Flows of geometric structures

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Geometric Structures

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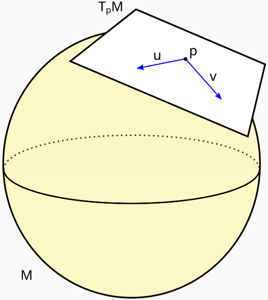

Credit: *Wikipedia*

Figure 1: Tangent space at a point in $\mathbb{S}^2$

An important question is the following:

What other “natural structures” can we put on Riemannian manifolds?

i.e., we would like to attach such a “natural structure” to each tangent space T_pM, for all $p \in M$, in a “smoothly varying” way.

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2) Hermitian structure on a vector space (manifold)
$n = 2m, \rightsquigarrow J \in \Gamma(End(TM))$, $J^2 = -\mathrm{Id}$, J compatible with g,
$\mathrm{U(m)} = \mathrm{SO(2m)} \cap \mathrm{GL(m,\mathbb{C})}$ preserves J.

Berger's list of possible holonomy groups

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n	$\mathrm{SO(n)}$	Generic Riemannian manifold
2m	$\mathrm{U(m)}$	Kähler
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4q	$\mathrm{Sp(q)}$	Hyper-Kähler
4q	$\mathrm{Sp(q) \cdot Sp(1)}$	Quaternionic-Kähler
7	G_2	G_2-holonomy
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- Immersion $i : L \to (M, g)$: "Best" could mean minimal: $H(i) = 0$.

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If we start with some u_0 that doesn't satisfy our "best" condition, we can try to evolve it in time as $u(t)$ in some geometric fashion, to hopefully improve it to be "closer to best". This is a geometric flow:

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Ricci Flow: $\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t))$,
Mean Curvature Flow: $\frac{\partial}{\partial t} i(t) = -H(i(t))$.

Given a geometric flow $\partial_t u(t) = P(u(t))$ with $u(0) = u_0$, we ask:

Q 1. Does it have short time existence and uniqueness? That is, does there exist an $\varepsilon > 0$ and a unique solution on $[0, \varepsilon)$?

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Q 3. If it has LTE, does it converge? That is, does $\lim_{t\to\infty} u(t)$ exist? Some flows exhibit stability: If we start "close enough" to a fixed point, then we have LTE/convergence to the fixed point modulo diffeomorhisms.

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Figure 2: Hamilton

Poincaré conjecture (1903)

Every three-dimensional topological manifold which is closed, connected, and has trivial fundamental group is homeomorphic to the three-dimensional sphere.

RF-contd.

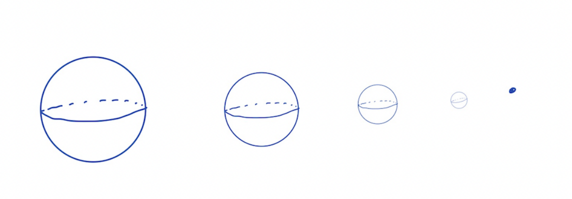
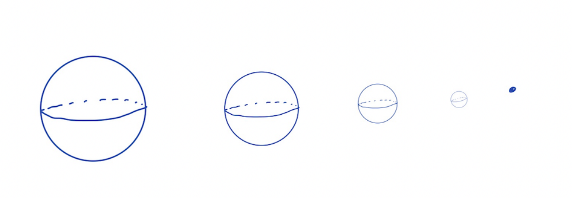


Figure 3: Shrinking $\mathbb{S}^2$ under Ricci flow



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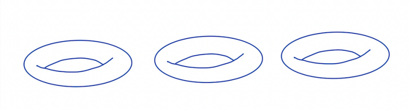
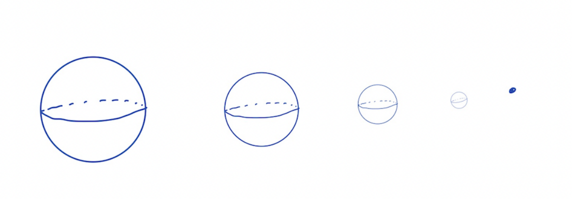


Figure 4: Static flat $\mathbb{T}^2$ under Ricci flow

RF-contd.



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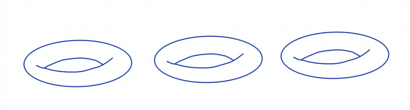

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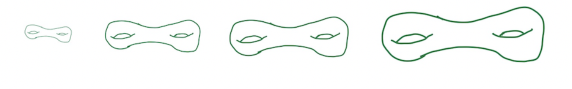


Figure 5: Expanding $\mathbb{T}^2 \# \mathbb{T}^2$ under Ricci flow

RF-contd.

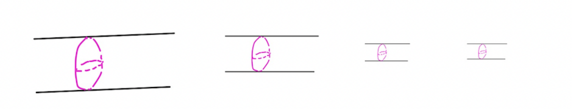
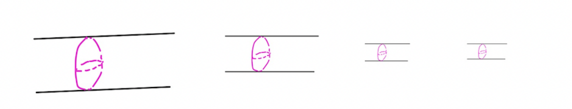


Figure 6: Product $\mathbb{S}^2 \times \mathbb{R}$ under Ricci flow

[1]credit: Prof. Peter Topping



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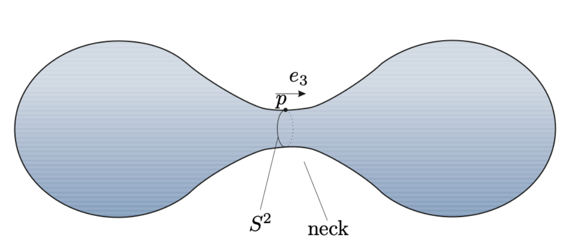

Figure 7: $\mathbb{S}^3$

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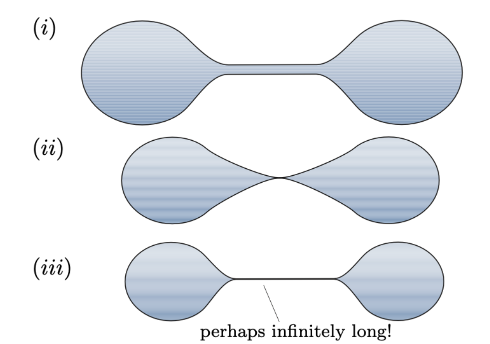

Figure 8: Neck pinch

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Analysis for the RF

Recall the questions which we asked for any geometric flow. What is the situation for the Ricci flow?

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Theorem (Perelman, 2003)

The Poincaré conjecture is true and the Ricci flow (with surgery!) on any connected, compact, simply connected 3-manifold converges to the round sphere $\mathbb{S}^3$.

Definition

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Let (M^8, Φ) be a manifold with a Spin(7)-structure Φ and let ∇ be the Levi-Civita connection of g_Φ. We call (M, Φ) a Spin(7)-manifold if $\nabla\Phi = 0$. This is a nonlinear equation on Φ. $\nabla\Phi$ is interpreted as the torsion T of the Spin(7)-structure

Spin(7)-manifolds, i.e., those having torsion-free Spin(7)-structure Φ are always Ricci-flat and have special holonomy contained in the Lie group Spin(7)$\subset \mathrm{SO}(8)$. Also admit parallel spinors $\rightsquigarrow$ important in physics.

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Thus, we answer **Q 1.** in the affirmative and the study of other problems related to the flow is work in progress.

Thank you for your attention.