NLO/NNLO - Higher Order computations

René Poncelet

THE HENRYK NIEWODNICZAŃSKI
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

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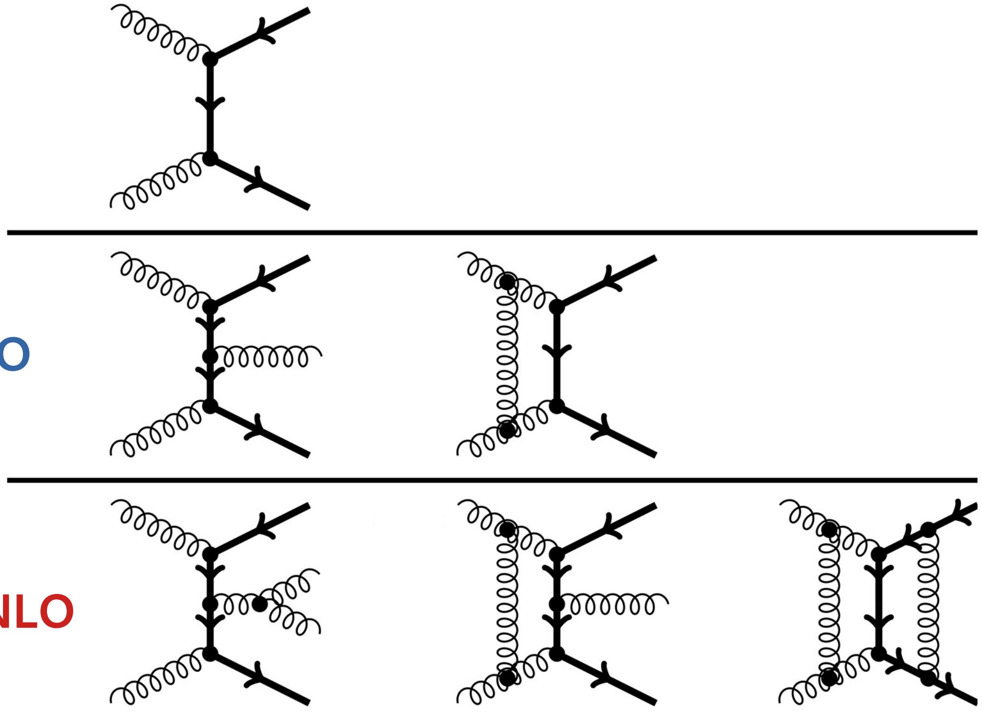
LO

NLO

NNLO



...

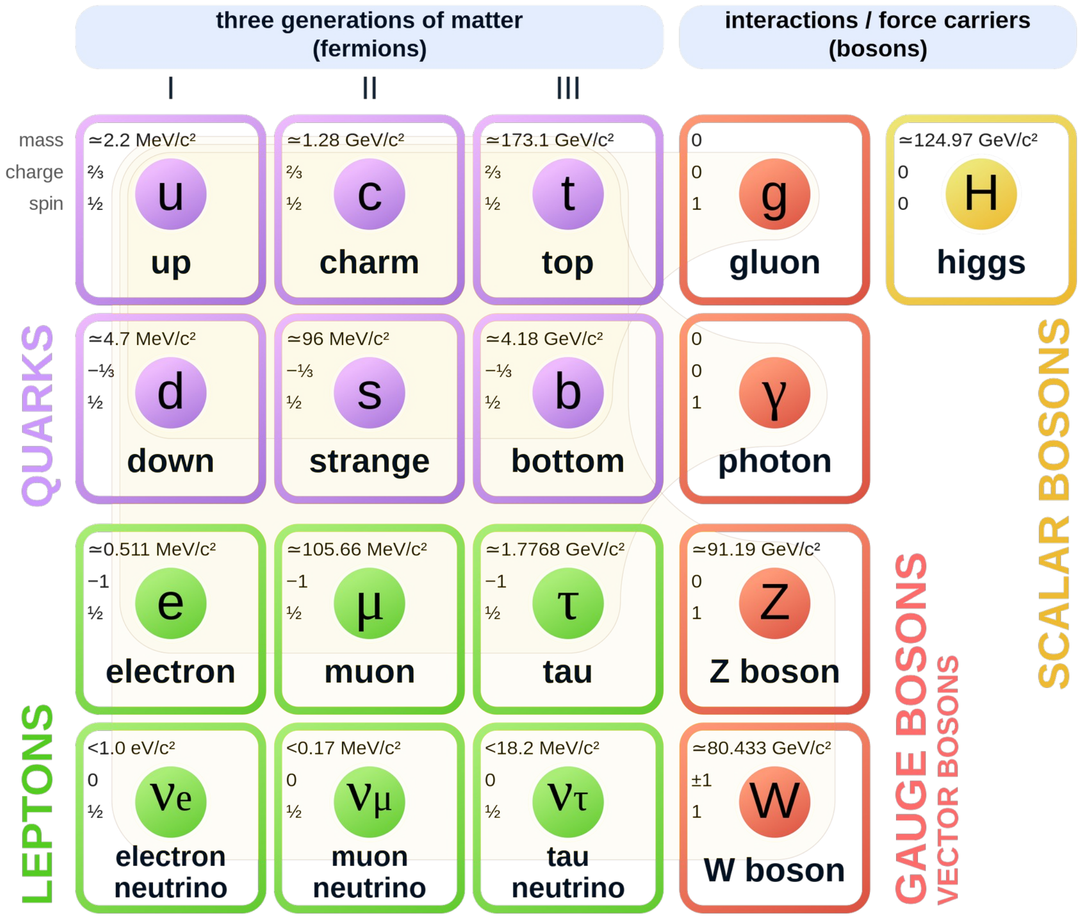
Outline

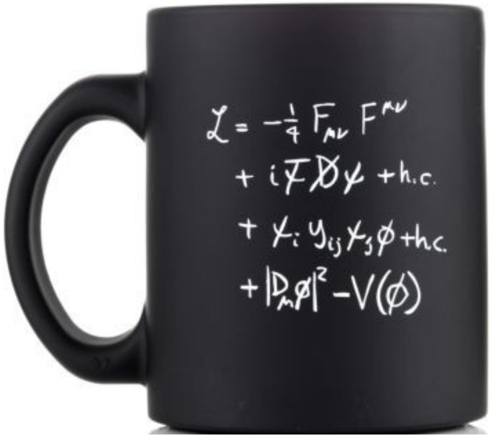
- **Motivation**
  + preliminaries: **Renormalisation** and **Regularisation**
- Anatomy of a higher-order QCD computation
  - **Virtual/Loop** corrections
  - **Real emission** corrections

  Infrared singularities in e+e- → jets
- Systematic higher-order computations:
  - **Multi-loop computations**
  - **Subtraction schemes** & **infrared safety**

  NLO, NNLO and beyond
- Higher-orders at **hadron colliders:**
  - **PDFs** & **factorization**
  - **Phenomenology**

Standard Model of Elementary Particles



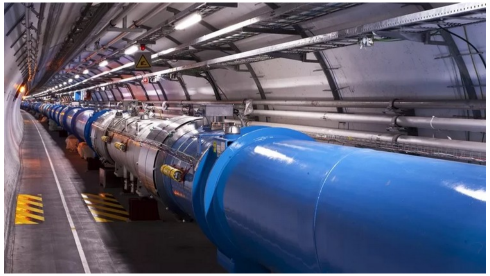


Credit: Wikipedia/CERN

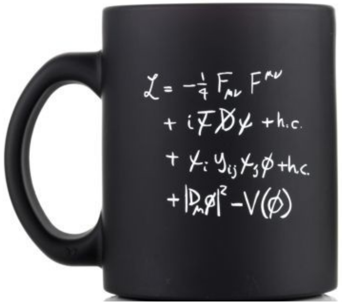
What are the fundamental building blocks of matter?

Scattering experiments

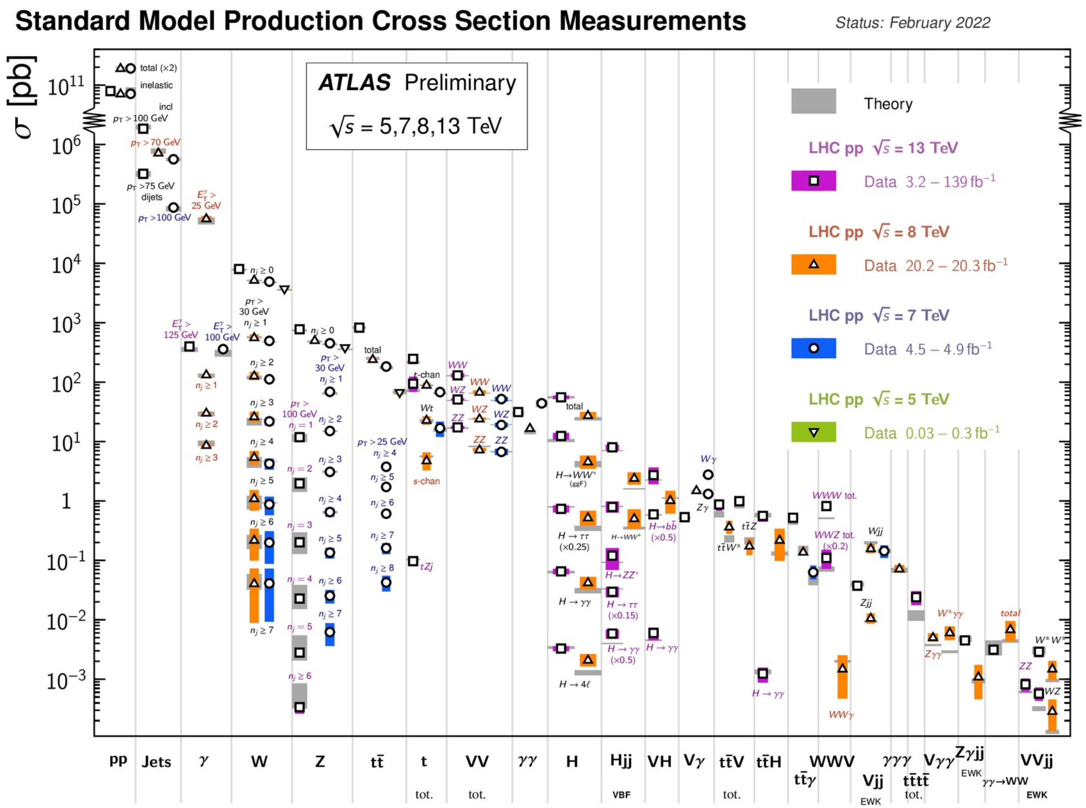
Large Hadron Collider (LHC)



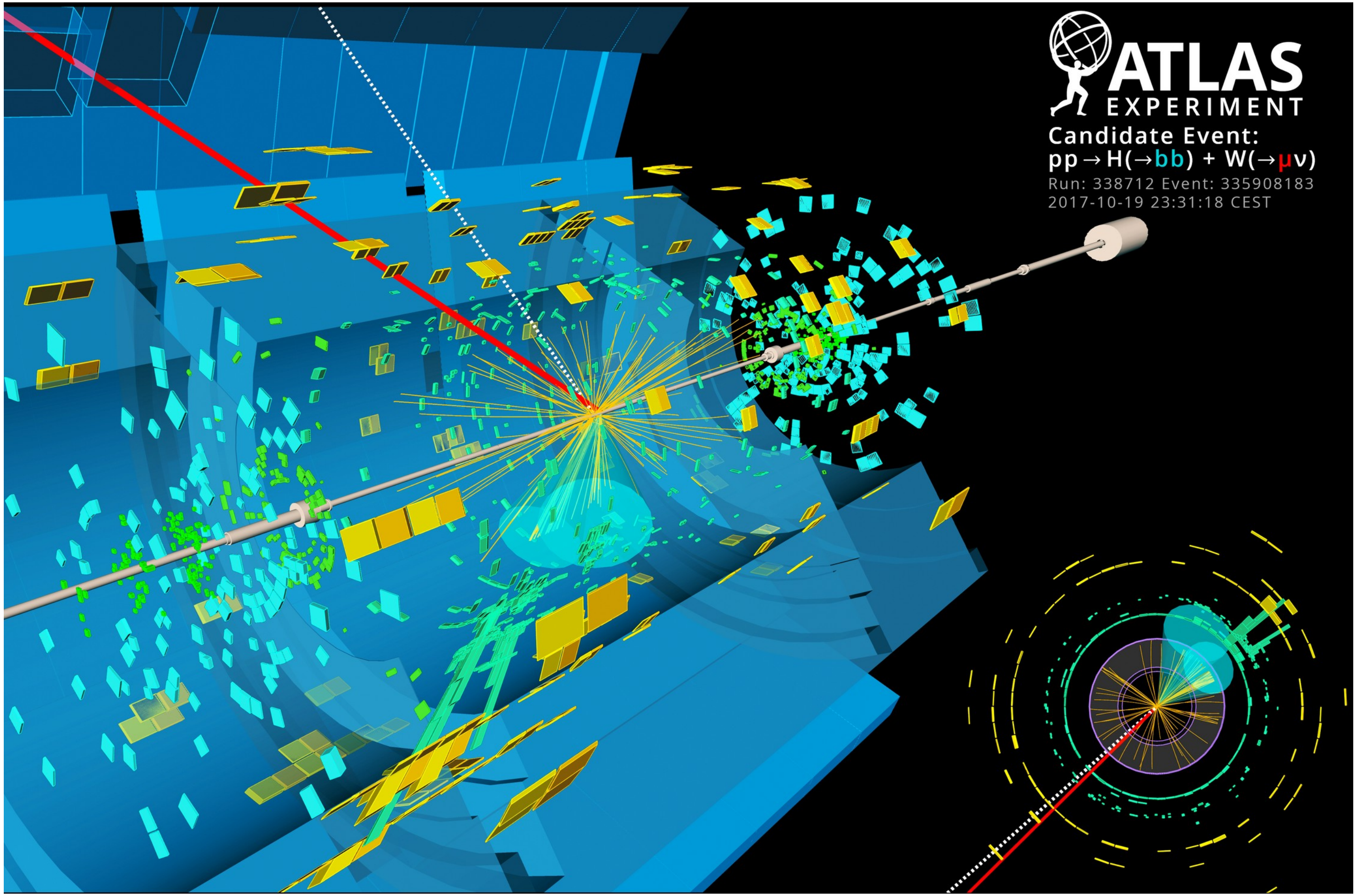
Credit: CERN



Theory/ Standard Model



Collision events



Theory picture of hadron collision events

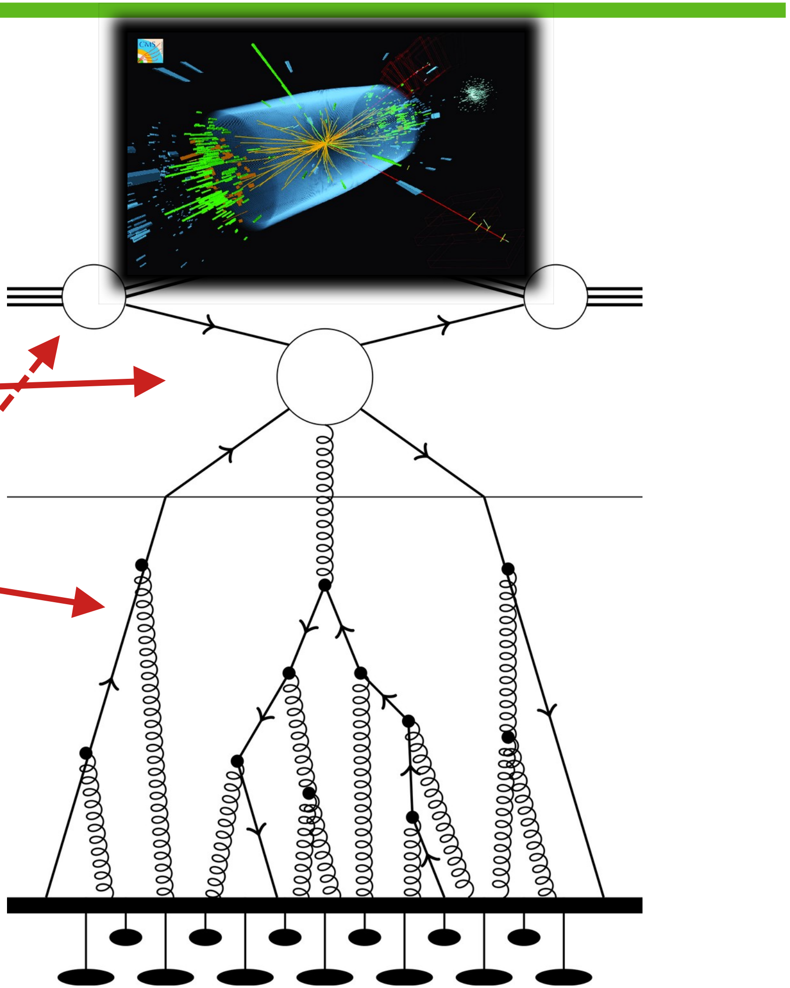
Guiding principle: factorization
"What you see depends on the energy scale"

In Quantum Chromodynamics (QCD):

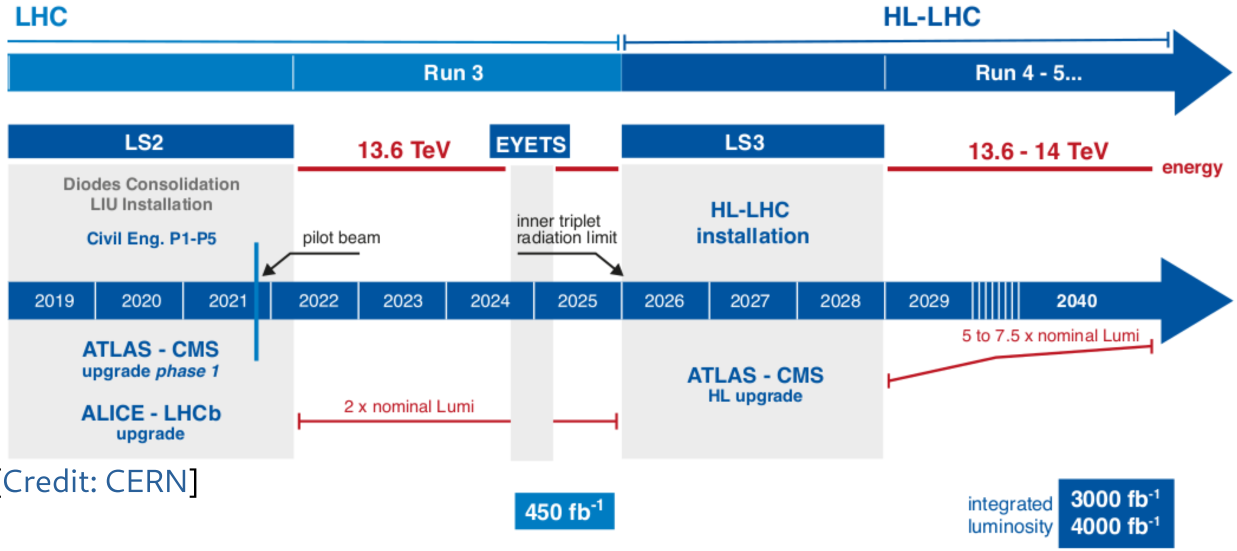
$Q \gg \Lambda_{\mathrm{QCD}}$ **Fixed-order perturbation theory**
scattering of individual partons

$Q \gtrsim \Lambda_{\mathrm{QCD}}$ **Parton-shower/Resummation**
all-order bridge between perturbative and non-perturbative physics
+ PDF factorization

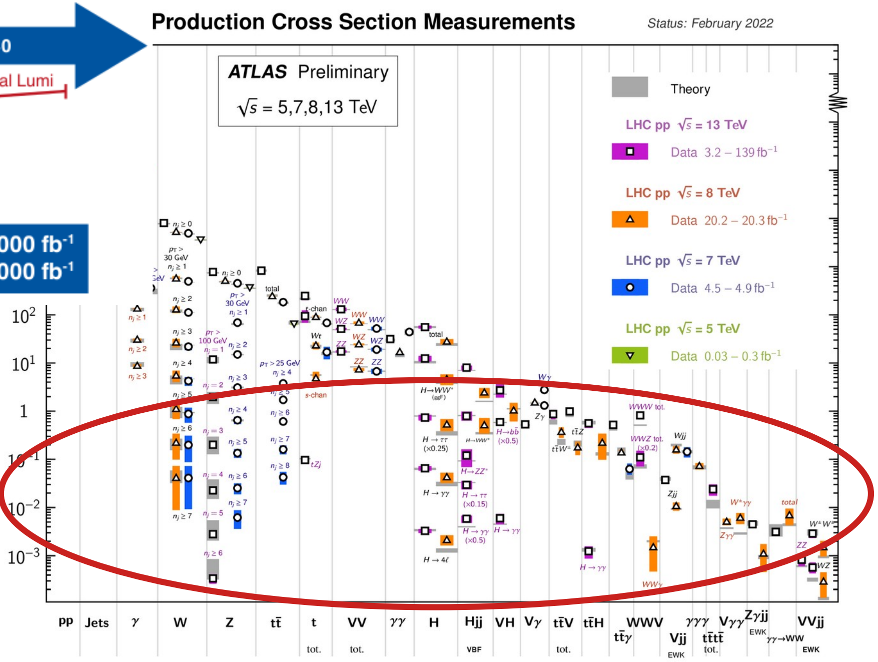
$Q \sim \Lambda_{\mathrm{QCD}}$ **"Hadronization"/MPI/...**
non-perturbative physics



LHC Precision era and future experiments



[Credit: CERN]



More precision for higher multiplicity processes:
→ theory needs to keep up!

Precision predictions

Resummation

Fixed order perturbation theory

Parton-showers

→ **Core element of event simulation**
→ **Describes high Q regime**

Precision theory predictions

Parametric input:
PDFs, couplings (α_s), ...

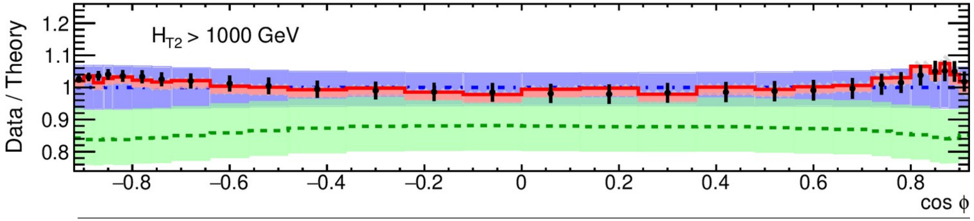
Soft physics:
MPI, colour reconnection,
...

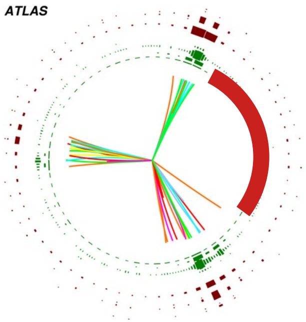
Fragmentation/hadronisation

Precision through higher-order perturbation theory

Example: ATLAS
multi-jet measurements

[ATLAS 2301.09351]





$\cos\phi$

Cross section = **LO** + **NLO** + **NNLO** + **O(α_s^3)**

$\sim (\alpha_s)^1$ $\sim (\alpha_s)^2$

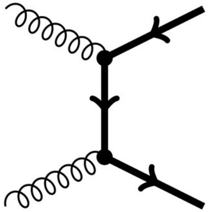
Theory uncertainty: **Order of magnitude** **O(10%)** **O(1%)**

Fixed-order expansion in the strong coupling $\alpha_s(m_Z) \approx 0.118$

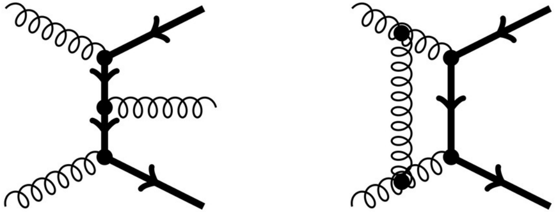
Experimental precision reaches percent-level already at LHC
next-to-next-to-leading order QCD needed on theory side!

NNLO QCD in collinear factorization

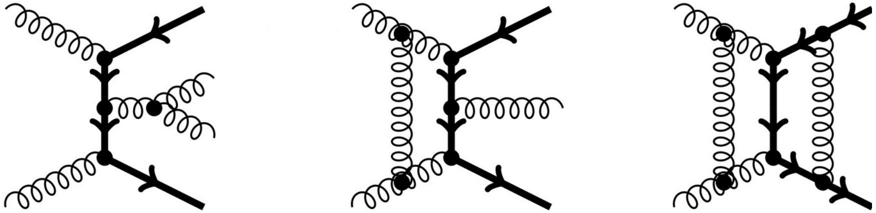
LO



NLO



NNLO



$$\sigma_{h_1 h_2 \to X} = \sum_{ij} \int_0^1 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \phi_{i,h_1}(x_1, \mu_F^2) \phi_{j,h_2}(x_2, \mu_F^2) \hat{\sigma}_{ij \to X}(\alpha_s(\mu_R^2), \mu_R^2, \mu_F^2)$$

Parton distribution functions
→ after the coffee break

Partonic cross section in perturbation theory:

$$\hat{\sigma}_{ab \to X} = \hat{\sigma}^{(0)}_{ab \to X} + \hat{\sigma}^{(1)}_{ab \to X} + \hat{\sigma}^{(2)}_{ab \to X} + \mathcal{O}(\alpha_s^3)$$

Focus on **higher-order QCD**
→ dominant corrections at the LHC

Preliminaries

Conventions: QCD Lagrangian

Quantum Chromodynamics is a **local gauge theory** of six quarks $q = d, u, s, c, b, t$

$$\mathcal{L} = \bar{\Psi}_{q,a}(\underbrace{i\not{\partial}\delta_{ab} - g_s t^A_{ab}\not{A}^A}_{(D_\alpha)_{ab} = \partial_\alpha \delta_{ab} + i g_s (t^C A^C_\alpha)_{ab}} - m_q \delta_{ab})\Psi_{q,b} - \frac{1}{4} F^A_{\mu\nu} F^{A,\mu\nu} \qquad F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_s f^{ABC} A^B_\mu A^C_\nu$$

$SU(3)$ $(SU(N_c))$

→ 3 fundamental indices or 'colours' (index: a) (N_c)

→ 8 adjoint indices/generators or gluons (index: A) $(N_c^2 - 1)$

'Colour' algebra:

$$[t^A, t^B]_{ab} = i f^{ABC} t^C_{ab}$$

structure constants

$$\mathrm{Tr}\, t^A t^B = T_F \delta^{AB} \text{ with } T_F = \frac{1}{2}$$

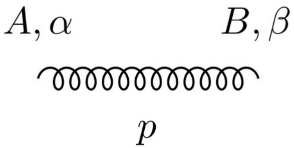
normalization

+ gauge fixing terms (for completeness): $L_{\text{gauge-fix}} = -\frac{1}{2\lambda}(\partial^\alpha A^A_\alpha)^2 \quad \mathcal{L}_{\text{ghost}} = \partial_\alpha \eta^{A\dagger}(D^\alpha_{AB}\eta^B)$

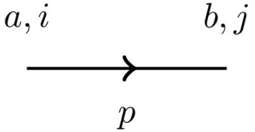
Covariant gauges (λ=1 Feynman, λ = 0 Landau)

Conventions: Feynman rules

Propagators

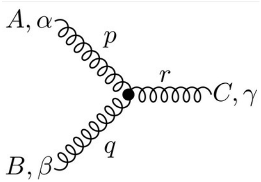


$$\delta^{AB}\left[-g^{\alpha\beta} + (1-\lambda)\frac{p^\alpha p^\beta}{p^2 + i\epsilon}\right]\frac{i}{p^2 + i\epsilon}$$

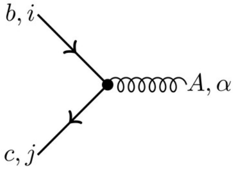


$$\delta^{ab}\frac{i}{(\not{p} - m + i\epsilon)_{ij}}$$

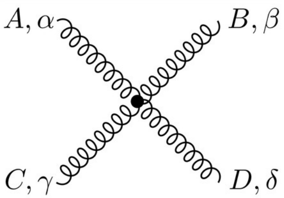
Vertices



$$-g_s f^{ABC}\left[(p-q)^\gamma g^{\alpha\beta} + (q-r)^\alpha g^{\beta\gamma} + (r-p)^\beta g^{\gamma\alpha}\right]$$



$$-ig_s t^A_{cb}\gamma^\alpha_{ji}$$



$$-ig_s^2 f^{XAC} f^{XBD}\left[g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right] + \dots$$

+ rule for loops:

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d}$$

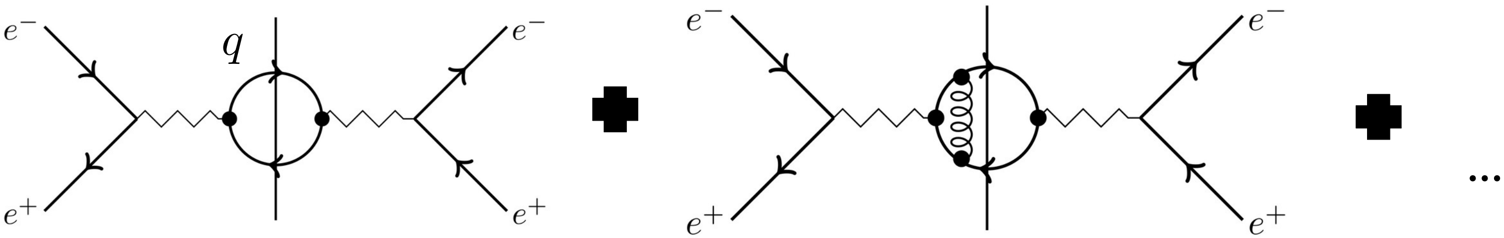
Ghost propagator: $g_s f^{ABC} q^\alpha$

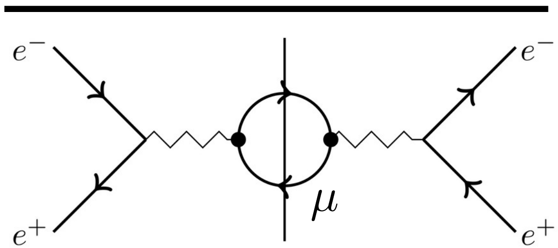
Gluon-ghost vertex: $\delta^{AB}\frac{i}{p^2 + i\epsilon}$

The strong coupling constant

In **massless** QCD the strong coupling constant is the **only free** parameter: $g_s = \sqrt{\alpha_s 4\pi}$

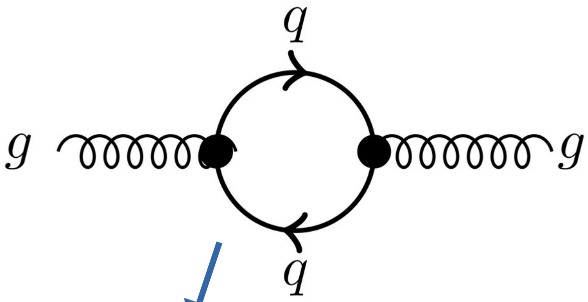
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \left[N_c \sum_q Q_q^2\right]\left(1 + \frac{\alpha_s}{\pi} + \dots\right)$$

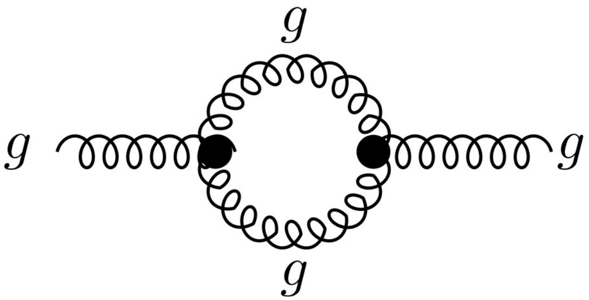




Measurement of R
→ measurement of α_s
(in practise not the best observable)

UV renormalisation of the strong coupling





+ ghosts

$$\sim \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu (q+m)\gamma^\nu (q+k+m)\right]}{(q^2-m^2)[(q+k)^2-m^2]} \xrightarrow{|q| \to \infty} \int \mathrm{d}q \frac{1}{q}$$

Ultra-violet divergence

Consider **cut-off regularization:** $\displaystyle\int_{q<\Lambda} \frac{\mathrm{d}q}{q} \sim \ln \Lambda$

Recover the full theory in the limit $\Lambda \to \infty$ breaks Lorentz invariance :(

Dimensional regularization

Working with a cut-off is cumbersome: broken Lorenz invariance of amplitudes...

Commonly used alternative: **dimensional regularization**

→ working in $d = 4 - 2\varepsilon$

→ Keeps **Lorentz** and **gauge invariance**

→ **Infrared divergences** can be treated in the same way

→ Instead of logarithms we find poles: $\frac{1}{\epsilon^n}$ for $\epsilon \to 0$

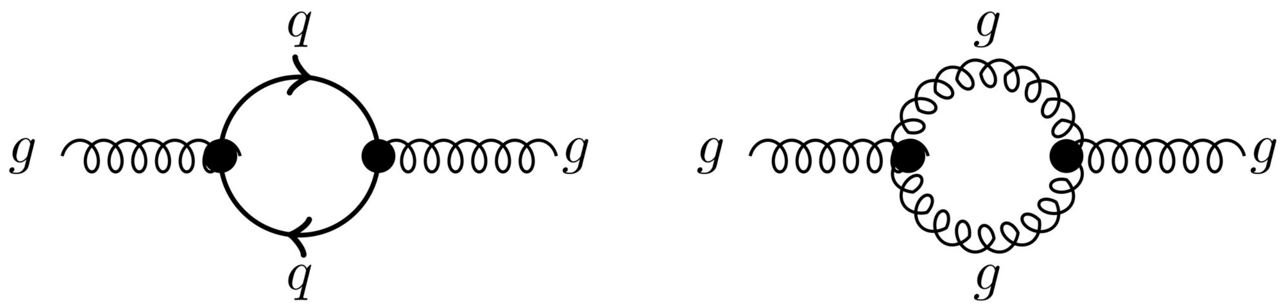
→ Implementation:

→ modify momentum integrals $\frac{\mathrm{d}^4 q}{(2\pi)^4} \to \frac{\mathrm{d}^d q}{(2\pi)^d}$

→ modified Lorentz and Dirac-algebra $\mathrm{Tr}\, g^{\mu\nu} = -2(1-\varepsilon)$

→ rescaled coupling (dimensionless) $g_s \to \mu^\varepsilon g_s$

Renormalisation of the strong coupling



➕ ghosts

$$\Pi^{gg}(k^2) = \frac{\alpha_s^{\text{bare}}}{2\pi}\left[-\beta_0\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu^2}{k^2}\right)\right) + \mathcal{O}(\varepsilon)\right] \qquad \beta_0 = \frac{11}{6}C_A - \frac{1}{3}n_f$$

Renormalization in QFTs:

- **bare Lagrangian parameters** are **not physical** quantities
- **absorb** (UV) divergences in **parameter definition** $\quad g_s^{\text{bare}} = Z_g g_s$
- **physical measurements** fix the renormalized value

UV renormalization in QCD

$\overline{\mathrm{MS}}$ Scheme (massless QCD, covariant gauge):

Wave functions:

$$Z_q = 1 - \lambda C_F \frac{\alpha_s S_\varepsilon}{4\pi\varepsilon} + \mathcal{O}(\alpha_s^2)$$

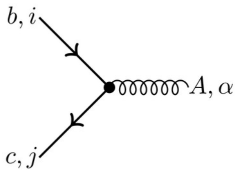
$$Z_g = 1 - \frac{\alpha_s S_\varepsilon}{4\pi\varepsilon} \left[\left(\frac{\lambda}{2} - \frac{13}{6} \right) C_A + \frac{4}{3} T_F n_f \right] + \mathcal{O}(\alpha_s^2)$$

Coupling:

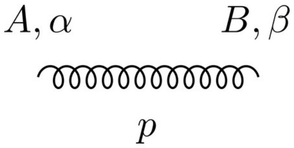
$$Z_{g_s} = 1 - \frac{\alpha_s S_\varepsilon}{4\pi\varepsilon} \frac{11 C_A - 4 n_f T_F}{6} + \dots$$

$$S_\varepsilon = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}$$

These introduce new diagrams in the perturbative expansions:
→ cancellation of all UV divergences



$\times (Z_{g_s} - 1)$



$\times (Z_g - 1)$

Renormalization group equations

The Lagrangian parameter now depend on an arbitrary scale μ

→ **physical quantities do not, for example the R-ratio**

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_s\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] R = 0 \qquad \left[-\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right] R(e^t, \alpha_s) = 0$$

$$t = \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{\mathrm{d}x}{\beta(x)}, \quad \alpha_s(\mu^2) = \alpha_s \qquad \longrightarrow \qquad R(1, \alpha_s(Q^2))$$

scale dependence only through α_s

The running coupling

Beta-function:

$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta(\alpha_s) \quad \beta(\alpha_s) = -\alpha_s^2 (b_0 + b_1 \alpha_s + \dots) \qquad b_0 = \frac{11 C_A - 2 n_f}{12\pi}$$

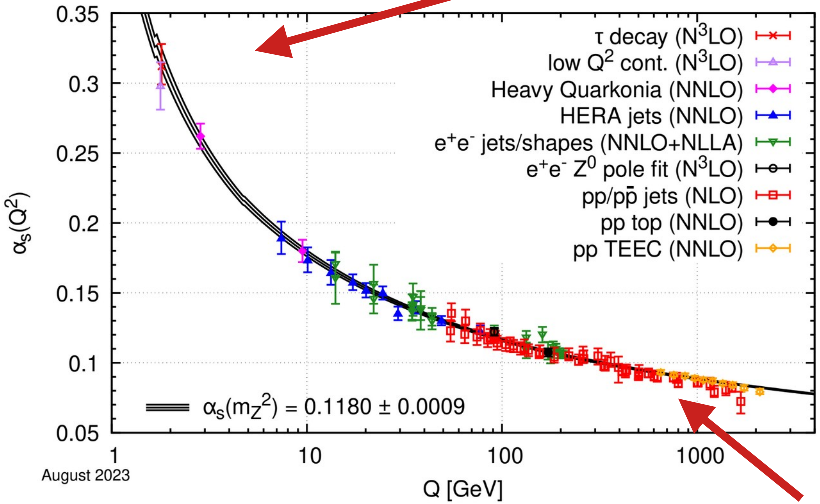
The renormalised coupling is fixed by experiment where we identify a reference scale

For example at the Z-pole: $\alpha_s^{\text{measured}} \equiv \alpha_s(m_Z^2)$

First order solution: $$\alpha_s(\mu^2) = \frac{\alpha_s(m_Z^2)}{1 + \beta_0 \frac{\alpha_s(m_Z^2)}{2\pi} \ln \frac{\mu^2}{m_Z^2}}$$

Asymptotic freedom & confinement

PDG



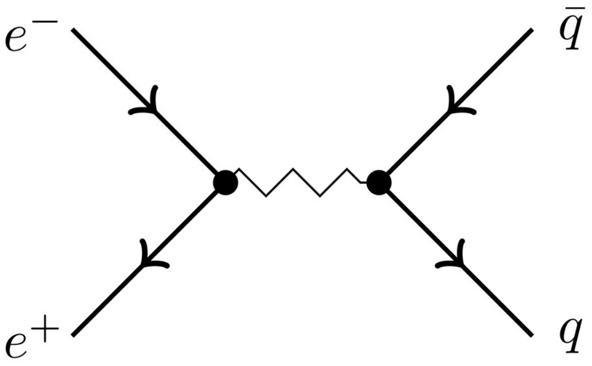
Confinement at small energy scales:
→ perturbation theory breaks downs
→ QCD bound states aka hadrons

$$\alpha_s(\mu^2) = \frac{\alpha_s(m_Z^2)}{1 + \beta_0 \frac{\alpha_s(m_Z^2)}{2\pi} \ln \frac{\mu^2}{m_Z^2}}$$

Asymptotic freedom at high energies:
→ dynamics of individual quarks and gluons
→ good regime for perturbation theory

Anatomy of higher order QCD computations

e+e- → jets: our higher-order QCD playground



$$a + b \to 1 + 2 + \cdots + n$$

Fermi's Golden Rule: $\mathrm{d}\sigma = \frac{1}{F}\langle|\mathcal{M}_n|^2\rangle \mathrm{d}\Phi_n$

Flux factor: $F = p_a \cdot p_b$

Lorentz Invariant Phase Space (LIPS):

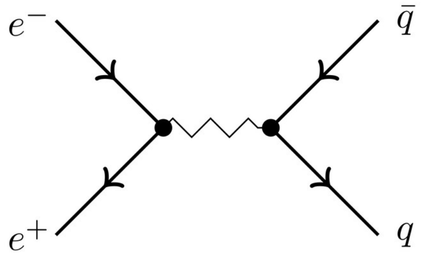
$$\mathrm{d}\Phi_n = \delta(p_a + p_b - \sum_{i=1}^{n} p_i) \prod_{i=1}^{n} \frac{\mathrm{d}^d p_i}{(2\pi)^d} 2\pi\theta(E_i)\delta(p_i^2 - m_i^2)$$

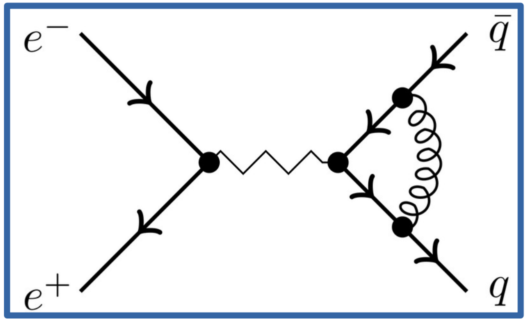
Leading order cross section:
(only photon exchange)

$$\sigma^{\mathrm{LO}}(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3s} N_c Q_q^2 \quad s = (p_a + p_b)^2$$

quark electric charge

Perturbative expansion





$$|\mathcal{M}_2|^2 = |\mathcal{M}_2^{(0)}|^2 + \boxed{2\,\mathrm{Re}\left[\left(|\mathcal{M}_2^{(0)}\right)^\dagger |\mathcal{M}_2^{(1)}\right]} + \boxed{2\,\mathrm{Re}\left[\left(|\mathcal{M}_2^{(0)}\right)^\dagger |\mathcal{M}_2^{(2)}\right] + |\mathcal{M}_2^{(1)}|^2} + \ldots$$

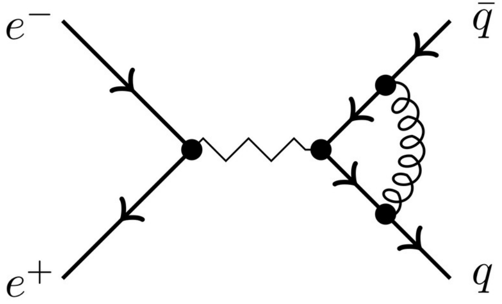
Next-to-leading order (NLO)

Next-to-next-to-leading order (NNLO)

$\mathcal{O}(\alpha^2)$ $\boxed{\mathcal{O}(\alpha^2\alpha_s)}$ $\boxed{\mathcal{O}(\alpha^2\alpha_s^2)}$

e+e-: the virtual corrections

→ UV counter terms not needed at NLO QCD



$$\int \mathrm{d}^4 q \frac{\cdots}{q^2 (q+p_1)^2 (q-p_2)^2} \quad |q| \to 0 \quad \sim \int_0 \frac{\mathrm{d}q}{q}$$

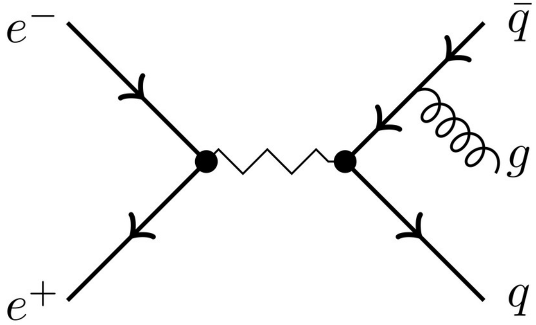
In dimensional regularization: $\sim \int_0 \frac{\mathrm{d}q}{q^{1+\varepsilon}}$

After some algebra:

$$\longrightarrow \quad \sigma^{\mathrm{LO}} \left[\frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\varepsilon)} \mathrm{Re} \left(\frac{4\pi\mu^2}{-s-i0} \right)^{\varepsilon} \left(\frac{-2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 \right) \right]$$

Real corrections

Consider the following matrix element:



$$|\mathcal{M}_3|^2 = |\mathcal{M}_3^{(0)}|^2 + 2\,\mathrm{Re}\left[\left(|\mathcal{M}_3^{(0)}\right)^\dagger |\mathcal{M}_3^{(1)}\right] + \dots$$

$\boxed{\mathcal{O}(\alpha^2\alpha_s)}$ $\boxed{\mathcal{O}(\alpha^2\alpha_s^2)}$

$$\rightarrow \frac{4\pi\alpha^2}{3s} N_c Q_q^2 \int \mathrm{d}\Phi_3 g_s^2 C_F 2\left(\frac{s_{13}+s_{23}+2s_{12}}{s s_{13} s_{23}}\right)$$

Real corrections in 4 dimensions → IR limits exposed

A bit of phase space magic: $\delta^d(p_a + p_b - p_1 - p_2 - p_3)$

3n-4=5 phase space dimensions → Integrate out two angles

$$x_i = \frac{2p_i \cdot p}{s} = \frac{s_{ij} + s_{ik}}{s} \to s_{ij} = 1 - x_k \qquad x_1 + x_2 + x_3 = 2$$

$$\mathrm{d}\Phi_3 = \frac{s}{2(4\pi)^3} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \delta(2 - x_1 - x_2 - x_3)$$

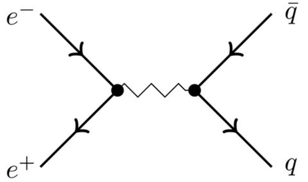
… reveals hidden treasure: $\sim \mathrm{d}\Phi_3 \left(\frac{x_q^2 + x_{\bar{q}}^2}{(1 - x_q)(1 - x_{\bar{q}})} \right)$

Singularities:
- collinear: $x_q \to 1 \text{ or } x_{\bar{q}} \to 1$
- soft: $x_q \to 1 \text{ and } x_{\bar{q}} \to 1$

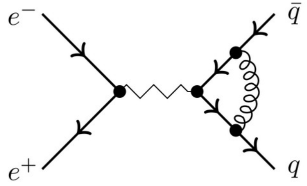
Regularize in dimensional regularization: $\sigma^{\mathrm{LO}} \left[\frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)} \left(\frac{4\pi\mu^2}{s} \right)^{\varepsilon} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} \right) \right]$

Combined NLO QCD

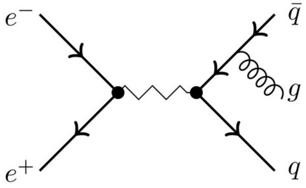
$$\sigma^{\mathrm{LO}}$$



- $$\sigma^{\mathrm{LO}}\left[\frac{\alpha_s}{2\pi}C_F\frac{1}{\Gamma(1-\varepsilon)}\,\mathrm{Re}\left(\frac{4\pi\mu^2}{-s-i0}\right)^{\varepsilon}\left(\frac{-2}{\varepsilon^2}-\frac{3}{\varepsilon}-8\right)\right]$$



- $$\sigma^{\mathrm{LO}}\left[\frac{\alpha_s}{2\pi}C_F\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)}\left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}\right)\right]$$



$$\sigma^{\mathrm{NLO}}=\sigma^{\mathrm{LO}}\left(1+\frac{\alpha_s}{\pi}\right)$$

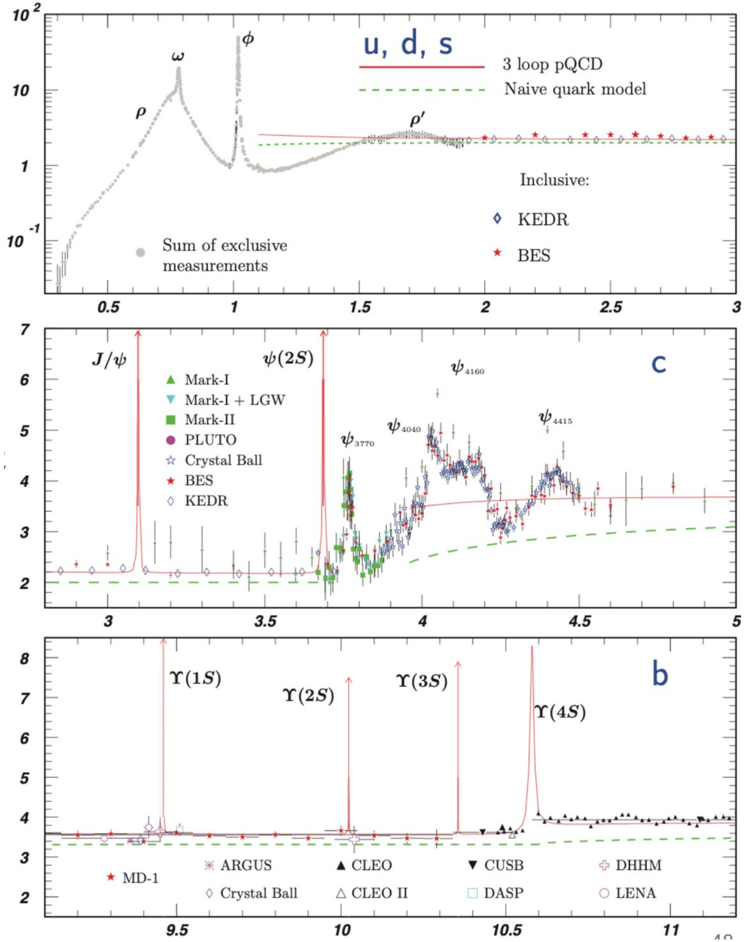
Kinoshita-Lee-Nauenberg Theorem

When calculating **sufficiently inclusive** observable quantities in quantum field theory (like cross sections or decay rates), **all infrared (soft and collinear) divergences cancel out** once you include every process that is **physically indistinguishable** within the detector resolution (i.e., summing over degenerate initial and final states).

This is ‘trivial’ for inclusive quantities like total cross sections,
but as soon we have to be careful as soon we want something more differential

Differential: not fully integrating over the phase space but only in some region of it

R-ratio



$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q Q_q^2 \theta(\sqrt{s} - 2m_q)$$

At the Z-pole:

(this includes now also Z-boson exchange)

$$R^{\text{LO}} = 20.09$$

$$R^{\text{NLO}} = 20.89$$

$$R^{\text{LEP}} = 20.79 \pm 0.09$$

Systematic higher order computations

Systematics of loop computation

In principle we have all the building blocks to compute any loop diagram

but in practise algebra challenging → own line of research:

- number of Feynman diagrams grows fast with the loop-order and number of legs
- more particles lead to **many** kinematic scales … even more algebra
- how to evaluate appearing Feynman integrals?

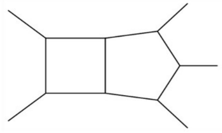
Two important techniques
(you often will hear about in TH talks)

→ **Integration-by-parts Identities**

→ **Master integral differential equations**

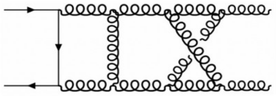
State-of-the-art box

Two-loop 5-point
[Abreu, Agarwal, Badger, Buccioni, Chawdhry, Chicherin,Czakon, de Laurenties, Febres-Cordero, Gambuti, Gehrmann, Henn, Lo Presti, Manteuffel, Ma, Mitov, Page, Peraro, Poncelet, Schabiner Sotnikov, Tancredi, Zhang,...]



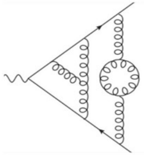
Three-loop 4-point

[Bargiela, Dobadilla, Canko, Caola, Jakubcik, Gambuti, Gehrmanr Lim, Mella, Mistleberger, Wasser, Manteuffel, Syrrakow, Smirnov Trancredi, ...]



Four-loop 3-point

[Henn, Lee, Manteuffel, Schabinger, Smirnov, Steinhauser,...]



Integration-By-Parts reduction

Rational functions of the kinematic invariants and ε

$$\sum_{\text{color,spin}} \left(\mathcal{M}^{(0)}\right)^{\dagger} \mathcal{M}^{(L)} = \sum_i \boxed{c_i(\{p\},\varepsilon)}\,\boxed{\mathcal{I}_i(\{p\},\varepsilon)}$$

Scalar (or Feynman) integrals:

Number of loops

Number of propagators

$$\mathcal{I}_i(\{p\},\epsilon) \equiv \mathcal{I}(\vec{n}_i,\{p\},\epsilon) = \int \prod_l^L \frac{\mathrm{d}^d k_l}{(2\pi)^d} \prod_{m=1}^N D_m^{-n_{i,m}}(\{p\},\{k\})$$

D_m : squared propagators, for example $D_1 = (p_1 - k_1)^2$, $D_2 = (k_1 - k_2)^2$, ...

Integration-By-Parts Identities and reduction

Bonus feature of dimensional regularization:

$k_i \to k_i + l$ does not change the integral since the boundaries are at infinity

$$\longrightarrow \quad 0 = \int \prod_l^L \frac{\mathrm{d}^d k_l}{(2\pi)^d} l_\mu \frac{\partial}{\partial l^\mu} \prod_{m=1}^N D_m^{-n_{i,m}}(\{p\},\{k\}) \qquad l \in \{p\} \cap \{k\}$$

This leads to large matrix (all coefficients are functions of the invariants and ε)
→ can be solved for small number of so-called master integrals
→ Gaussian elimination → Laporta algorithm

Non-trivial in practise, many refinements in state-of-the-art computations
→ huge computers...

$$\longrightarrow \quad \sum_{\text{color,spin}} \left(\mathcal{M}^{(0)}\right)^\dagger \mathcal{M}^{(L)} = \sum_i c_i'(\{p\},\varepsilon)\, \mathrm{MI}_i(\{p\},\varepsilon)$$

The number of master integrals is typically **much smaller** ($10^6 \to 10^2$)

Master integrals

Typically with smaller $\max\{n_i\}$ then generic integrals → easier integrals

$$\mathrm{MI}_i(\{p\},\epsilon) \equiv \mathrm{MI}(\vec{n_i},\{p\},\epsilon) = \int \prod_l^L \frac{\mathrm{d}^d k_l}{(2\pi)^d} \prod_{m=1}^N D_m^{-n_{i,m}}(\{p\},\{k\})$$

How to deal with the master integrals?

→ **Feynman parameters** and related techniques

→ **Method by regions**: decompose the integral into its IR singularity structure

However, direct analytical integration is **often not possible**:

→ numerics

→ **differential equations**

Differential equations for master integrals

$$\mathrm{MI}_i(\{p\},\epsilon) \equiv \mathrm{MI}(\vec{n_i},\{p\},\epsilon) = \int \prod_l^L \frac{\mathrm{d}^d k_l}{(2\pi)^d} \prod_{m=1}^N D_m^{-n_{i,m}}(\{p\},\{k\})$$

Consider the derivative with respect to external kinematic invariants:

$$\frac{\mathrm{dMI_i}}{\mathrm{d}s_{ij}} = \sum_l \frac{\partial p_l}{\partial s_{ij}} \frac{\partial \mathrm{MI_i}}{\partial p_l} = \sum_k c_k \mathrm{MI_k}$$

$$D_1^{-1} = (p_1 - k_1)^{-2} \to \frac{\partial D_1^{-1}}{\partial p_1} \sim D_1^{-2}$$

Recover Feynman integrals
→ use IBPs again

System of differential equations: $\mathrm{dMI}_i = \mathrm{d}A(\{p\},\epsilon)_{ij}\mathrm{MI}_j$

→ needs boundary conditions: kinematic limits, special points, numerics,….

→ often solved as expansion in ε

→ analytic solutions → special functions (Polylogarithms, Hypergeometric functions, …)

→ solve numerically as evolution from the boundary

Automation at one-loop

Only part of loop amplitudes is need in practise

→ needs understanding of the basis of all functions contributing to ε^0 : box, triangles, bubbles

→ ‘projection’ directly on this basis (up to rational terms)

→ **numerical implementation**: One-Loop-Providers (OLP)

→ OpenLoops

→ Recola

→ MadLoop

→ …

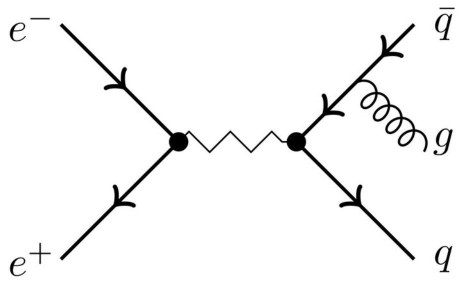
→ **automation at two-loops**?

→ one of the toughest problems of the higher order theory community at the moment

Systematics of real emissions

How to deal with the real radiation contributions?

- phase space constraints make computations more complicated
- more particles imply more soft and collinear limits
- 'observables' might depend on the kinematics
  → how to reconcile this with the phase space integration?



→ eventually only a numerical approach can be practical!
→ but numerics in $d = 4 - 2\varepsilon$ is tricky…

But **how to get rid of the IR divergences**?

Two types of limits: soft and collinear

factorization comes to the rescue…

Systematics of real emissions: soft limits

Factorization of matrix elements in the **soft limit** (only for gluons)

$$q \to \lambda q \text{ with } \lambda \to 0$$

$$\left|\mathcal{M}^{(0)}_{g,a_1,\ldots}(q,p_1,\ldots)\right|^2 \simeq -4\pi\alpha_s \sum_{ij} \mathcal{S}_{ij}(q) \left\langle \mathcal{M}^{(0)}_{a_1,\ldots}(p_1,\ldots) \right| \mathbf{T}_i \cdot \mathbf{T}_j \left| \mathcal{M}^{(0)}_{a_1,\ldots}(p_1,\ldots) \right\rangle$$

$$\mathcal{S}_{ij}(q) = \frac{p_i \cdot p_j}{(p_i \cdot q)\,(p_j \cdot q)}$$

sum over colour-correlators: emissions from colour dipoles

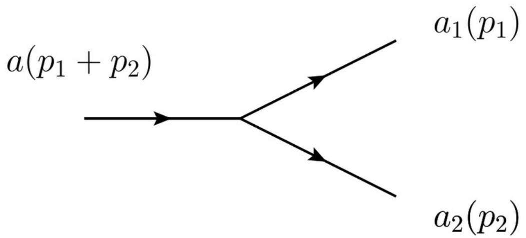
In our e+e- example: $-4\pi\alpha_s \mathcal{S}_{12}(p_3) C_F |\mathcal{M}_2|^2 \text{ for } p_3 \to 0$

$$\mathcal{S}_{12}(p_3) = \frac{1}{2}\frac{s}{s_{13}s_{23}} \sim \frac{1}{\lambda^2}$$

[Keep in mind a factor of λ from the phase space measure]

Systematics of real emissions: collinear limits

Collinear limits in the Sudakov parametrization:



$$p_1^\mu = zp^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2p\cdot n}, \qquad p_2^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\frac{n^\mu}{2p\cdot n},$$

$$s_{12} = 2p_1\cdot p_2 = -\frac{k_\perp^2}{z(1-z)}, \qquad p^2 = n^2 = p\cdot k_\perp = n\cdot k_\perp = 0,$$

$$k_\perp^\mu \to 0.$$

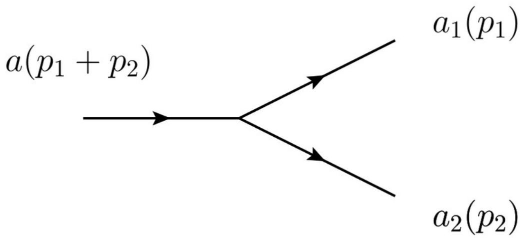
$$\left|\mathcal{M}^{(0)}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\right|^2 \simeq 4\pi\alpha_s\,\frac{2}{s_{12}}\langle\mathcal{M}^{(0)}_{a,\ldots}(p,\ldots)\big|\hat{\mathbf{P}}^{(0)}_{a_1a_2}(z,k_\perp;\epsilon)\big|\mathcal{M}^{(0)}_{a,\ldots}(p,\ldots)\rangle$$

diagonal in colour space but gluon – splitting kernels induce spin-correlations

Systematics of real emissions: collinear limits

$$\left|\mathcal{M}^{(0)}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\right|^2 \simeq 4\pi\alpha_s \frac{2}{s_{12}}\left\langle\mathcal{M}^{(0)}_{a,\ldots}(p,\ldots)\right|\hat{\mathbf{P}}^{(0)}_{a_1a_2}(z,k_\perp;\epsilon)\left|\mathcal{M}^{(0)}_{a,\ldots}(p,\ldots)\right\rangle$$

The splitting functions depend on the flavours involved:



$$\hat{P}^{(0),\,\mu\nu}_{gg}(z,k_\perp;\epsilon) = 2C_A\left[-g^{\mu\nu}\left(\frac{z}{1-z}+\frac{1-z}{z}\right)-2(1-\epsilon)z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right],$$

$$\hat{P}^{(0),\,\mu\nu}_{q\bar{q}}(z,k_\perp;\epsilon) = \hat{P}^{(0),\,\mu\nu}_{\bar{q}q}(z,k_\perp;\epsilon) = T_F\left[-g^{\mu\nu}+4z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right],$$

$$\hat{P}^{(0),\,ss'}_{qg}(z,k_\perp;\epsilon) = \hat{P}^{(0),\,ss'}_{\bar{q}g}(z,k_\perp;\epsilon) = \delta^{ss'}\,C_F\left[\frac{1+z^2}{1-z}-\epsilon(1-z)\right],$$

$$\hat{P}^{(0),\,ss'}_{gq}(z,k_\perp;\epsilon) = \hat{P}^{(0),\,ss'}_{g\bar{q}}(z,k_\perp;\epsilon) = \hat{P}^{(0),\,ss'}_{qg}(1-z,k_\perp;\epsilon).$$

For e+e- example:
g collinear with quark

$$P_{qg}(z)$$

$$z = E_q/(E_q+E_g) = x_q/(x_q+x_g)$$

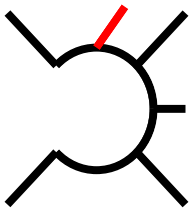
Next-to-leading order case

$$\hat{\sigma}_{ab}^{(1)} = \hat{\sigma}_{ab}^{\mathrm{R}} + \hat{\sigma}_{ab}^{\mathrm{V}} + \hat{\sigma}_{ab}^{\mathrm{C}}$$

KLN theorem
sum is finite for sufficiently inclusive observables
and regularization scheme independent

Each term separately infrared (IR) divergent:

Real corrections:

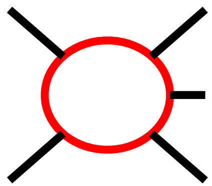


Measurement function

$$\hat{\sigma}_{ab}^{\mathrm{R}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}_{n+1}^{(0)} \middle| \mathcal{M}_{n+1}^{(0)} \right\rangle \boxed{\mathrm{F}_{n+1}}$$

Phase space integration over unresolved configurations

Virtual corrections:

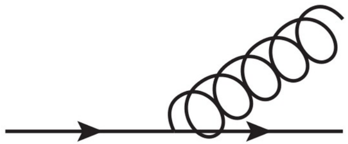


$$\hat{\sigma}_{ab}^{\mathrm{V}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, 2\mathrm{Re} \left\langle \mathcal{M}_n^{(0)} \middle| \mathcal{M}_n^{(1)} \right\rangle \mathrm{F}_n$$

Integration over loop-momentum
(UV divergences cured by renormalization)

IR singularities in real radiation

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1}$$

$\ni \mathrm{d}\Phi_1$ 

Finite function

$$\sim \int_0 \mathrm{d}E\mathrm{d}\theta \frac{1}{E(1-\cos\theta)} f(E, \cos(\theta))$$

Divergent

Regularization in Conventional Dimensional Regularization (CDR) $d = 4 - 2\epsilon$

$$\to \int_0 \mathrm{d}E\mathrm{d}\theta \frac{1}{E^{1-2\epsilon}(1-\cos\theta)^{1-\epsilon}} f(E, \cos(\theta)) \sim \frac{1}{\epsilon^2} + \ldots$$

Cancellation against similar divergences in

$$\hat{\sigma}^{\mathrm{V}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, 2\mathrm{Re} \left\langle \mathcal{M}^{(0)}_n \middle| \mathcal{M}^{(1)}_n \right\rangle \mathrm{F}_n$$

How to extract these poles? Slicing and Subtraction

Central idea: Divergences arise from infrared (IR, soft/collinear) limits → Factorization!

Slicing

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int_{\delta(\Phi)\geq\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} + \frac{1}{2\hat{s}} \int_{\delta(\Phi)<\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1}$$

$$\approx \frac{1}{2\hat{s}} \int_{\delta(\Phi)\geq\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} + \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, \tilde{M}(\delta_c)\mathrm{F}_n + \mathcal{O}(\delta_c)$$

$$\ldots + \hat{\sigma}^{\mathrm{V}}_{ab} = \mathrm{finite}$$

Subtraction

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int \left(\mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} - \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n \right) + \frac{1}{2\hat{s}} \int \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n$$

$$\frac{1}{2\hat{s}} \int \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n = \frac{1}{2\hat{s}} \int \underline{\mathrm{d}\Phi_n \, \mathrm{d}\Phi_1} \, \mathcal{S}\mathrm{F}_n$$

Phase space factorization
→ momentum mappings

Most popular NLO QCD schemes:
CS [hep-ph/9605323],
FKS [hep-ph/9512328]

→ Basis of modern event simulation

Slicing and Subtraction

Slicing

- Conceptually simple
- Recycling of lower computations
- Non-local cancellations/power-corrections
  → computationally expensive
- Comparatively easy to extend to N3LO

Subtraction

- Conceptually more difficult
- Local subtraction → efficient
- Better numerical stability
- Choices:
  - Momentum mapping
  - Subtraction terms
  - Numerics vs. analytic

NNLO QCD schemes

qT-slicing [Catani'07],
N-jettiness slicing [Gaunt'15/Boughezal'15]

Antenna [Gehrmann'05-'08],
Colorful [DelDuca'05-'15],
Sector-improved residue subtraction [Czakon'10-'14'19]
Projection [Cacciari'15],
Nested collinear [Caola'17],
Geometric [Herzog'18],
Unsubtraction [Aguilera-Verdugo'19],
...

Infrared safety of the measurement function

$$\sigma \sim \int \mathrm{d}\Phi_n |\mathcal{M}_n|^2 \to \int \mathrm{d}\Phi_n |\mathcal{M}_n|^2 \mathcal{F}_n(p_1, \ldots, p_n)$$

'Measurement function' defines
→ observables → cuts → jets
→ histograms → ...

KLN theorem: average over sufficient unresolved degrees of freedom
→ IR safe observables: well behaved* in the soft and collinear regions!

*the precise notion what properties are sufficient or equivalent is sometimes still a matter of debate

Soft limits:

$$\mathcal{F}_{n+1}(p_1, \ldots, p_{n+1}) \to \mathcal{F}_n(p_1, \ldots, \not{p}_i, \ldots, p_{n+1})$$

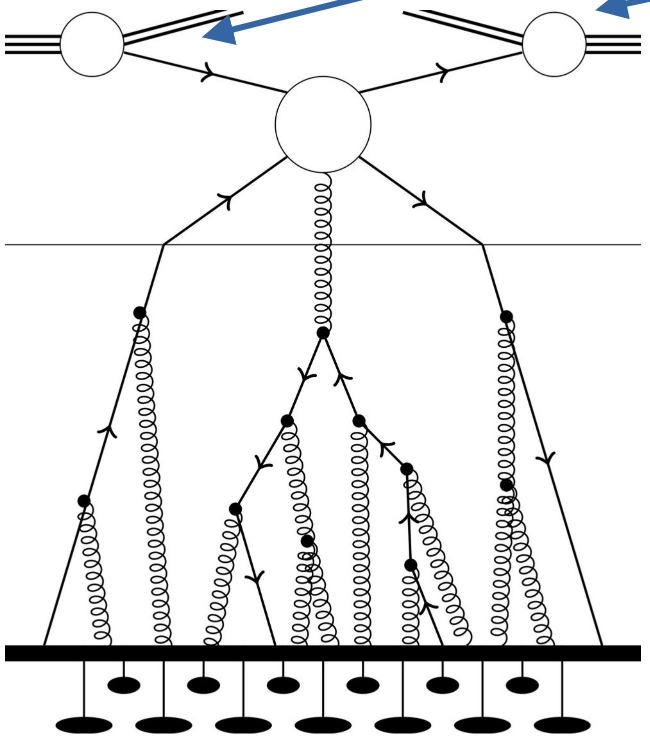
Collinear limits:

$$\mathcal{F}_{n+1}(p_1, \ldots, p_{n+1}) \to \mathcal{F}_n(p_1, \ldots, \not{p}_i, \ldots, \not{p}_j, \ldots, p_{n+1}, p_i + p_j)$$

Higher-orders at hadron colliders

Hadron-hadron collisions

$$\sigma_{h_1 h_2 \to X} = \sum_{ij} \int_0^1 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 f_{i,h_1}(x_1, \mu_F^2) f_{j,h_2}(x_2, \mu_F^2) \hat{\sigma}_{ij\to X}(\alpha_s(\mu_R^2), \mu_R^2, \mu_F^2) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$



- Bound state interactions in the proton: typical time scale $t_{\mathrm{had}} \sim 1/m_p$
- Scattering at a **high energy**

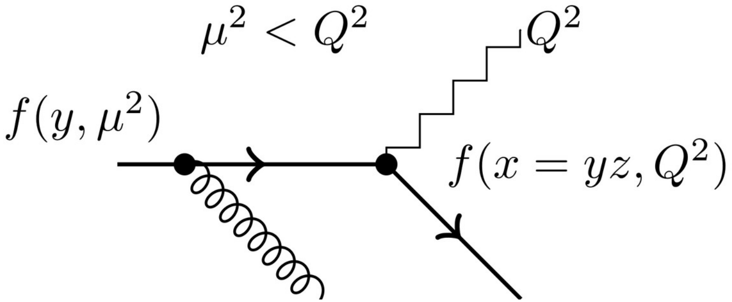
$$Q \gg m_p \leftrightarrow t_{\mathrm{hard}} \ll t_{\mathrm{had}}$$

- Asymptotic freedom at high Q:
  → small coupling
  → proton a collection of free quarks/gluons
- Momentum distribution described by **parton distribution function** (PDF)
  → extracted from data

Evolution of PDFs

The PDFs depend on the scale they are probed at $f_{a,h}(x, \mu_F^2)$

→ the origin are emissions at a lower energy scale
consider the striking a parton with a virtual photon Q^2



splitting functions!

$$f(x, Q^2) = f(x, \mu^2) + \int_x^1 \mathrm{d}y f(y, \mu^2) \int_0^1 \mathrm{d}x \delta(x - yz) \int_{\mu^2}^{Q^2} \mathrm{d}k^2 P(z, k^2)$$

no emission

all possibilities of emissions

DGLAP

$$f(x,Q^2) = f(x,\mu^2) + \int_x^1 \mathrm{d}y f(y,\mu^2) \int_0^1 \mathrm{d}x \delta(x - yz) \int_{\mu^2}^{Q} {}^2\mathrm{d}k^2 P(z,k^2)$$

$$\longrightarrow \qquad \frac{\mathrm{d}f(x,\mu^2)}{\mathrm{d}\mu^2} = \int_x^1 \mathrm{d}z P(z,\mu^2) f(x/z,\mu^2)$$

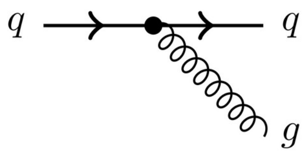
Taking into account that the splitting may change the parton flavour:
DGLAP [Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 77]

$$\frac{\mathrm{d}f_a(x,\mu^2)}{\mathrm{d}\mu^2} = \int_x^1 \mathrm{d}z \frac{\alpha_s}{2\pi} P_{ab}(z,\mu^2) f_b(x/z,\mu^2)$$

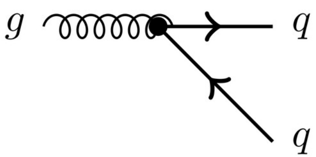
Prediction of the running of PDFs (universal, i.e. not dependent on the collider)
→ still needs input at a given scale (similar to coupling)

DGLAP: splitting functions

$$\frac{\mathrm{d}f_a(x,\mu^2)}{\mathrm{d}\mu^2} = \int_x^1 \mathrm{d}z \frac{\alpha_s}{2\pi} P_{ab}(z,\mu^2) f_b(x/z,\mu^2)$$



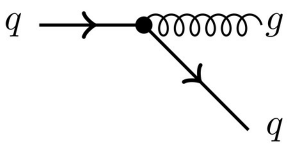
$$P_{qq}(z) = C_F \left[\boxed{\left(\frac{1+z^2}{1-z}\right)_+} + \frac{3}{2}\delta(1-z) \right]$$



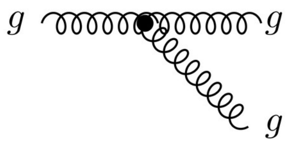
$$P_{qg}(z) = T_F(z^2 + (1-z)^2)$$

‘+’ distribution:

$$\int_0^1 \mathrm{d}x (f(x))_+ g(x) = \int_0^1 \mathrm{d}x f(x)(g(x) - g(1))$$



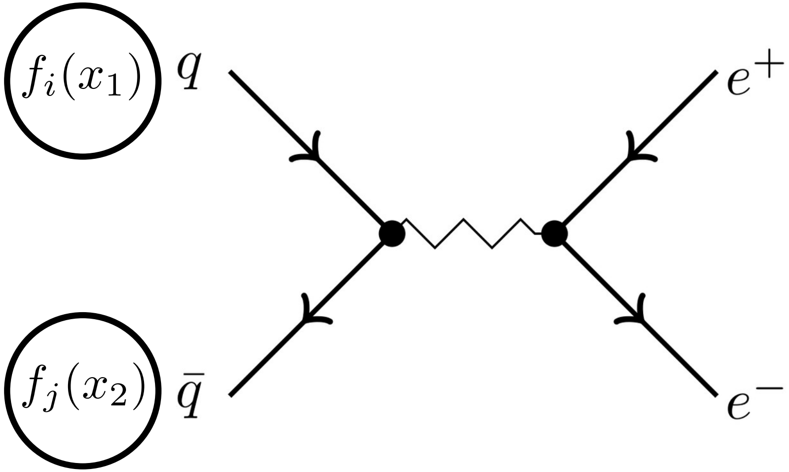
$$P_{gq}(z) = C_F \left(\frac{1+(1-z)^2}{z} \right)$$



$$2C_A \left(\left(\left(\frac{z}{1-z} \right)_+ + \frac{1-z}{z} + z(1-z) \right) + \delta(1-z) \left(\frac{11N_c - 2n_f}{6} \right)$$

The Drell-Yan process

Simplest and probably best understood hadron-hadron process



Kinematics of the boson:

$$q^\mu = (p_1 + p_2)^\mu$$

$$Q = \sqrt{q^2}\,, \quad Y = \frac{1}{2}\ln\frac{q^0+q^3}{q^0-q^3}$$

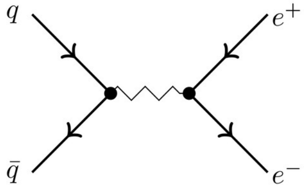
$$\xi_{i/j} = \frac{Q}{\sqrt{s}}e^{\pm Y}$$

$$\sigma_{pp\to e^+e^-} = \sum_{ij}\int_0^1\int_0^1 \mathrm{d}x_1\mathrm{d}x_2 f_i(x_1) f_j(x_2)\hat{\sigma}_{ij\to e^+e^-}(\alpha_s(\mu_R^2),\mu^2) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

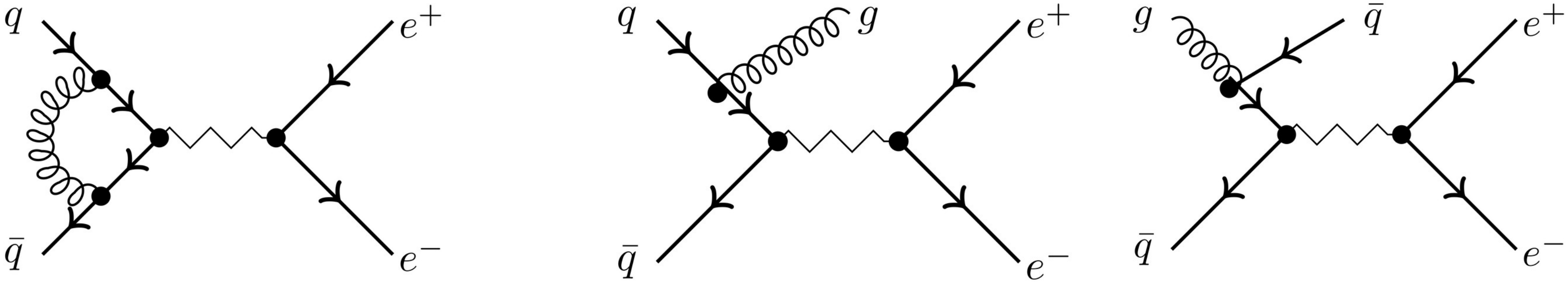
$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q\mathrm{d}Y} = \sum_{ij}\frac{1}{n_i n_j}\frac{Q}{\xi_i\xi_j s}\int_{\xi_i}^1\frac{\mathrm{d}z_i}{z_i}f_i(\frac{\xi_i}{z_i},\mu_F^2)\int_{\xi_j}^1\frac{\mathrm{d}z_j}{z_j}f_j(\frac{\xi_j}{z_j},\mu_F^2)F_{ij}(Q^2,z_i,z_j,\mu_R^2,\mu_F^2)$$

Partonic channels

At leading order we have: $(ij) = (q\bar{q}), (\bar{q}q) \quad \text{for } q \in \{d, u, s, \dots\}$



At next-to-leading order we have: $(ij) = (q\bar{q}), (\bar{q}q), (qg), (gq), (\bar{q}g), (g\bar{q}) \quad \text{for } q \in \{d, u, s, \dots\}$



Collinear initial state singularities

$q\bar{q}$ channel: LO: $F^{(0)}_{q\bar{q}} = \delta(1-z_i)\delta(1-z_j)\bar{F}^0_{q\bar{q}}(Q^2)$

$$F^{(1),\text{real}}_{q\bar{q}} = \frac{\alpha_s}{2\pi} C_F c_\varepsilon \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \left[\delta(1-z_i)\delta(1-z_j)\left(\frac{2}{\varepsilon^2}\right) - \delta(1-z_i)\frac{1}{\varepsilon}\left(\frac{1+z_j^2}{1-z_j}\right)_+ - (z_i \leftrightarrow z_j) + \text{finite}\right] \bar{F}^{(0)}_{q\bar{q}}$$

$$F^{(1),\text{virtual}}_{q\bar{q}} = \frac{\alpha_s}{2\pi} C_F c_\varepsilon \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \left[\delta(1-z_i)\delta(1-z_j)\left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon}\right) + \text{finite}\right]$$

$$F^{(1)}_{q\bar{q}} = \frac{\alpha_s}{2\pi} c_\varepsilon \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \frac{1}{\varepsilon}\left[\delta(1-z_i)\boxed{P_{qq}(z_j)} + (z_i \leftrightarrow z_j)\right] \bar{F}^{(0)}_{q\bar{q}}$$

single poles do not cancel but ~ splitting functions….

qg channels:

$$F^{(1),\text{real}}_{qg} = \frac{\alpha_s}{2\pi} C_F c_\varepsilon \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \left[-\delta(1-z_i)\frac{1}{\varepsilon}\left(z_j^2 + (1-z_j)^2\right) + \text{finite}\right] \bar{F}^{(0)}_{q\bar{q}}$$

$$F^{(1),\text{virtual}}_{qg} = 0$$

$$F^{(1)}_{qg} = \frac{\alpha_s}{2\pi} \frac{N_c^2-1}{N_c} c_\varepsilon \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \left[-\delta(1-z_i)\frac{1}{\varepsilon}\boxed{P_{qg}(z_j)}\right] \bar{F}^{(0)}_{q\bar{q}}$$

$$c_\varepsilon = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}$$

NLO PDFs

Absorb the singularity in the definition of the PDF → renormalised PDFs

$$\frac{1}{\varepsilon}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon} = \frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{Q^2}\right) + \mathcal{O}(\varepsilon^2) = \underbrace{\frac{1}{\varepsilon} + \ln\left(\frac{Q^2}{\mu_F^2}\right)}_{\text{part of the PDF}} + \ln\left(\frac{\mu_F^2}{Q^2}\right) + \mathcal{O}(\varepsilon^2)$$

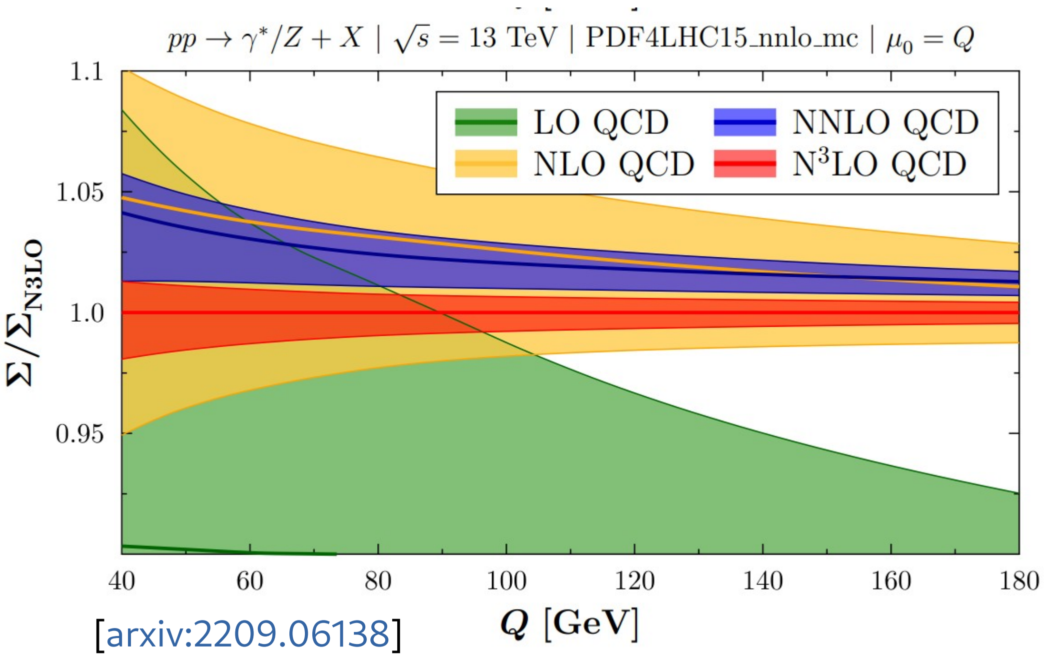
part of the partonic cross section

Renormalised PDFs

$$f_a(x) \to f_a(x,\mu_F^2) = f_a(x) + \sum_b \frac{\alpha_s}{2\pi} c_\varepsilon \left(\frac{\mu^2}{\mu_F^2}\right)^{\varepsilon} \frac{1}{\varepsilon} \int_x^1 P_{ab}(z) f_b(x/z)$$

Important: the PDF depends now also on the order of the computation
For example: NNPDF31_nnlo_as0118 vs. NNPDF31_nlo_as0118

Drell-Yan higher order QCD cross sections



[arxiv:2209.06138]

Perturbative convergence:

- O(10%) correction at NLO
- O(1%) correction at NNLO
- O(1%) correction at N3LO?
  (PDF consistency?)

Being evermore precise is good, but

how to derive uncertainties?

Theory uncertainties from scale variations

$$d\sigma = d\sigma^{(0)} + \alpha_s d\sigma^{(1)} + \alpha_s^2 d\sigma^{(2)} + \ldots \xrightarrow{\text{RGE}} \mu \frac{d\sigma^{(n)}}{d\mu} = \mathcal{O}\left(\alpha_s^{(n_0+n+1)}\right)$$

Of same order as the next dominant term → exploiting this to estimate size of $d\sigma^{(n+1)}$

Scale variation prescription (ad-hoc and heuristic choice)

- choose 'sensible' μ_0

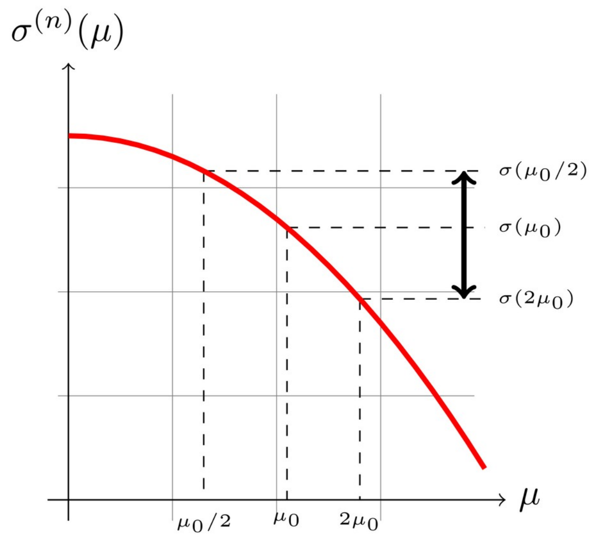
→ principle of fasted apparent convergence: $\sigma^{(n)}(\mu_{\mathrm{FAC}}) = 0$

→ principle of minimal sensitivity: $\mu \frac{d\sigma(\mu)}{d\mu}\Big|_{\mu=\mu_{\mathrm{PMC}}} = 0$

→ ...

- vary with a factor (typically 2)

- take envelope as uncertainty



Scale variation approach

Change of scale = change of renormalisation scheme: $\tilde{\alpha}(\alpha) = \alpha(1 + b_0\alpha + b_1\alpha^2 + b_2\alpha^3 + \dots)$

$f(\alpha) = f_0 + f_1\alpha + f_2\alpha^2 + f_3\alpha^3 + \dots$

For QCD: $\alpha = \alpha_s(\mu_0)$ $\quad \tilde{\alpha} = \alpha_s(\mu)$

$$b_0 = \frac{\beta_0}{2\pi}L \qquad b_1 = \frac{\beta_0^2}{4\pi^2}L^2 + \frac{\beta_1}{8\pi^2}L \qquad b_2 = \frac{\beta_0^3}{8\pi^3}L^3 + \frac{5\beta_0\beta_1}{32\pi^2}L^2 + \frac{\beta_2}{32\pi^3}L \qquad L = \ln\frac{\mu_0}{\mu}$$

$$\tilde{f}^{\mathrm{LO}}(\tilde{\alpha}) = \tilde{f}_0 = f_0$$

$$\tilde{f}^{\mathrm{NLO}}(\tilde{\alpha}) = \tilde{f}_0 + \tilde{f}_1\tilde{\alpha} = f_0 + \alpha f_1 + \boxed{\alpha^2 b_0 f_1} + \mathcal{O}(\alpha^3)$$

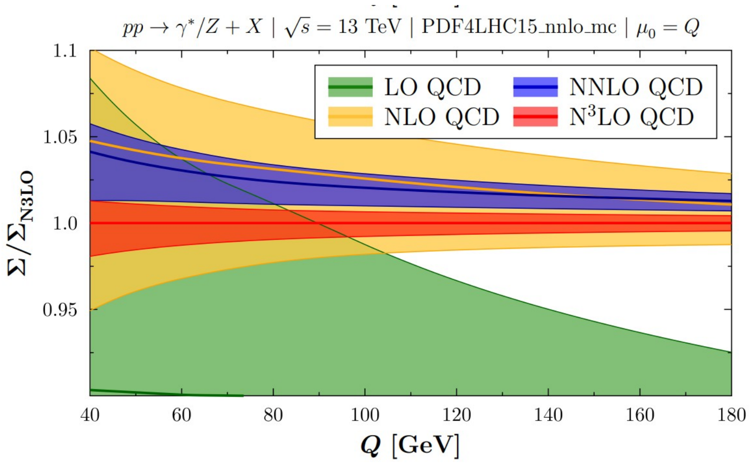
$$\tilde{f}^{\mathrm{NNLO}}(\tilde{\alpha}) = \tilde{f}_0 + \tilde{f}_1\tilde{\alpha} + \tilde{f}_2\tilde{\alpha}^2 = f_0 + \alpha f_1 + \alpha^2 f_2 + \boxed{\alpha^3(2b_0(f_2 - b_0 f_1) + b_1 f_1)} + \mathcal{O}(\alpha^4)$$

Scale variations as uncertainties can work ...

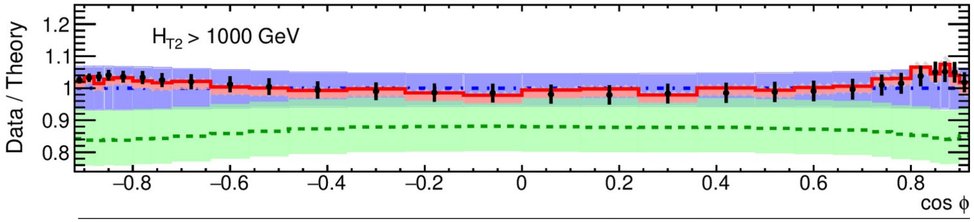
$$\sigma_{h_1h_2\to X} = \sum_{ij} \int_0^1 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 f_{i,h_1}(x_1,\mu_F^2) f_{j,h_2}(x_2,\mu_F^2) \hat{\sigma}_{ij\to X}(\alpha_s(\mu_R^2),\mu_R^2,\mu_F^2) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

→ two scales: renormalisation and factorisation scale
→ conventional 7-point variations by a factor of 2

"Agreement within the variation envelope"

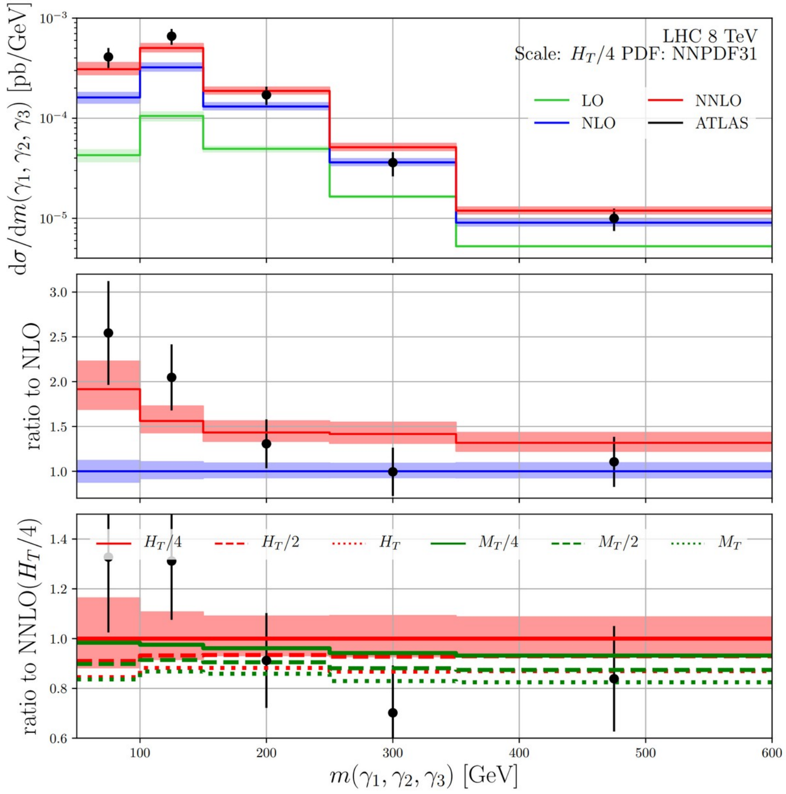


multi-jet cross section (TEEC)

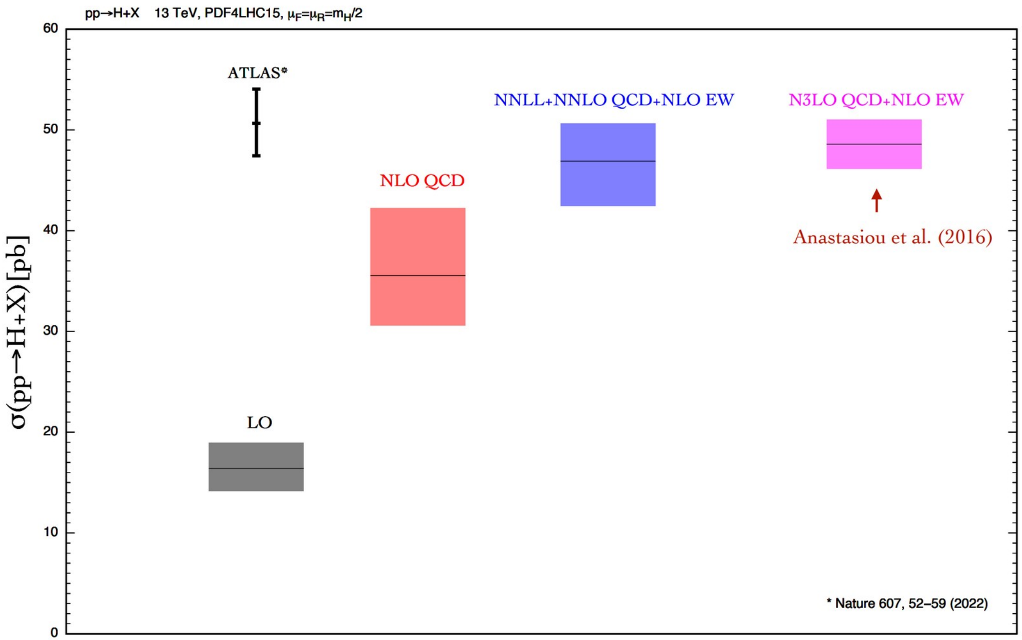


...sometimes :/

Three photon production



Higgs production



[talk by Grazzini]

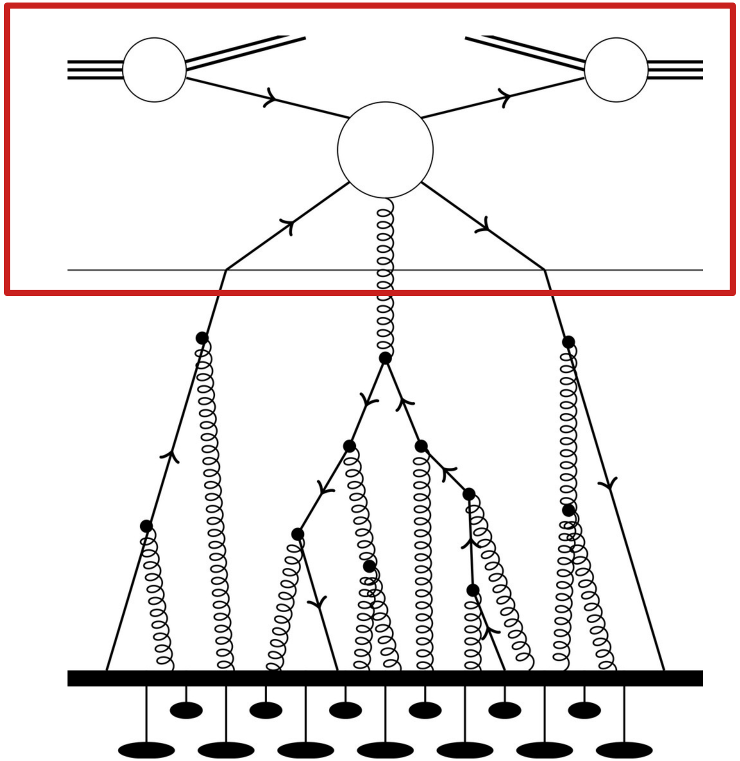
NNLO QCD needed before "convergence" kicks in...

Short comings of scale variations

- not always reliable … however in most cases issues are understood/expected:
  new channels, phase space constraints, etc. → often we can design workarounds
- however, some issues are more fundamental:
  → how to choose the central scale? → not a physical parameter, no ‘true’ value
    (Principle of fasted apparent convergence, principle of minimal sensitivity,...)
  → how to propagate the estimated uncertainty, no statistical interpretation!
  → what about correlations? Based on ‘fixed form’ of the lower orders and RGE.
- Alternatives:
  - Bayesian methods
  - Theory Nuisance Parameters
- with increasing precision this becomes more relevant...

End of lecture

- Overview over various aspects of higher order QCD
- Importance of infrared behaviour of massless gauge theories
- Pointers to current research topics



Things I didn't cover but are important for LHC pheno:

- NLO electro-weak corrections
- Matching to parton-shower and resummation (see lectures by)
- Jet algorithms and jet physics