Introduction to Event Generators

Part 2: Parton Showers and Jet Physics

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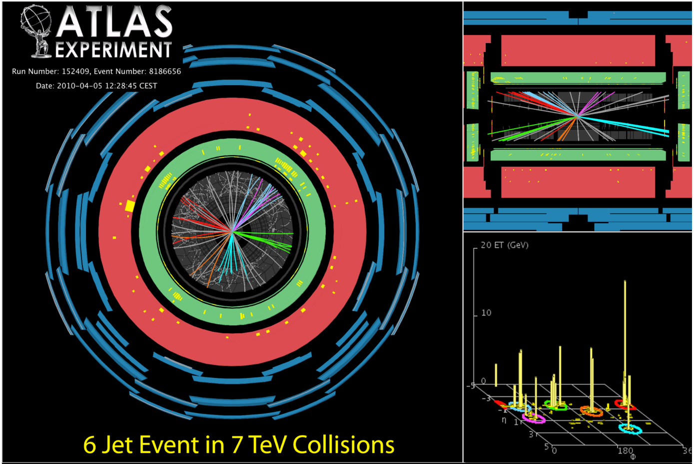
Multijets – the need for Higher Orders



An event with 6 jets taken on April 4th, 2010. The jets have calibrated transverse momenta between 30 GeV and 70 GeV and are well separated in the detector. 12

$2 \to 6$ process or $2 \to 2$ dressed up by bremsstrahlung!?

Perturbative QCD

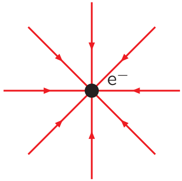
Perturbative calculations $\Rightarrow$ **Matrix Elements**.
Improved calculational techniques allows
$\star$ more **legs** (= final-state partons)
$\star$ more **loops** (= virtual partons not visible in final state)
but with limitations, especially for loops.
See presentations by Rene Poncelet and Andrea Banfi.

Parton Showers:
approximations to matrix element behaviour,
most relevant for multiple emissions at low energies and/or angles.
Main topic of this lecture.

Matching and Merging:
methods to combine matrix elements (at high scales)
with parton showers (at low scales),
with a consistent and smooth transition.
Huge field at LHC.

In the beginning: Electrodynamics

An electrical charge, say an electron,
is surrounded by a field:



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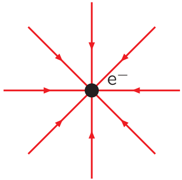
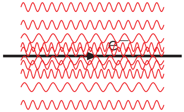


For a rapidly moving charge
this field can be expressed in terms of
an equivalent flux of photons:



$$\mathrm{dn}_\gamma \approx \frac{2\alpha_{\mathrm{em}}}{\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega}$$

Equivalent Photon Approximation,
or method of virtual quanta (e.g. Jackson)
(Bohr; Fermi; Weiszäcker, Williams ∼1934)

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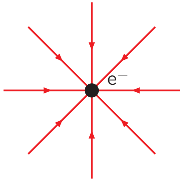

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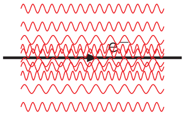


Equivalent Photon Approximation,
or method of virtual quanta (e.g. Jackson)
(Bohr; Fermi; Weiszäcker, Williams $\sim$1934)

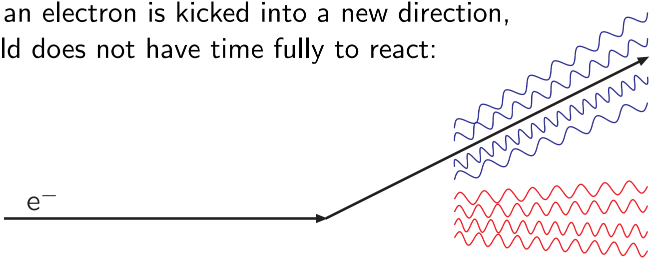
θ: collinear divergence, saved by $m_{\mathrm{e}} > 0$ in full expression.

ω: true divergence, $n_\gamma \propto \int \mathrm{d}\omega/\omega = \infty$, but $E_\gamma \propto \int \omega \, \mathrm{d}\omega/\omega$ finite.

These are virtual photons: continuously emitted and reabsorbed.

In the beginning: Bremsstrahlung

When an electron is kicked into a new direction,
the field does not have time fully to react:



- Initial State Radiation (ISR):
  part of it continues $\sim$ in original direction of e
- Final State Radiation (FSR):
  the field needs to be regenerated around outgoing e,
  and transients are emitted $\sim$ around outgoing e direction

Emission rate provided by equivalent photon flux in both cases.
Approximate cutoffs related to timescale of process:
the more violent the hard collision, the more radiation!

In the beginning: Exponentiation

Assume $\sum E_\gamma \ll E_\mathrm{e}$ such that energy-momentum conservation is not an issue. Then

$$\mathrm{d}\mathcal{P}_\gamma = \mathrm{dn}_\gamma \approx \frac{2\alpha_\mathrm{em}}{\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega}$$

is the probability to find a photon at ω and θ,
irrespectively of which other photons are present.

Uncorrelated $\Rightarrow$ Poissonian number distribution:

$$\mathcal{P}_i = \frac{\langle n_\gamma \rangle^i}{i!} e^{-\langle n_\gamma \rangle}$$

with

$$\langle n_\gamma \rangle = \int_{\theta_\mathrm{min}}^{\theta_\mathrm{max}} \int_{\omega_\mathrm{min}}^{\omega_\mathrm{max}} \mathrm{dn}_\gamma \approx \frac{2\alpha_\mathrm{em}}{\pi} \ln\left(\frac{\theta_\mathrm{max}}{\theta_\mathrm{min}}\right) \ln\left(\frac{\omega_\mathrm{max}}{\omega_\mathrm{min}}\right)$$

Note that $\int \mathrm{d}\mathcal{P}_\gamma = \int \mathrm{dn}_\gamma > 1$ is not a problem:
proper interpretation is that *many* photons are emitted.

Exponentiation: reinterpretation of $\mathrm{d}\mathcal{P}_\gamma$ into Poissonian.

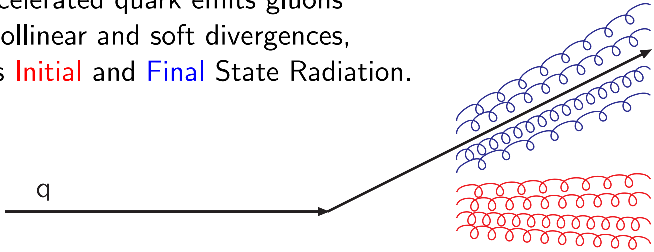
So how is QCD the same?

- A quark is surrounded by a gluon field

$$\mathrm{d}\mathcal{P}_{\mathrm{g}} = \mathrm{dn}_{\mathrm{g}} \approx \frac{8\alpha_{\mathrm{s}}}{3\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega}$$

  i.e. only differ by substitution $(-1)^2\alpha_{\mathrm{em}} \to (4/3)\alpha_{\mathrm{s}}$.

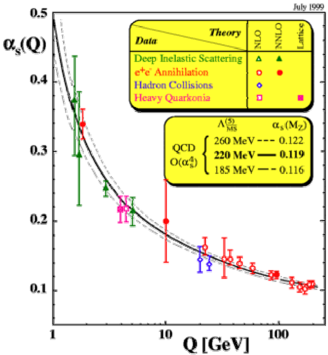
- An accelerated quark emits gluons with collinear and soft divergences, and as Initial and Final State Radiation.



- Typically $\langle n_{\mathrm{g}} \rangle = \int \mathrm{dn}_{\mathrm{g}} \gg 1$ since $\alpha_{\mathrm{s}} \gg \alpha_{\mathrm{em}}$
  $\Rightarrow$ even more pressing need for exponentiation.

So how is QCD different?

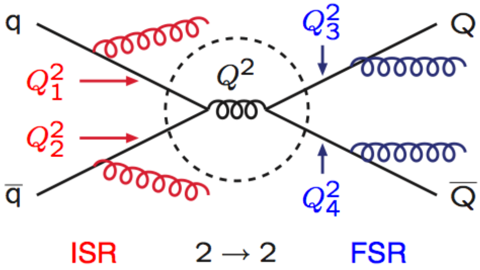
- QCD is non-Abelian, so a gluon is charged and is surrounded by its own field: emission rate $(4/3)\alpha_\mathrm{s} \to 3\alpha_\mathrm{s}$, field structure more complicated, interference effects more important.
- $\alpha_s(Q^2)$ diverges for $Q^2 \to \Lambda^2_\mathrm{QCD}$, with $\Lambda_\mathrm{QCD} \sim 0.2\,\mathrm{GeV} = 1\,\mathrm{fm}^{-1}$.
- Confinement: gluons below Λ_QCD not resolved $\Rightarrow$ de facto cutoffs.



Unclear separation between "accelerated charge" and "emitted radiation": many possible Feynman graphs $\approx$ histories.

The Parton-Shower Approach

$$2 \to n \quad = \quad (2 \to 2) \oplus \text{ISR} \oplus \text{FSR}$$

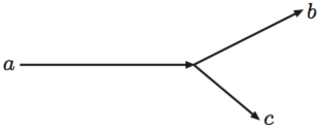


FSR = Final-State Radiation = timelike shower
$Q_i^2 \sim m^2 > 0$ decreasing
ISR = Initial-State Radiation = spacelike showers
$Q_i^2 \sim -m^2 > 0$ increasing

Why "time" like and "space" like?

Consider four-momentum conservation in a branching $a \to b\,c$

$$\mathbf{p}_{\perp a} = 0 \quad \Rightarrow \quad \mathbf{p}_{\perp c} = -\mathbf{p}_{\perp b}$$
$$p_+ = E + p_\mathrm{L} \quad \Rightarrow \quad p_{+a} = p_{+b} + p_{+c}$$
$$p_- = E - p_\mathrm{L} \quad \Rightarrow \quad p_{-a} = p_{-b} + p_{-c}$$



Define $p_{+b} = z\,p_{+a}, \quad p_{+c} = (1-z)\,p_{+a}$

Use $p_+ p_- = E^2 - p_\mathrm{L}^2 = m^2 + p_\perp^2$

$$\frac{m_a^2 + p_{\perp a}^2}{p_{+a}} = \frac{m_b^2 + p_{\perp b}^2}{z\,p_{+a}} + \frac{m_c^2 + p_{\perp c}^2}{(1-z)\,p_{+a}}$$

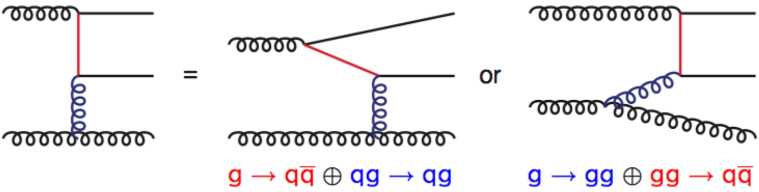
$$\Rightarrow \quad m_a^2 = \frac{m_b^2 + p_\perp^2}{z} + \frac{m_c^2 + p_\perp^2}{1-z} = \frac{m_b^2}{z} + \frac{m_c^2}{1-z} + \frac{p_\perp^2}{z(1-z)}$$

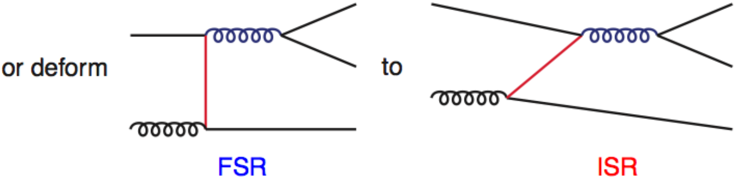
Final-state shower: $m_b = m_c = 0 \Rightarrow m_a^2 = \frac{p_\perp^2}{z(1-z)} > 0 \Rightarrow$ timelike

Initial-state shower: $m_a = m_c = 0 \Rightarrow m_b^2 = -\frac{p_\perp^2}{1-z} < 0 \Rightarrow$ spacelike

Doublecounting

A $2 \to n$ graph can be "simplified" to $2 \to 2$ in different ways:



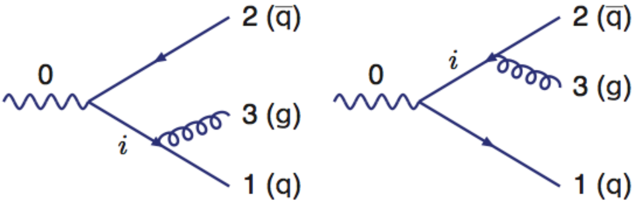


Do not doublecount: $2 \to 2$ = most virtual = shortest distance

(detailed handling of borders ⇒ **match & merge**)

Final-state radiation

Standard process $\boxed{\mathrm{e}^+\mathrm{e}^- \to \mathrm{q}\overline{\mathrm{q}}\mathrm{g}}$ by two Feynman diagrams:



$$x_i = \frac{2E_i}{E_{\mathrm{cm}}}$$

$$x_1 + x_2 + x_3 = 2$$

$$\frac{\mathrm{d}\sigma_{\mathrm{ME}}}{\sigma_0} = \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{4}{3}\,\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

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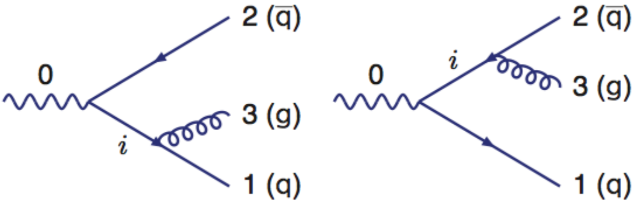

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Convenient (but arbitrary) subdivision to “split” radiation:

$$\frac{1}{(1-x_1)(1-x_2)}\,\frac{(1-x_1)+(1-x_2)}{x_3} = \frac{1}{(1-x_2)x_3} + \frac{1}{(1-x_1)x_3}$$

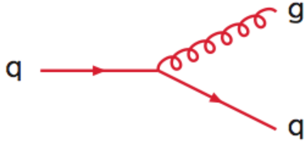
From matrix elements to parton showers

Rewrite for $x_2 \to 1$, i.e. q–g collinear limit:

$$1 - x_2 = \frac{m_{13}^2}{E_{\mathrm{cm}}^2} = \frac{Q^2}{E_{\mathrm{cm}}^2} \Rightarrow \mathrm{d}x_2 = \frac{\mathrm{d}Q^2}{E_{\mathrm{cm}}^2}$$

define z as fraction q retains in branching $\mathrm{q} \to \mathrm{qg}$

$$\begin{aligned} x_1 &\approx z \quad \Rightarrow \quad \mathrm{d}x_1 \approx \mathrm{d}z \\ x_3 &\approx 1 - z \end{aligned}$$



$$\Rightarrow \mathrm{d}\mathcal{P} = \frac{\mathrm{d}\sigma}{\sigma_0} = \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{\mathrm{d}x_2}{(1-x_2)} \frac{4}{3} \frac{x_2^2 + x_1^2}{(1-x_1)} \mathrm{d}x_1 \approx \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{\mathrm{d}Q^2}{Q^2} \frac{4}{3} \frac{1+z^2}{1-z} \mathrm{d}z$$

In limit $x_1 \to 1$ same result, but for $\overline{\mathrm{q}} \to \overline{\mathrm{q}}\mathrm{g}$.

$\mathrm{d}Q^2/Q^2 = \mathrm{d}m^2/m^2$: "mass (or collinear) singularity"

$\mathrm{d}z/(1-z) = \mathrm{d}\omega/\omega$ "soft singularity"

The DGLAP equations

Generalizes to

DGLAP (Dokshitzer–Gribov–Lipatov–Altarelli–Parisi)

$$
\begin{aligned}
\mathrm{d}\mathcal{P}_{a\to bc} &= \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z \\
P_{\mathrm{q}\to\mathrm{qg}} &= \frac{4}{3}\,\frac{1+z^2}{1-z} \\
P_{\mathrm{g}\to\mathrm{gg}} &= 3\,\frac{(1-z(1-z))^2}{z(1-z)} \\
P_{\mathrm{g}\to\mathrm{q}\overline{\mathrm{q}}} &= \frac{n_f}{2}\,(z^2+(1-z)^2) \quad (n_f = \text{no. of quark flavours})
\end{aligned}
$$

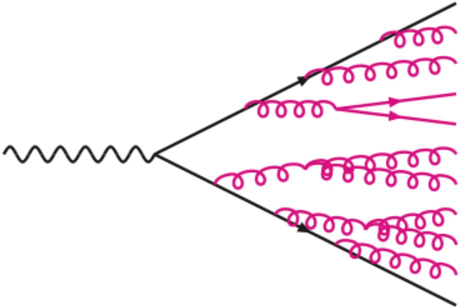
Universality: any matrix element reduces to DGLAP in collinear limit.

$$
\text{e.g.}\quad \frac{\mathrm{d}\sigma(\mathrm{H}^0\to\mathrm{q}\overline{\mathrm{q}}\mathrm{g})}{\mathrm{d}\sigma(\mathrm{H}^0\to\mathrm{q}\overline{\mathrm{q}})} = \frac{\mathrm{d}\sigma(\mathrm{Z}^0\to\mathrm{q}\overline{\mathrm{q}}\mathrm{g})}{\mathrm{d}\sigma(\mathrm{Z}^0\to\mathrm{q}\overline{\mathrm{q}})} \quad \text{in collinear limit}
$$

The iterative structure

Generalizes to many consecutive emissions if strongly ordered, $Q_1^2 \gg Q_2^2 \gg Q_3^2 \ldots$ ($\approx$ time-ordered).
To cover "all" of phase space use DGLAP in whole region $Q_1^2 > Q_2^2 > Q_3^2 \ldots$.

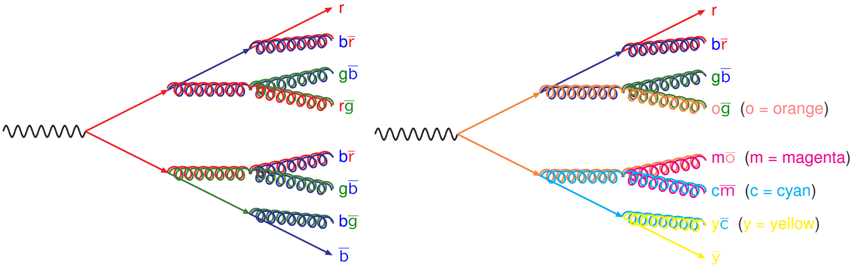
Iteration gives final-state parton showers:



Need soft/collinear cuts to stay away from nonperturbative physics.
Details model-dependent, but around 1 GeV scale.

Planar QCD

With $N_C = 3$ you need to reuse colours, but not if $N_C \to \infty$:



Colour lines crossed between $\mathcal{M}$ and $\mathcal{M}^\dagger$ scale like $1/N_C^2$ in $|\mathcal{M}|^2$, so vanish for $N_C \to \infty \Rightarrow$ planar QCD. Thus

$$\sigma = \sigma_{\mathrm{LC}} + \frac{1}{N_C^2}\sigma_{\mathrm{NLC}} + \frac{1}{N_C^4}\sigma_{\mathrm{NNLC}} + \cdots$$

Also showers and hadronization become simpler in this limit. Still use correct $N_c = 3$ for exact calculations, but $N_C \to \infty$ for colour connections in hard process and shower history.

The Sudakov form factor – 1

Time evolution, conservation of total probability:
$\mathcal{P}(\mathrm{no\ emission}) = 1 - \mathcal{P}(\mathrm{emission})$.

Multiplicativeness, with $T_i = (i/n)T$, $0 \leq i \leq n$:

$$\begin{aligned}
\mathcal{P}_{\mathrm{no}}(0 \leq t < T) &= \lim_{n\to\infty} \prod_{i=0}^{n-1} \mathcal{P}_{\mathrm{no}}(T_i \leq t < T_{i+1}) \\
&= \lim_{n\to\infty} \prod_{i=0}^{n-1} (1 - \mathcal{P}_{\mathrm{em}}(T_i \leq t < T_{i+1})) \\
&= \exp\left(- \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathcal{P}_{\mathrm{em}}(T_i \leq t < T_{i+1}) \right) \\
&= \exp\left(- \int_0^T \frac{\mathrm{d}\mathcal{P}_{\mathrm{em}}(t)}{\mathrm{d}t} \mathrm{d}t \right) \\
\Longrightarrow \quad \mathrm{d}\mathcal{P}_{\mathrm{first}}(T) &= \mathrm{d}\mathcal{P}_{\mathrm{em}}(T) \exp\left(- \int_0^T \frac{\mathrm{d}\mathcal{P}_{\mathrm{em}}(t)}{\mathrm{d}t} \mathrm{d}t \right)
\end{aligned}$$

cf. radioactive decay in lecture 1.

The Sudakov form factor – 2

Expanded, with $Q \sim 1/t$ (Heisenberg)

$$\begin{aligned} \mathrm{d}\mathcal{P}_{a\to bc} &= \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z \\ &\times \exp\left(-\sum_{b,c}\int_{Q^2}^{Q^2_\mathrm{max}}\frac{\mathrm{d}Q'^2}{Q'^2}\int\frac{\alpha_\mathrm{s}}{2\pi}\,P_{a\to bc}(z')\,\mathrm{d}z'\right) \end{aligned}$$

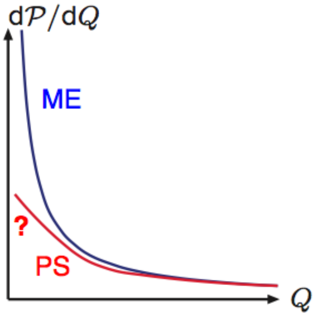
where the exponent is (one definition of) the Sudakov form factor

A given parton can only branch once, i.e. if it did not already do so

Note that $\sum_{b,c}\int\int \mathrm{d}\mathcal{P}_{a\to bc} \equiv 1 \Rightarrow$ *convenient for Monte Carlo*
($\equiv 1$ if extended over whole phase space, else possibly nothing happens before you reach $Q_0 \approx 1$ GeV).

The Sudakov form factor – 3

Sudakov regulates singularity for *first* emission ...



...but in limit of *repeated soft* emissions $\mathrm{q} \to \mathrm{qg}$ (but no $\mathrm{g} \to \mathrm{gg}$) one obtains the same inclusive Q emission spectrum as for ME,

i.e. divergent ME spectrum
$\Longleftrightarrow$ infinite number of PS emissions

More complicated in reality:

- energy-momentum conservation effects big since α_{s} big, so hard emissions frequent
- $\mathrm{g} \to \mathrm{gg}$ branchings leads to accelerated multiplication of partons

The ordering variable

In the evolution with

$$\mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z$$

Q^2 orders the emissions (memory).

If $Q^2 = m^2$ is one possible evolution variable
then $Q'^2 = f(z)Q^2$ is also allowed, since

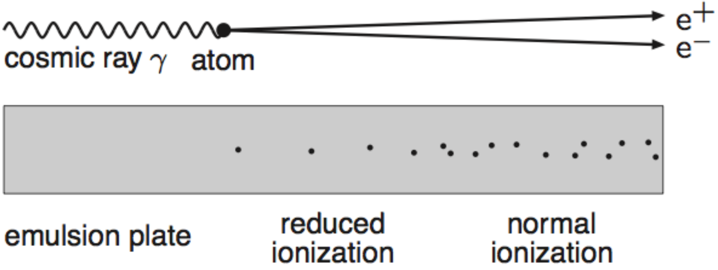
$$\left|\frac{\mathrm{d}(Q'^2,z)}{\mathrm{d}(Q^2,z)}\right| = \left|\begin{array}{cc} \frac{\partial Q'^2}{\partial Q^2} & \frac{\partial Q'^2}{\partial z} \\ \frac{\partial z}{\partial Q^2} & \frac{\partial z}{\partial z} \end{array}\right| = \left|\begin{array}{cc} f(z) & f'(z)Q^2 \\ 0 & 1 \end{array}\right| = f(z)$$

$$\Rightarrow \mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{f(z)\mathrm{d}Q^2}{f(z)Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z = \frac{\alpha_{\mathrm{s}}}{2\pi}\,\frac{\mathrm{d}Q'^2}{Q'^2}\,P_{a\to bc}(z)\,\mathrm{d}z$$

- $Q'^2 = E_a^2\theta_{a\to bc}^2 \approx m^2/(z(1-z))$; angular-ordered shower
- $Q'^2 = p_\perp^2 \approx m^2 z(1-z)$; transverse-momentum-ordered

Coherence

QED: Chudakov effect (mid-fifties)



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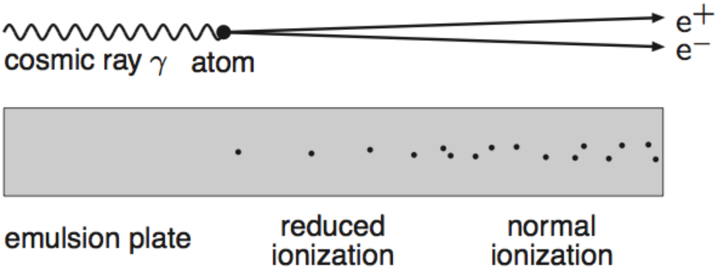
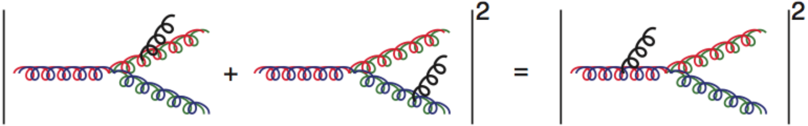


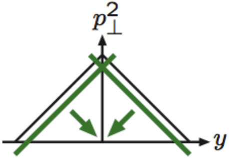
QCD: colour coherence for **soft** gluon emission



solved by • requiring emission angles to be decreasing
or • requiring transverse momenta to be decreasing

Ordering variables in the LEP/Tevatron era

PYTHIA: $Q^2 = m^2$



large mass first

$\Rightarrow$ "hardness" ordered

coherence brute force

covers phase space

ME merging simple

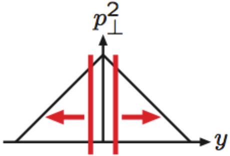
g $\to$ q$\overline{\text{q}}$ simple

not Lorentz invariant

no stop/restart

ISR: $m^2 \to -m^2$

HERWIG: $Q^2 \sim E^2\theta^2$



large angle first

$\Rightarrow$ **hardness not ordered**

coherence inherent

gaps in coverage

ME merging messy

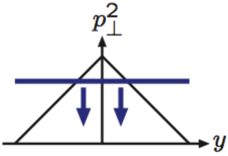
g $\to$ q$\overline{\text{q}}$ simple

not Lorentz invariant

no stop/restart

ISR: $\theta \to \theta$

ARIADNE: $Q^2 = p_\perp^2$



large $p_\perp$ first

$\Rightarrow$ "hardness" ordered

coherence inherent

covers phase space

ME merging simple

g $\to$ q$\overline{\text{q}}$ **messy**

Lorentz invariant

can stop/restart

ISR: more messy

The HERWIG algorithm

Basic ideas, to which much has been added over the years:

1. Evolution in $Q_a^2 = E_a^2 \xi_a$ with $\xi_a \approx 1 - \cos\theta_a$, i.e.

$$\mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_\mathrm{s}}{2\pi} \frac{\mathrm{d}(E_a^2 \xi_a)}{E_a^2 \xi_a} P_{a\to bc}(z)\,\mathrm{d}z = \frac{\alpha_\mathrm{s}}{2\pi} \frac{\mathrm{d}\xi_a}{\xi_a} P_{a\to bc}(z)\,\mathrm{d}z$$

   Require ordering of consecutive ξ values, i.e. $(\xi_b)_\mathrm{max} < \xi_a$ and $(\xi_c)_\mathrm{max} < \xi_a$.

2. Reconstruct masses backwards in algorithm
   $m_a^2 = m_b^2 + m_c^2 + 2E_b E_c \xi_a$
   Note: $\xi_a = 1 - \cos\theta_a$ only holds for $m_b = m_c = 0$.

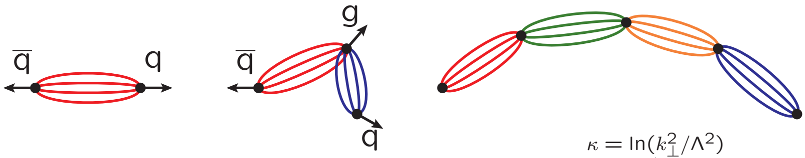
3. Reconstruct complete kinematics of shower (forward again).

\+ angular ordering built in from start

− total jet/system mass not known beforehand ($\Rightarrow$ boosts)

− some wide-angle regions never populated, “dead zones”

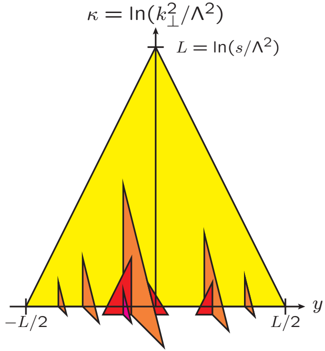
Dual description of partonic state:
partons connected by dipoles ⇔ dipoles stretched between partons
parton branching ⇔ dipole splitting



$p_\perp$-ordered dipole emissions ⇒ **coherence** (cf. angular ordering).

2 → 3 on-shell parton branchings with local $(E, \mathbf{p})$ conservation.
ARIADNE shower + many more.

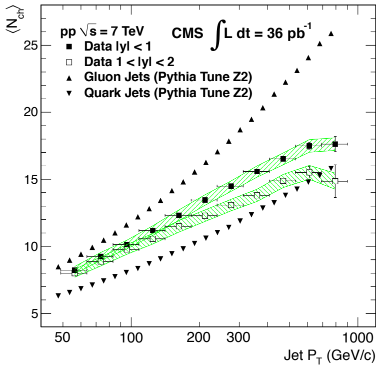
Neat representation in **Lund plane** (hot topic today).

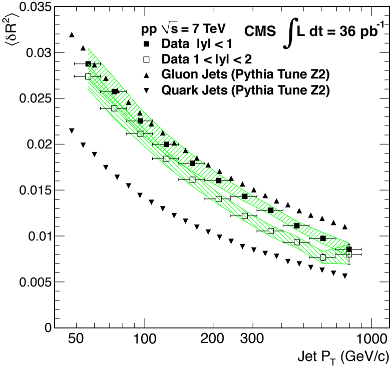


Quark vs. gluon jets

$$\frac{P_{\mathrm{g}\to\mathrm{gg}}}{P_{\mathrm{q}\to\mathrm{qg}}} \approx \frac{N_c}{C_F} = \frac{3}{4/3} = \frac{9}{4} \approx 2$$

$\Rightarrow$ gluon jets are softer and broader than quark ones
(also helped by hadronization models, lecture 4).





Note transition g jets $\to$ q jets for increasing $p_\perp$.

Heavy flavours: the dead cone

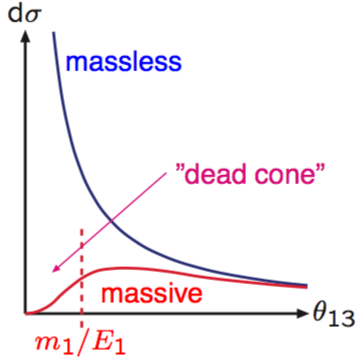
Matrix element for $\mathrm{e}^+\mathrm{e}^- \to \mathrm{q\overline{q}g}$ for small θ_{13}

$$\frac{\mathrm{d}\sigma_{\mathrm{q\overline{q}g}}}{\sigma_{\mathrm{q\overline{q}}}} \propto \frac{x_1^2 + x_2^2}{(1-x_1)\,(1-x_2)} \approx \frac{\mathrm{d}\omega}{\omega}\,\frac{\mathrm{d}\theta_{13}^2}{\theta_{13}^2}$$

is modified for heavy quark Q:

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathrm{q\overline{q}g}}}{\sigma_{\mathrm{q\overline{q}}}} &\propto \frac{\mathrm{d}\omega}{\omega}\,\frac{\mathrm{d}\theta_{13}^2}{\theta_{13}^2}\left(\frac{\theta_{13}^2}{\theta_{13}^2 + m_1^2/E_1^2}\right)^2 \\ &= \frac{\mathrm{d}\omega}{\omega}\,\frac{\theta_{13}^2\,\mathrm{d}\theta_{13}^2}{(\theta_{13}^2 + m_1^2/E_1^2)^2}\end{aligned}$$

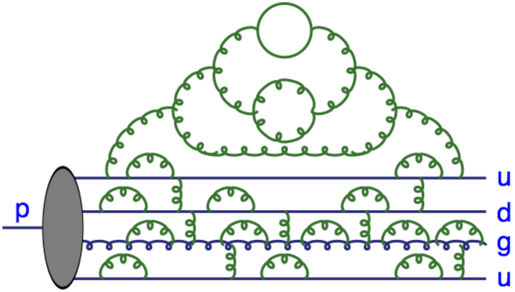
so "dead cone" for $\theta_{13} < m_1/E_1$



For charm and bottom lagely filled in by their decay products.

Parton Distribution Functions

Hadrons are composite, with time-dependent structure:

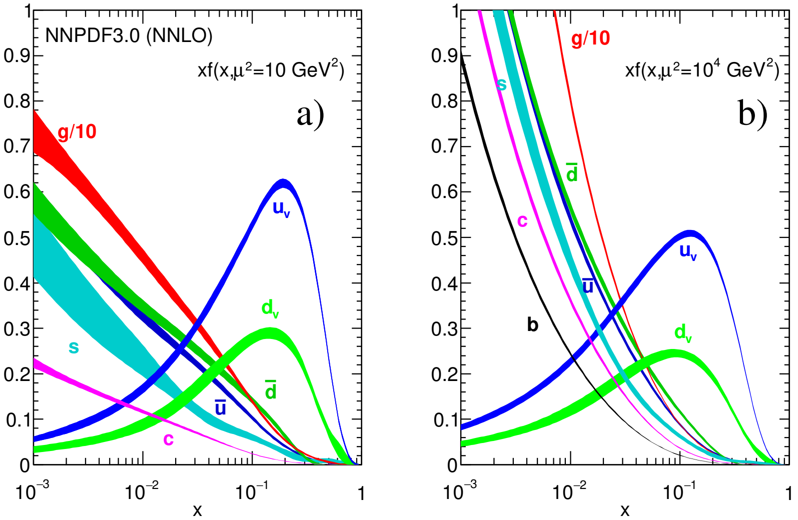


$f_i(x, Q^2)$ = number density of partons i
at momentum fraction x and probing scale Q^2.
Linguistics (example):

$$F_2(x, Q^2) \quad = \quad \sum_i e_i^2 \, x f_i(x, Q^2)$$

structure function — parton distributions

PDF example



Several PDF collaborations: CTEQ, MMHT, NNPDF, ...

See presentation by Katerina Lipka tomorrow.

PDF evolution

Initial conditions at small Q_0^2 unknown: nonperturbative.

Resolution dependence perturbative, by DGLAP:

DGLAP (Dokshitzer–Gribov–Lipatov–Altarelli–Parisi)

$$\frac{\mathrm{d}f_b(x,Q^2)}{\mathrm{d}(\ln Q^2)} = \sum_a \int_x^1 \frac{\mathrm{d}z}{z}\, f_a(y,Q^2)\, \frac{\alpha_\mathrm{s}}{2\pi}\, P_{a\to bc}\left(z=\frac{x}{y}\right)$$

DGLAP already introduced for (final-state) showers:

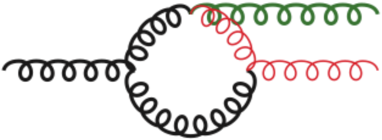
$$\mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z$$

Same equation, but different context:

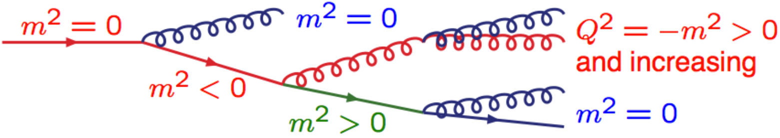
- $\mathrm{d}\mathcal{P}_{a\to bc}$ is probability for the individual parton to branch; while
- $\mathrm{d}f_b(x,Q^2)$ describes how the ensemble of partons evolve by the branchings of individual partons as above.

Initial-State Shower Basics

- Parton cascades in p are continuously born and recombined.
- Structure at Q is resolved at a time $t \sim 1/Q$ *before* collision.
- A hard scattering at Q^2 probes fluctuations up to that scale.
- A hard scattering inhibits full recombination of the cascade.



- Convenient reinterpretation:



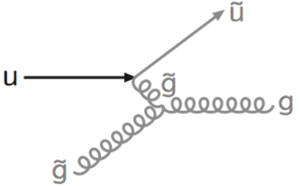
Forwards vs. backwards evolution

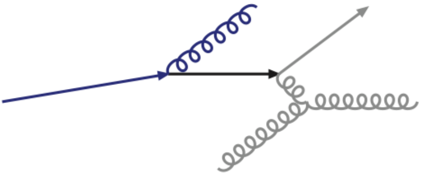
Event generation could be addressed by **forwards evolution**:
pick a complete partonic set at low Q_0 and evolve,
consider collisions at different Q^2 and pick by σ of those.

Inefficient:

1. have to evolve and check for *all* potential collisions, but 99.9...% inert
2. impossible (or at least very complicated) to steer the production, e.g. of a narrow resonance (Higgs)

Backwards evolution is viable and ∼equivalent alternative:
start at hard interaction and trace what happened "before"





Backwards evolution master formula

Monte Carlo approach, based on *conditional probability*: recast

$$\frac{\mathrm{d}f_b(x,Q^2)}{\mathrm{d}t} = \sum_a \int_x^1 \frac{\mathrm{d}z}{z}\, f_a(x',Q^2)\, \frac{\alpha_\mathrm{s}}{2\pi}\, P_{a\to bc}(z)$$

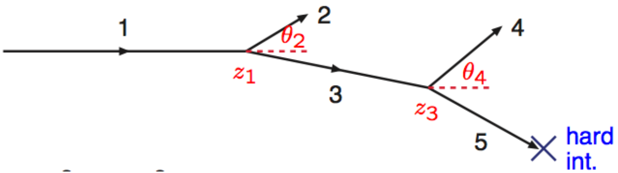
with $t = \ln(Q^2/\Lambda^2)$ and $z = x/x'$ to

$$\mathrm{d}\mathcal{P}_b = \frac{\mathrm{d}\mathrm{f}_b}{f_b} = |\mathrm{d}t| \sum_a \int \mathrm{d}z\, \frac{x' f_a(x',t)}{x f_b(x,t)}\, \frac{\alpha_\mathrm{s}}{2\pi}\, P_{a\to bc}(z)$$

then solve for *de*creasing t, i.e. backwards in time,
starting at high Q^2 and moving towards lower,
with Sudakov form factor $\exp(-\int \mathrm{d}\mathcal{P}_b)$.

Extra factor $x' f_a / x f_b$ relative to final-state equations.

Coherence in spacelike showers

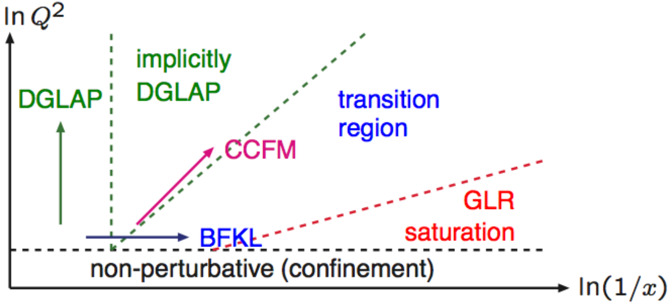


$z_1 = E_3/E_1$
$z_3 = E_5/E_3$
$\theta_2 = \theta_{12}$
$\theta_4 = \theta_{14}$!!

with $Q^2 = -m^2 =$ spacelike virtuality

- kinematics only:
  $Q_3^2 > z_1 Q_1^2,\ Q_5^2 > z_3 Q_3^2, \ldots$
  i.e. Q_i^2 need not even be ordered
- coherence of leading collinear singularities:
  $Q_5^2 > Q_3^2 > Q_1^2$, i.e. Q^2 ordered
- coherence of leading soft singularities (more messy):
  $E_3\theta_4 > E_1\theta_2$, i.e. $z_1\theta_4 > \theta_2$
  $z \ll 1$: $E_1\theta_2 \approx p_{\perp 2}^2 \approx Q_3^2,\ E_3\theta_4 \approx p_{\perp 4}^2 \approx Q_5^2$
  i.e. reduces to Q^2 ordering as above
  $z \approx 1$: $\theta_4 > \theta_2$, i.e. angular ordering of soft gluons
  $\Longrightarrow$ reduced phase space

Evolution procedures



DGLAP: Dokshitzer–Gribov–Lipatov–Altarelli–Parisi
evolution towards larger Q^2 and (implicitly) towards smaller x
BFKL: Balitsky–Fadin–Kuraev–Lipatov
evolution towards smaller x (with small, unordered Q^2)
CCFM: Ciafaloni–Catani–Fiorani–Marchesini
interpolation of DGLAP and BFKL
GLR: Gribov–Levin–Ryskin
nonlinear equation in dense-packing (saturation) region,
where partons recombine, not only branch

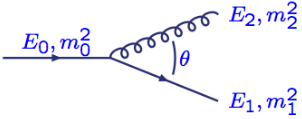
Initial- vs. final-state showers

Both controlled by same evolution equations

$$\mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{\mathrm{d}Q^2}{Q^2}\,P_{a\to bc}(z)\,\mathrm{d}z\;\cdot\;(\mathrm{Sudakov})$$

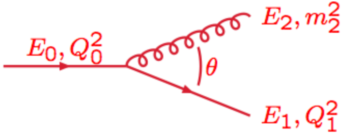
but

Final-state showers:
Q^2 timelike ($\sim m^2$)



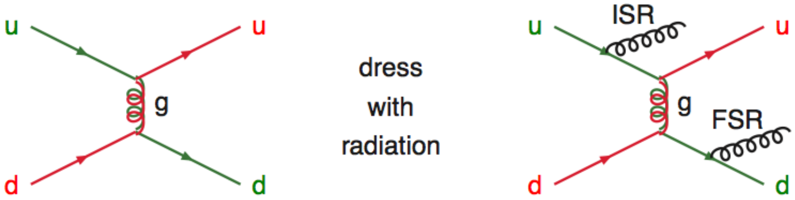
decreasing E, m^2, θ
both daughters $m^2 \geq 0$
physics relatively simple
$\Rightarrow$ "minor" variations:
Q^2, shower vs. dipole, ...

Initial-state showers:
Q^2 spacelike ($\approx -m^2$)



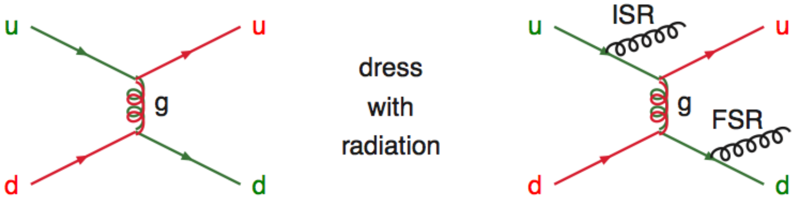
decreasing E, increasing Q^2, θ
one daughter $m^2 \geq 0$, one $m^2 < 0$
physics more complicated
$\Rightarrow$ more formalisms:
DGLAP, BFKL, CCFM, GLR, ...

Combining FSR with ISR

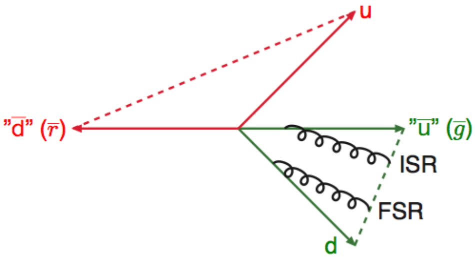


Separate processing of ISR and FSR misses interference
($\sim$ colour dipoles)

Combining FSR with ISR



Separate processing of ISR and FSR misses interference ($\sim$ colour dipoles)



ISR+FSR add coherently in regions of colour flow and destructively else

in "normal" shower by azimuthal anisotropies

automatic in dipole (by proper boosts)

Next-to-leading log showers

$$\mathrm{d}\mathcal{P}_{\mathrm{g}} = \mathrm{dn}_{\mathrm{g}} \approx \frac{8\alpha_{\mathrm{s}}}{3\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega} \mapsto \alpha_{\mathrm{s}} L^2$$

gives leading-log answer $P_n \propto (\alpha_{\mathrm{s}} L^2)^n = \alpha_{\mathrm{s}}^n L^{2n}$.
Resummation/exponentiation gives Sudakov $P_0 \propto \exp(-\alpha_{\mathrm{s}} L^2)$.
(Transverse momentum cuts both θ and $\omega \Rightarrow \alpha_{\mathrm{s}}^n L^n$.)

More careful handling of kinematics, α_{s} running, splitting kernels (also $\mathrm{g} \to \mathrm{ggg}$), etc., give subleading corrections $\propto \alpha_{\mathrm{s}}^n L^{2n-1}$.
All showers have some elements of NLL, e.g. momentum conservation, but some dedicated ongoing projects:

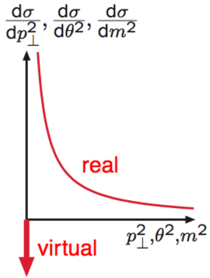
- Deductor (Nagy, Soper)
- PanScales (Salam et al.)
- Herwig 7 (Plätzer et al.)
- Vincia (Skands et al.)
- Alaric (Krauss et al.)

see presentation by
Daniel Reichelt
on Thursday

Matrix elements vs. parton showers

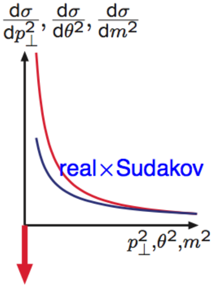
ME : Matrix Elements

- $+$ systematic expansion in α_{s} ('*exact*')
- $+$ powerful for multiparton Born level
- $+$ flexible phase space cuts
- $-$ loop calculations very tough
- $-$ negative cross section in collinear regions
  $\Rightarrow$ unpredictive jet/event structure
- $-$ *no easy match to hadronization*



PS : Parton Showers

- $-$ approximate, to LL (or NLL)
- $-$ main topology not predetermined
  $\Rightarrow$ inefficient for exclusive states
- $+$ process-generic $\Rightarrow$ simple multiparton
- $+$ Sudakov form factors/resummation
  $\Rightarrow$ sensible jet/event structure
- $+$ *easy to match to hadronization*



Matrix elements and parton showers

Recall complementary strengths:

- ME's good for well separated jets
- PS's good for structure inside jets

Marriage desirable! But how?

Problems:
- gaps in coverage?
- doublecounting of radiation?
- Sudakov?
- NLO (+NLL) consistency?

First attempt 40 years ago — Matrix Element Corrections.

Key topic of event generator development in last 30 years, with impressive progress.

See presentations by Tomas Jezo on Thursday and Friday.

Matrix Element Corrections (MEC)

= cover full phase space with smooth transition ME/PS.

Want to reproduce $$W^{\mathrm{ME}} = \frac{1}{\sigma(\mathrm{LO})} \frac{\mathrm{d}\sigma(\mathrm{LO}+\mathrm{g})}{\mathrm{d}(\mathrm{phasespace})}$$

by shower generation with $W^{\mathrm{PS}} > W^{\mathrm{ME}}$ + correction procedure

$$\overbrace{W^{\mathrm{ME}}}^{\text{wanted}} = \overbrace{W^{\mathrm{PS}}}^{\text{generated}} \quad \overbrace{\frac{W^{\mathrm{ME}}}{W^{\mathrm{PS}}}}^{\text{correction}}$$

- Exponentiate ME correction by shower Sudakov form factor:

$$W^{\mathrm{PS}}_{\mathrm{actual}}(Q^2) = W^{\mathrm{ME}}(Q^2) \exp\left(-\int_{Q^2}^{Q^2_{\mathrm{max}}} W^{\mathrm{ME}}(Q'^2)\,\mathrm{d}Q'^2\right)$$

- Memory of shower remains in Q^2 choice, i.e. "time" ordering.
  - ME regularized: probability ≤ 1 instead of divergent.
  - NLO correction simple for FSR, more messy for ISR: replace $\sigma(\mathrm{LO}) \to \sigma(\mathrm{NLO})$ in prefactor (POWHEG).

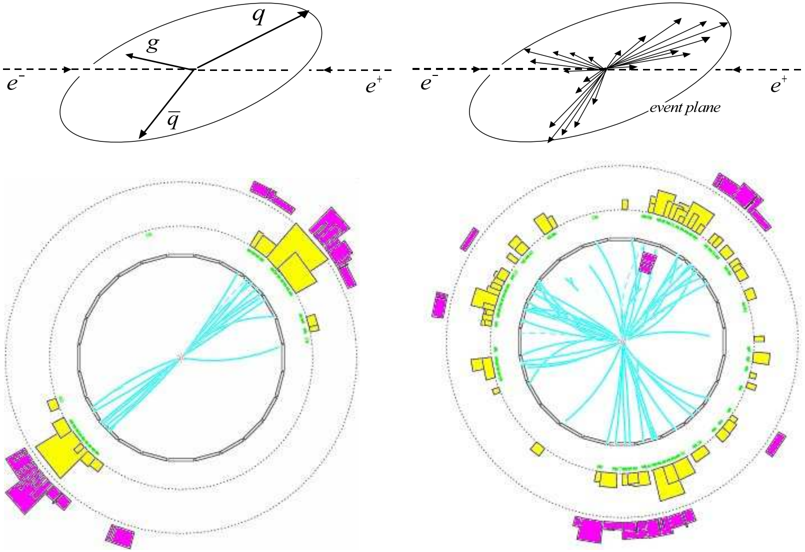
Event and jet characterization

Key difference between $\mathrm{e^+e^-}$ and pp:
• $\mathrm{e^+e^-} \to \mathrm{q\overline{q}}$ is rotationally symmetric on unit sphere.
• pp has "irrelevant" beam remnants along collision axis, requiring "true jets" to stick out in $p_\perp$.

Brief history:

- Spear (SLAC): find event axis in $\mathrm{e^+e^-} \to \mathrm{q\overline{q}} \Rightarrow$ Sphericity.
- Fixed-target pp experiments collision alignment $\Rightarrow$ Thrust.
- PETRA (DESY): early 80'ies, $\mathrm{e^+e^-} \to \mathrm{q\overline{q}g}$, establish g.
  1) S, T; extend Sphericity and Thrust families to 3 axes.
  2) clustering algorithms, e.g. JADE, Durham $k_\perp$.
- S$\mathrm{p\overline{p}}$S (CERN): cone jets in (η, φ) space, e.g. UA1.
- Tevatron (Fermilab): cone algorithms, increasingly messy.
- LHC: return of clustering with new safer and faster algorithms.
  Anti-$k_\perp$ "is" infrared safe return to UA1 cone algorithm.

Two- and three-jet events in e^+e^-



Sphericity

View as eigenvector problem, e.g. rotation axes of irregular 3D body. Here spanned up by the $\mathbf{p}_i$ of "all" particles in event.

$$S^{ab} = \frac{\sum_i p_i^a p_i^b}{\sum_i p_i^2} \quad a, b = x, y, z$$

S^{ab} has three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Sphericity $S = \frac{3}{2}(\lambda_2 + \lambda_3)$, $0 \leq S \leq 1$.
$S = 0$: two back-to-back pencil jets, e.g. $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$.
$S = 1$: spherically symmetric distribution.

Aplanarity $A = \frac{3}{2}\lambda_3$, $0 \leq A \leq \frac{1}{2}$.
$A = 0$: all particles in one plane.
$A = 1/2$: like $S = 1$.

Problem: collinear unsafe!
E.g. different answer if $\pi^0 \to \gamma\gamma$ counted as one or two particles.

Linearized Sphericity

Collinear safe alternative, used in same way but with

$$L^{ab} = \frac{\sum_i \frac{p_i^a p_i^b}{|\mathbf{p}_i|}}{\sum_i |\mathbf{p}_i|} \quad a, b = x, y, z$$

No proper name: some confusion!

Additional measures:
$C = 3(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$
$D = 27\lambda_1\lambda_2\lambda_3$
used to characterize 3- and 4-jet topologies, respectively.

(Linearized) Sphericity family not normally used in pp,
since beam jets dominate structure.
Solution: set all $p_i^z = 0$ so only transverse structure studied.
Modified "2D" $S = 2\lambda_2$ and no A.

Thrust

Thrust is computationally more demanding optimization

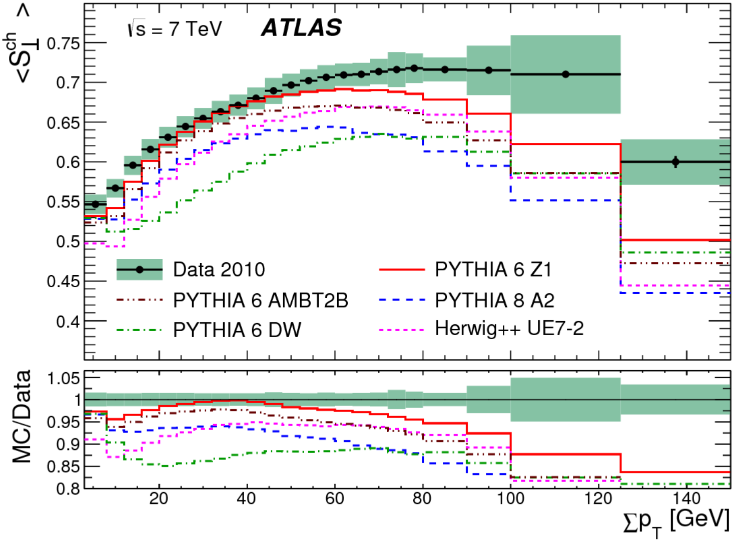
$$T = \max_{|\mathbf{n}|=1} \frac{\sum_i |\mathbf{p}_i \mathbf{n}|}{\sum_i |\mathbf{p}_i|}$$

with **n** for maximum is called Thrust axis.
$1/2 < T < 1$, with $T = 1$ for two back-to-back pencil jets and $T = 1/2$ for a spherically symmetric distribution.

$$\begin{aligned}
\mathrm{Major} &= \max_{|\mathbf{n}'|=1, \mathbf{n}'\mathbf{n}=0} \frac{\sum_i |\mathbf{p}_i \mathbf{n}'|}{\sum_i |\mathbf{p}_i|} \\
\mathrm{Minor} &= \frac{\sum_i |\mathbf{p}_i \mathbf{n}''|}{\sum_i |\mathbf{p}_i|} \quad \text{with } \mathbf{n}''\mathbf{n} = \mathbf{n}''\mathbf{n}' = 0 \\
\mathrm{Oblateness} &= \mathrm{Major} - \mathrm{Minor}
\end{aligned}$$

Major and Oblateness again useful for 3-jet structure,
Minor for 4-jet one.

2D Sphericity at the LHC



Competition between more $\sum p_\perp$ by more particles or by jets?

Clustering algorithms — basics

Most clustering algorithms are based on sequential recombination:

- Define a distance measure d_{ij} between to objects i and j, partons or particles, where $d_{ij} = 0$ is closest possible.
- Define a procedure whereby any objects i and j can be joined into a new object k, e.g. $p_k = p_i + p_j$.
- Define a stopping criterion, e.g. that all $d_{ij} > d_{\mathrm{min}}$ or that only n_{min} objects remain.
- Start out with a list of n objects.
- Calculate all d_{ij} and find pair i_{min} and j_{min} with smallest value.
- Remove i_{min} and j_{min} from list and insert joined object k.
- Iterate last two steps until the stopping criterion is met.
- Jets $=$ the objects that now remain.

$2 \to 1$ joining can be viewed as undoing $1 \to 2$ parton branchings.
Less obvious interpretation of hadronization step.

Clustering algorithms in $\mathrm{e^+e^-}$

Naive thought $d_{ij} = m_{ij}^2$, but allows clustering of opposite objects.

JADE is almost like invariant mass:

$$d_{ij} = \frac{2 E_i E_j (1 - \cos\theta_{ij})}{E_{\mathrm{vis}}^2}$$

where $E_{\mathrm{vis}} \approx E_{\mathrm{CM}}$ is visible energy.

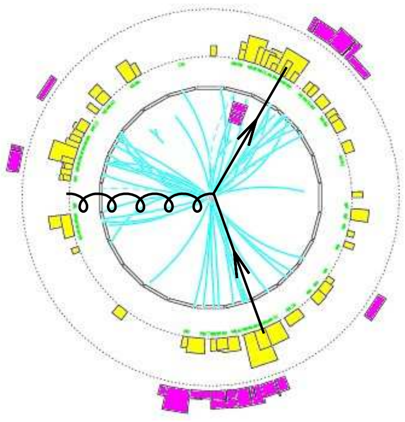
Durham offers a theoretically preferred alternative

$$d_{ij} = \frac{2 \min(E_i^2, E_j^2)(1 - \cos\theta_{ij})}{E_{\mathrm{CM}}^2}$$

which can be viewed as the (scaled) $p_\perp^2$ of the softer object with respect to the harder one:
$2(1 - \cos\theta) \approx \sin^2\theta$ for small θ and $p_\perp = p \sin\theta$.
Undoes $p_\perp$-ordered branchings (to some approximation).

Clustering algorithm ambiguities

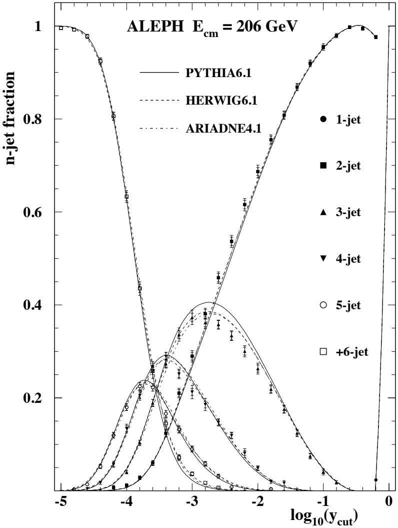
Interpretation is in the eye of the beholder:

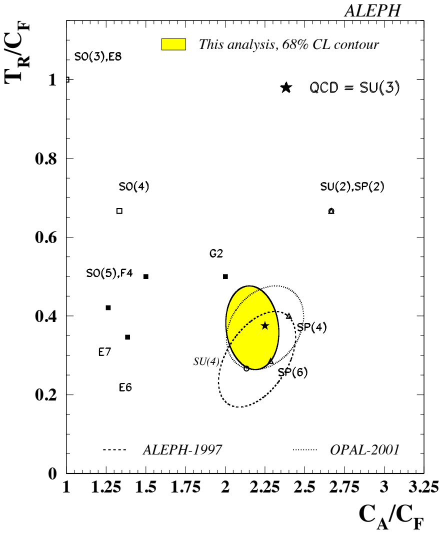


How many jets?
Which are quarks and which gluons?

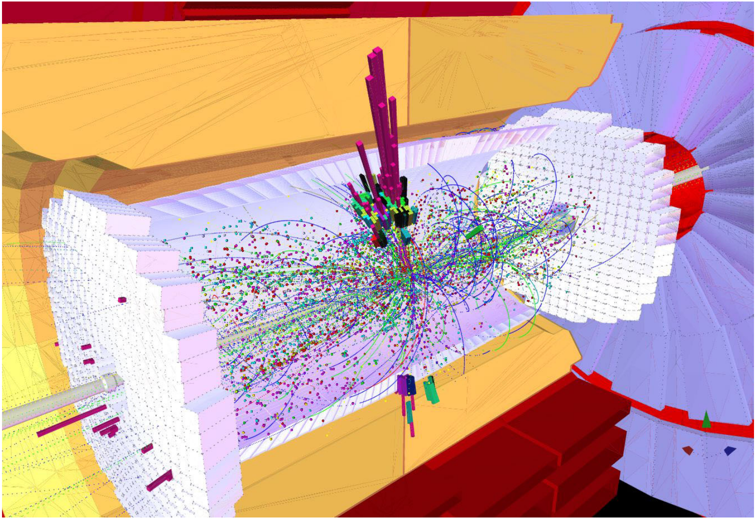
Clustering algorithm results

Most LEP QCD physics based on jet finding, e.g.:





Clustering conditions in hadron collisions



Most particles are at small $p_\perp$, say below 1 GeV, and at small angles with respect to beam axis, outside central tracking region.

Cylindrical symmetry and rapidity

Cylindrical coordinates:

$$\begin{aligned}\frac{\mathrm{d}^3 p}{E} &= \frac{\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z}{E} = \frac{\mathrm{d}^2 p_\perp\,\mathrm{d}p_z}{E} = \mathrm{d}^2 p_\perp\,\mathrm{d}y \\ &= p_\perp \mathrm{d}p_\perp\,\mathrm{d}\varphi\,\mathrm{d}y = \frac{1}{2}\mathrm{d}p_\perp^2\,\mathrm{d}\varphi\,\mathrm{d}y\end{aligned}$$

with rapidity y given by

$$\begin{aligned}y &= \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \frac{1}{2}\ln\frac{(E+p_z)^2}{(E+p_z)(E-p_z)} = \frac{1}{2}\ln\frac{(E+p_z)^2}{m^2+p_\perp^2} \\ &= \ln\frac{E+p_z}{m_\perp} = \ln\frac{m_\perp}{E-p_z}\end{aligned}$$

Exercise: show that $\mathrm{dp}_z/E = \mathrm{d}y$ by showing that $\mathrm{d}y/\mathrm{d}p_z = 1/E$.

Hint: use that $E = \sqrt{m_\perp^2 + p_z^2}$.

Lightcone kinematics and boosts

Introduce (lightcone) $p^+ = E + p_z$ and $p^- = E - p_z$.
Note that $p^+ p^- = E^2 - p_z^2 = m_\perp^2$.
Consider boost along z axis with velocity β and $\gamma = 1/\sqrt{1-\beta^2}$

$$\left\{ \begin{array}{l} p'_z = \gamma(p_z + \beta\, E) \\ E' = \gamma(E + \beta\, p_z) \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} p'^+ = kp^+ \\ p'^- = p^-/k \end{array} \right. \quad \text{with} \quad k = \sqrt{\frac{1+\beta}{1-\beta}}$$

$$\begin{aligned} y' &= \frac{1}{2} \ln \frac{p'^+}{p'^-} = \frac{1}{2} \ln \frac{k\, p^+}{p^-/k} = y + \ln k \\ y'_2 - y'_1 &= (y_2 + \ln k) - (y_1 + \ln k) = y_2 - y_1 \end{aligned}$$

Note how integration of cross section nicely separates into rapidity:

$$\sigma^{AB} = \sum_{i,j} \iint \mathrm{d}x_1\, \mathrm{d}x_2\, f_i^{(A)}(x_1, Q^2)\, f_j^{(B)}(x_2, Q^2) \int \mathrm{d}\hat{\sigma}_{ij}(\hat{s} = x_1 x_2 s)$$

$$\iint \mathrm{d}x_1\, \mathrm{d}x_2 = \iint \mathrm{d}\tau\, \mathrm{d}y \quad \text{with} \quad \tau = x_1 x_2 \quad \text{and} \quad y = \frac{1}{2} \ln \frac{x_1}{x_2}$$

Pseudorapidity

If experimentalists cannot measure m they may assume $m = 0$.
Instead of rapidity y they then measure pseudorapidity η:

$$y = \frac{1}{2} \ln \frac{\sqrt{m^2 + \mathbf{p}^2} + p_z}{\sqrt{m^2 + \mathbf{p}^2} - p_z} \quad \Rightarrow \quad \eta = \frac{1}{2} \ln \frac{|\mathbf{p}| + p_z}{|\mathbf{p}| - p_z} = \ln \frac{|\mathbf{p}| + p_z}{p_\perp}$$

or

$$\begin{aligned} \eta &= \frac{1}{2} \ln \frac{|\mathbf{p} + |\mathbf{p}|\, \cos\theta}{|\mathbf{p}| - |\mathbf{p}|\, \cos\theta} = \frac{1}{2} \ln \frac{1 + \cos\theta}{1 - \cos\theta} \\ &= \frac{1}{2} \ln \frac{2\cos^2\theta/2}{2\sin^2\theta/2} = \ln \frac{\cos\theta/2}{\sin\theta/2} = -\ln\tan\frac{\theta}{2} \end{aligned}$$

which thus only depends on polar angle.

η is **not** simple under boosts: $\eta'_2 - \eta'_1 \neq \eta_2 - \eta_1$.
You may even flip sign!

The pseudorapidity dip

By analogy with $\mathrm{d}y/\mathrm{d}p_z = 1/E$ it follows that $\mathrm{d}\eta/\mathrm{d}p_z = 1/|\mathbf{p}|$.

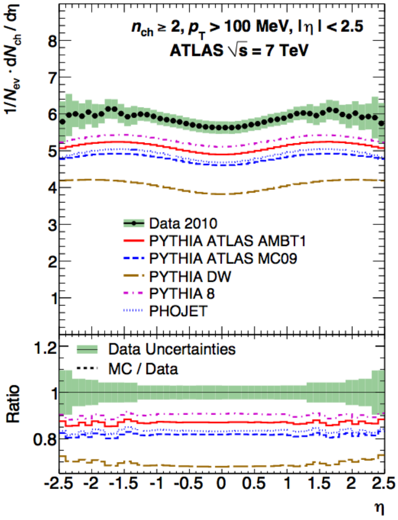
Thus

$$\frac{\mathrm{d}\eta}{\mathrm{d}y} = \frac{\mathrm{d}\eta/\mathrm{d}p_z}{\mathrm{d}y/\mathrm{d}p_z} = \frac{E}{|\mathbf{p}|} > 1$$

with limits

$$\frac{\mathrm{d}\eta}{\mathrm{d}y} \to \frac{m_\perp}{p_\perp} \text{ for } p_z \to 0$$
$$\frac{\mathrm{d}\eta}{\mathrm{d}y} \to 1 \text{ for } p_z \to \pm\infty$$

so if $\mathrm{d}n/\mathrm{d}y$ is flat for $y \approx 0$
then $\mathrm{d}n/\mathrm{d}\eta$ has a dip there.



The R separation

Massless four-vectors can be written in cylindrical coordinates like

$$p = p_{\perp}(\cosh y; \cos\varphi, \sin\varphi, \sinh y) \, .$$

The invariant mass of two massless four-vectors is

$$\begin{aligned}
m_{ij}^2 &= (p_i + p_j)^2 = 2 p_i p_j \\
&= 2 p_{\perp i} p_{\perp j} \, (\cosh(y_i - y_j) - \cos(\varphi_i - \varphi_j)) \\
&\approx 2 p_{\perp i} p_{\perp j} \left(1 + \frac{1}{2}(y_i - y_j)^2 - (1 - \frac{1}{2}(\varphi_i - \varphi_j)^2) \right) \\
&= p_{\perp i} p_{\perp j} \, (\Delta y_{ij}^2 + \Delta\varphi_{ij}^2) = p_{\perp i} p_{\perp j} R_{ij}^2
\end{aligned}$$

so a circle in the (y, φ) plane is a meaningful concept.

The $k_\perp$ algorithm

- Each original particle defines a cluster, with well-defined four-momentum $\Rightarrow (p_\perp, y, \varphi)$.
- Define distance measures of all clusters i to the beam and of all cluster pairs (i, j) relative to each other

$$d_{iB} = p_{\perp i}^2$$

$$d_{ij} = \min\left(p_{\perp i}^2, p_{\perp j}^2\right) \frac{R_{ij}^2}{R^2}$$

- Find the smallest of all d_{iB} and d_{ij}.
  a) If a d_{iB} and $p_{\perp i} < p_{\perp \mathrm{min}}$ then throw it.
  b) Else if a d_{iB} then call i a jet and remove it from cluster list.
  c) Else if a d_{ij} then combine i and j to a new cluster with four-momentum $p_i + p_j$.
- Repeat until no clusters remain.

Two key parameters R and $p_{\perp \mathrm{min}}$, where $p_{\perp \mathrm{min}} = 0$ is allowed simplification.

The $k_\perp$ family

Generalize the d_{iB} and d_{ij} measures to

$$d_{iB} = p_{\perp i}^{2p}$$
$$d_{ij} = \min\left(p_{\perp i}^{2p}, p_{\perp j}^{2p}\right) \frac{R_{ij}^2}{R^2}$$

- $p = 1$ is $k_\perp$ algorithm; preferentially clusters soft particles.
- $p = 0$ is Cambridge–Aachen or no-$k_\perp$ algorithm.
- $p = -1$ is anti-$k_\perp$ algorithm; preferentially clusters around hardest particle and give round jet catchment areas.

All three are infrared and collinear safe; i.e. the addition of a soft particle, or the splitting of a particle into two collinear ones, do not alter the outcome.

These, and many more jet algorithms, are available in the FastJet package. (Faster than naive step-by-step clustering.)

Clustering results