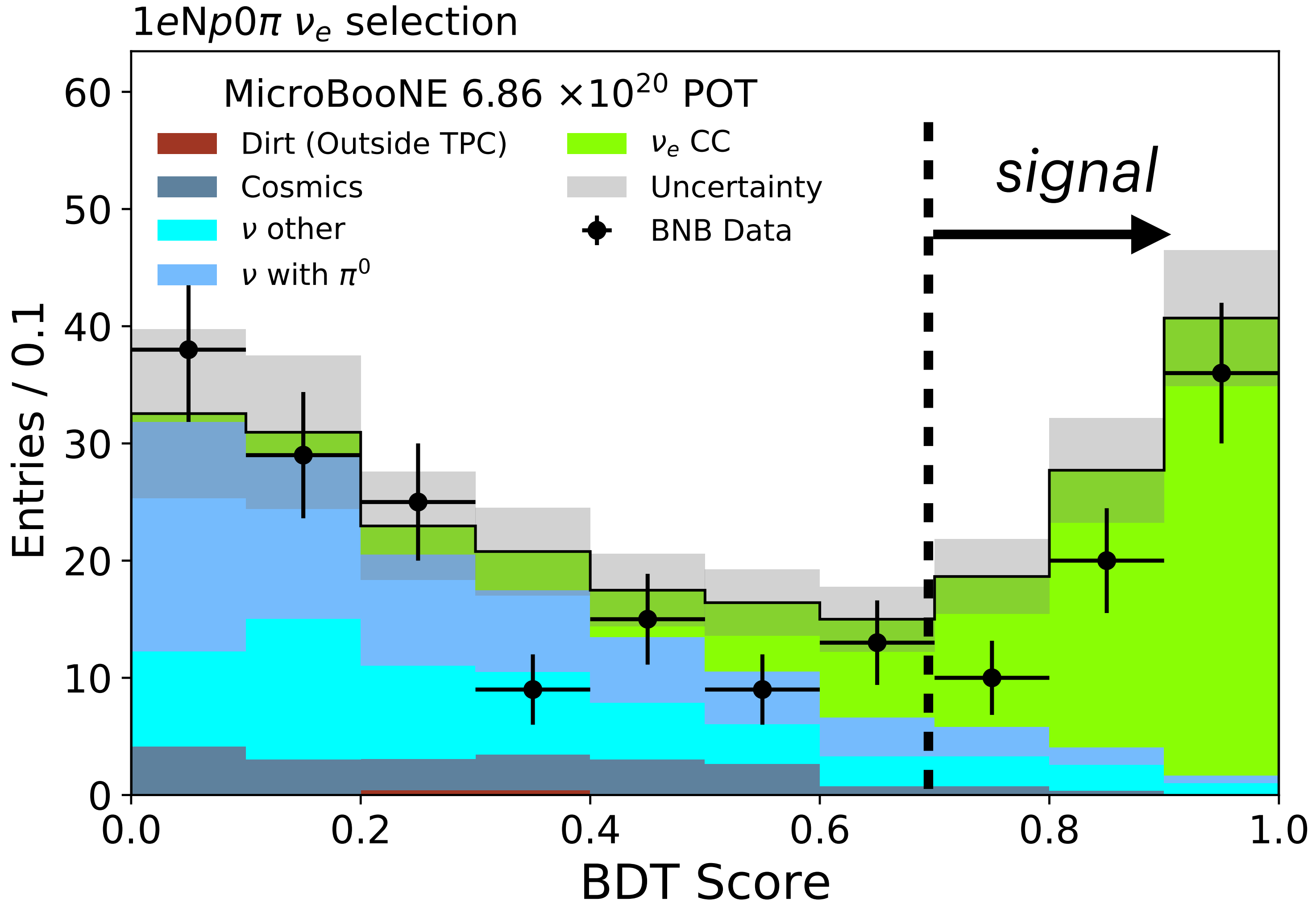
Event Selection for Electron Neutrinos

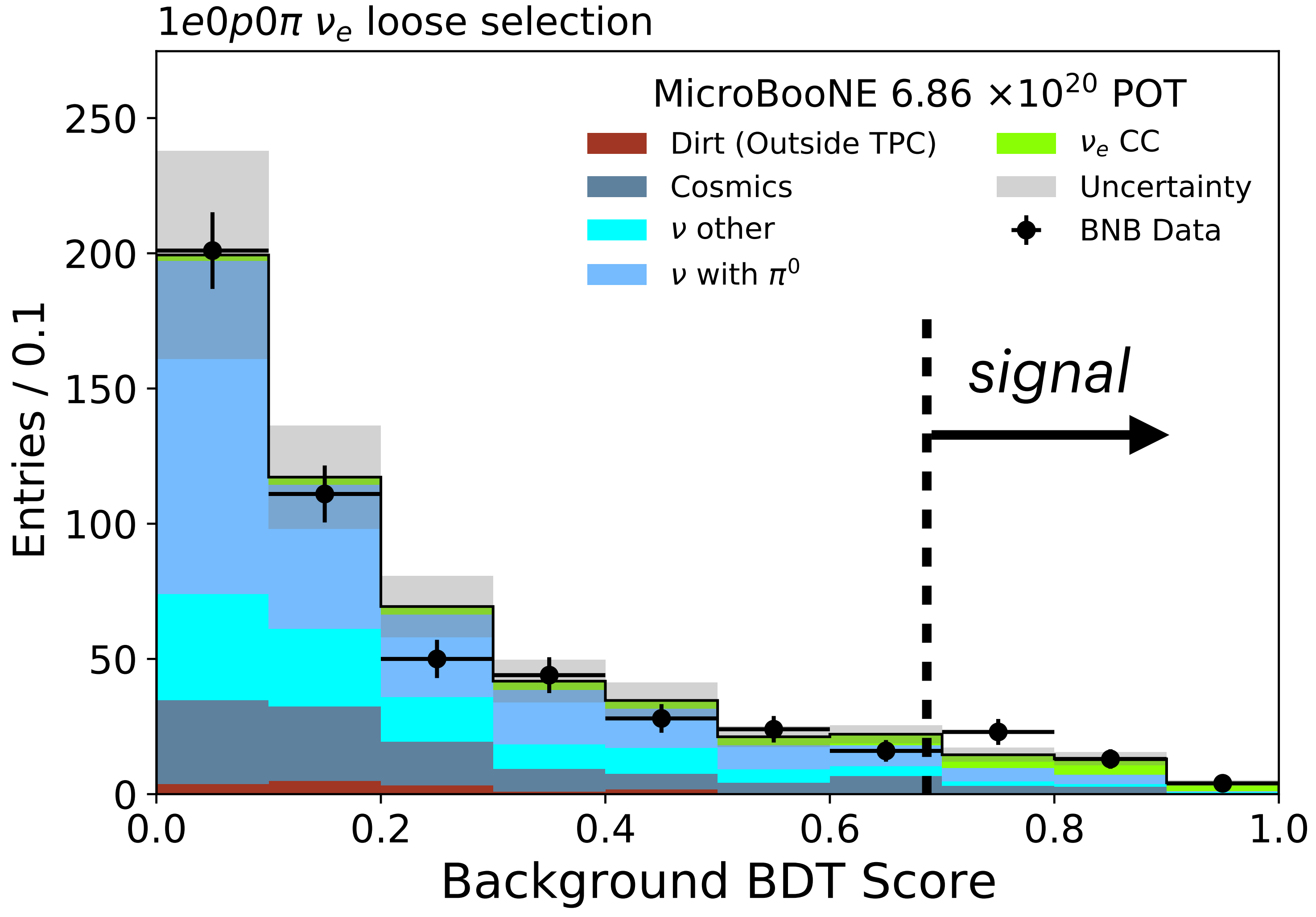
Final states corresponding to the MiniBooNE signal

Signal Events





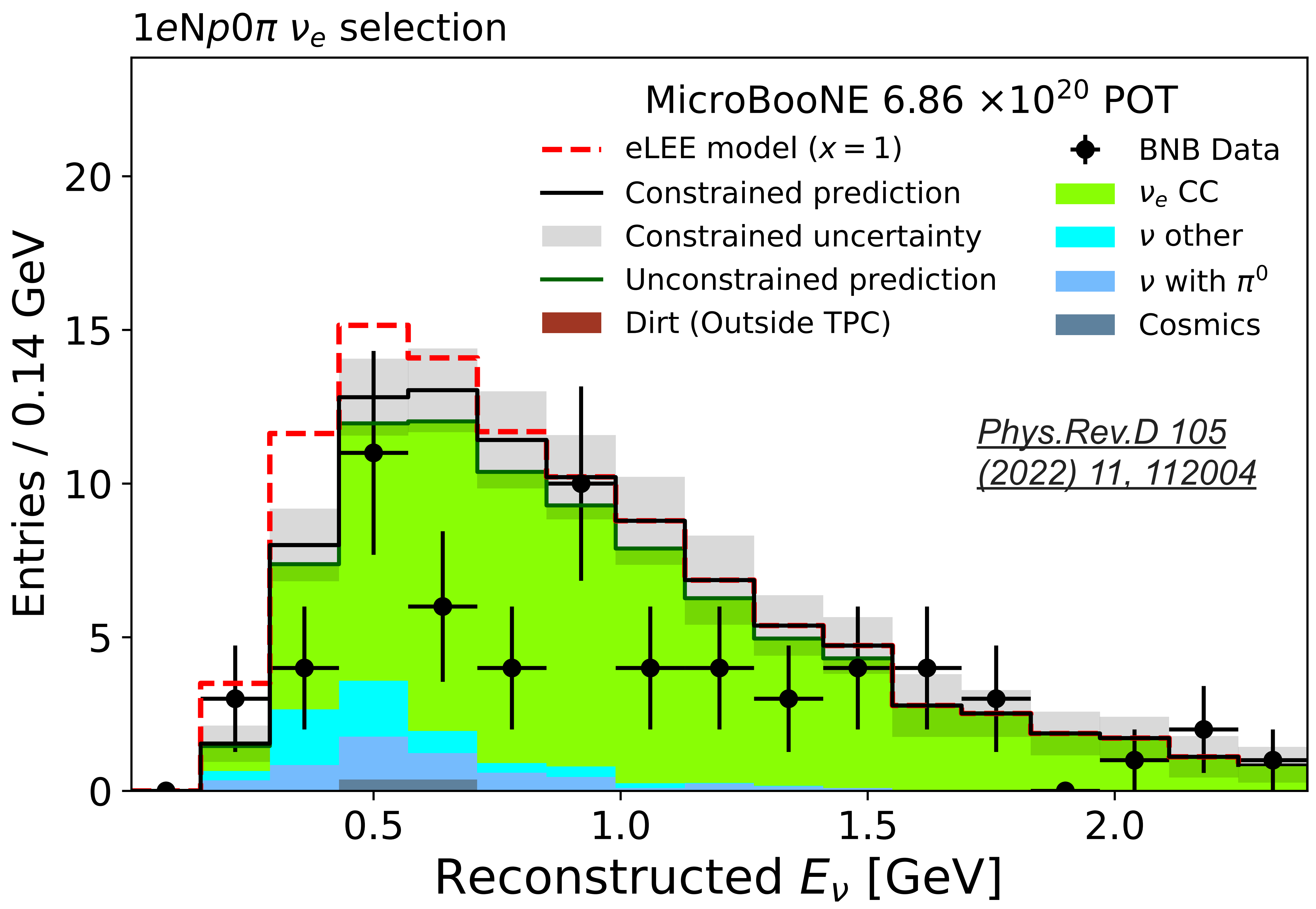


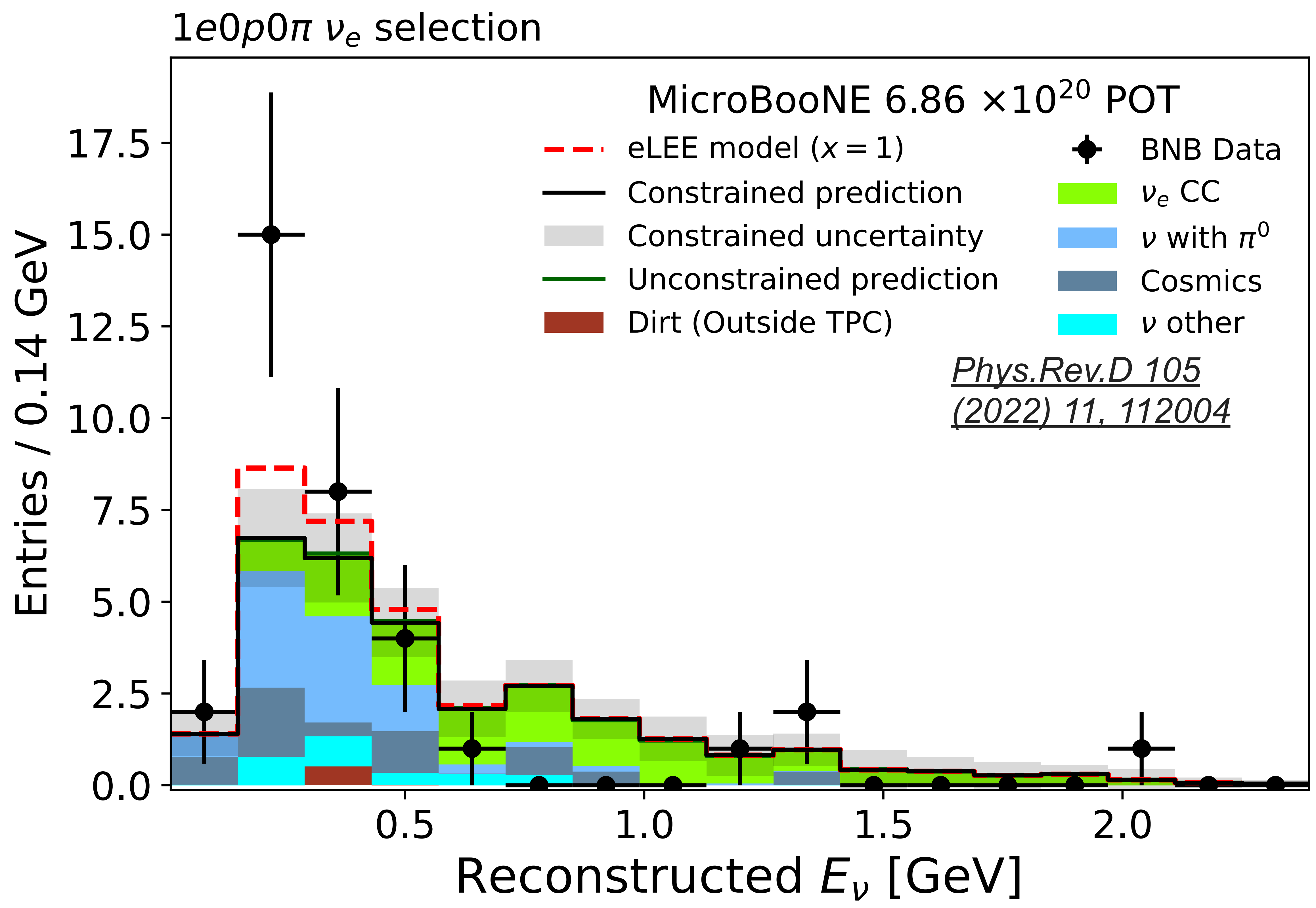


- Reconstruction using Pandora toolkit [Eur. Phys. J. C 78, 82 (2018)]
- First three runs of data (2015 — 2018), corresponding to 6.86×10^{20} protons on target (POT)
- Final states *without* pions (would have been visible to MiniBooNE), split by number of protons (zero vs. greater than zero)

Signal Histograms

Final selection level with the first three runs of data

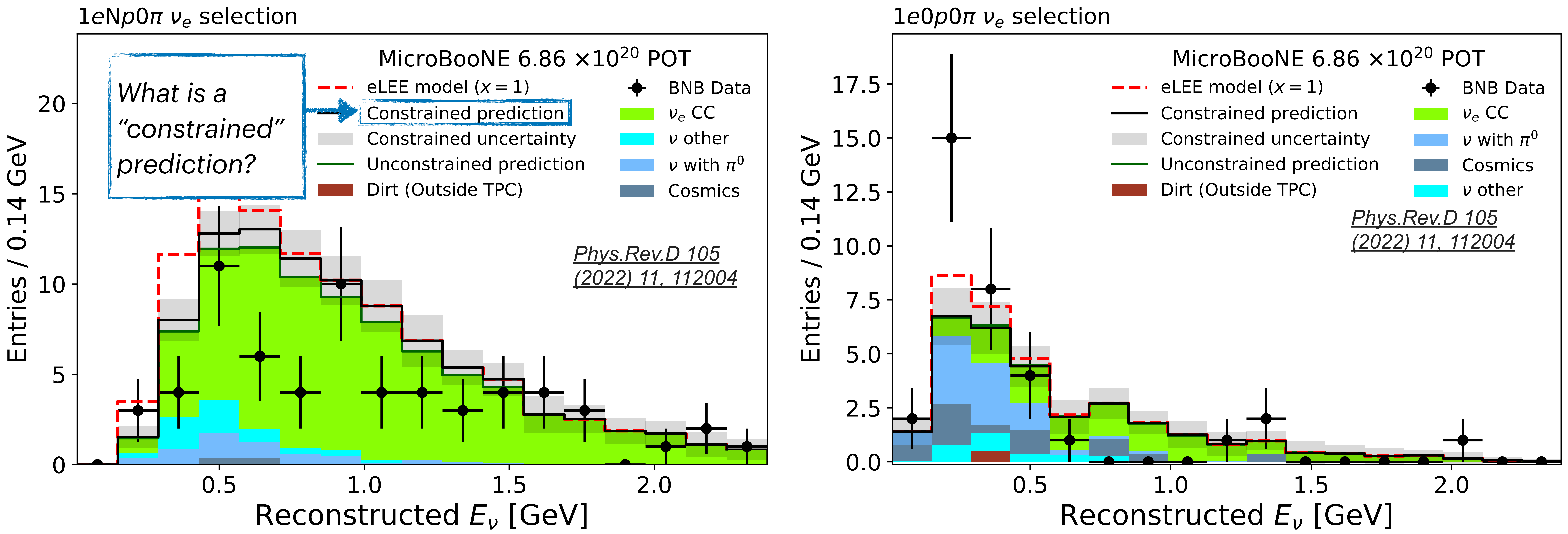




- Data did not confirm low-energy excess, but results were inconclusive → more statistics needed!

Signal Histograms

Final selection level with the first three runs of data



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Signal Histograms

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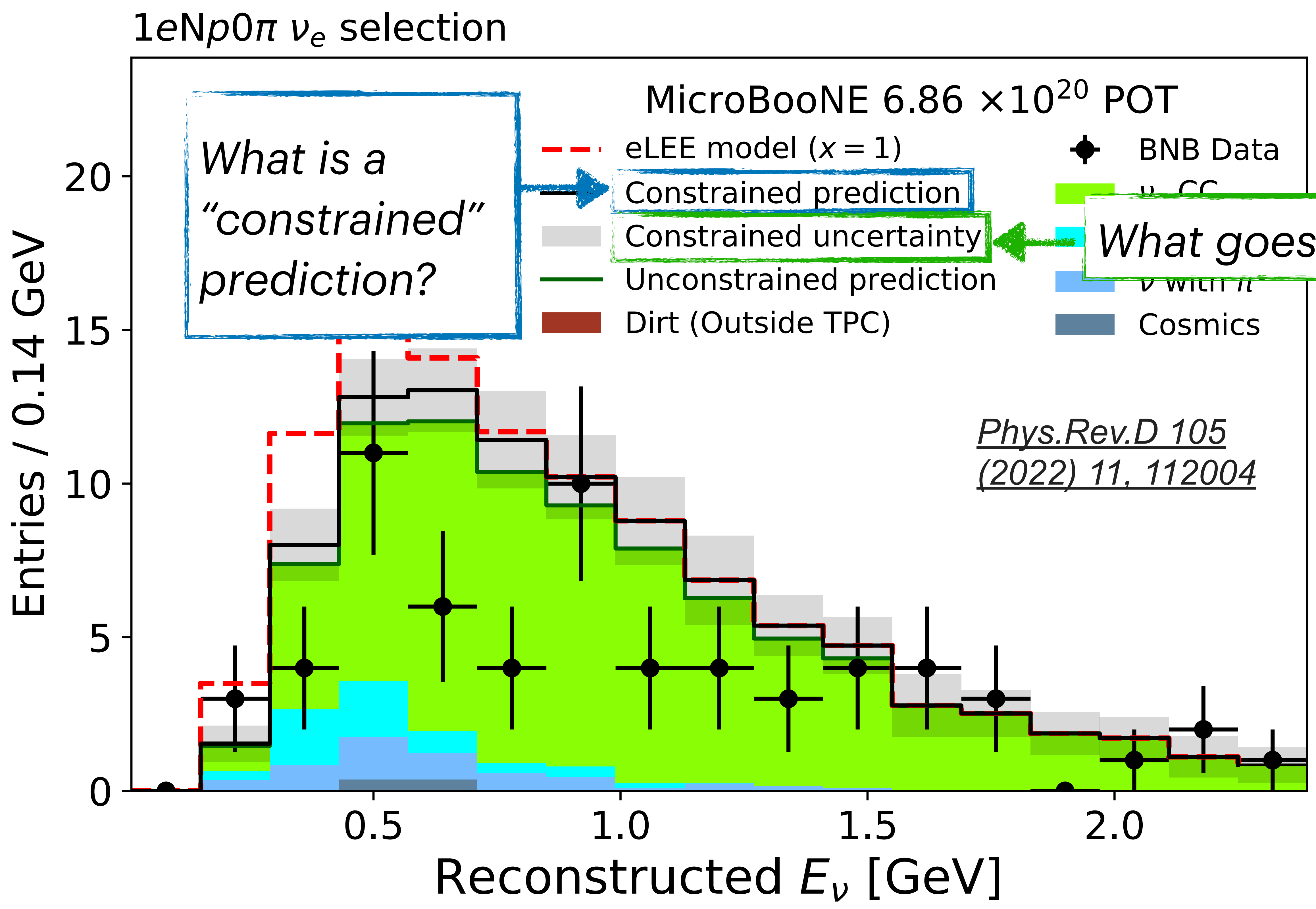

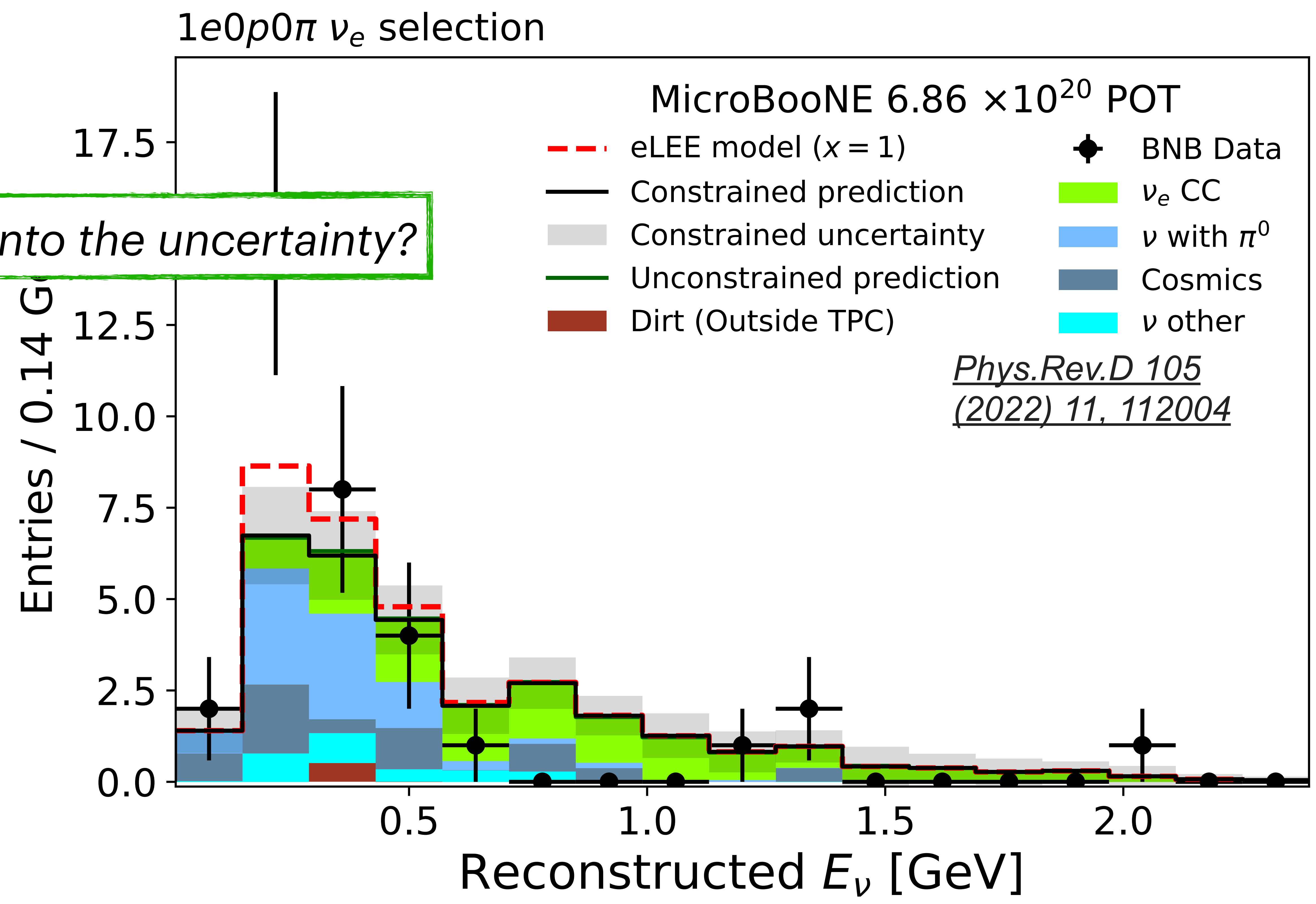

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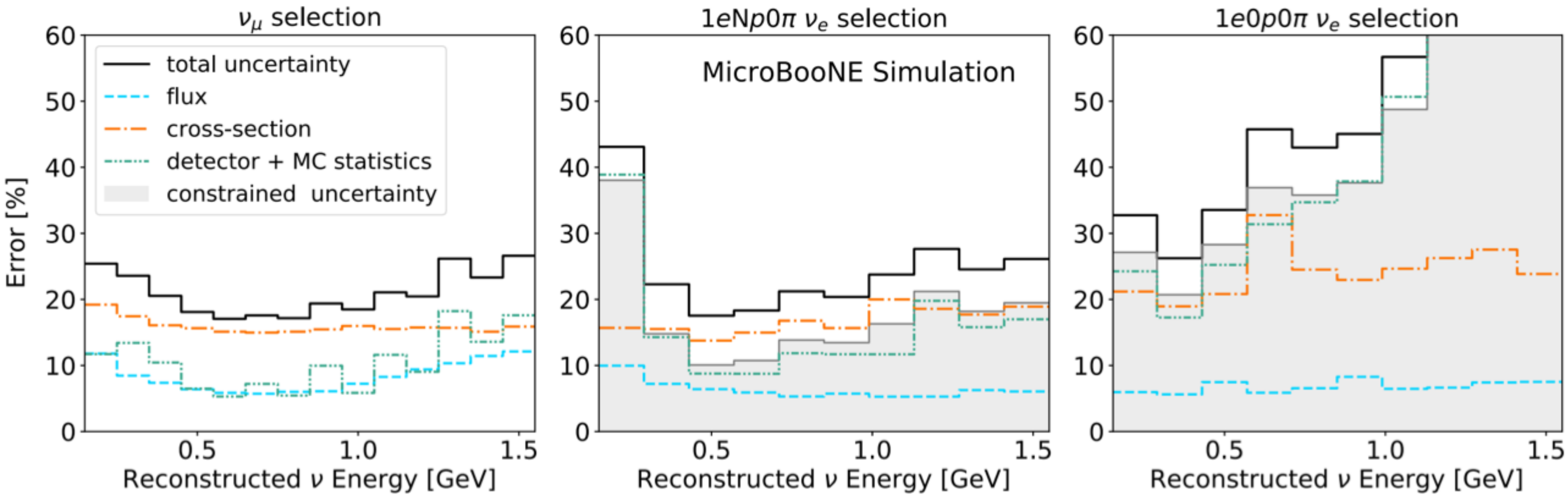
Systematic Uncertainties in MicroBooNE

Error sources considered in this work



- Considering flux, cross-section and detector variations
- Cross-sections are largest source of systematic uncertainties

Systematic Uncertainties in MicroBooNE

Error sources considered in this work

Flux (from BNB)

- Same flux prediction as was used in MiniBooNE
- 13 systematic parameters

Cross-sections

- GENIE Reweight used to fluctuate cross-section
- 53 systematic parameters

Hadronic Re-interaction

- Hadrons can interact strongly with Argon nuclei
- 3 systematic parameters

Detector

- Changing various properties of the detector
- 10 systematic parameters

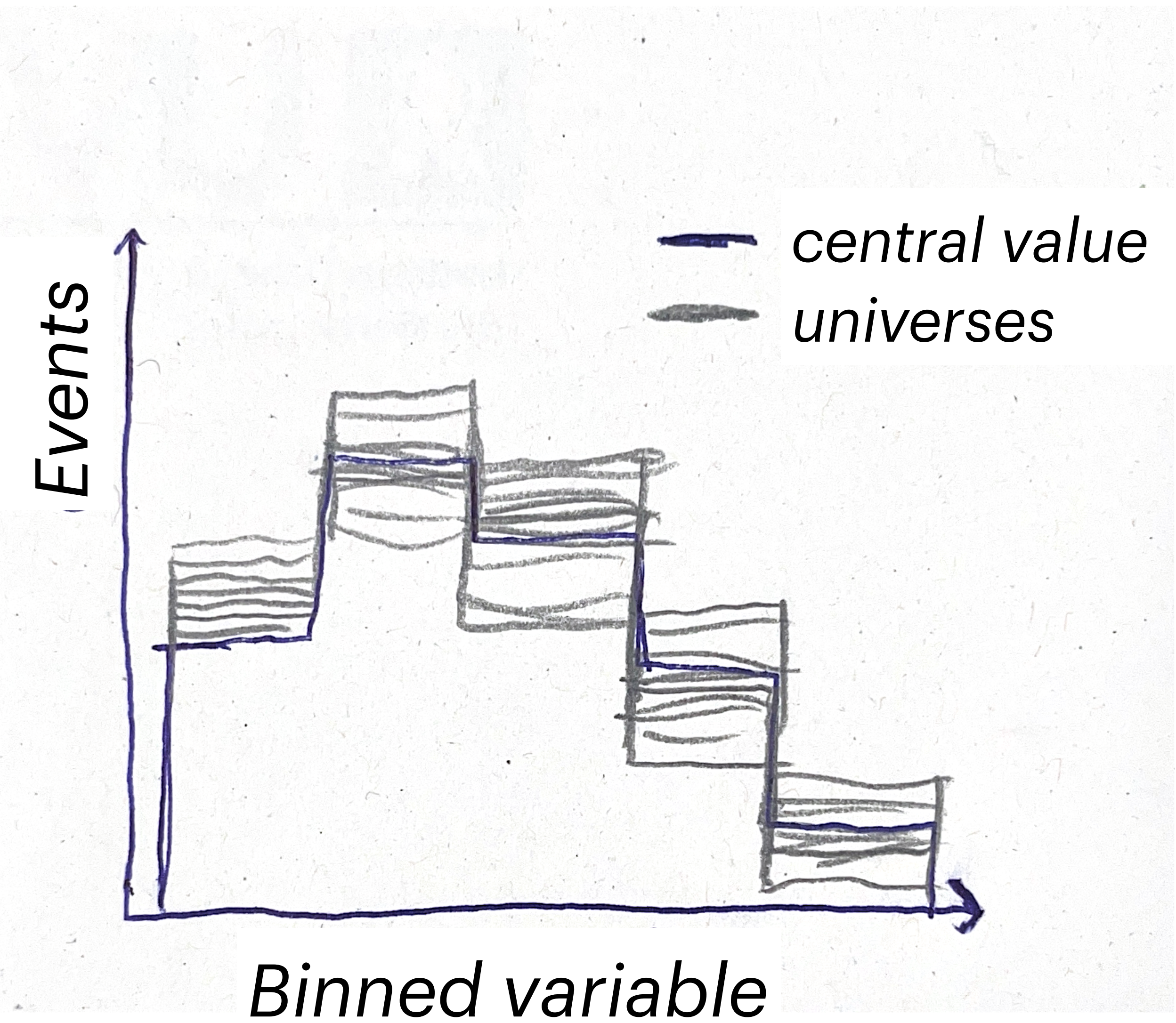
In total 79 nuisance parameters to consider!

➡ Fitting all of these with e.g. MINUIT would be horrendous…

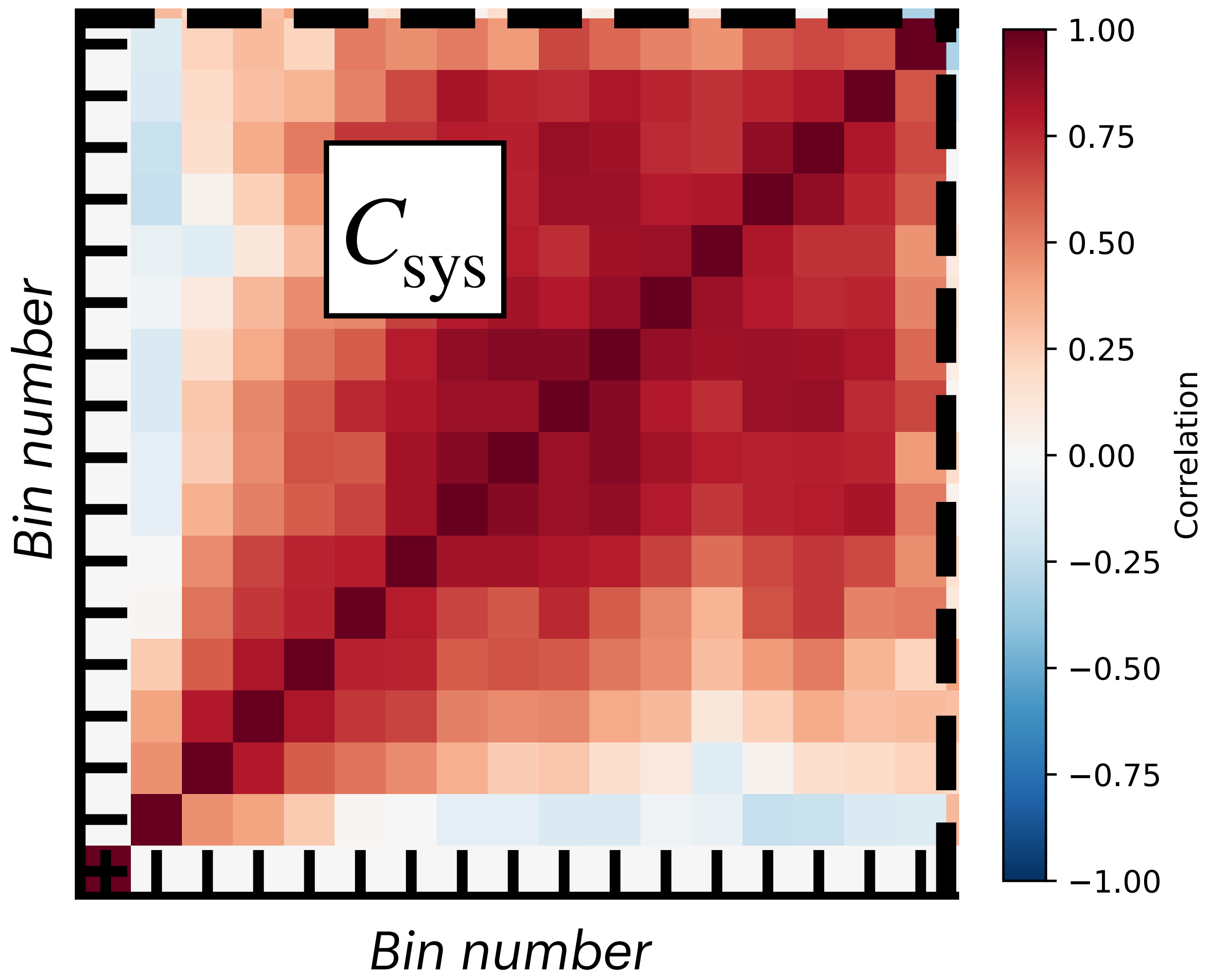
➡ With a few assumptions, we can be much more efficient!

Covariance Matrix Method for Systematic Uncertainties

1. (For each source of uncertainty) fluctuate parameters randomly according to their priors



2. Calculate covariance matrix from universes



Correlation matrix for the systematic uncertainties from the 1eNp0π selection.

3. Add covariance to chi-square loss function

observed bin counts; total covariance; expectation value

$$\chi^2 = (\mathbf{x} - \mu) C^{-1} (\mathbf{x} - \mu)$$

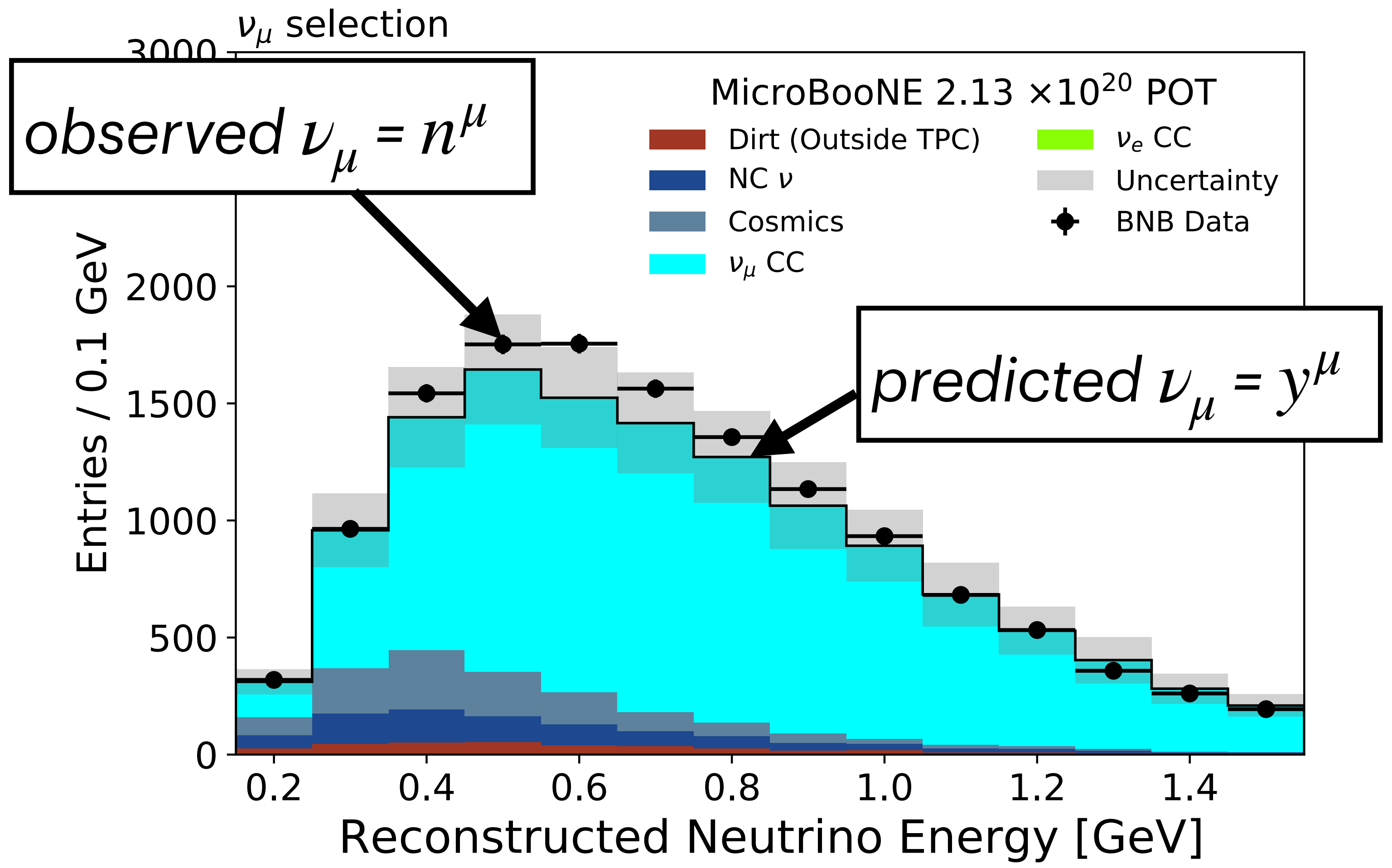
statistical uncertainty (approximates Poisson errors); systematic uncertainties from universes, with correlations

$$C = C_{\text{stat}} + C_{\text{sys}}$$

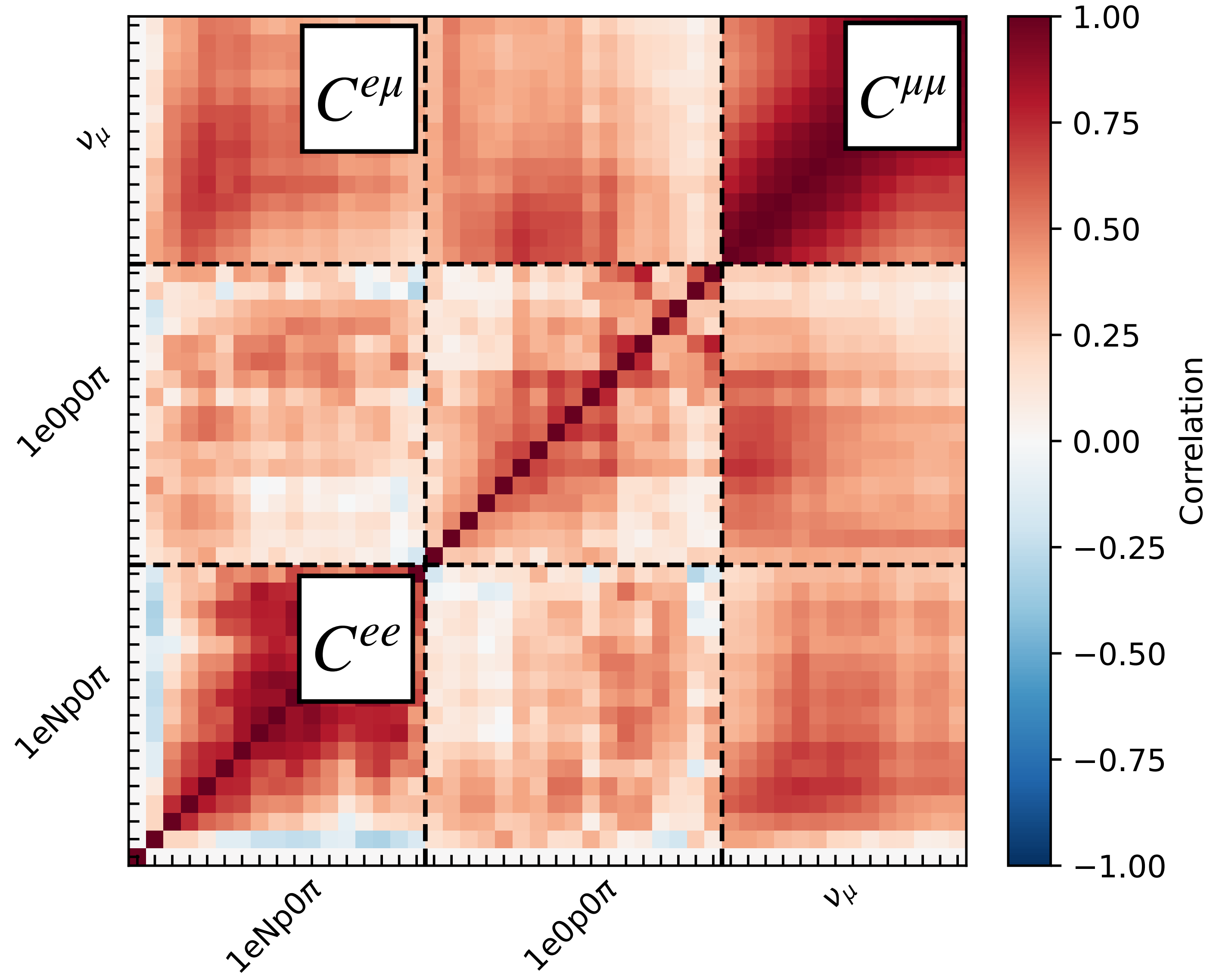
➡ Under assumption that systematic effects are linear, this χ^2 is the same as if we *had* fit all nuisance parameters!

Sideband Constraints

Using muon data to update the electron neutrino prediction



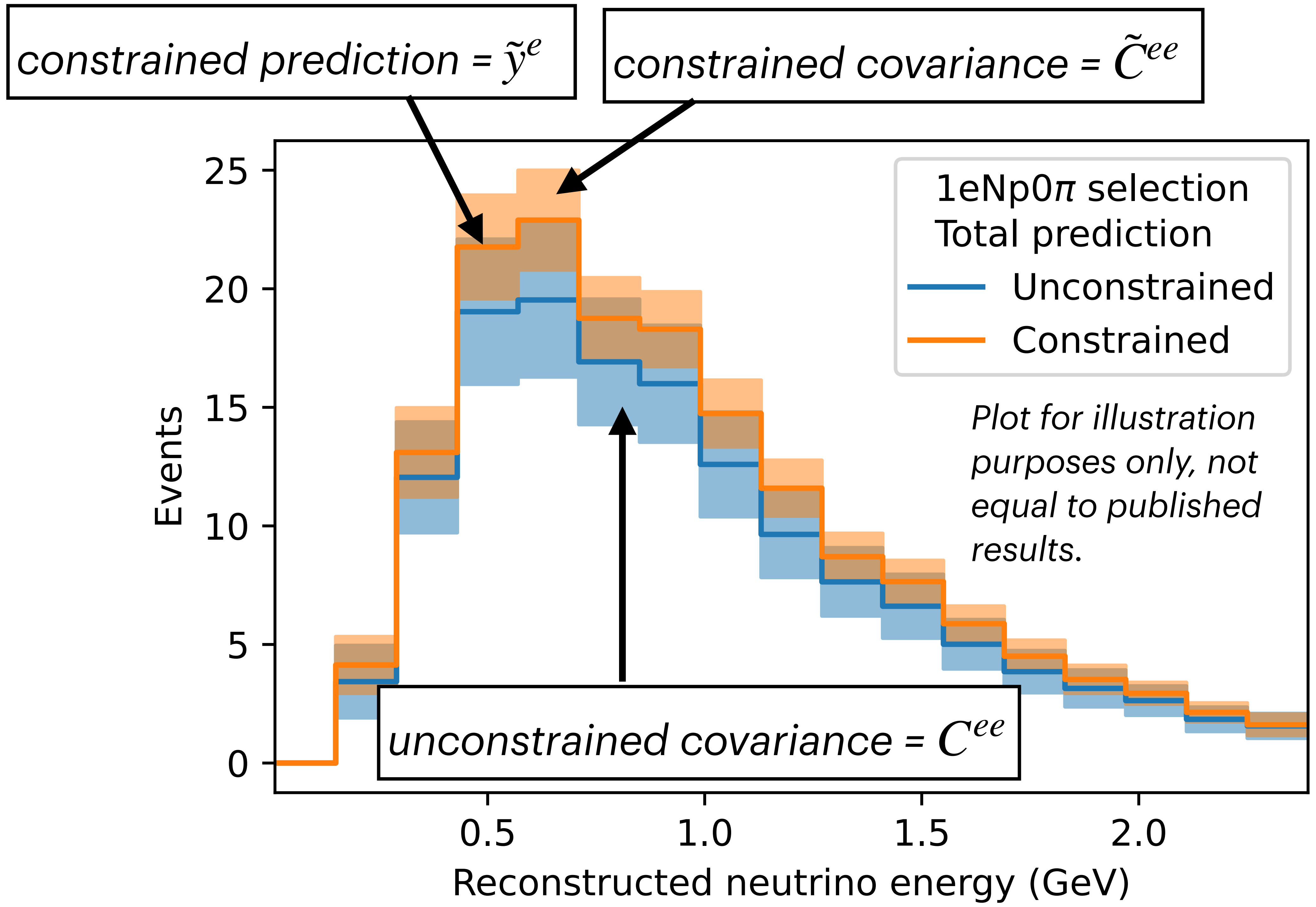
Fluctuate systematic uncertainties in all channels



Block-matrix update

$$\tilde{y}^e = y^e + C^{e\mu}(C^{\mu\mu})^{-1}(n^\mu - y^\mu)$$

$$\tilde{C}^{ee} = C^{ee} - C^{e\mu}(C^{\mu\mu})^{-1}C^{\mu e}$$

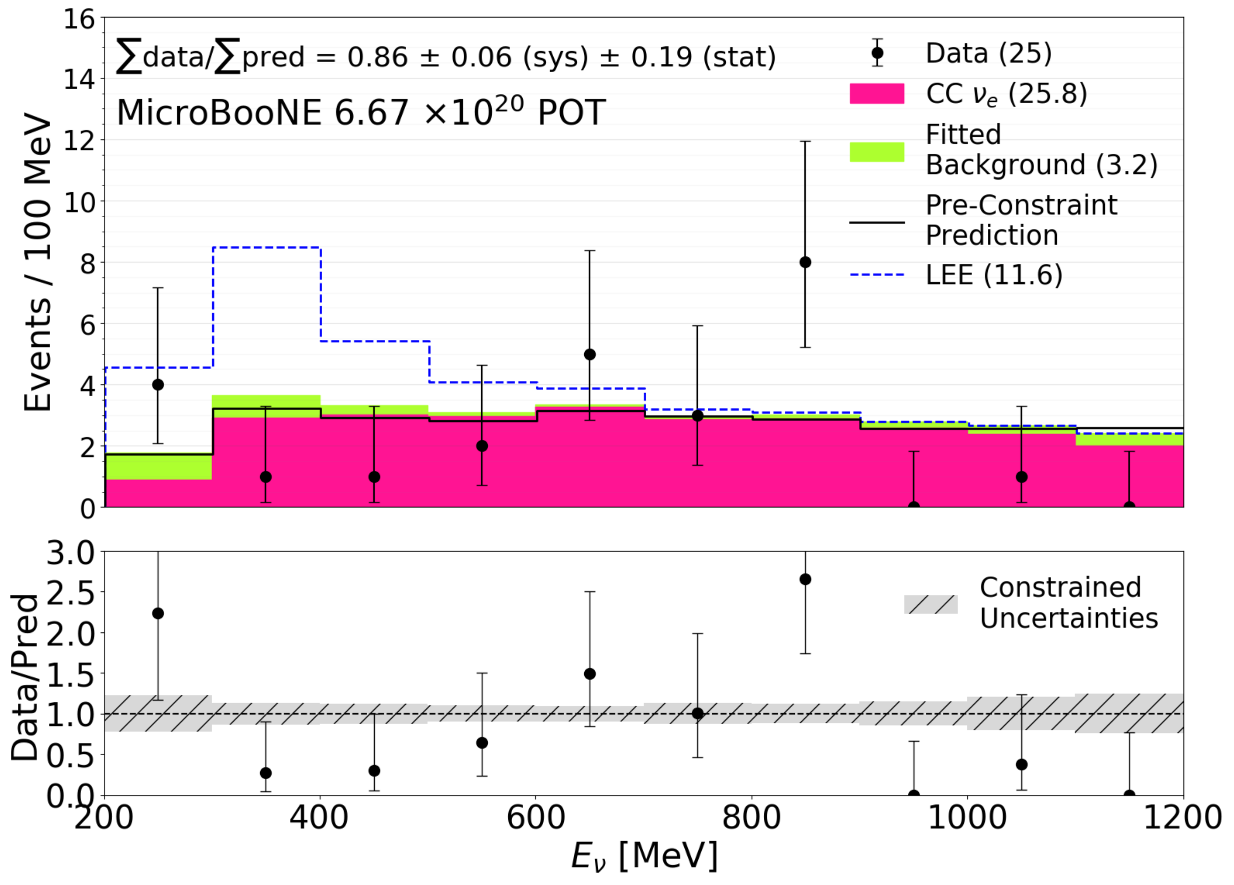


Update corresponds to the Bayesian posterior for the ν_e prediction given the ν_μ observations and their correlation.

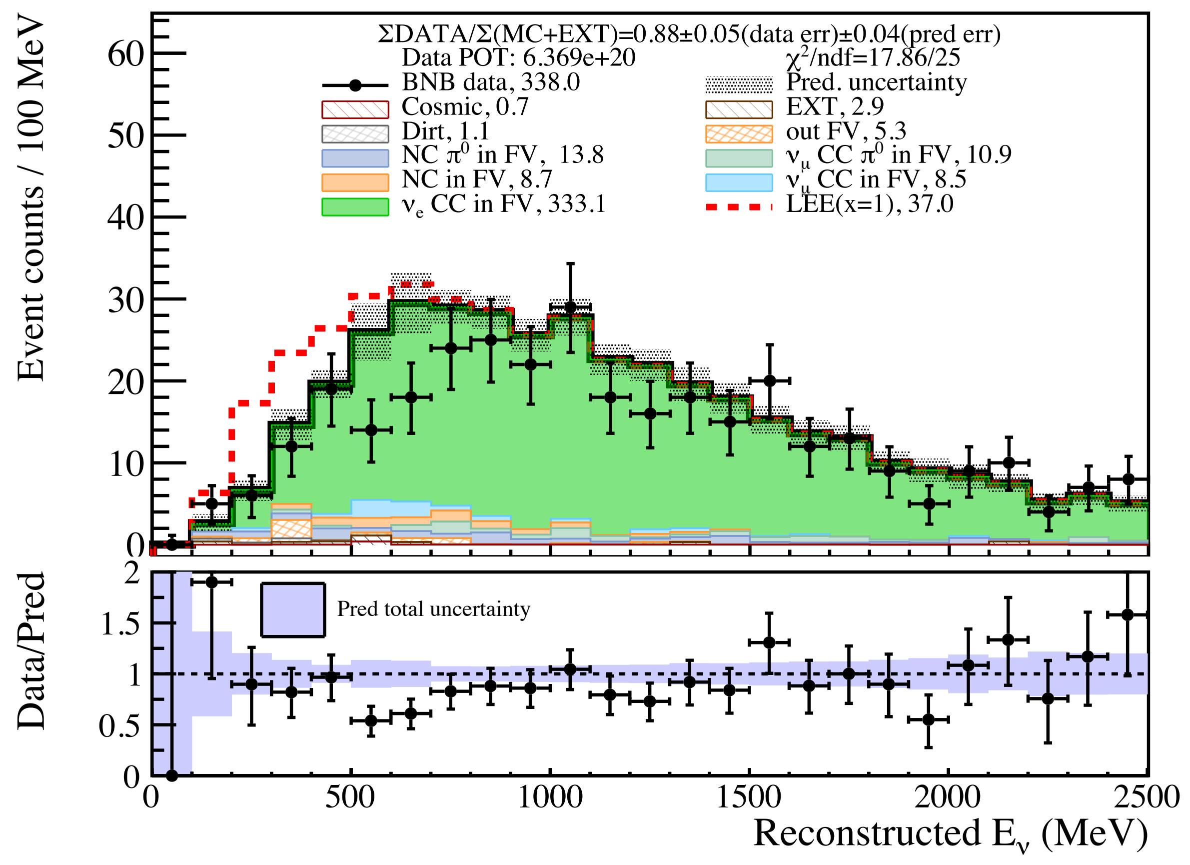
First Generation of LEE Results

Analyzing runs 1-3 (2015 — 2018) of MicroBooNE data

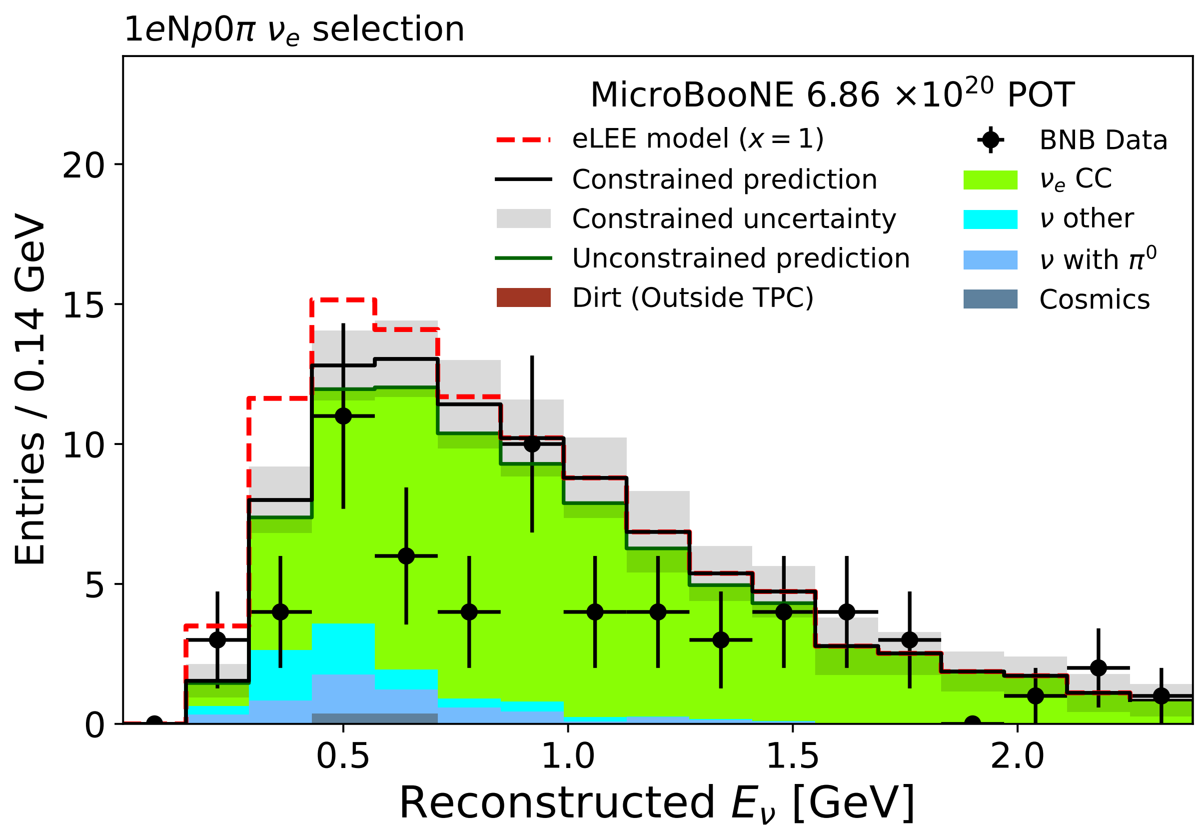
Electron neutrino model



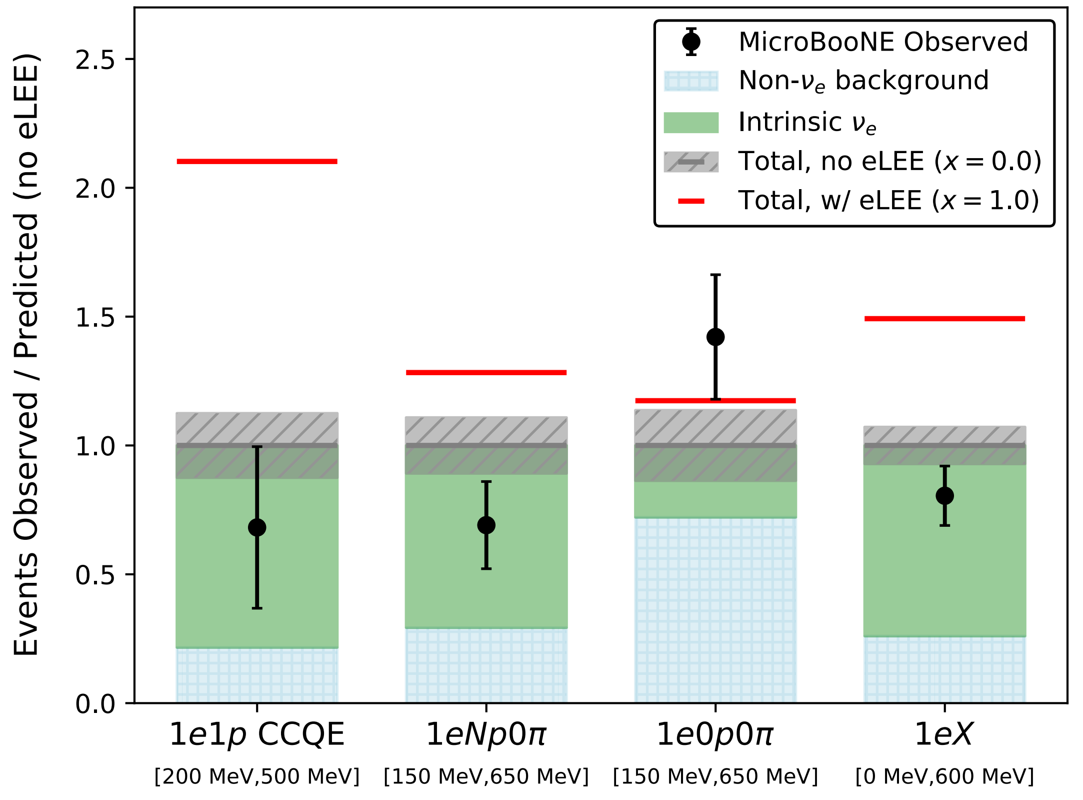
Phys.Rev.D 105 (2022) 11, 112003



Phys.Rev.D 105 (2022) 11, 112005

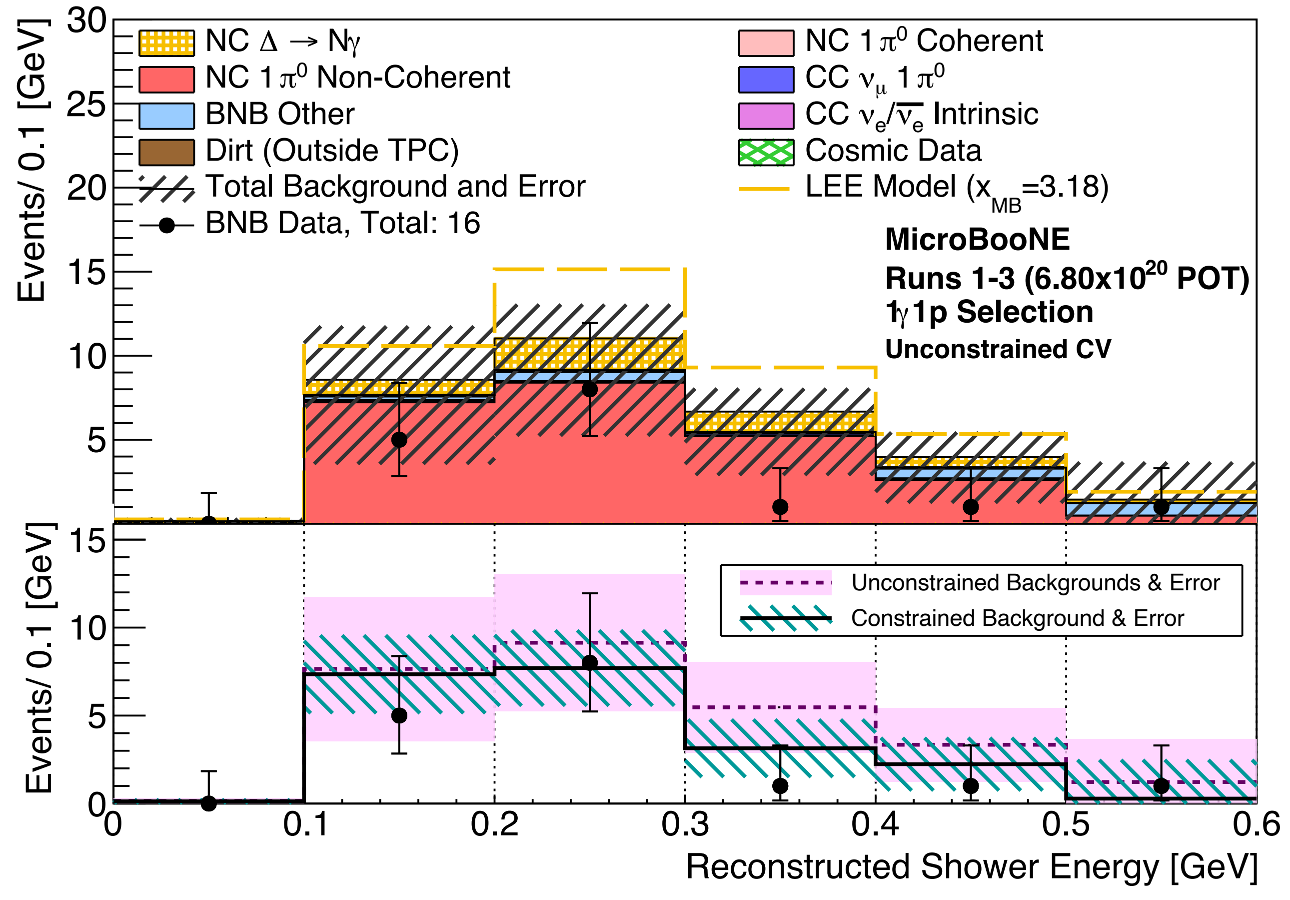


Phys.Rev.D 105 (2022) 11, 112004



Phys.Rev.Lett. 128 (2022) 24, 241801

NC π^0 → Nɣ



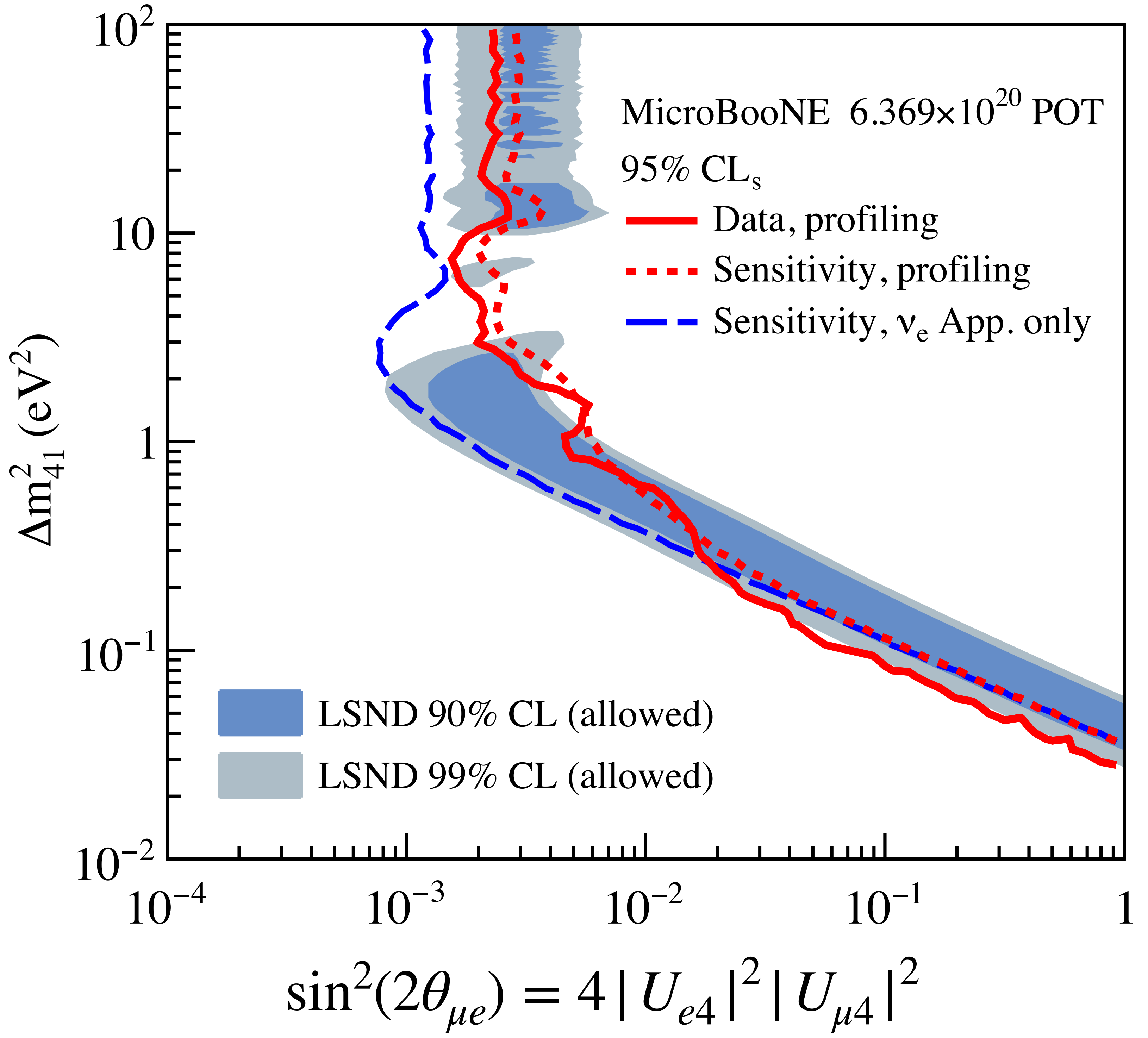
Phys.Rev.Lett. 128 (2022) 111801

- First three years of data (6.8×10^{20} POT) **do not confirm** LEE
  - ➡ Exclusion statistics limited in all analyses so far

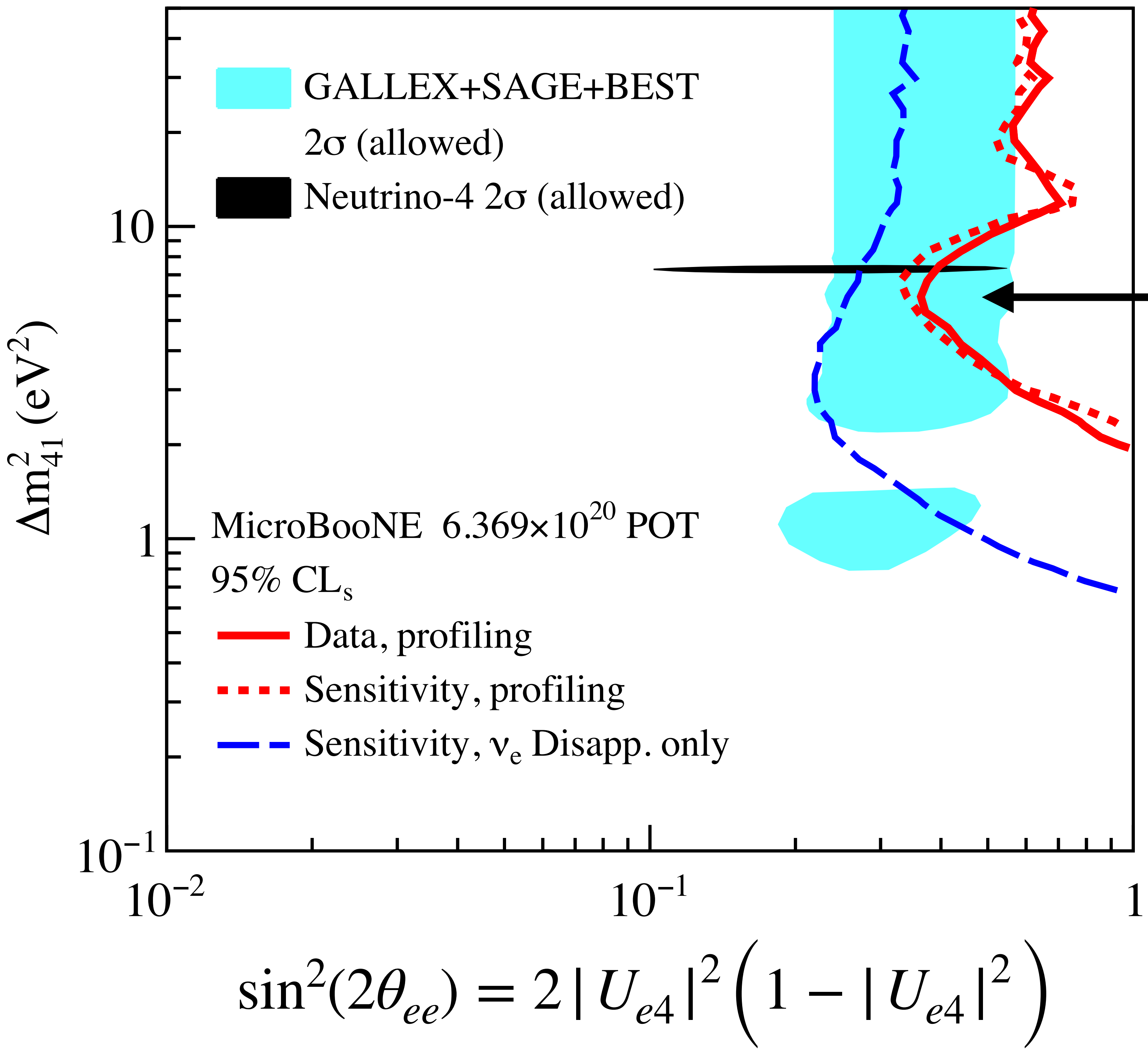
Sterile Neutrino Results

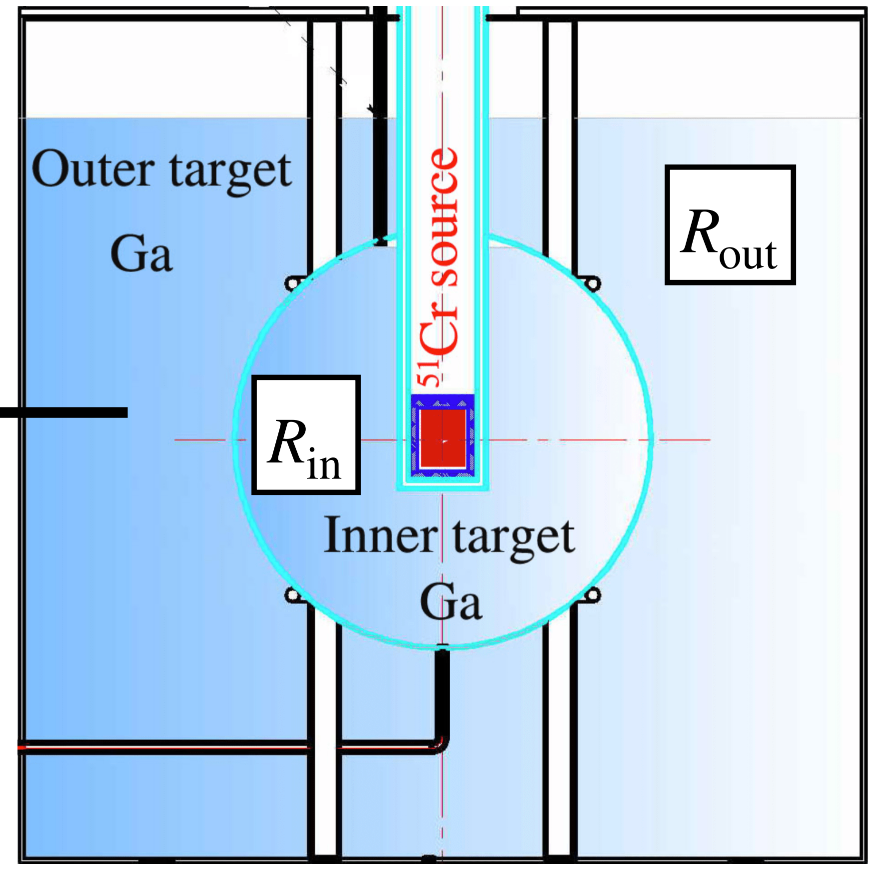
Analyzing runs 1-3 (2015 — 2018) of MicroBooNE data

$\nu_\mu \rightarrow \nu_e$ **appearance channel**



$\nu_e \rightarrow \nu_e$ **disappearance channel**





Experimental setup of the BEST experiment. Phys.Rev.Lett. 128 (2022) 23, 232501

- Deficit in $\nu_e \rightarrow \nu_e$ channel of ~20%

- Result using only the BNB beam published in Phys.Rev.Lett. 130 (2023) 1, 011801

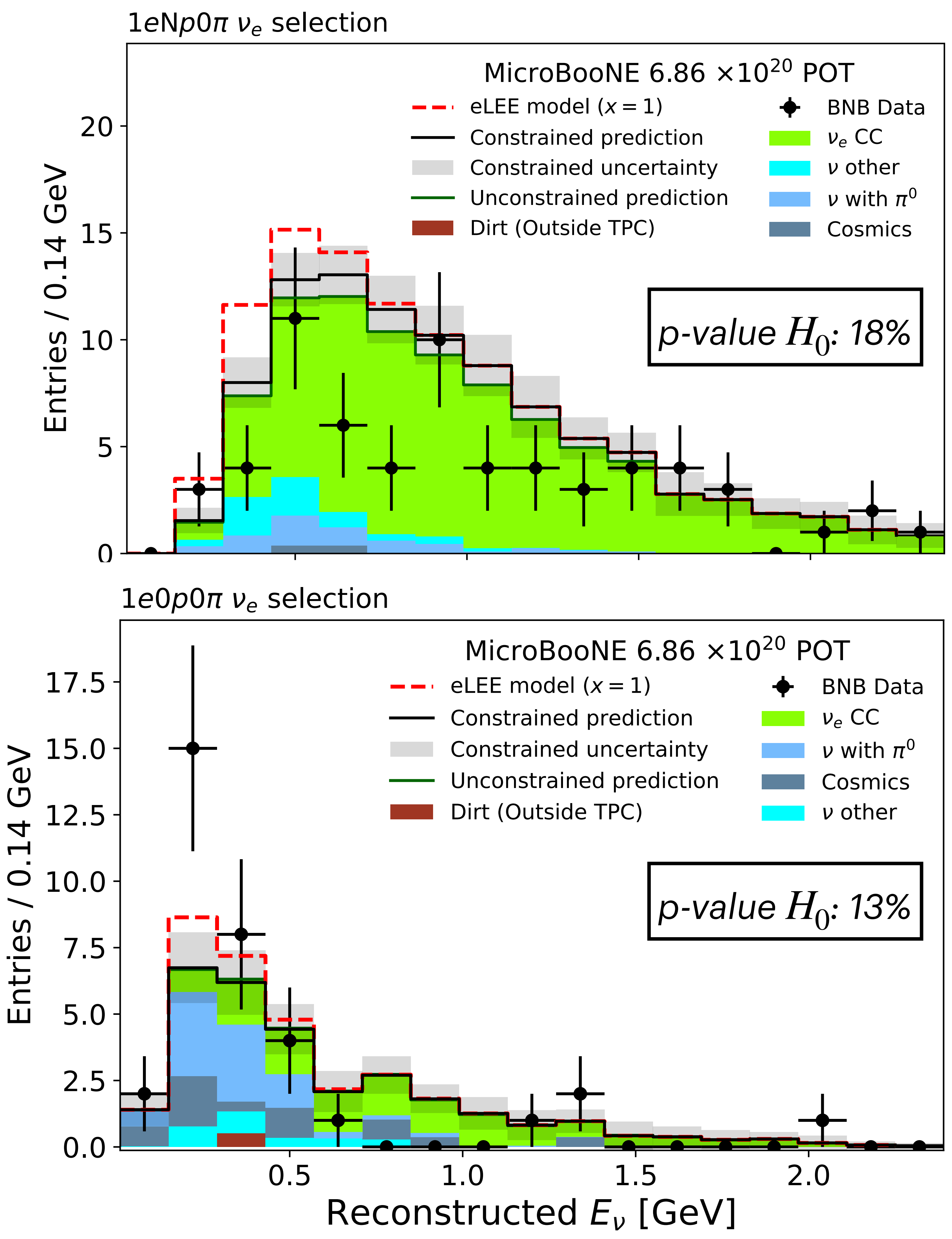
New: Analyzing Runs 1-5

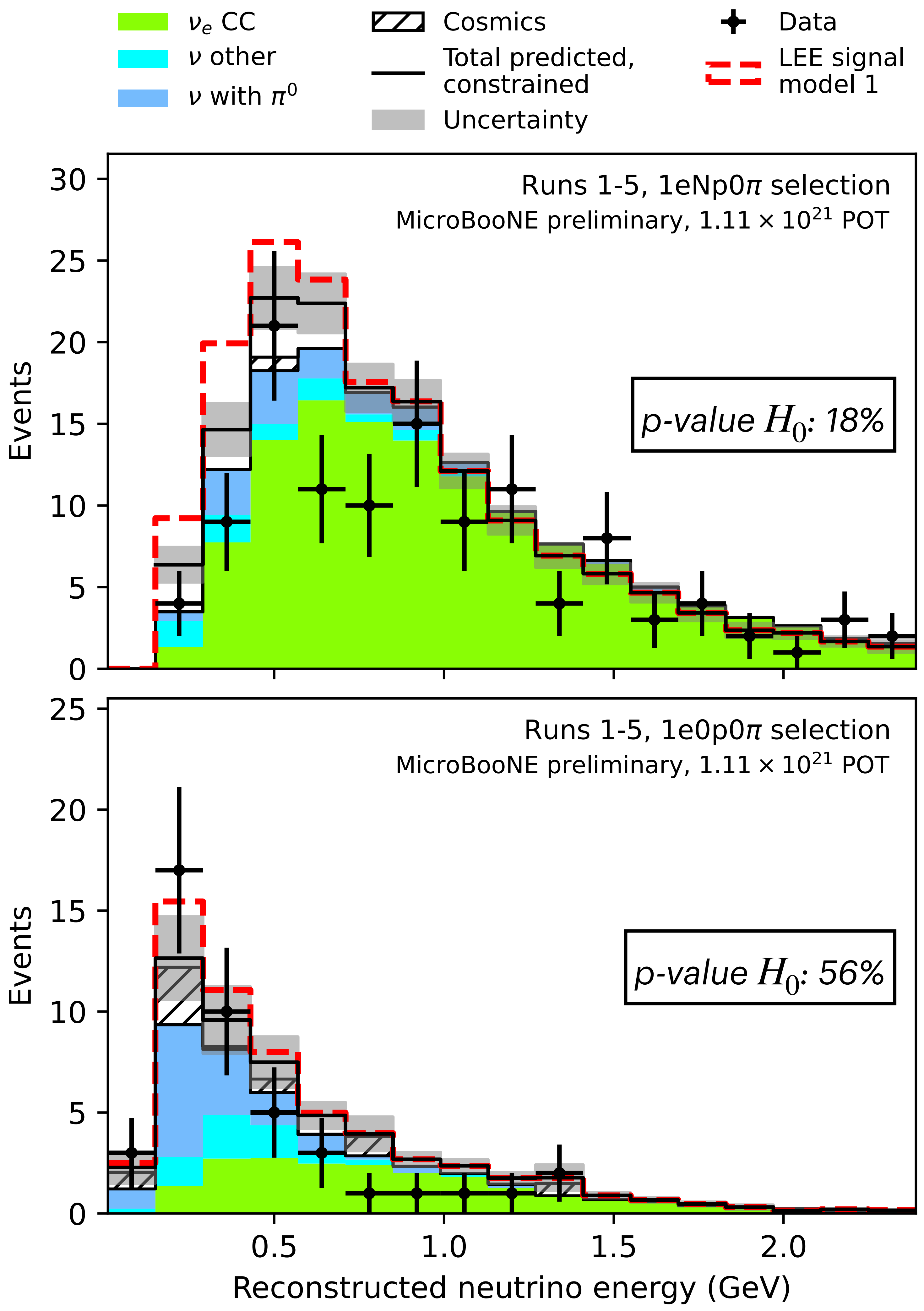
First time use of the full dataset from years 2015 — 2020

First LEE search with runs 1-5

Testing the electron neutrino hypothesis with doubled statistics

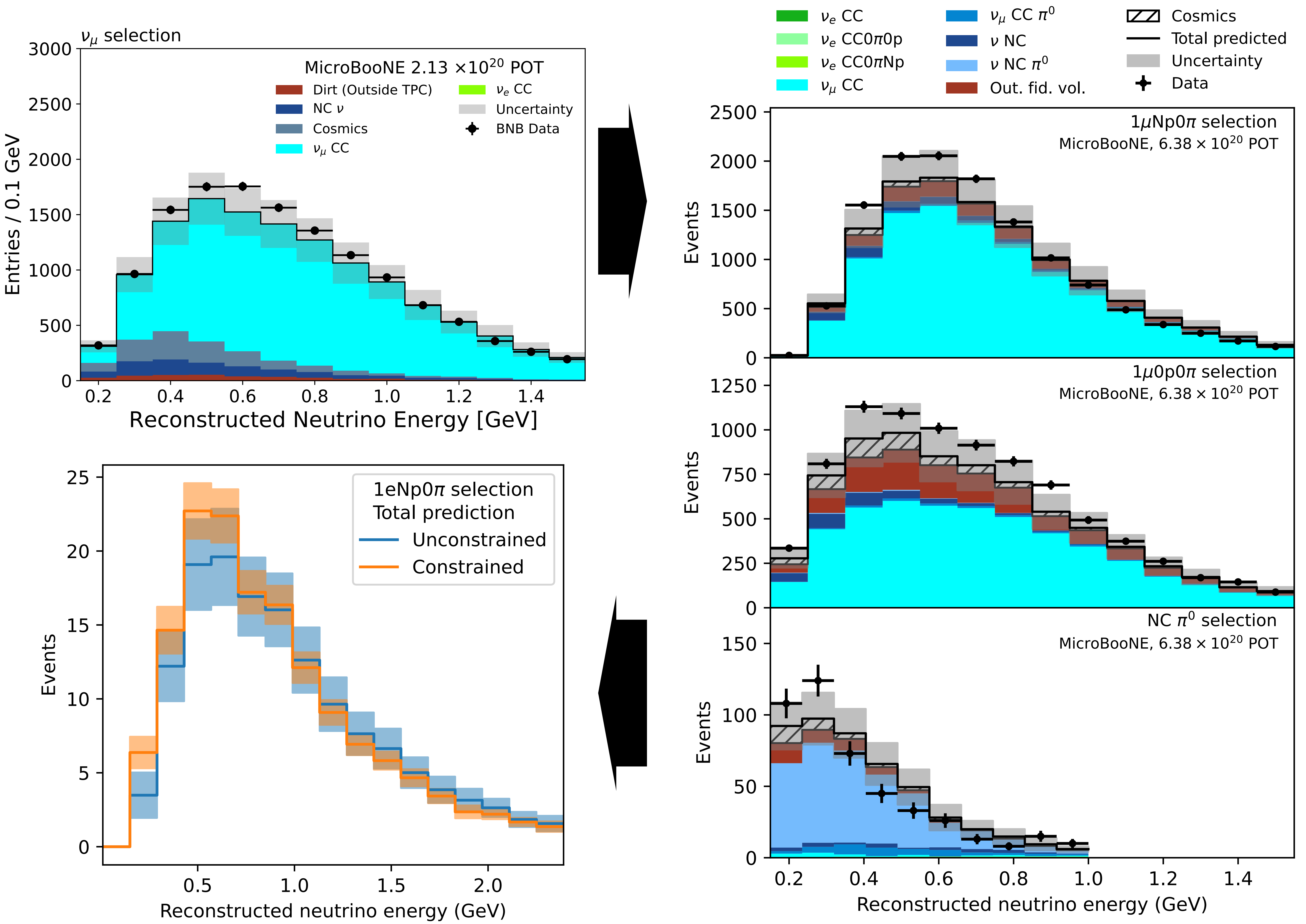
- Updated statistics: 6.8×10^{20} POT $\rightarrow 1.1 \times 10^{21}$ POT
- Same reconstruction & event selection as first Pandora-based result
- Data/MC compatibility (assuming H_0) stayed the same in 1eNp, improved in 1e0p channel





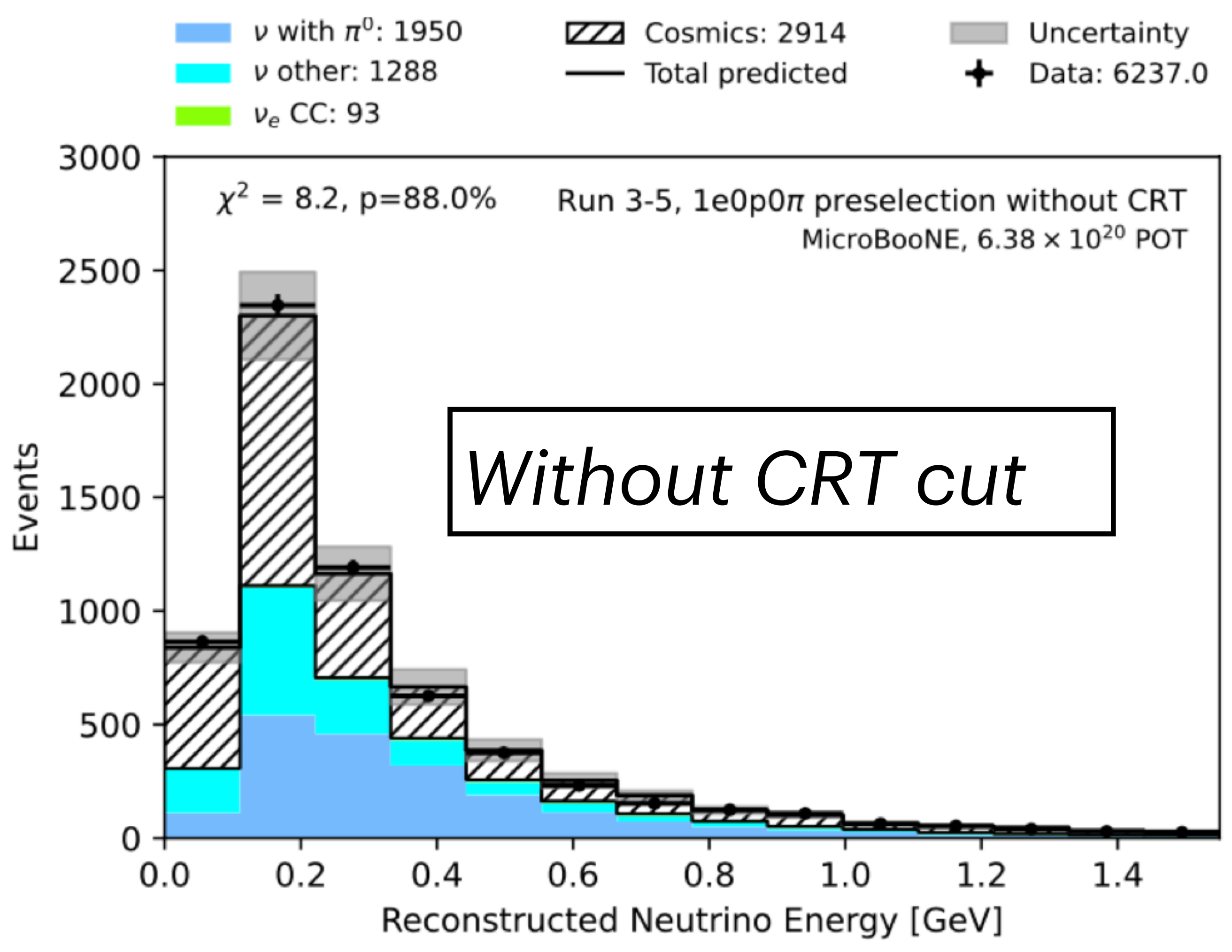
Updated Sideband Constraints

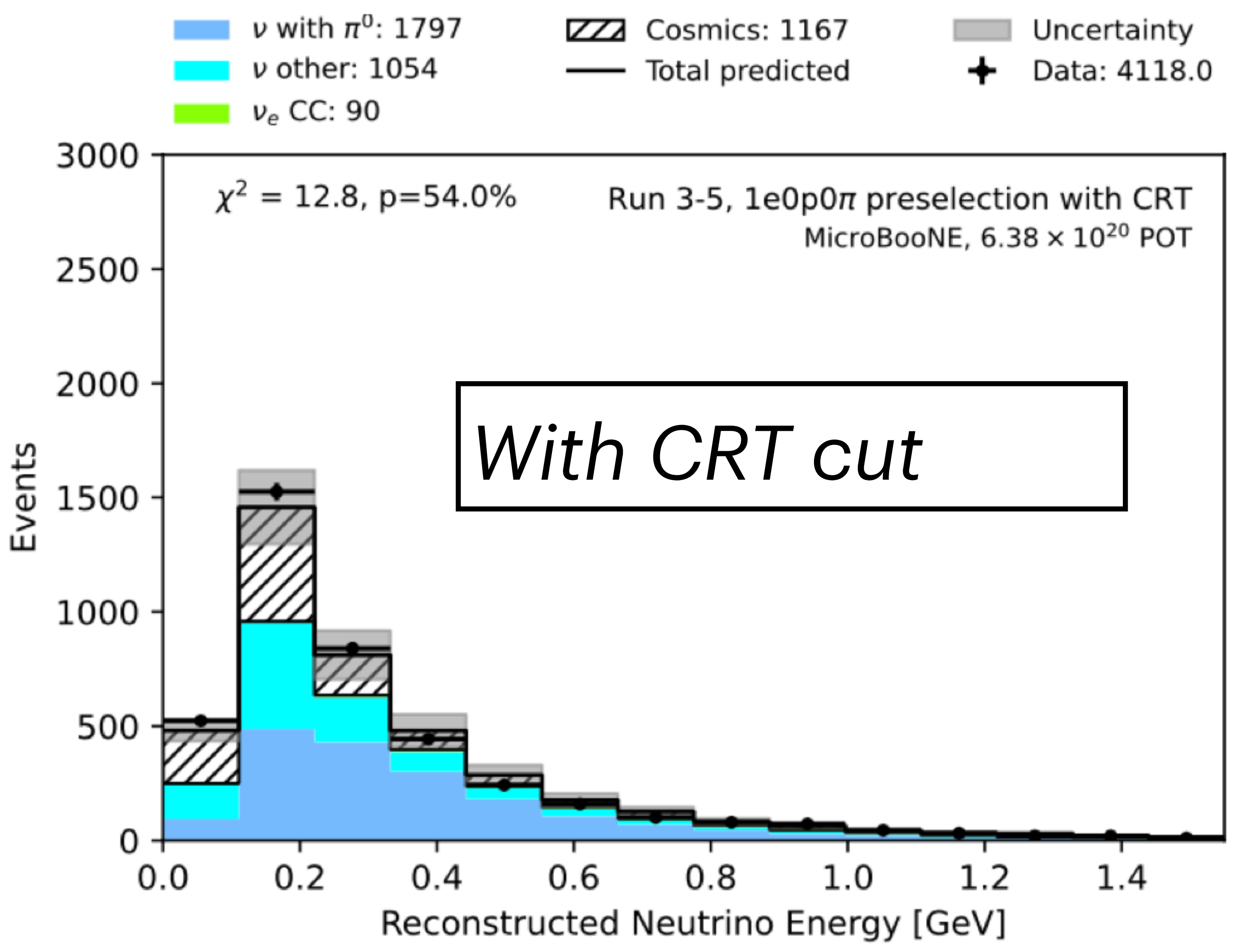
- 3x increased statistics in sideband selection: 2.1×10^{20} POT → 6.4×10^{20} POT
- Split ν_μ selection into 1μ0p and 1μNp channels
- Added NC π^0 selection to constrain background
- Improved sensitivity nearly as much as statistics increase

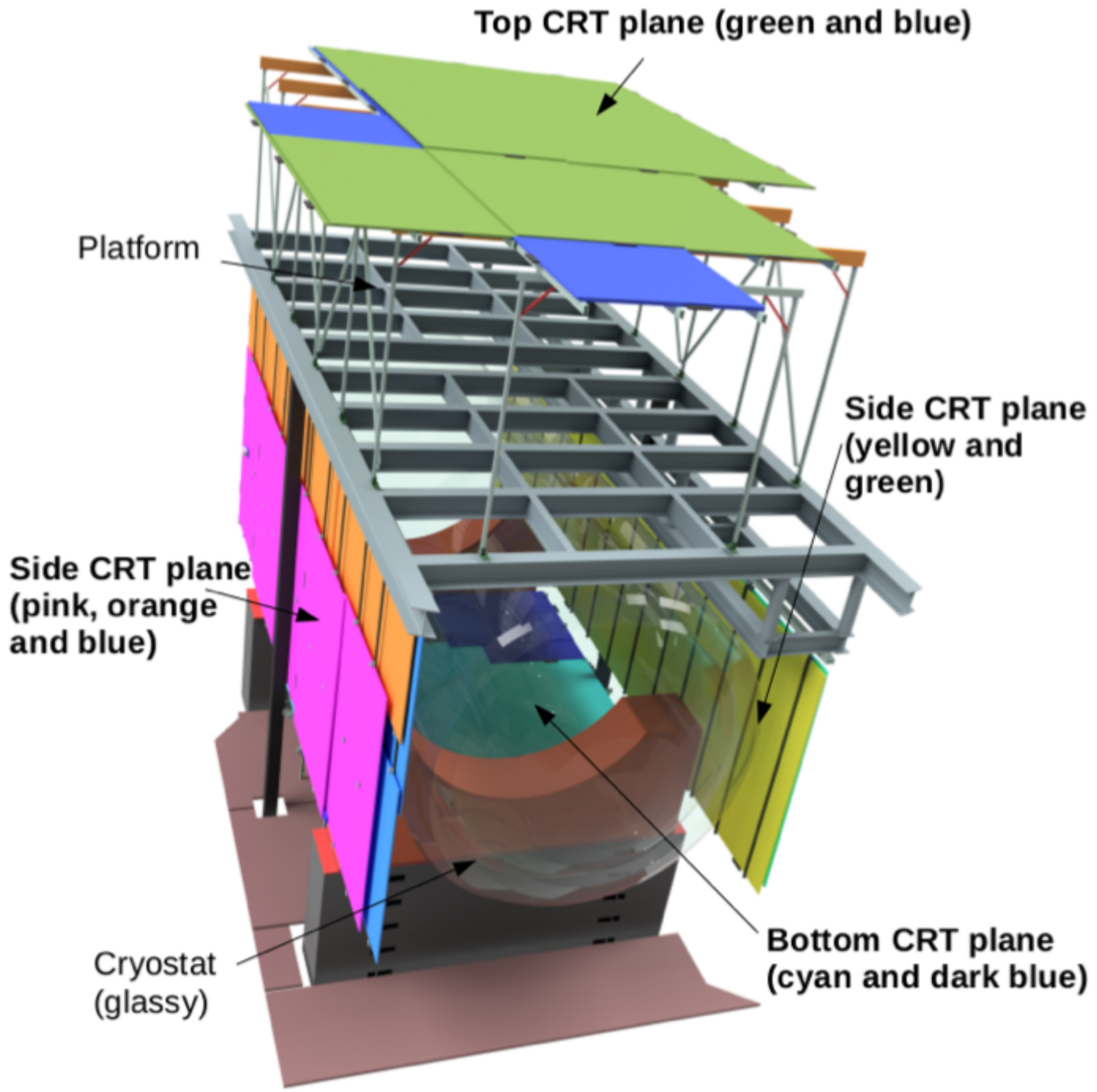


Cosmic Ray Tagger

- Scintillator modules on top of detector to veto cosmic rays
- First round of analysis only used for sidebands, this time also applied to signal
- Removes 60% of background in sidebands, 24% in signal bands

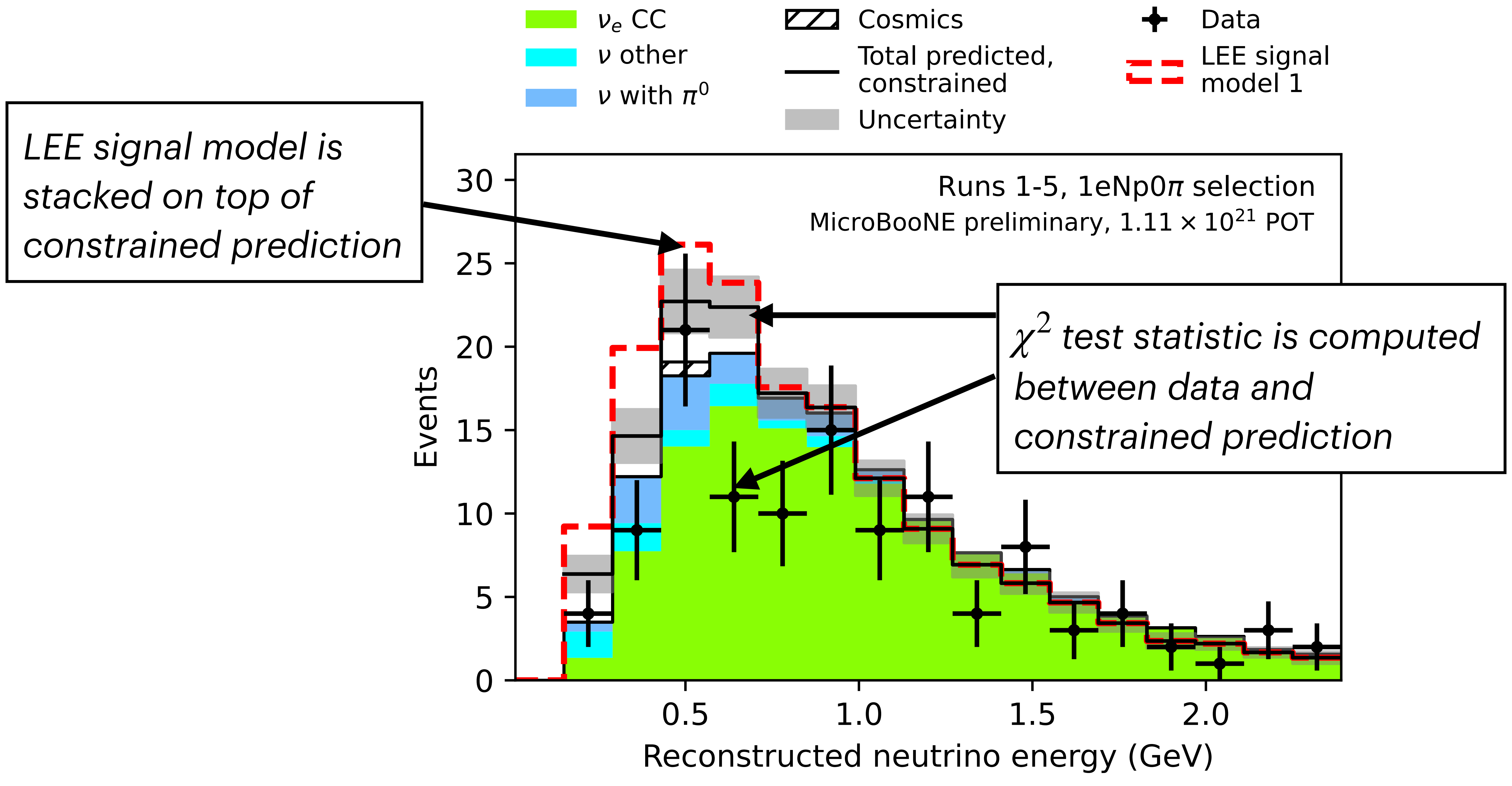






Statistical Tests

How we compute the significance of the LEE exclusion



χ^2 test statistics for 1eNp0π and 1e0p0π selections are added

Two-hypothesis test

- Comparing χ^2 between GENIE pred. and GENIE + LEE
- ***Reject LEE hypothesis at*** 2.5σ CL_s

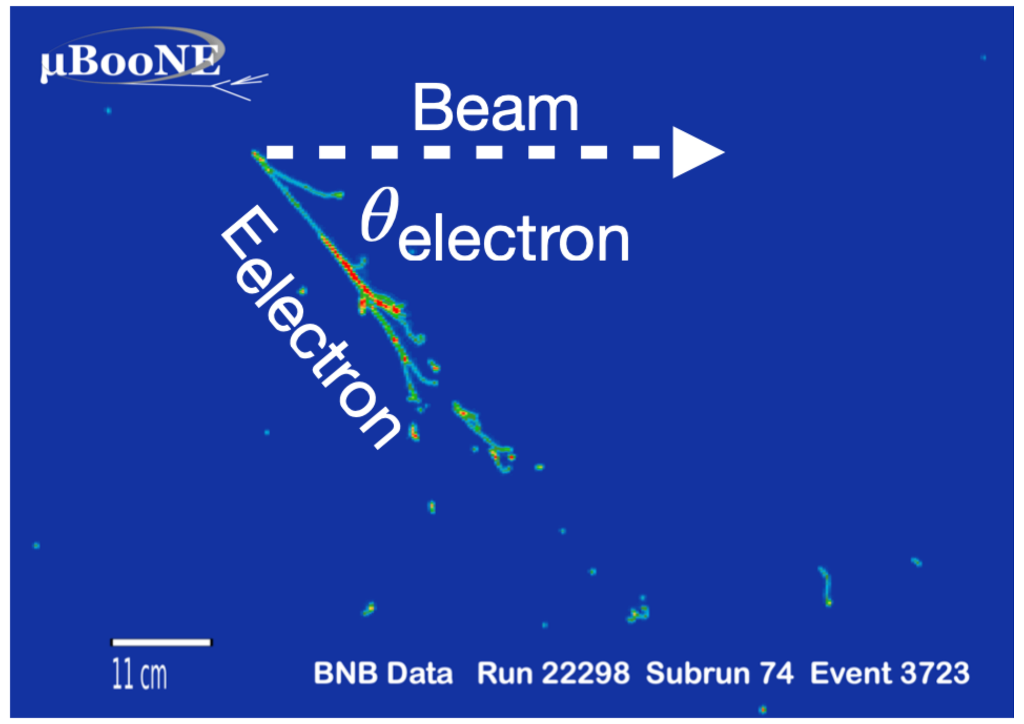
LEE scale fit

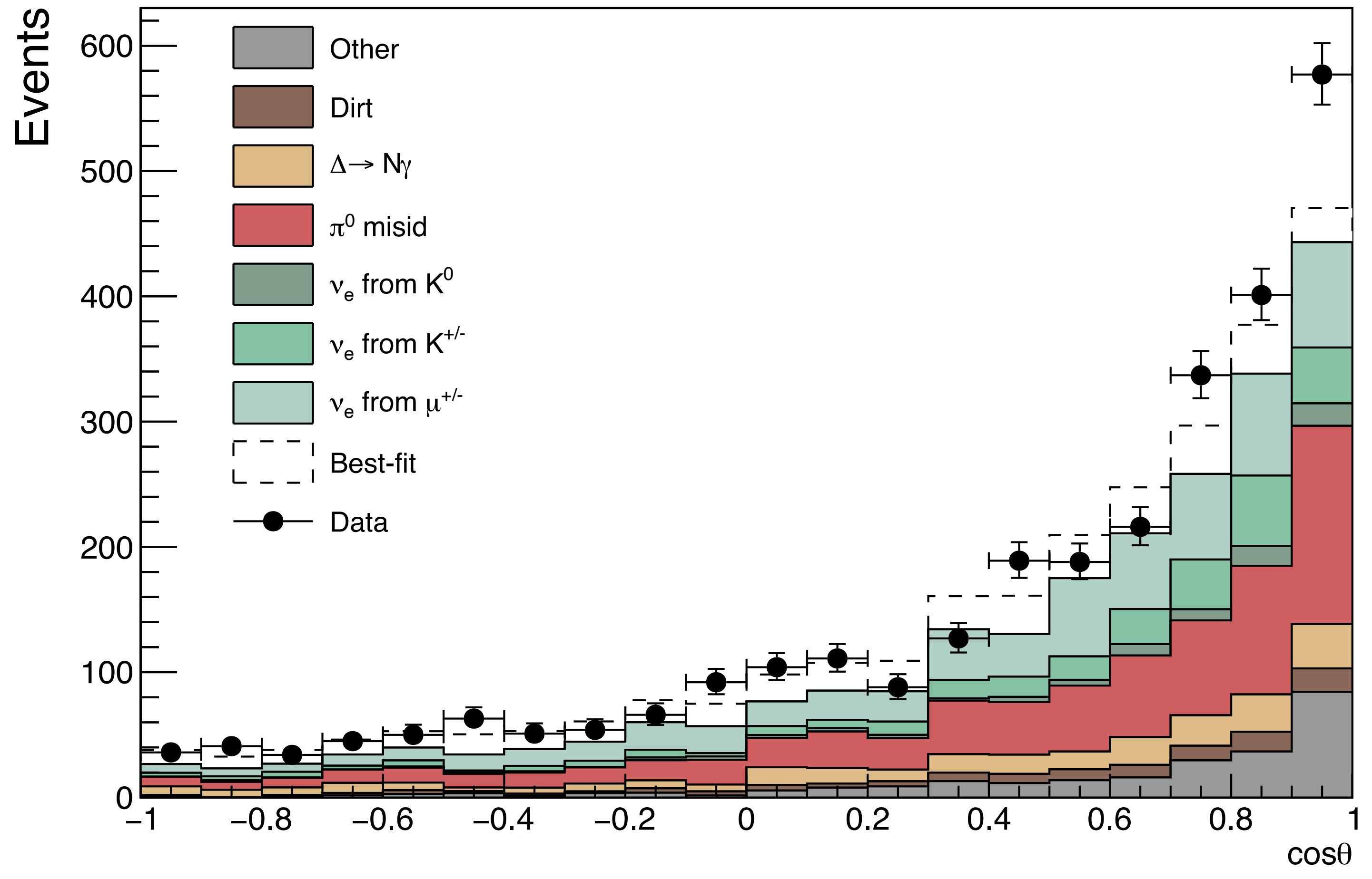
- Let amplitude of LEE signal float freely
- Using Feldman-Cousins method, we ***reject LEE hypothesis at*** $> 99\%$ CL
- 2σ upper limit at x=0.47

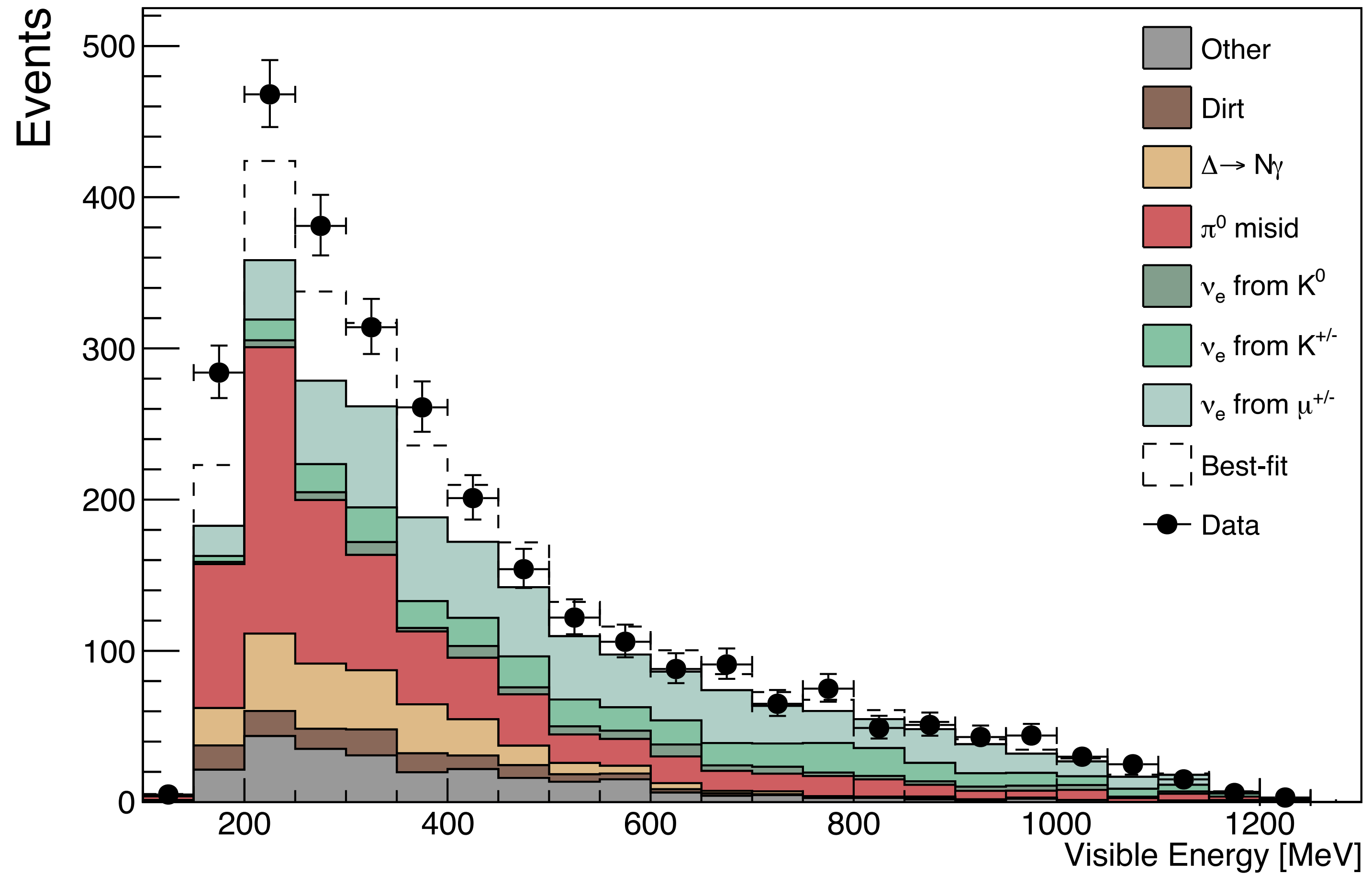
Kinematic Signal Model

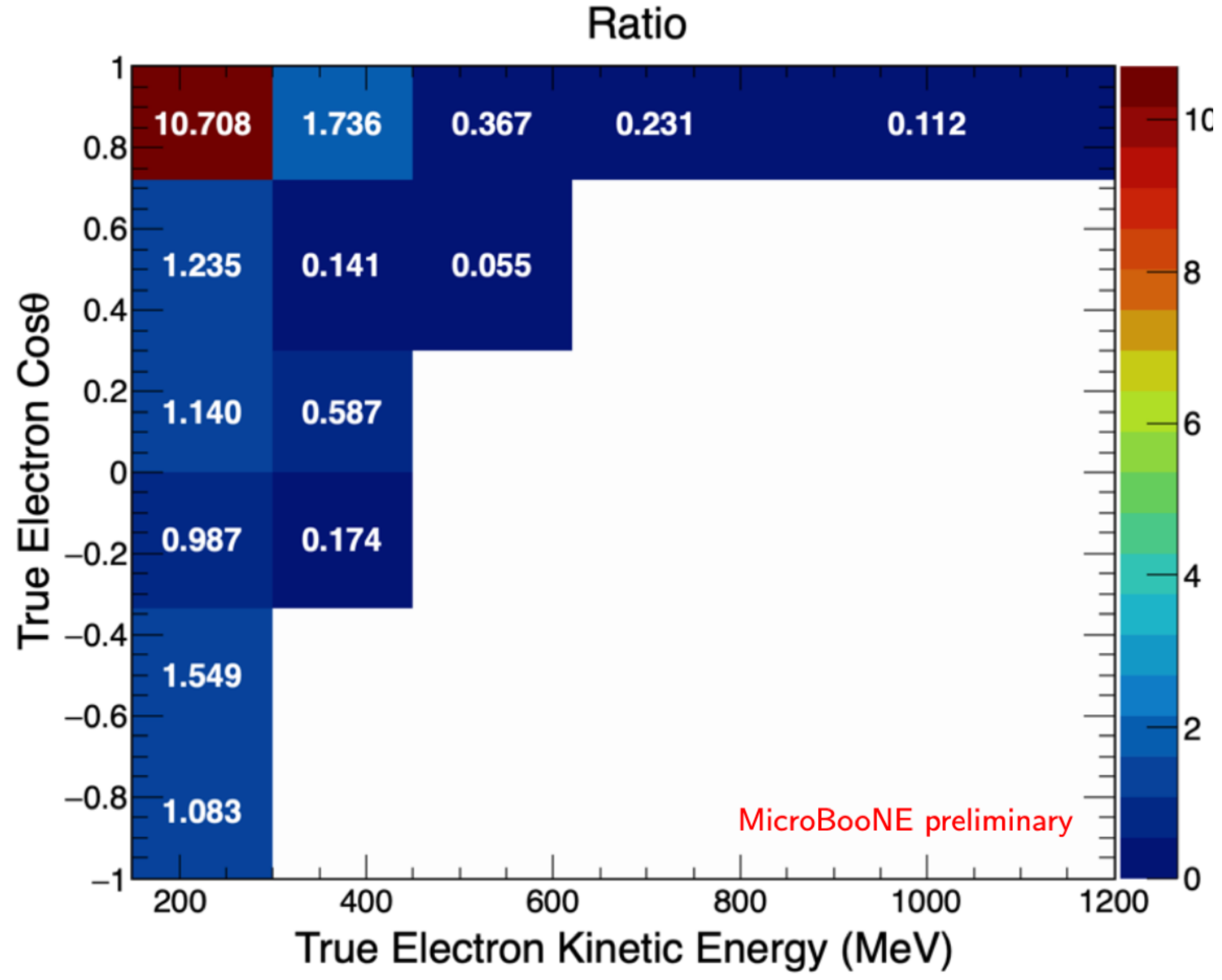
Signal model based on direct observables

- Reconstructed CCQE energy not directly measured by detector, relies on modeling
- Fitting nu energy doesn't match signature in observables $E_{vis}, \cos(\theta)$



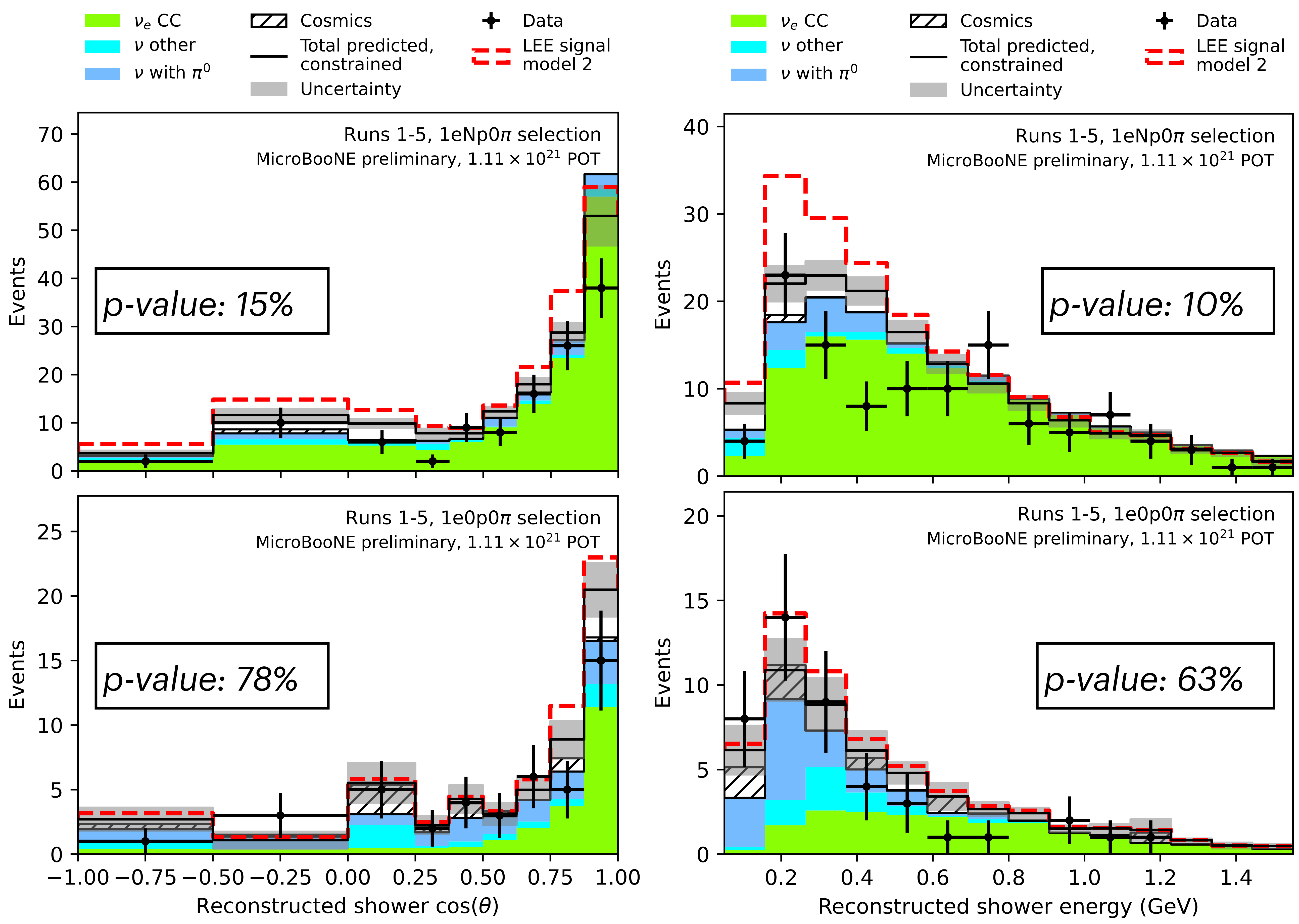






Kinematic Signal Model

Signal model based on direct observables



Analysis setup

- Binning data in reconstructed shower energy, reconstructed shower $\cos(\theta)$
- Performing statistical tests independently for both variables

Two hypothesis test results:

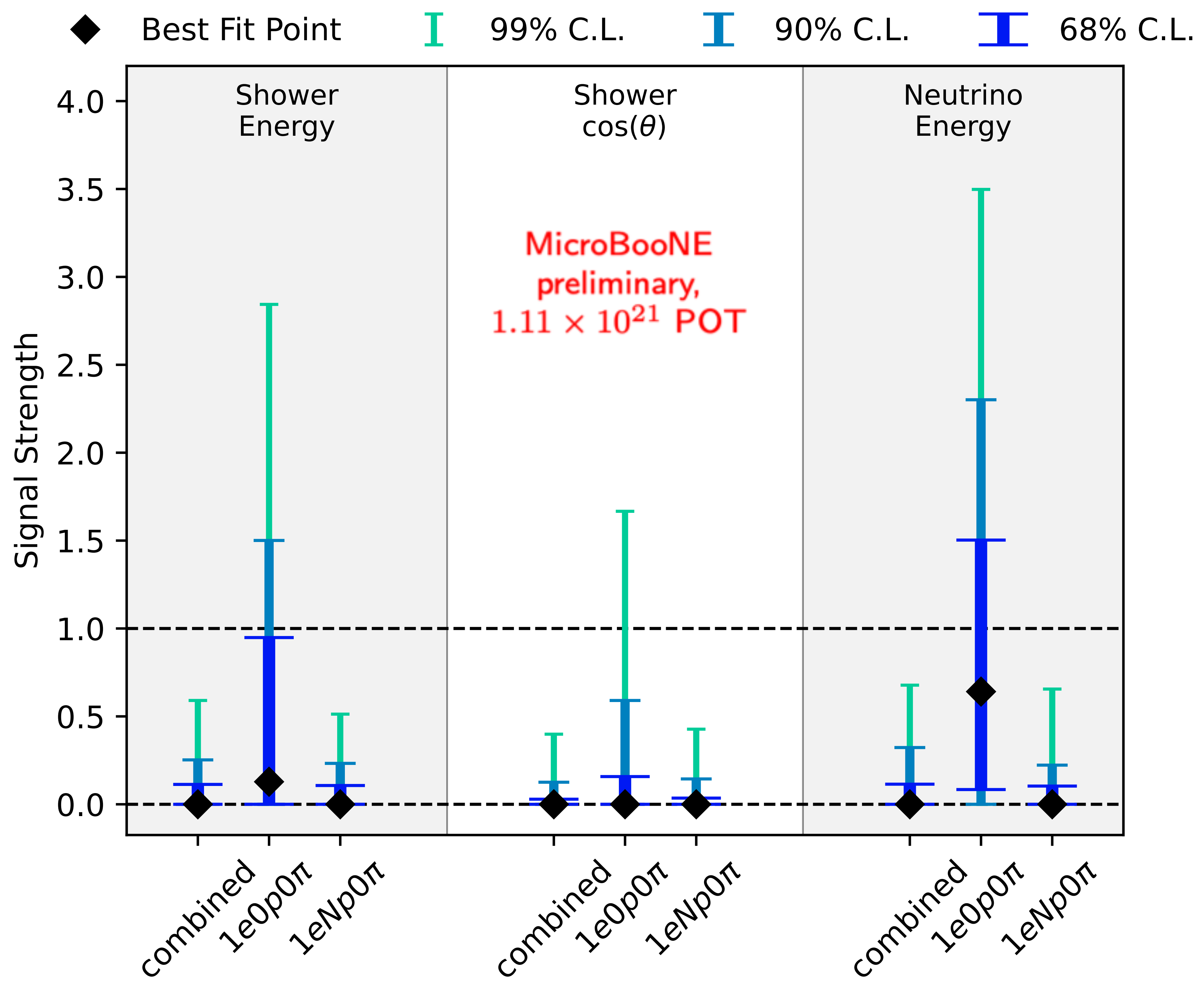
➡ Rejects LEE at 3.5σ in E_{shr}, 3.8σ in $\cos(\theta)$

LEE amplitude fit result

➡ LEE rejected at > 99% CL

Results Summary

- LEE hypothesis tested for the first time using all five runs of MicroBooNE
- LEE hypothesis is rejected at
  - ➡ 2.5σ when using neutrino energy model with 2σ upper limit at x=0.47
  - ➡ 3.5σ (3.8σ) when using kinematic signal model binned in shower energy (angle) with 2σ upper limits at x=0.39 (x=0.22)
- Observed histograms largely compatible with MC prediction within statistical + systematic uncertainties with p-values consistently > 10%

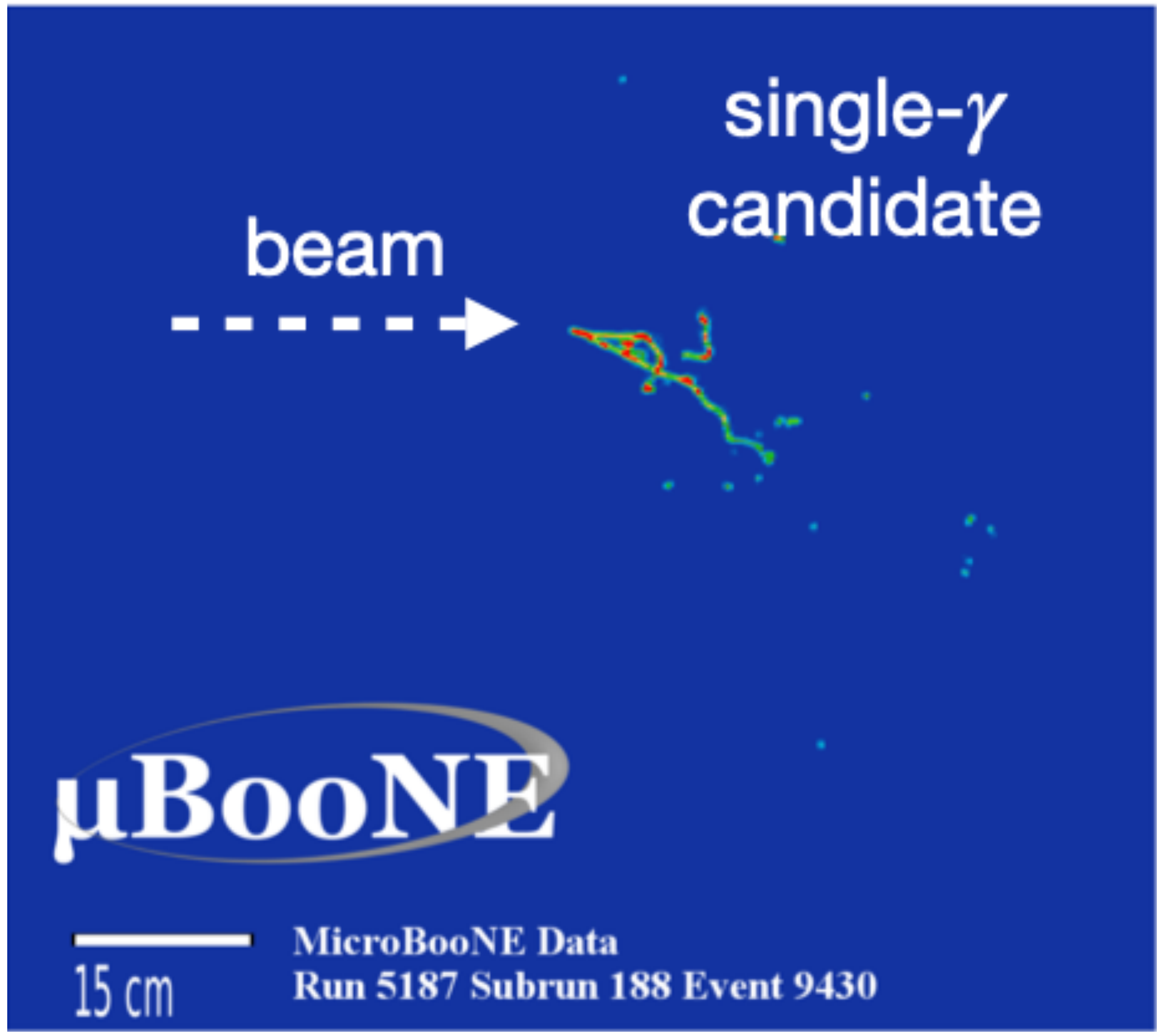


FC-corrected limits on the LEE amplitude from all signal models and channels tested.

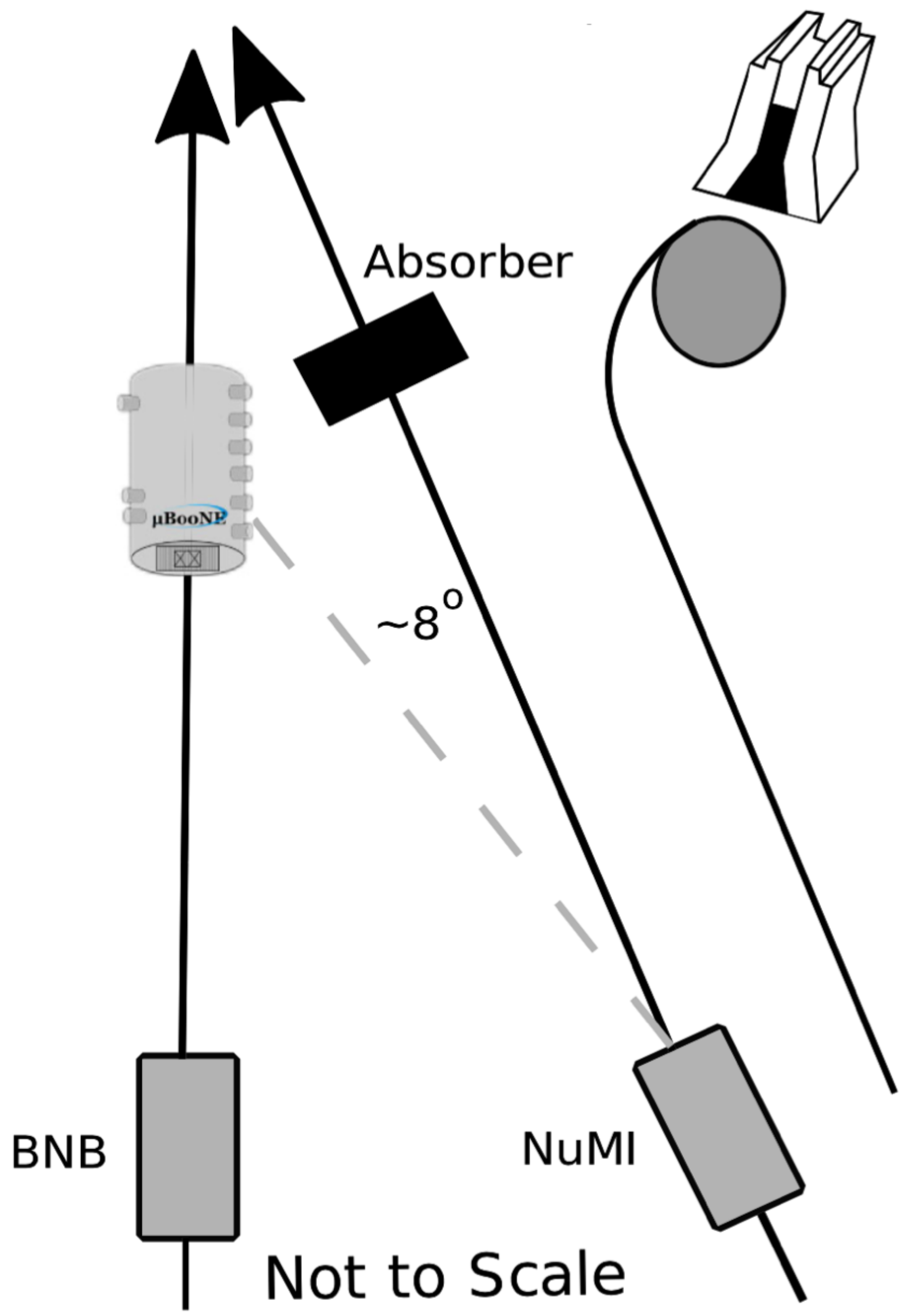
Upcoming Analyses

Single-photon LEE searches

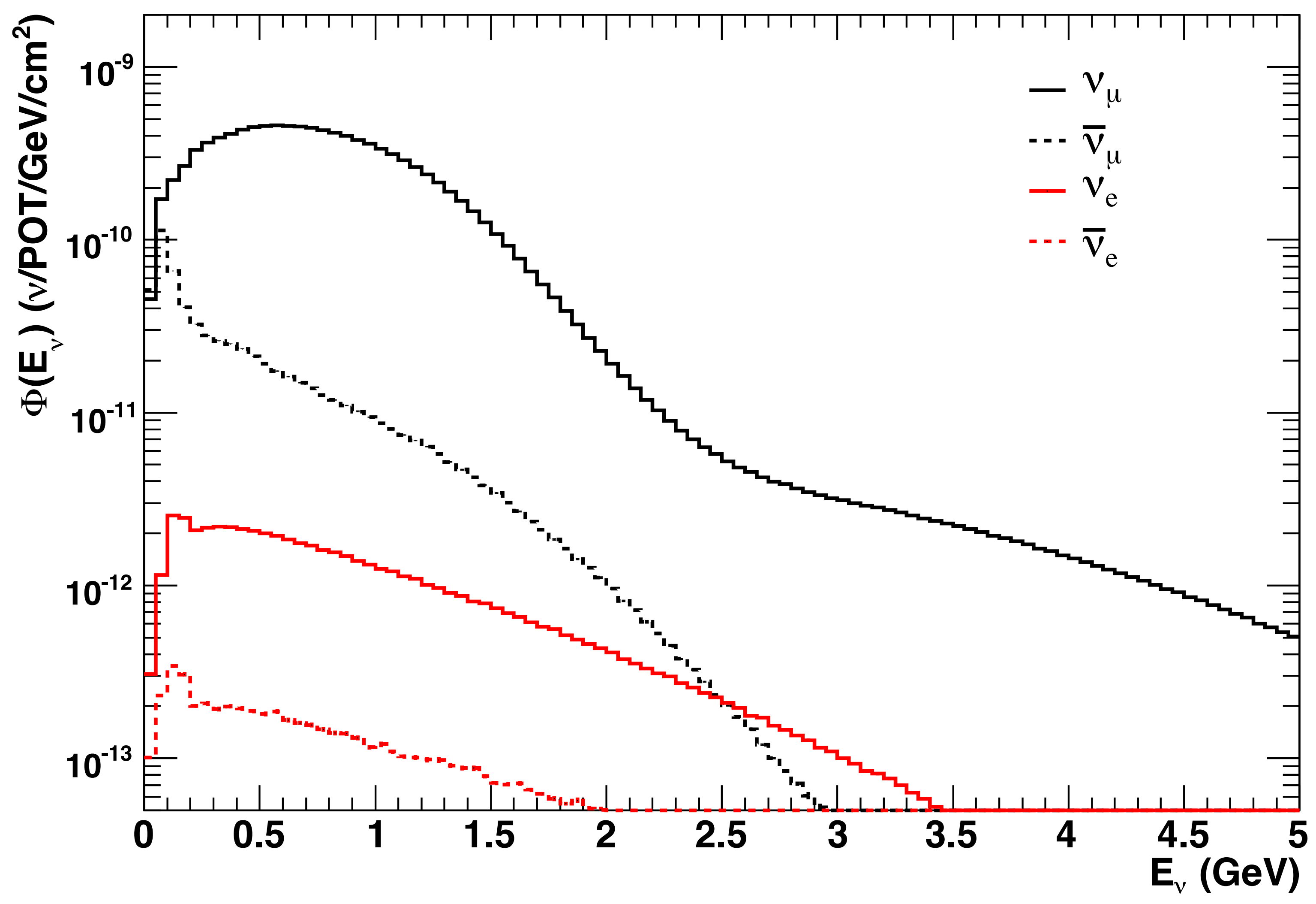
- Past search for photon LEE origins assumed NC π° processes as signal origin
  - ➡ Future analysis will search for excessive photons from any source
- Three new analyses:
  - ➡ NC coherent production: MICROBOONE-NOTE-1131-PUB
  - ➡ Inclusive single-photon search: MICROBOONE-NOTE-1125-PUB
  - ➡ Updated NC π° → Nɣ search: MICROBOONE-NOTE-1126-PUB



One Detector, Two Beams



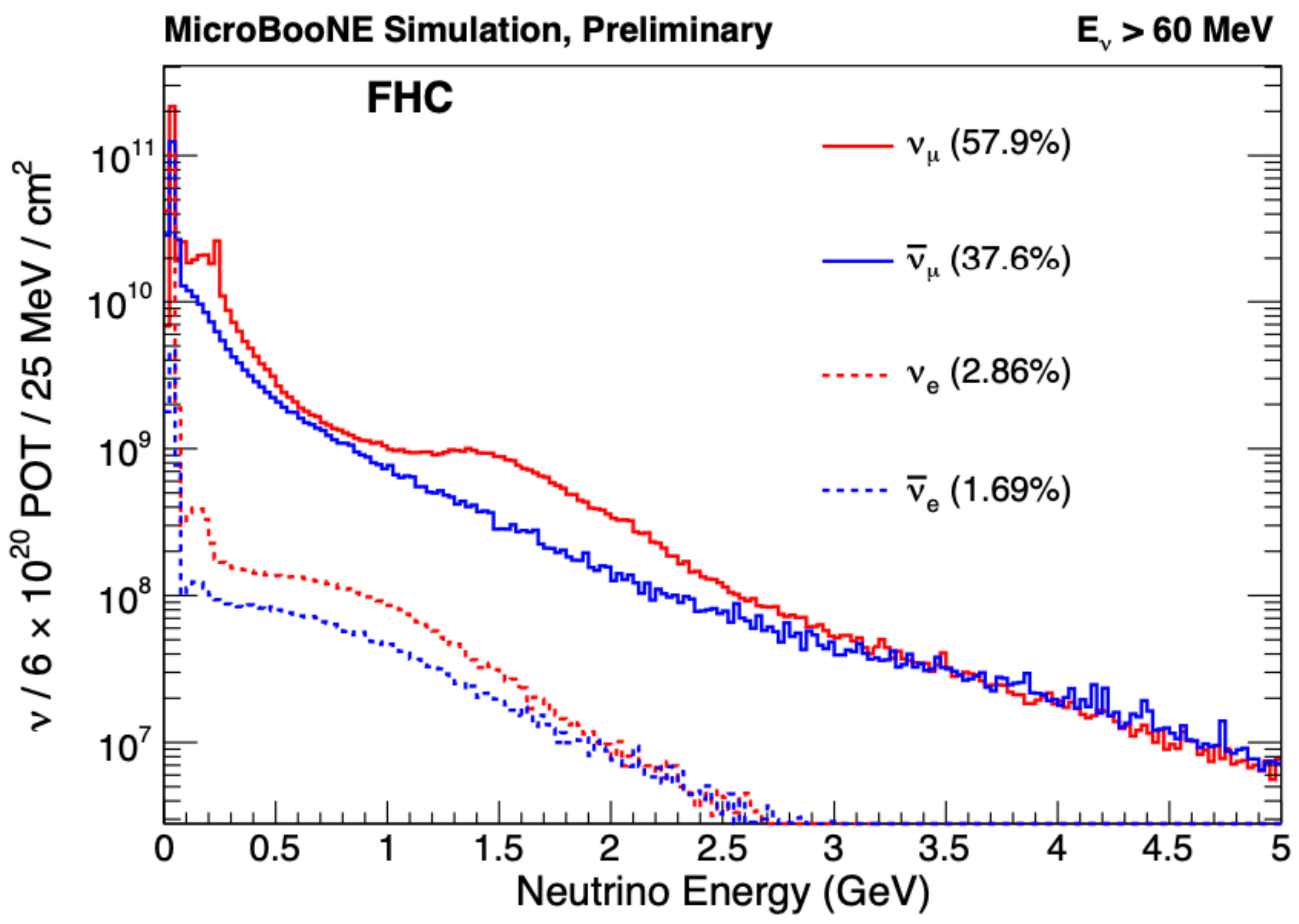
BNB Neutrino Beam



Total predicted flux at the MiniBooNE detector by neutrino species with horn in neutrino mode. Phys.Rev.D 79 (2009) 072002

- Peak energy: 700 MeV
- 99.5% ν_μ/ 0.5% ν_e
- On-axis

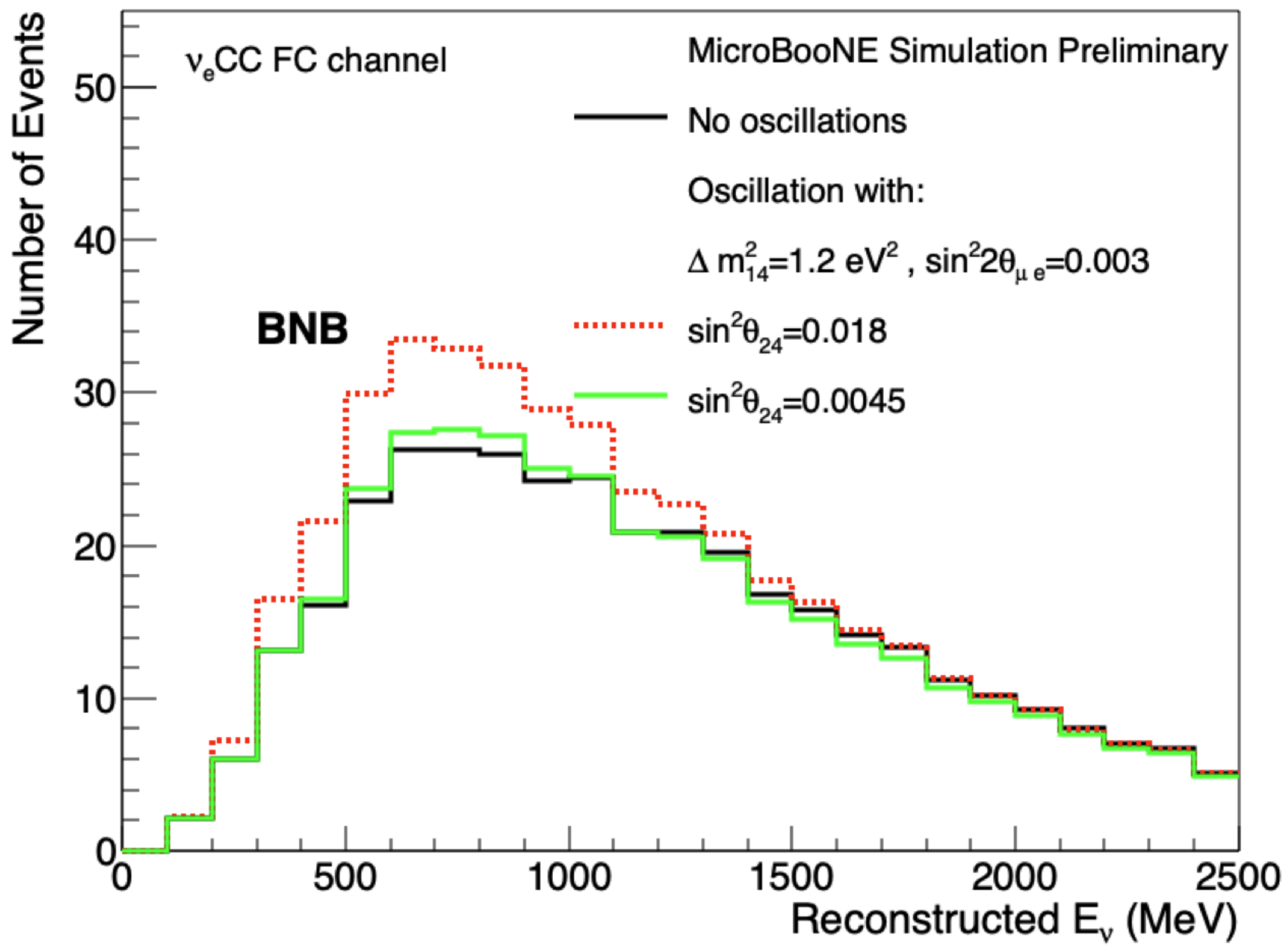
NuMI Beam

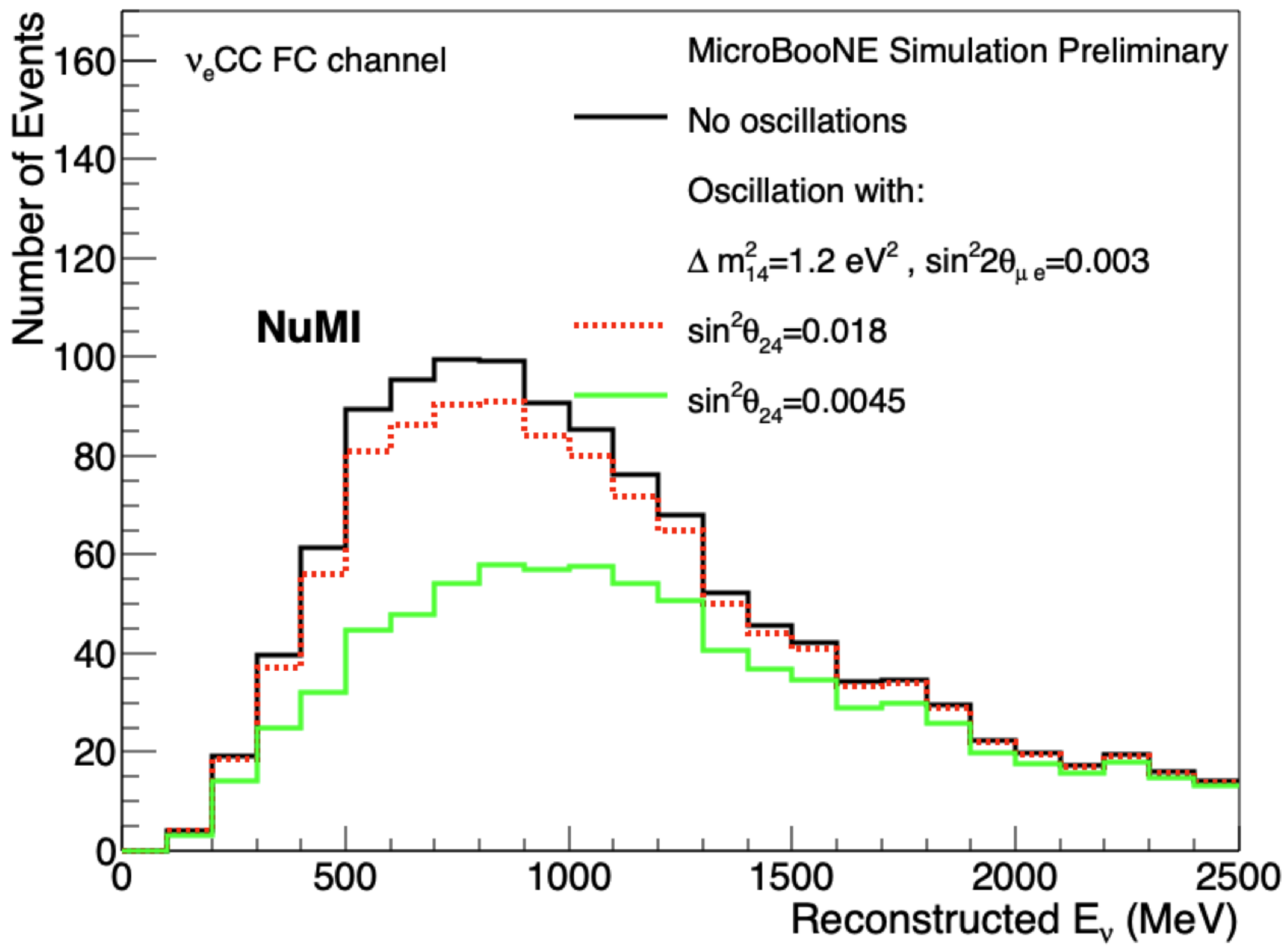


- 95% ν_μ / 5% ν_e
- 8° off-axis
- Flux from target and absorber

3+1 Sterile Neutrino Analysis

Breaking the degeneracy with two beams

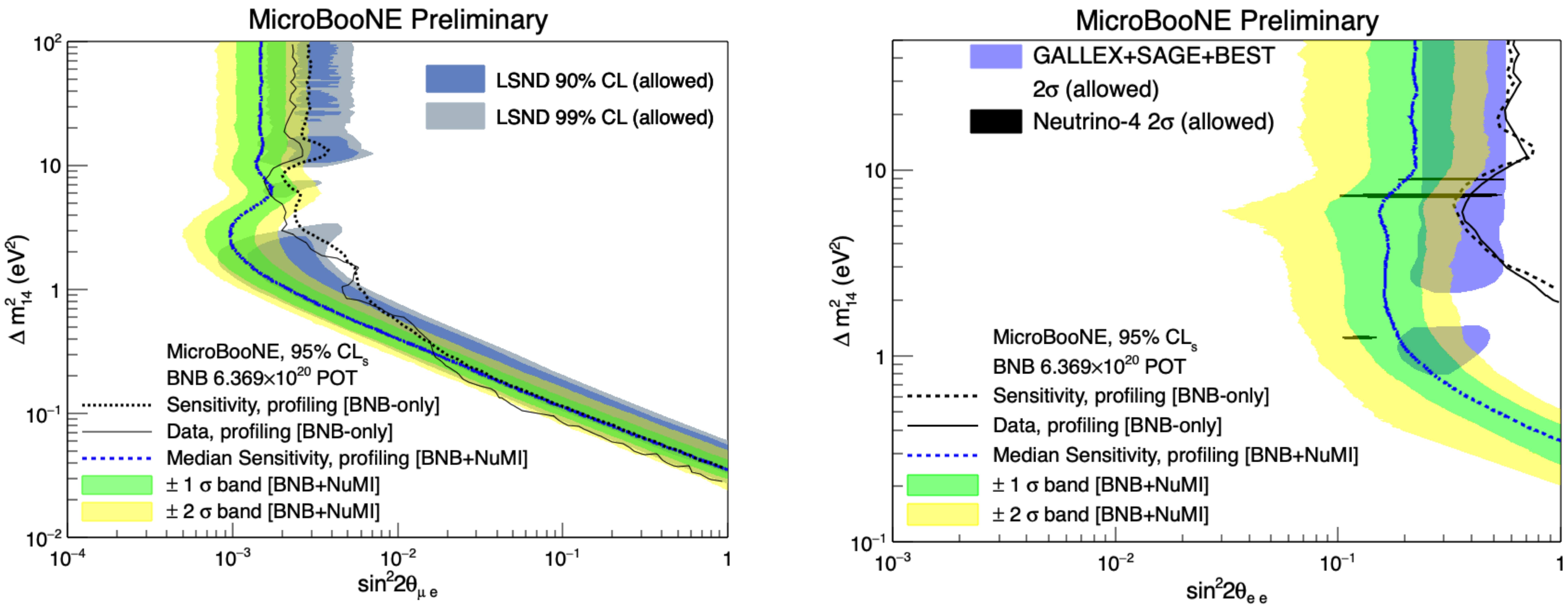




- Oscillation effect in 3+1 model can be hidden in appearance/disappearance degeneracy when using only one beam
- Adding NuMI beam breaks degeneracy (different ν_e/ν_μ mixture)

3+1 Sterile Neutrino Analysis

Expected Sensitivity

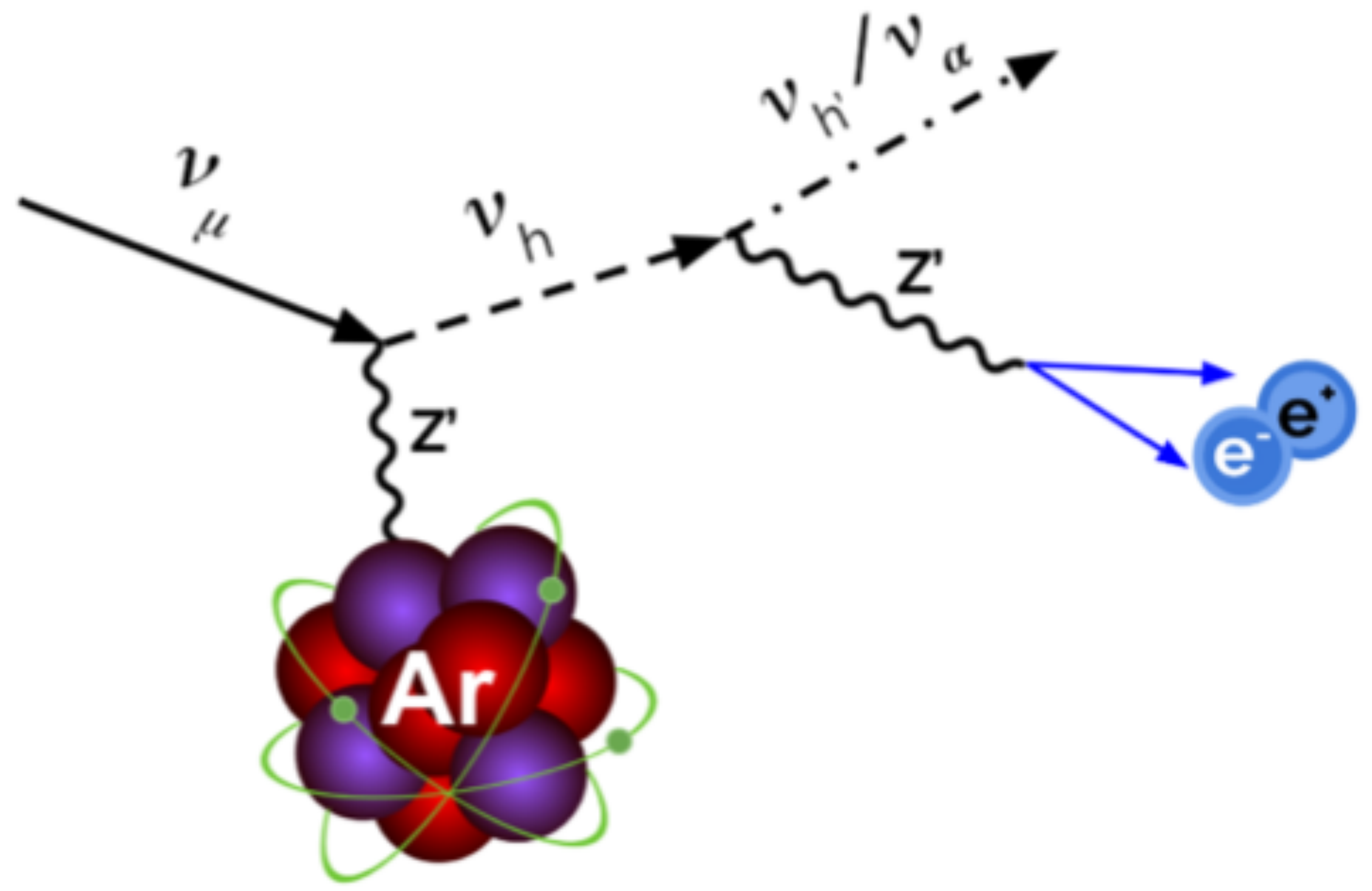


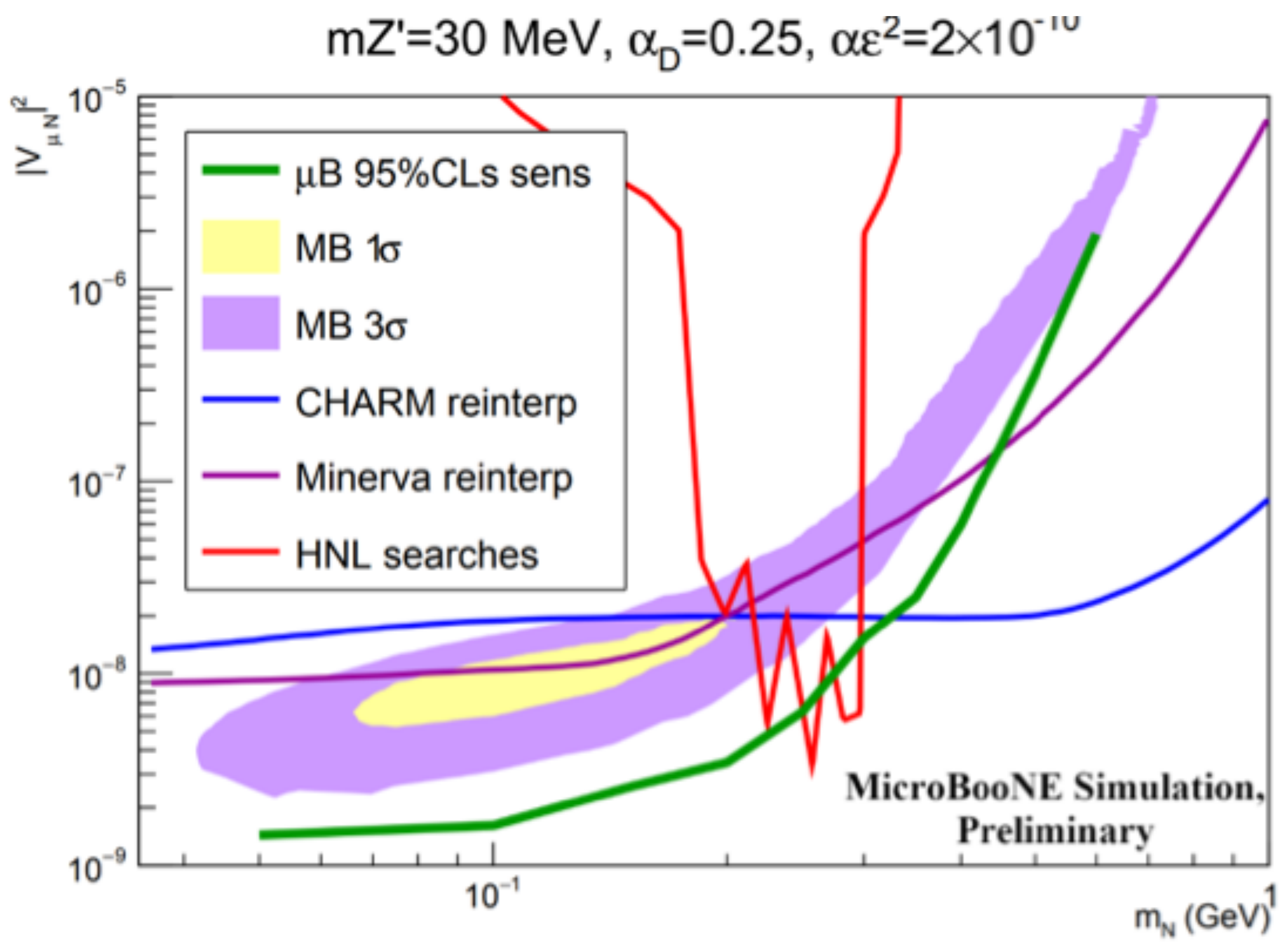
- Sensitivity expected to be sufficient to exclude parameter space preferred by LSND, Gallium anomaly
- More information in public note: MICROBOONE-NOTE-1132-PUB
- Results to be made public soon!

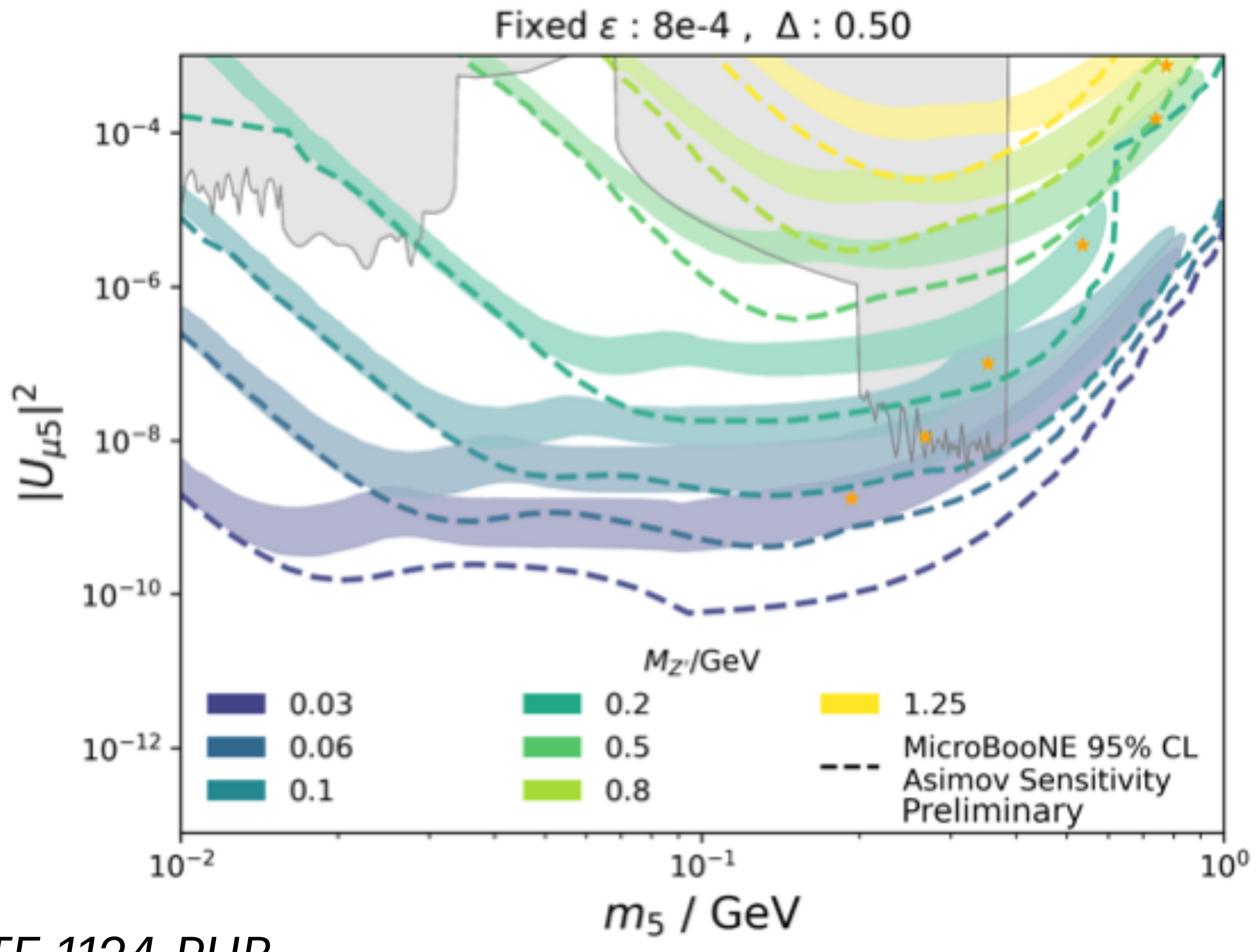
BSM Hypotheses to Explain the LEE

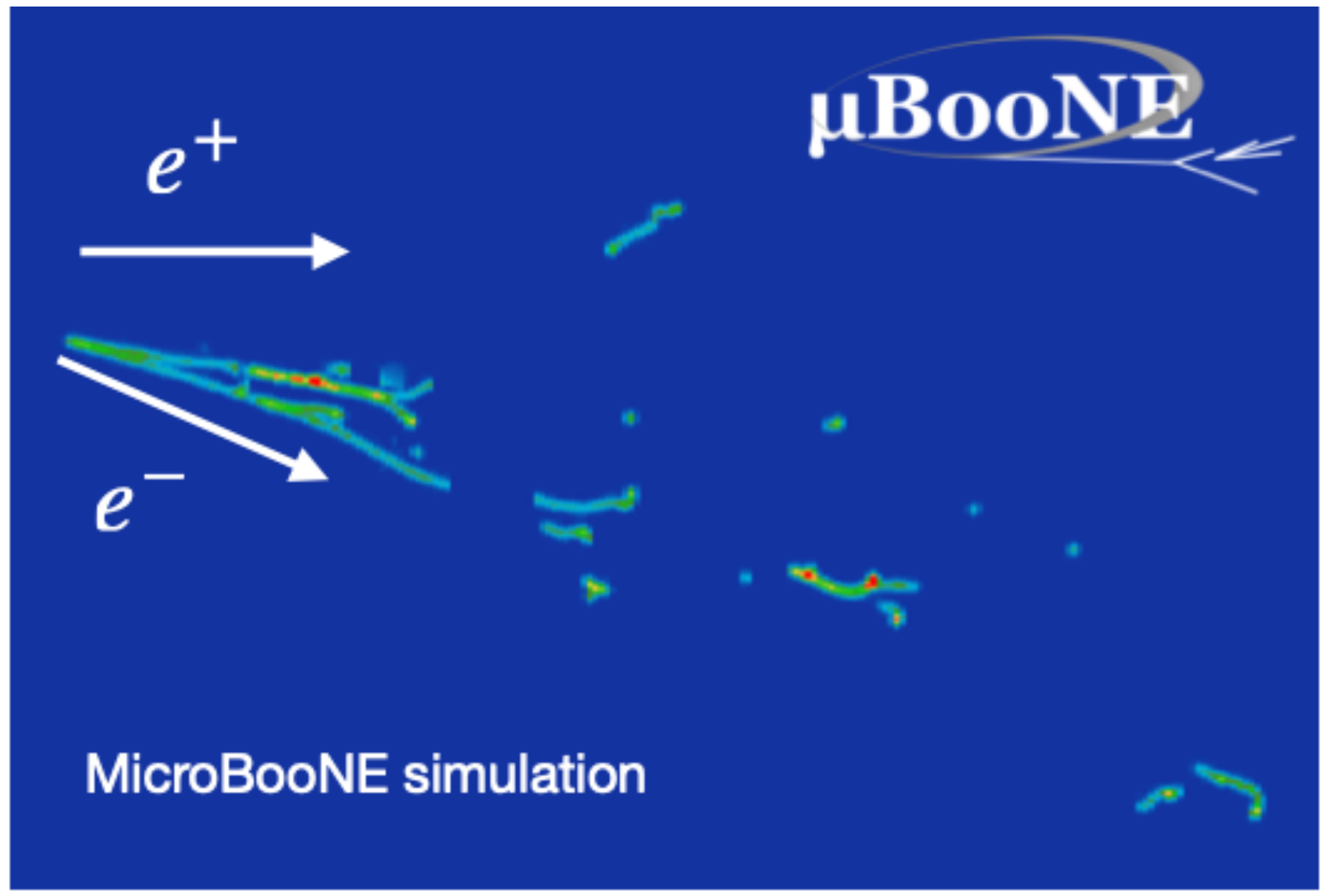
Collimated e+/e- pairs

- Proton/antiproton pairs would look identical to MB (fuzzy ring) if sufficiently collimated
- “Dark neutrino” decay proposed as mechanism
- MicroBooNE expected to exclude MB preferred region at > 95% CL









MICROBOONE-NOTE-1124-PUB

Summary

New results using the full dataset for the first time

- Repetition of low-energy electron neutrino excess search based on Pandora reconstruction toolkit
- Improved statistics and other innovations greatly increased significance of results
- Electron neutrino interpretation of MiniBooNE excess *rejected* at 2.5σ CL_s or higher

Upcoming analyses

- Powerful 3+1 sterile search with sensitivity to reject long-baseline anomaly parameter space
- Extensive tests of wide range of possible explanations for MiniBooNE excess

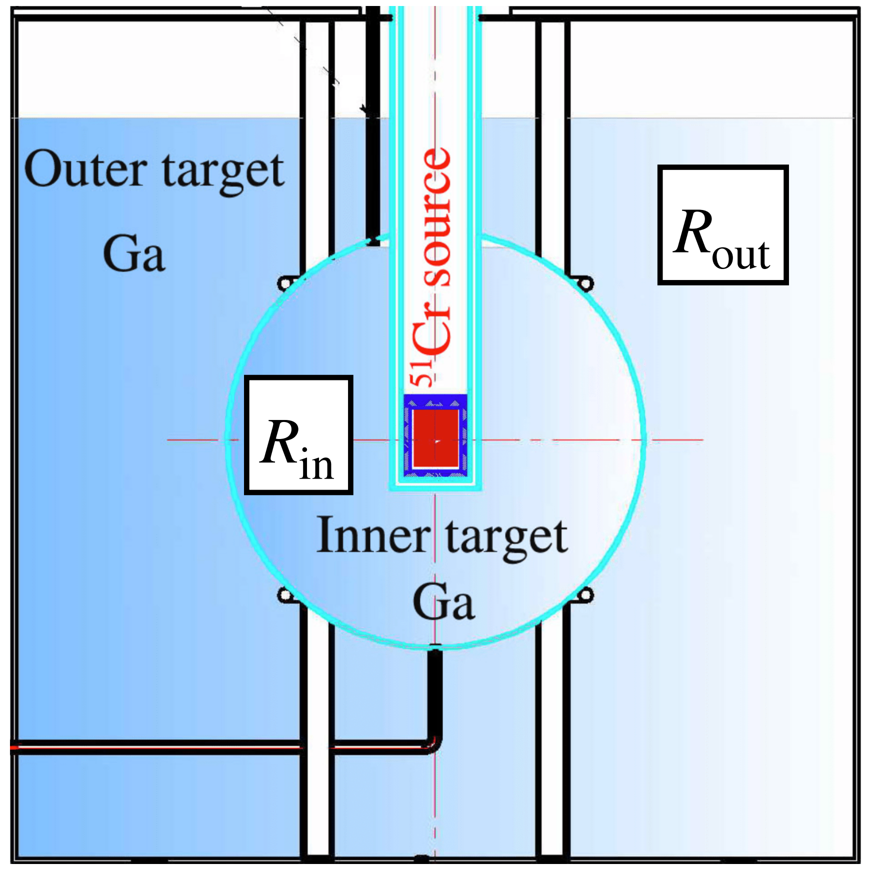
There is a lot to learn from the full MicroBooNE dataset, and we are only getting started!


MicroBooNE collaboration meeting
Michigan State University, May 2024

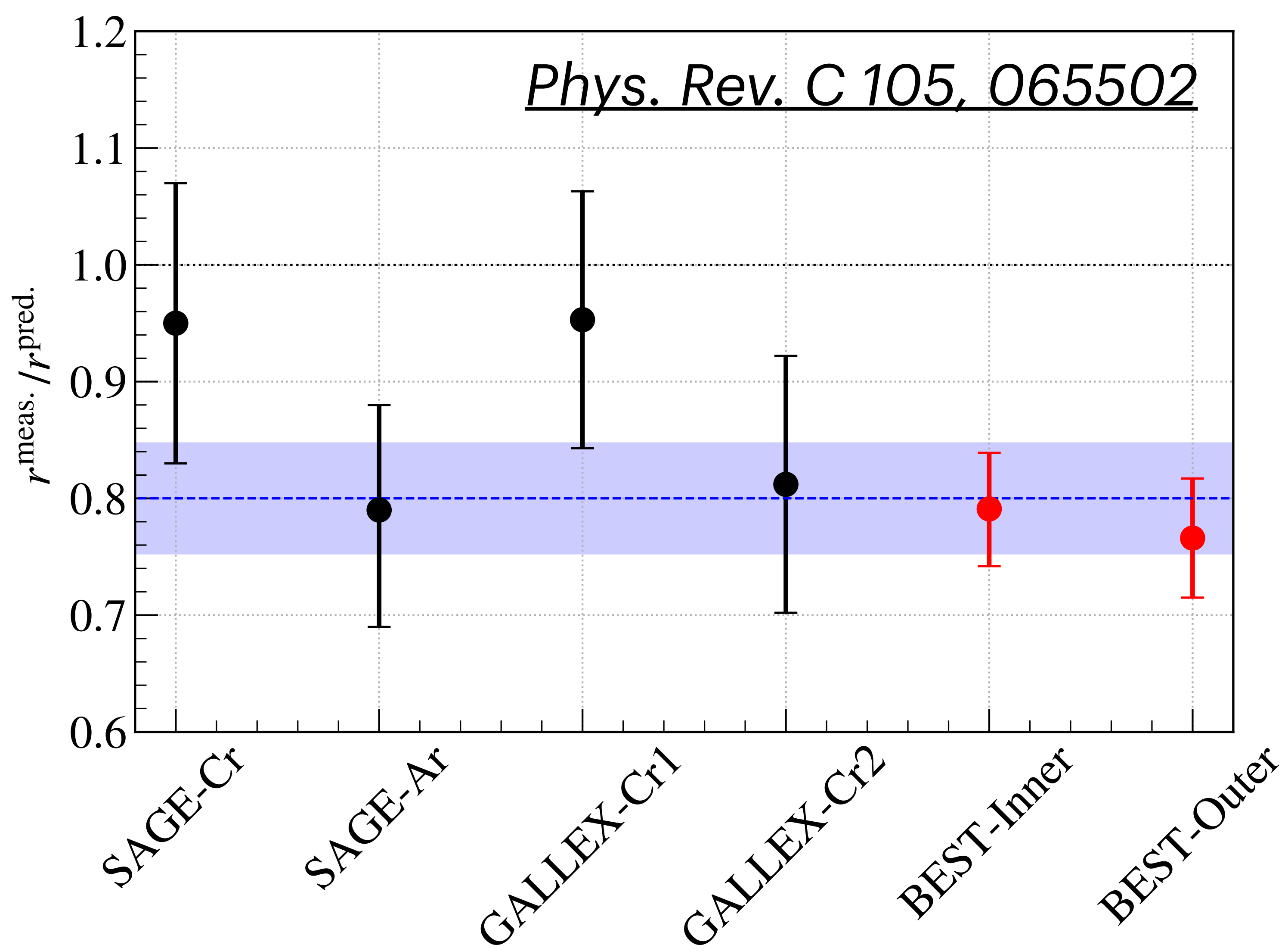
Thank you!

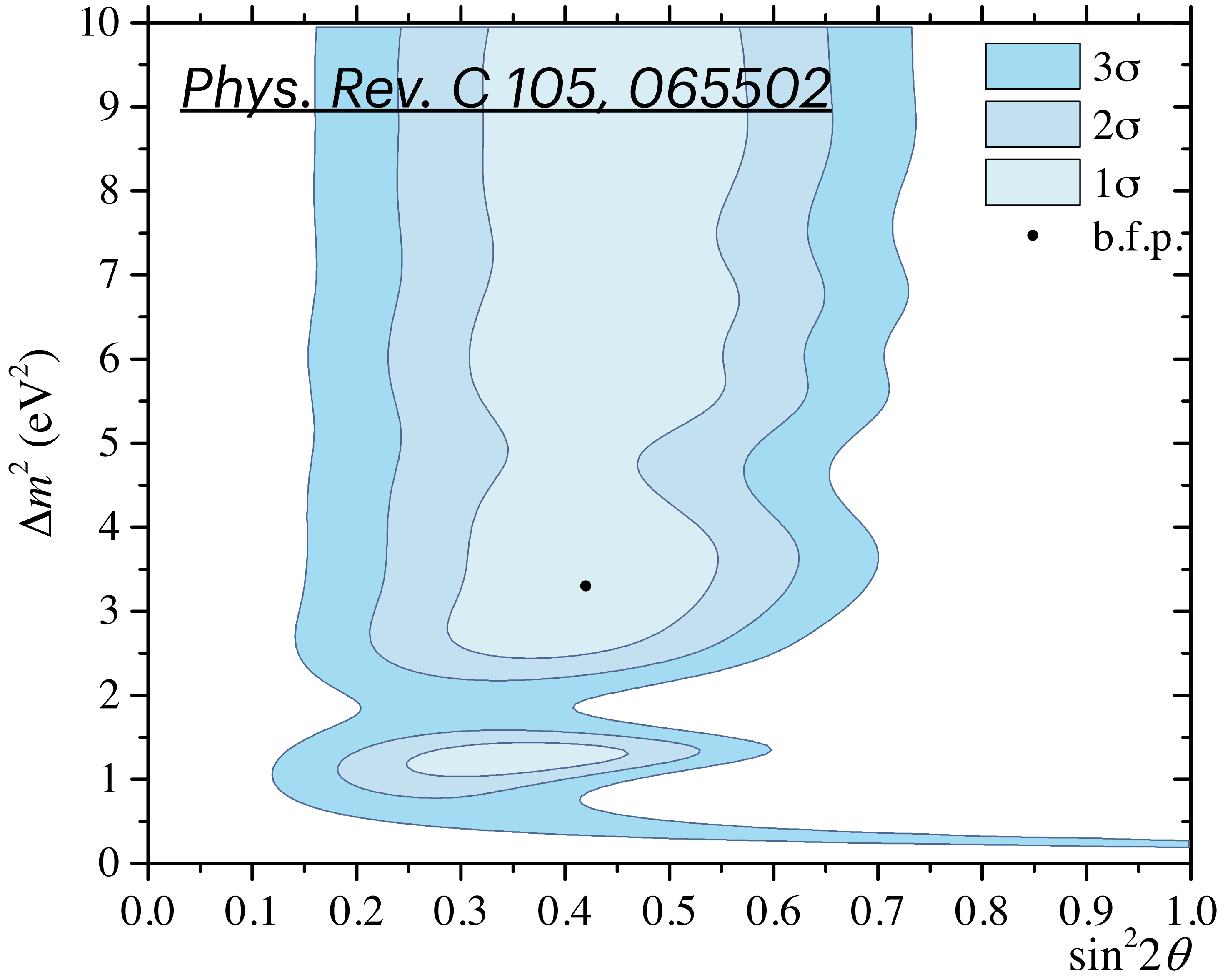
Backup

Gallium Anomaly



Experimental setup of the BEST experiment. Phys.Rev.Lett. 128 (2022) 23, 232501

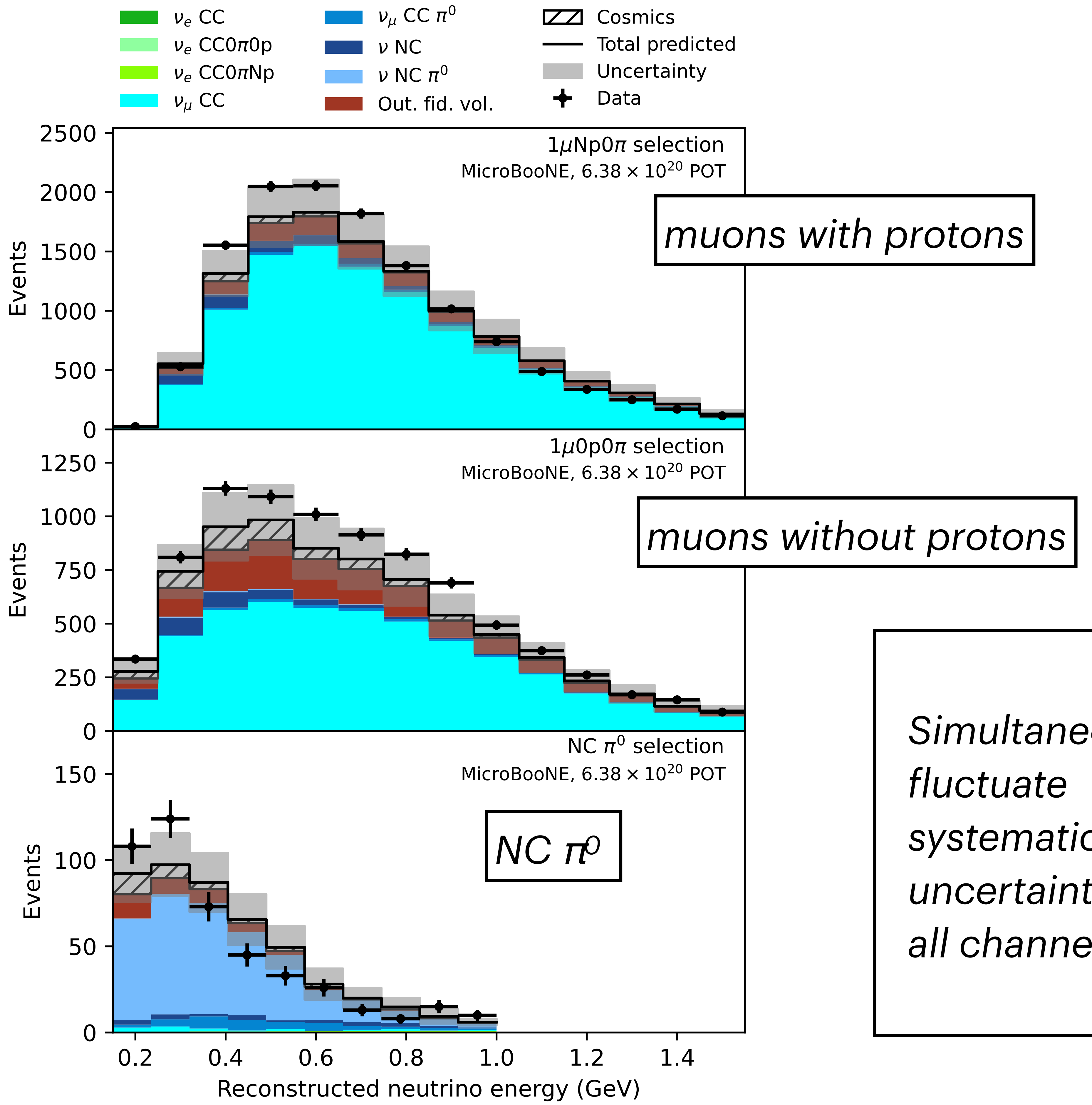




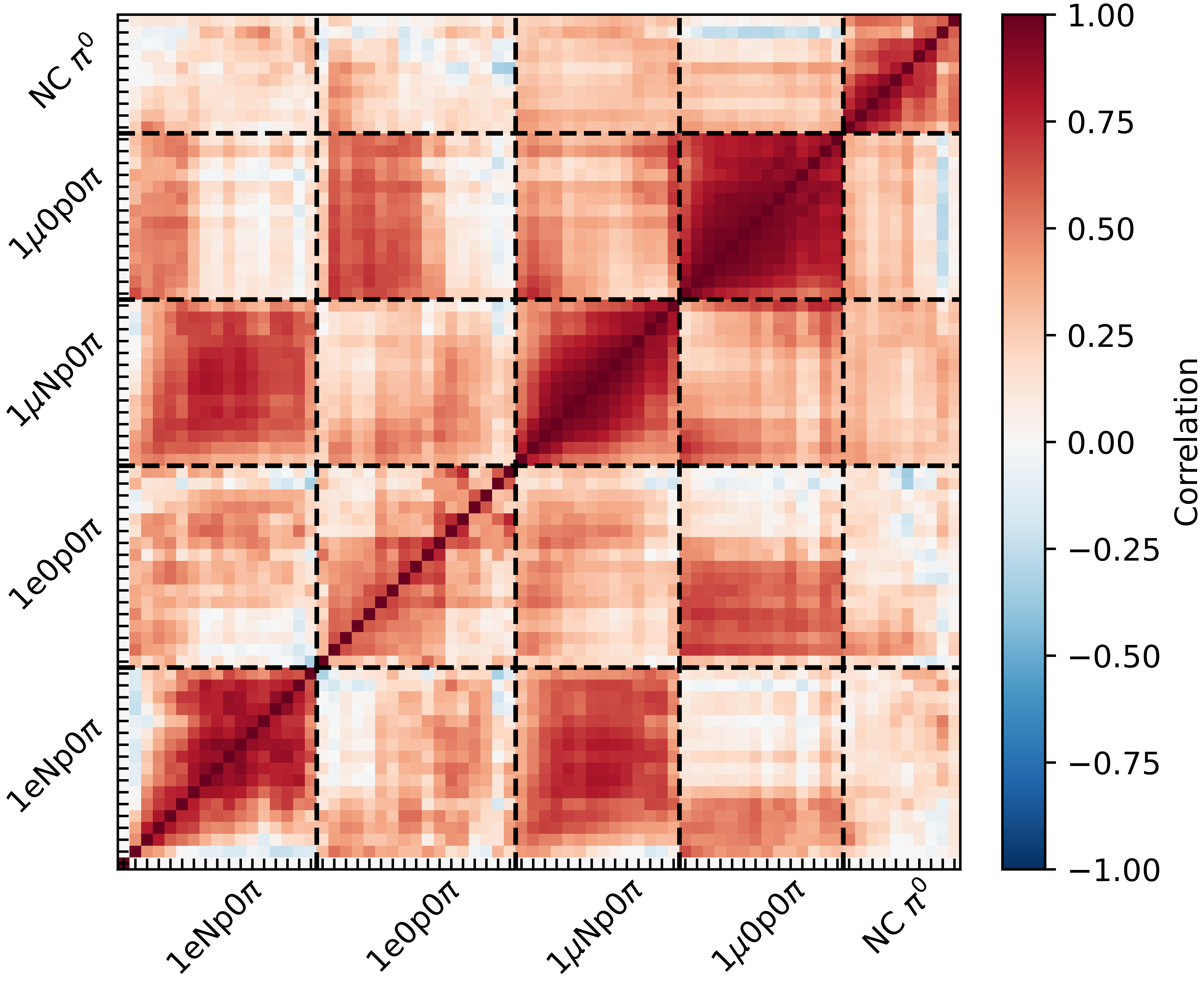
- Deficit in $\nu_e \to \nu_e$ channel of ~20%
- If interpreted in 3+1 model, then $\Delta m^2_{41} > 1 \text{ eV}^2$ and $\sin^2 2\theta_{ee} = 4|U_{e4}|^2\left(1 - |U_{e4}|^2\right) \approx 0.4$

Sideband Constraints

Using muon data to update the electron neutrino prediction



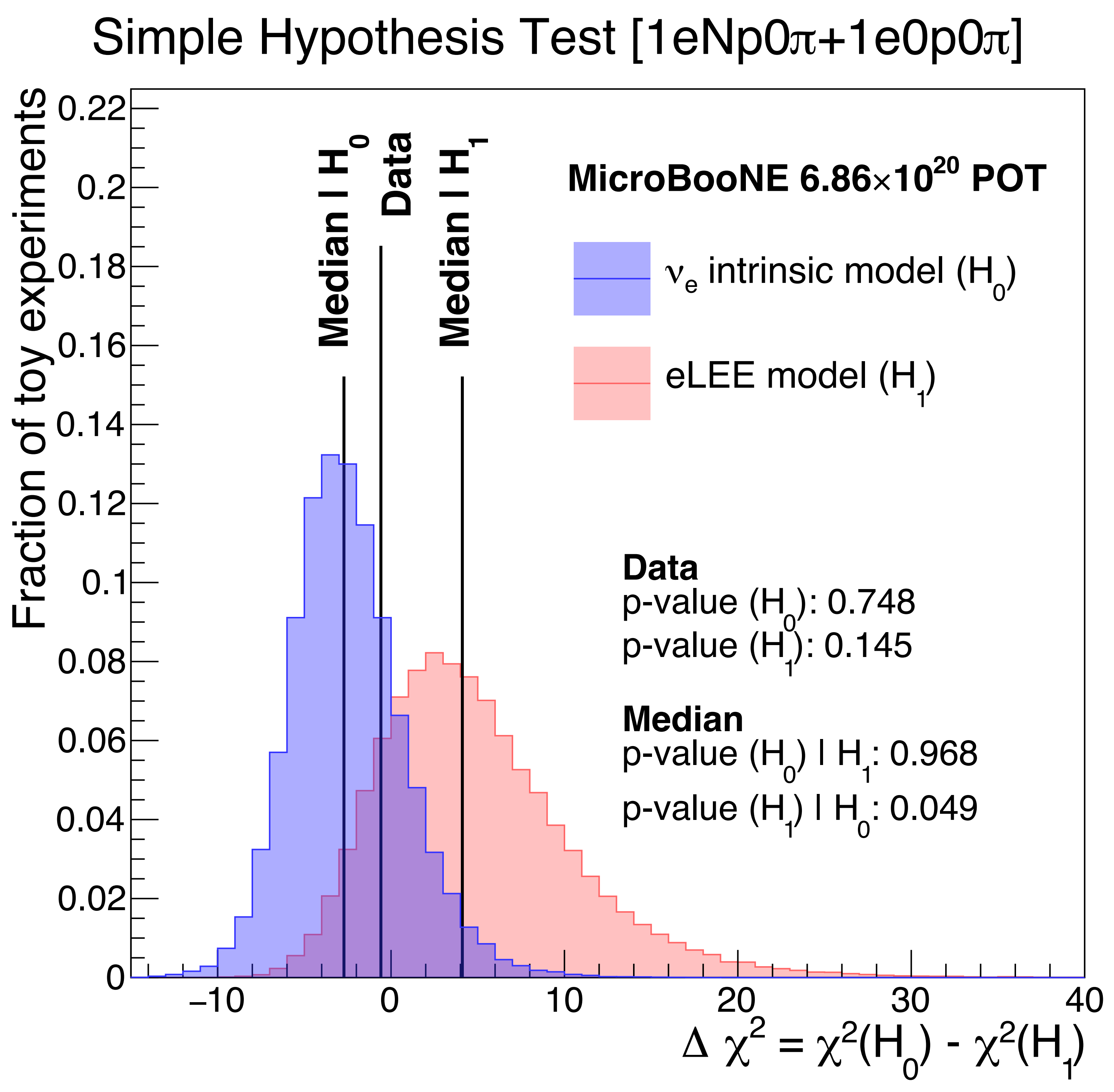
Simultaneously fluctuate systematic uncertainties in all channels



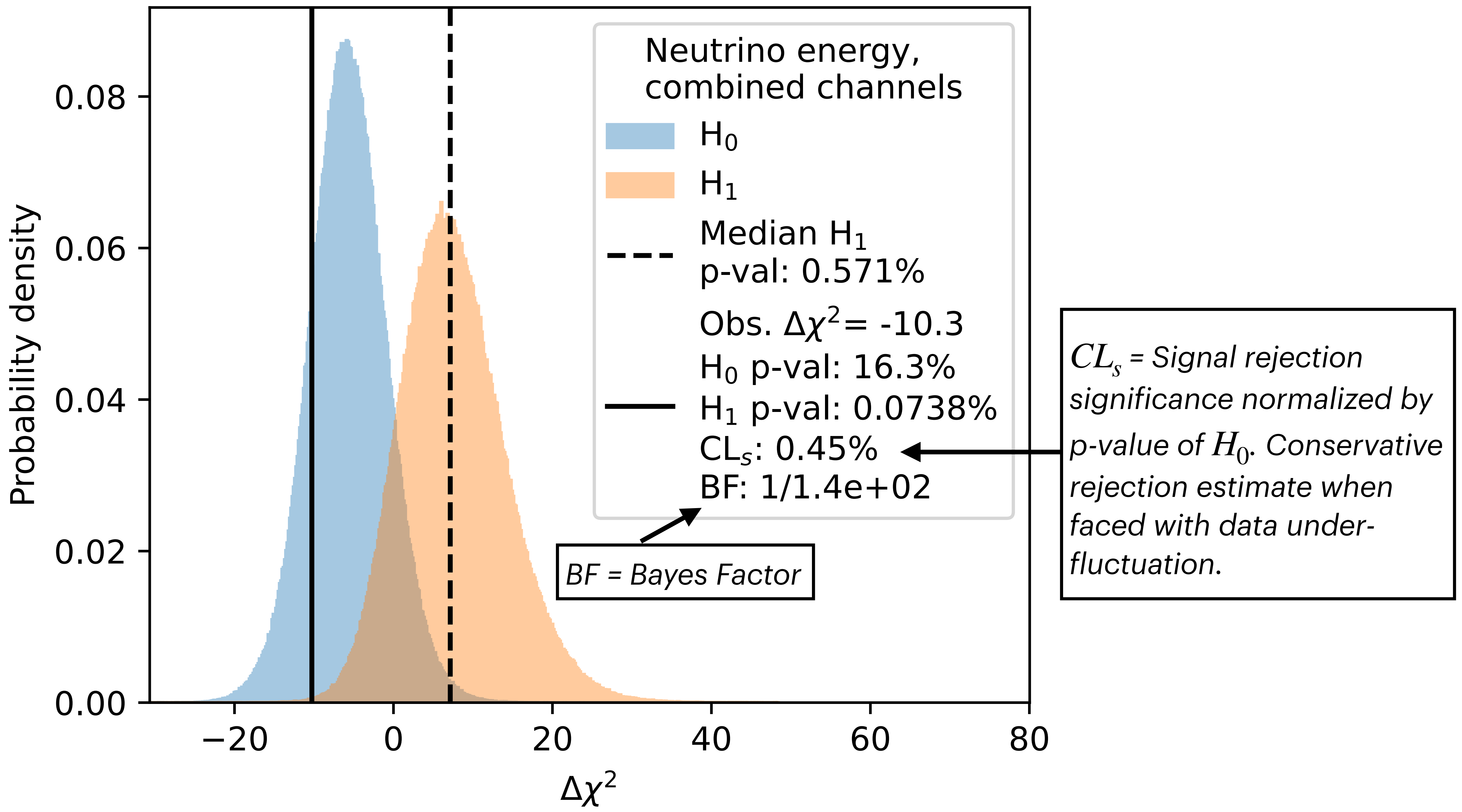
Correlation between all signal and sideband channels of the analysis.

Comparison of Statistical Power

First analysis, runs 1-3



This analysis, runs 1-5



- Greatly increased rejection significance of the LEE hypothesis (H_1) with respect to the first round of the analysis!

Old Analysis Results

Tables from Phys. Rev. D 105 (2022) 11, 112004

TABLE IV. Summary of the simple hypothesis tests. Reported p value (H_0) [p value (H_1)] results reflect the probability for the H_0 (H_1) hypothesis to give $\Delta\chi^2 = \chi^2(H_0) - \chi^2(H_1)$ smaller than the observed value. The observed value of $\chi^2(H_0)$ is reported in Table III. The median sensitivity in terms of these p values is also reported under the assumption that the eLEE model H_1 (no-signal scenario H_0) is true. The fraction of toy experiments generated under the H_0 hypothesis with $\Delta\chi^2$ larger than the median value obtained for the eLEE model H_1 is $1 - p$ value (H_0) so the combined $1eNp0\pi + 1e0p0\pi$ median sensitivity to reject H_0 if H_1 is true is 0.032.

Channel	Obs. $\Delta\chi^2$	$\Delta\chi^2 <$ obs. p value (H_0)	$\Delta\chi^2 <$ obs. p value (H_1)	Sensitivity p value $(H_0)\vert H_1$	Sensitivity p value $(H_1)\vert H_0$
$1eNp0\pi$	−3.89	0.285	0.021	0.957	0.061
$1e0p0\pi$	3.11	0.984	0.928	0.759	0.249
$1eNp0\pi + 1e0p0\pi$	−0.58	0.748	0.145	0.968	0.049

TABLE V. Best-fit eLEE model signal strength (μ) and 90% confidence intervals. The sensitivity is quantified by reporting the expected upper limits assuming $\mu = 0$.

Channel	Data μ_{BF}	Data 90% CL interval on μ	Sensitivity 90% upper limit on μ
$1eNp0\pi$	0.00	[0.00, 0.82]	1.16
$1e0p0\pi$	4.00	[1.13, 15.01]	3.41
$1eNp0\pi + 1e0p0\pi$	0.36	[0.00, 1.57]	1.07

Two-Hypothesis Test Results

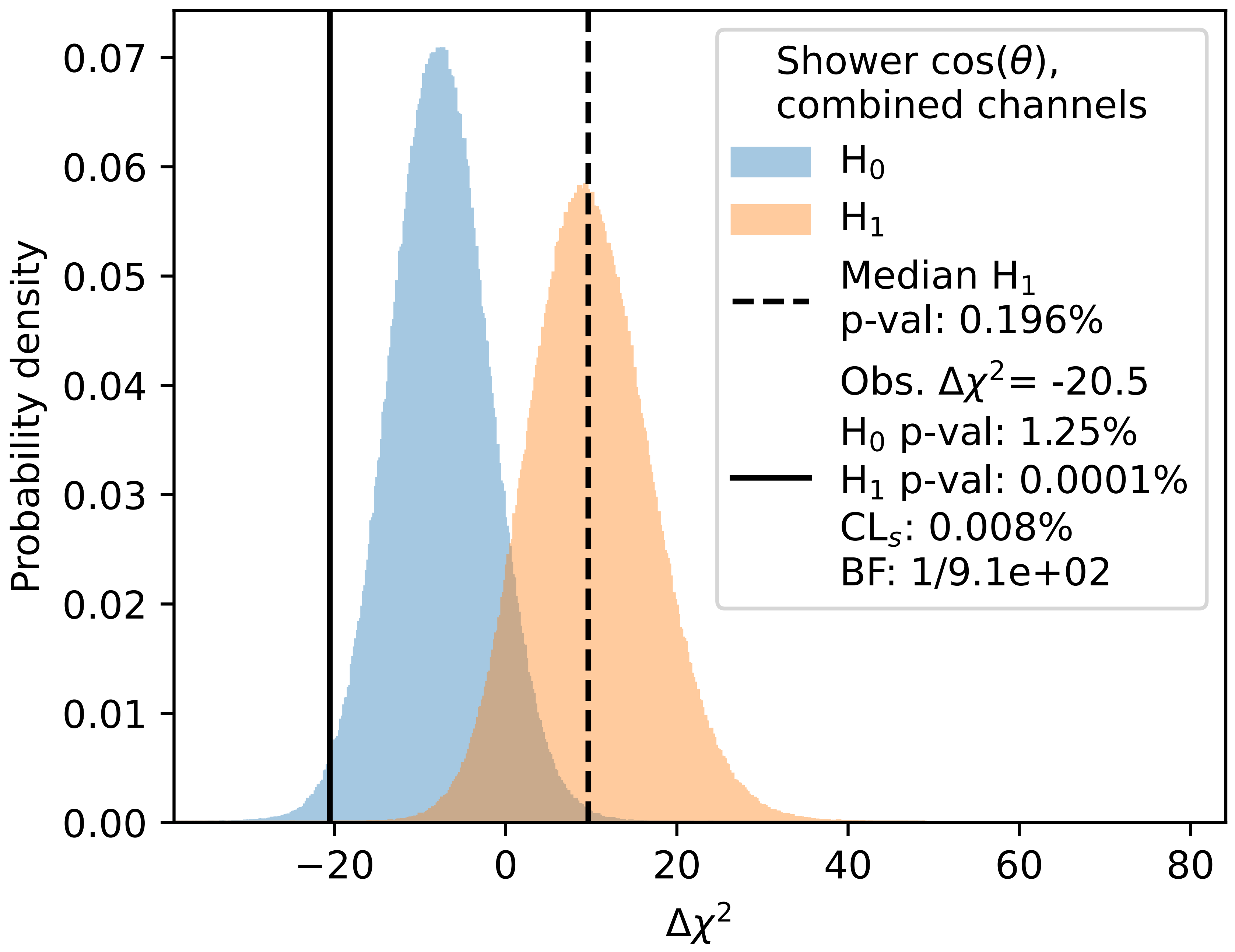
Neutrino Energy Model

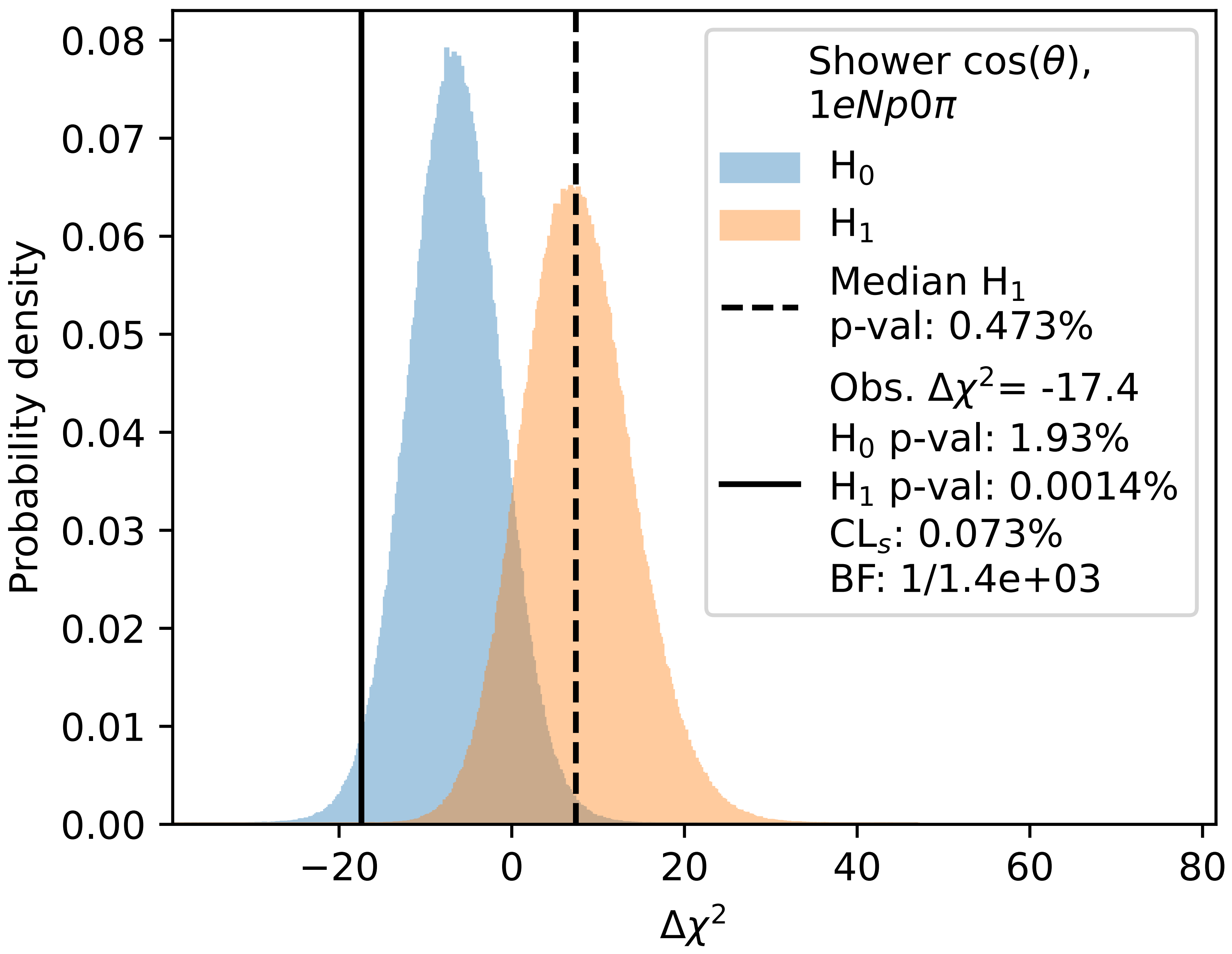
Two-Hypothesis Test Results

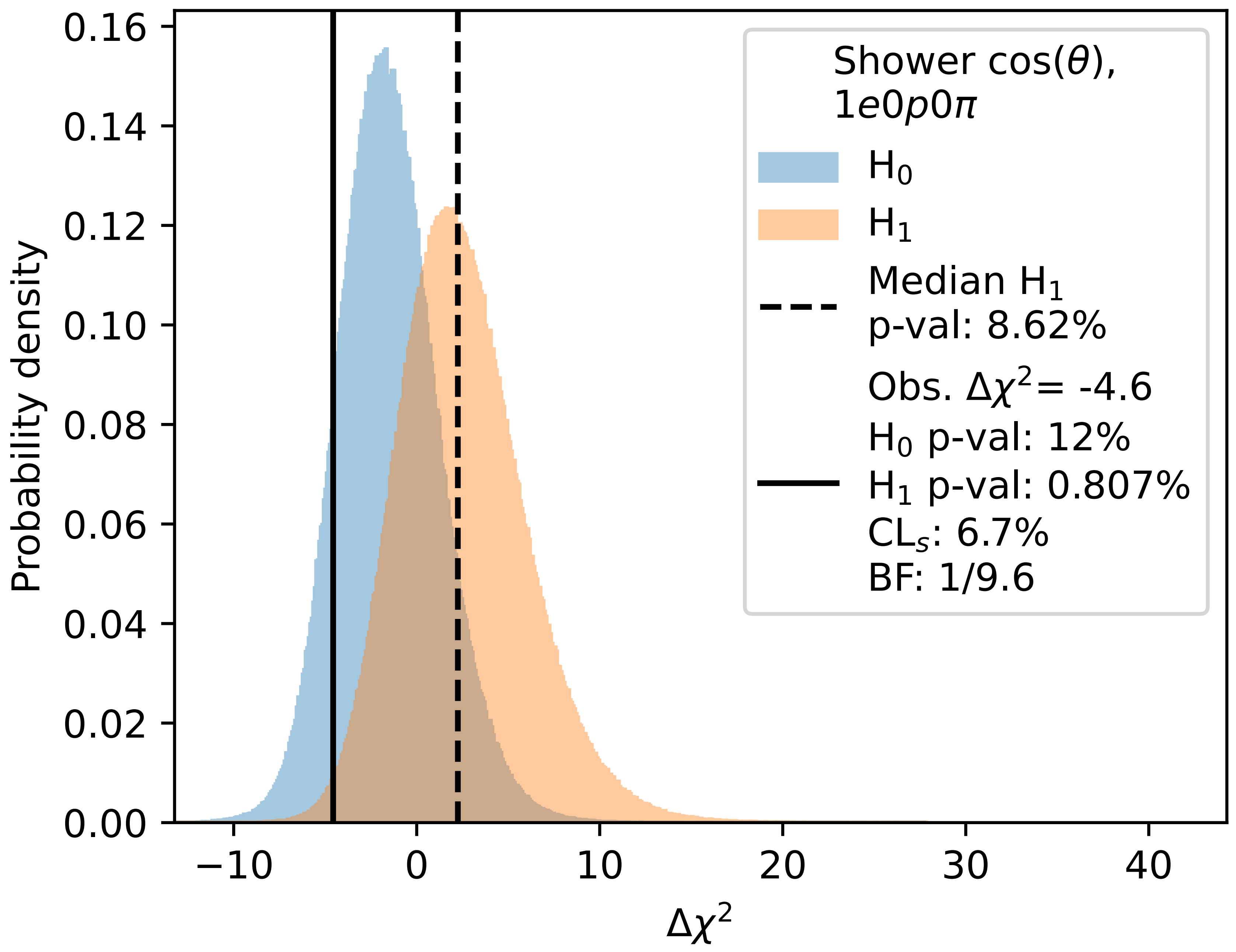
Kinetic Signal Model, Shower Energy Binning

Two-Hypothesis Test Results

Kinetic Signal Model, Shower Angle Binning







Statistical Test Results

TABLE I. Results with data corresponding to 1.11×10^{21} POT. The first three rows show the χ^2 between the data and the null hypothesis after constraint (H_0) and its corresponding p-value. Rows 4 through 8 show the results of the two-hypothesis test in which H_0 is compared to the signal model hypotheses (H_1). The median sensitivity gives the confidence level at which we would be able to reject the null hypothesis at the median $\Delta\chi^2$ expected under H_1. Finally, the confidence level for rejecting H_1 using the CL_s method is reported. The last three rows show the best fit point of the fitted signal strength, μ_{BF}, its upper limit at 2σ CL and the expected upper limit for the case that the data corresponded exactly to the prediction at H_0.

Signal Model	Signal Model 1			Signal Model 2						
Variable	Neutrino Energy			Electron Energy			Electron $\cos(\theta)$			
Channel	$1eNp0\pi$	$1e0p0\pi$	Combined	$1eNp0\pi$	$1e0p0\pi$	Combined	$1eNp0\pi$	$1e0p0\pi$	Combined	Row
observed χ^2	15.0	9.9	24.9	23.3	13.3	35.9	14.4	6.2	19.8	1
ndof	10	10	20	14	14	28	9	9	18	2
$P(\chi^2 >\mathrm{obs.}\|H_0)$ [%]	18.4	56.1	31.3	10.4	62.5	26.0	15.3	77.6	43.4	3
obs. $H_0 - H_1$ $\Delta\chi^2$	-11.3	0.4	-10.3	-15.3	-1.2	-15.2	-17.4	-4.6	-20.5	4
$P(\Delta\chi^2 < \mathrm{obs.}\|H_0)$ [%]	5.5	78.9	16.3	9.5	59.3	16.9	1.9	12.0	1.25	5
$P(\Delta\chi^2 < \mathrm{obs.}\|H_1)$ [%]	0.04	34.2	0.07	0.002	13.8	0.003	0.001	0.8	0.0001	6
Median sensitivity [%]	1.21	11.1	0.57	0.06	8.7	0.03	0.47	8.6	0.20	7
1 - CL_s [%]	0.65	43	0.45	0.023	23	0.02	0.07	6.7	0.008	8
μ_{BF}	0.0	0.6	0.0	0.0	0.14	0.0	0.0	0.0	0.0	9
2σ CL upper limit on μ	0.34	2.64	0.47	0.34	1.90	0.39	0.24	0.88	0.22	10
Exp. 2σ CL limit	1.03	1.88	0.88	0.71	1.80	0.64	0.84	1.80	0.74	11

Covariance Method Proof

Chi-square

Basic χ^2 test statistic:

$$\chi^2 = (\mathbf{x} - \mu)\mathbf{C}^{-1}(\mathbf{x} - \mu)$$

$$\mathbf{\Sigma}_{\text{stat}} \equiv \mathbf{\Sigma}_{\text{MC}} + \text{diag}(\mu)$$

$\chi^2 = \mathbf{y}^T \mathbf{\Sigma}_{\text{eff}}^{-1} \mathbf{y}$

Incorporating Nuisance Parameters into a Chi-Squared Fit

Step 1: Initial Chi-Squared Formulation

Begin with the chi-squared test statistic that includes the nuisance parameter vector θ:

$$\chi^2(\mathbf{y}, \theta) = (\mathbf{y} - \mathbf{G}\theta)^T \mathbf{\Sigma}_{\text{stat}}^{-1} (\mathbf{y} - \mathbf{G}\theta) + (\theta - \theta_0)^T \mathbf{\Sigma}_{\theta}^{-1} (\theta - \theta_0)$$

Here, $\mathbf{y} = \mathbf{x} - \mu$, and $\mathbf{\Sigma}_{\text{stat}}$ is the statistical covariance matrix.

Covariance Method Proof

Step 2: Derive the Best-Fit Point

To find the best-fit value of θ, differentiate χ^2 with respect to θ and set it to zero. The result is:

$$\theta_{\text{best}} = \left(\mathbf{G}^T \mathbf{\Sigma}_{\text{stat}}^{-1} \mathbf{G} + \mathbf{\Sigma}_{\theta}^{-1}\right)^{-1} \mathbf{G}^T \mathbf{\Sigma}_{\text{stat}}^{-1} \mathbf{y}$$

Step 3: Insert the Best-Fit Value into χ^2

Upon inserting the best-fit value and simplifying, we obtain:

$$\chi^2_{\text{best}} = \mathbf{y}^T \left(\mathbf{\Sigma}_{\text{stat}}^{-1} - \mathbf{\Sigma}_{\text{stat}}^{-1} \mathbf{G} (\mathbf{G}^T \mathbf{\Sigma}_{\text{stat}}^{-1} \mathbf{G} + \mathbf{\Sigma}_{\theta}^{-1})^{-1} \mathbf{G}^T \mathbf{\Sigma}_{\text{stat}}^{-1}\right) \mathbf{y}$$

Covariance Method Proof

Step 4: Utilize the Matrix Inversion Lemma

We use the Matrix Inversion Lemma,

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1},$$

and identify the matrices as follows:

- $A \equiv \mathbf{\Sigma}_{\text{stat}}$
- $U \equiv \mathbf{G}$
- $C \equiv \mathbf{\Sigma}_{\theta}$
- $V \equiv \mathbf{G}^T$

  Using the lemma, we can rewrite the effective inverse covariance matrix $\mathbf{\Sigma}_{\text{eff}}^{-1}$ as:

$$\begin{aligned} \mathbf{\Sigma}_{\text{eff}}^{-1} &= \mathbf{\Sigma}_{\text{stat}}^{-1} - \mathbf{\Sigma}_{\text{stat}}^{-1}\mathbf{G}(\mathbf{\Sigma}_{\theta}^{-1} + \mathbf{G}^T\mathbf{\Sigma}_{\text{stat}}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{\Sigma}_{\text{stat}}^{-1} \\ &= \left(\mathbf{\Sigma}_{\text{stat}} + \mathbf{G}\mathbf{\Sigma}_{\theta}\mathbf{G}^T\right)^{-1} \end{aligned}$$

  Here, we see that the term $\mathbf{G}\mathbf{\Sigma}_{\theta}\mathbf{G}^T$ acts as an additional term being added to the covariance matrix.

Covariance Method Proof

Step 5: Identify Additional matrix as covariance of universes

Consider randomly sampling "universes" by varying the parameters according to their Gaussian prior $\theta \sim \mathcal{N}(\theta_0, \mathbf{\Sigma}_\theta)$. The histogram in each "universe" k would then be $\mu_k = \mu + \mathbf{G}\theta_k$.

The covariance matrix of these histograms across different "universes" $\mathbf{\Sigma}_\mu$ would then be:

$$\mathbf{\Sigma}_\mu = \mathrm{E}[(\mu_k - \mu)(\mu_k - \mu)^T] = \mathbf{G}\mathrm{E}[\theta_k \theta_k^T]\mathbf{G}^T = \mathbf{G}\mathbf{\Sigma}_\theta \mathbf{G}^T.$$

This is precisely the term that we identified using the matrix inversion lemma to act as an additional term being added to the covariance matrix $\mathbf{\Sigma}_{\text{stat}}$.

We have therefore proven that, if

$$\chi^2(\mathbf{y}, \theta) = (\mathbf{y} - \mathbf{G}\theta)^T \mathbf{\Sigma}_{\text{stat}}^{-1}(\mathbf{y} - \mathbf{G}\theta) + (\theta - \theta_0)^T \mathbf{\Sigma}_\theta^{-1}(\theta - \theta_0)$$

then

$$\begin{aligned} \min_\theta \chi^2(\mathbf{y}, \theta) &= \mathbf{y}^T \mathbf{\Sigma}_{\text{eff}}^{-1} \mathbf{y} \\ &= \mathbf{y}^T \left(\mathbf{\Sigma}_{\text{stat}} + \mathbf{G}\mathbf{\Sigma}_\theta \mathbf{G}^T\right)^{-1} \mathbf{y} \\ &= \mathbf{y}^T \left(\mathbf{\Sigma}_{\text{stat}} + \mathbf{\Sigma}_\mu\right)^{-1} \mathbf{y} \ , \end{aligned}$$

where $\mathbf{\Sigma}_\mu$ is the systematic covariance matrix obtained by randomly sampling the systematic uncertainties according to their prior distribution.