NLL Shower Accuracy

Terascale Monte Carlo School,
DESY Hamburg, 27 November 2025

Daniel
Reichelt

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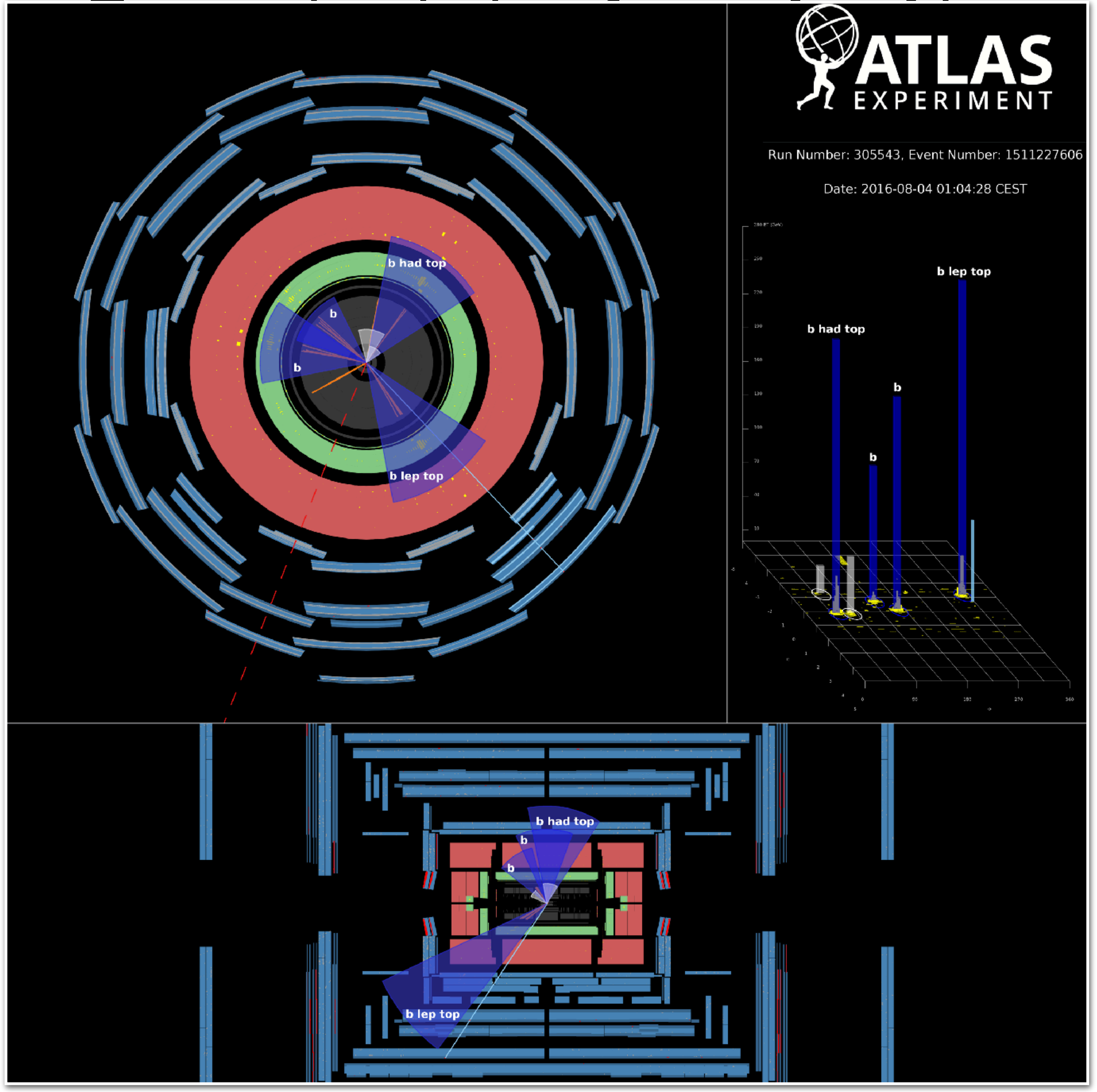
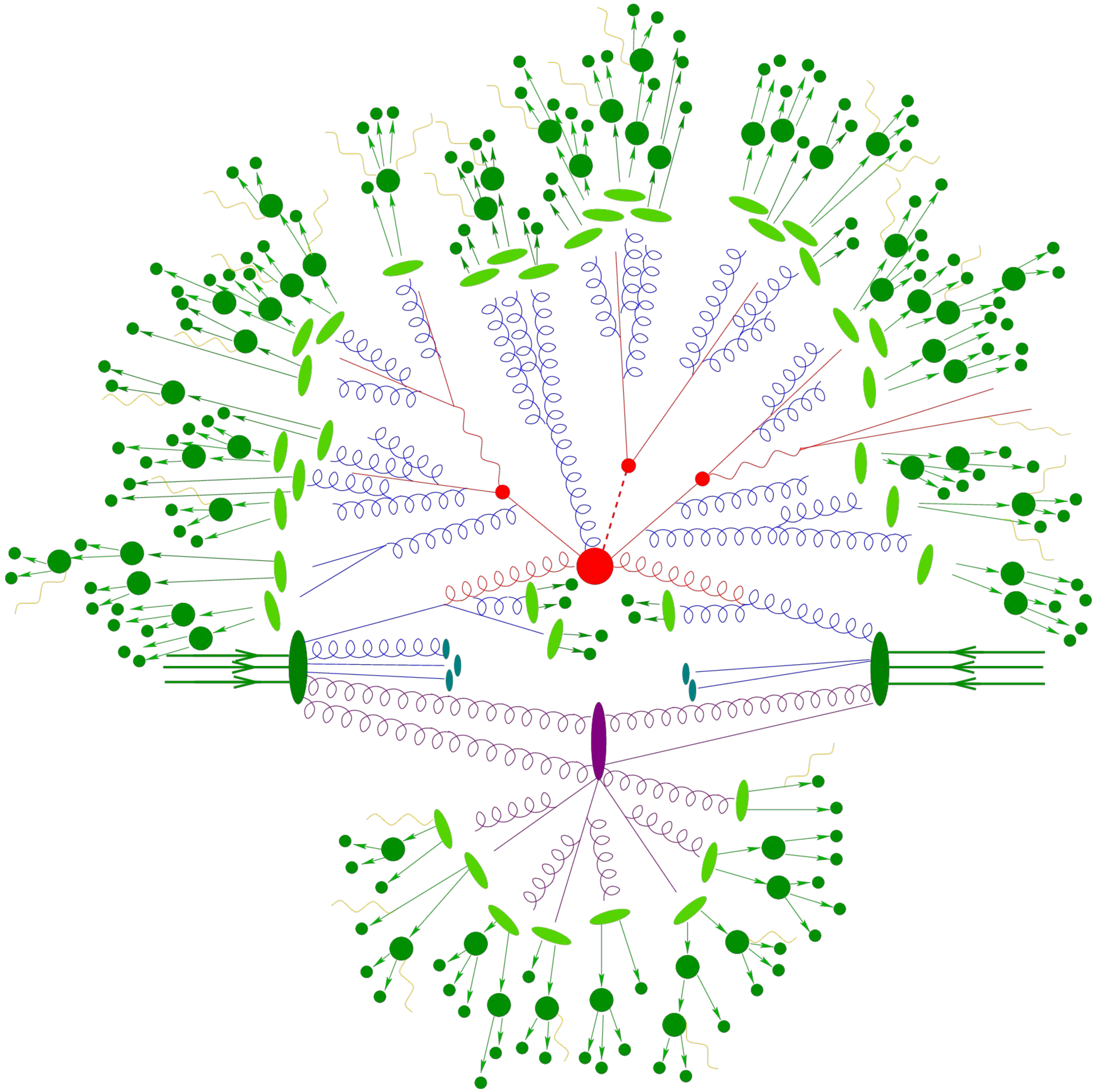
by the
European Union

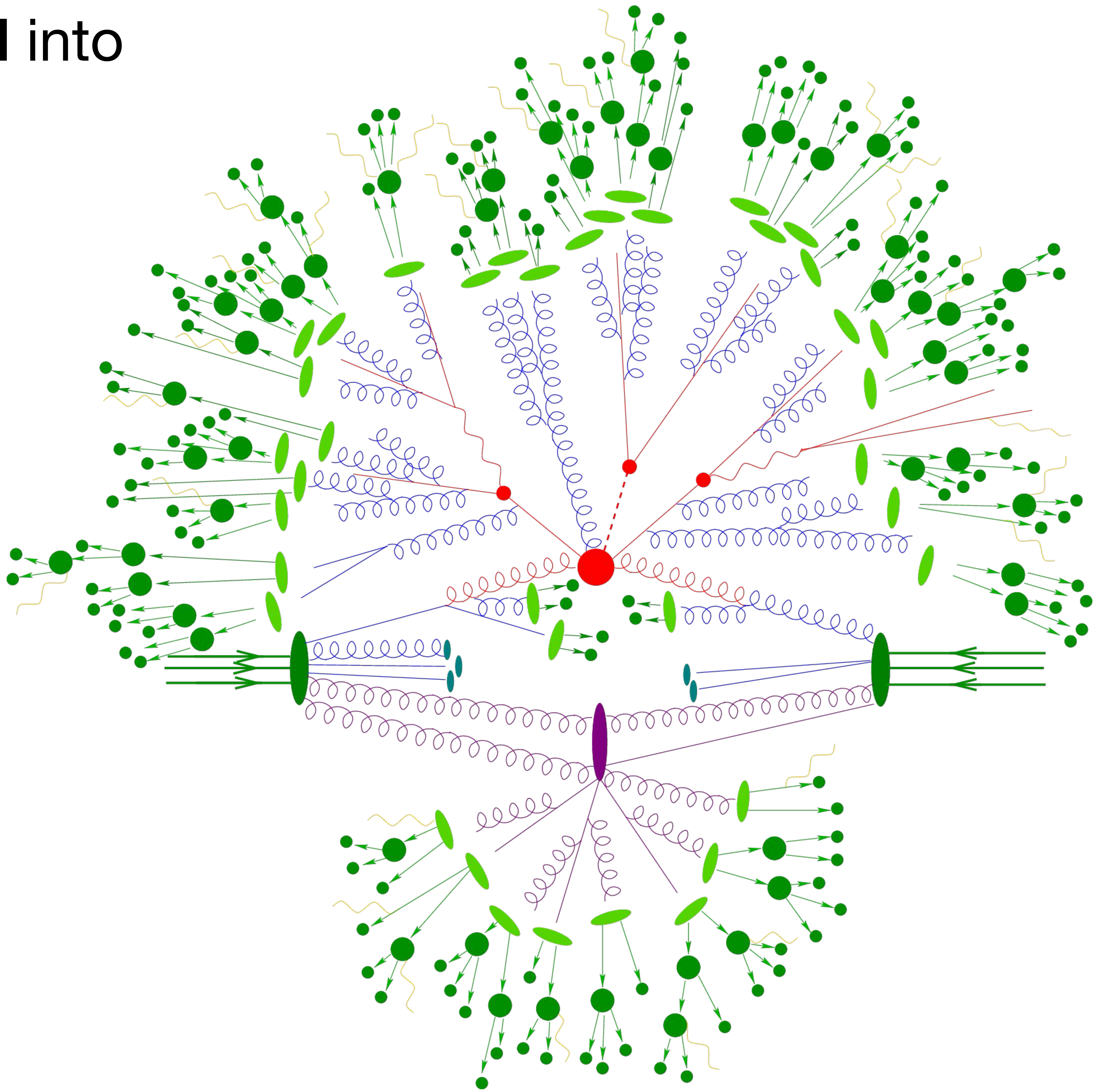
Event Generators





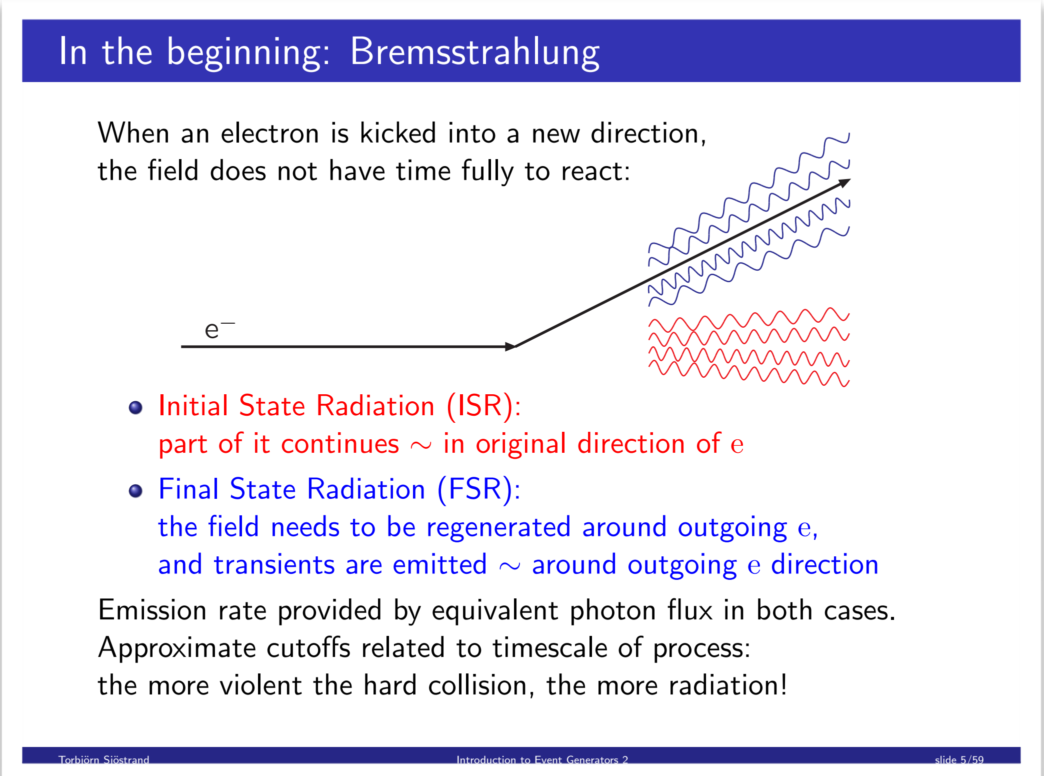
Event Generators

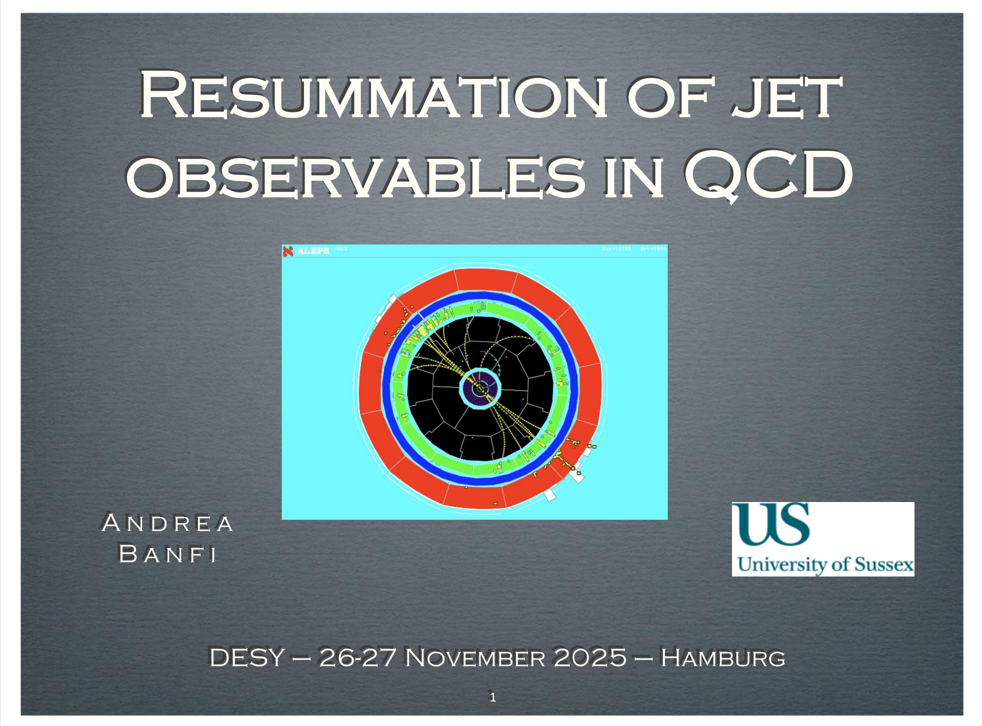
- Event simulation **factorised** into
  - **Hard Process**
  - **Parton Shower**
  - **PDF/Underlying event**
  - **Hadronisation**
  - **Hadron Decays**



Intro

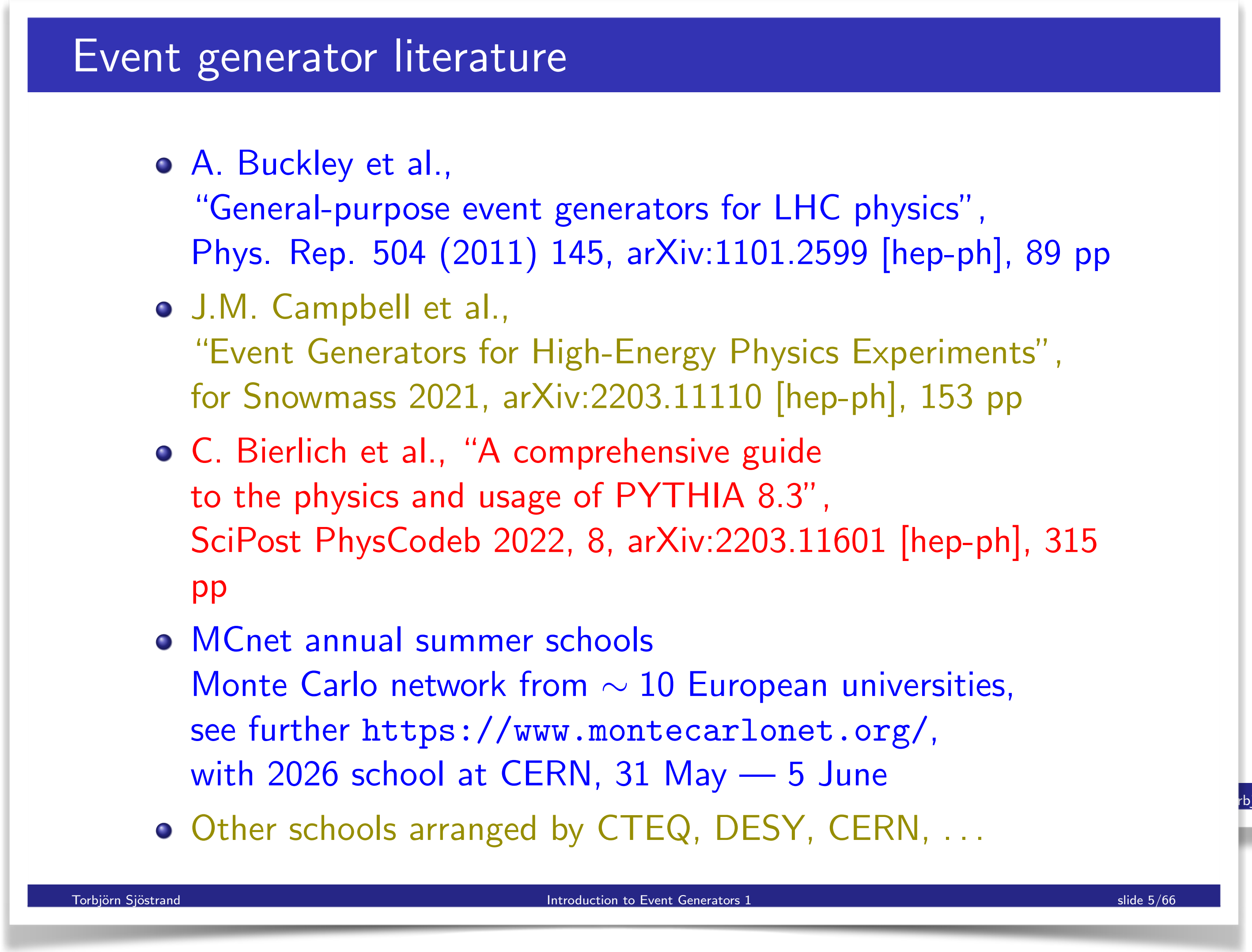
- We know a lot about filed theory and QCD in certain cases and limits, PS is an attempt to do our best in all limits at once
    - Compatibility with semi-classical Bremsstrahlung
    - General behavior of FO matrix elements
    - Collinear limits → fragmentation functions follow DGLAP evolution
    - Soft+Collinear limits → reproduce results from resummed calculations for certain (classes of) observables
- What I will try here:
    - Go through theoretical arguments entering shower development, with eye on previous lectures, try to pin-point relations





Reading (and sources for this lecture)

- See lecture 1 for standard particle/ collider physics books:


Event generator literature

- A. Buckley et al., "General-purpose event generators for LHC physics", Phys. Rep. 504 (2011) 145, arXiv:1101.2599 [hep-ph], 89 pp
- J.M. Campbell et al., "Event Generators for High-Energy Physics Experiments", for Snowmass 2021, arXiv:2203.11110 [hep-ph], 153 pp
- C. Bierlich et al., "A comprehensive guide to the physics and usage of PYTHIA 8.3", SciPost PhysCodeb 2022, 8, arXiv:2203.11601 [hep-ph], 315 pp
- MCnet annual summer schools Monte Carlo network from ∼ 10 European universities, see further `https://www.montecarlonet.org/`, with 2026 school at CERN, 31 May — 5 June
- Other schools arranged by CTEQ, DESY, CERN, . . .

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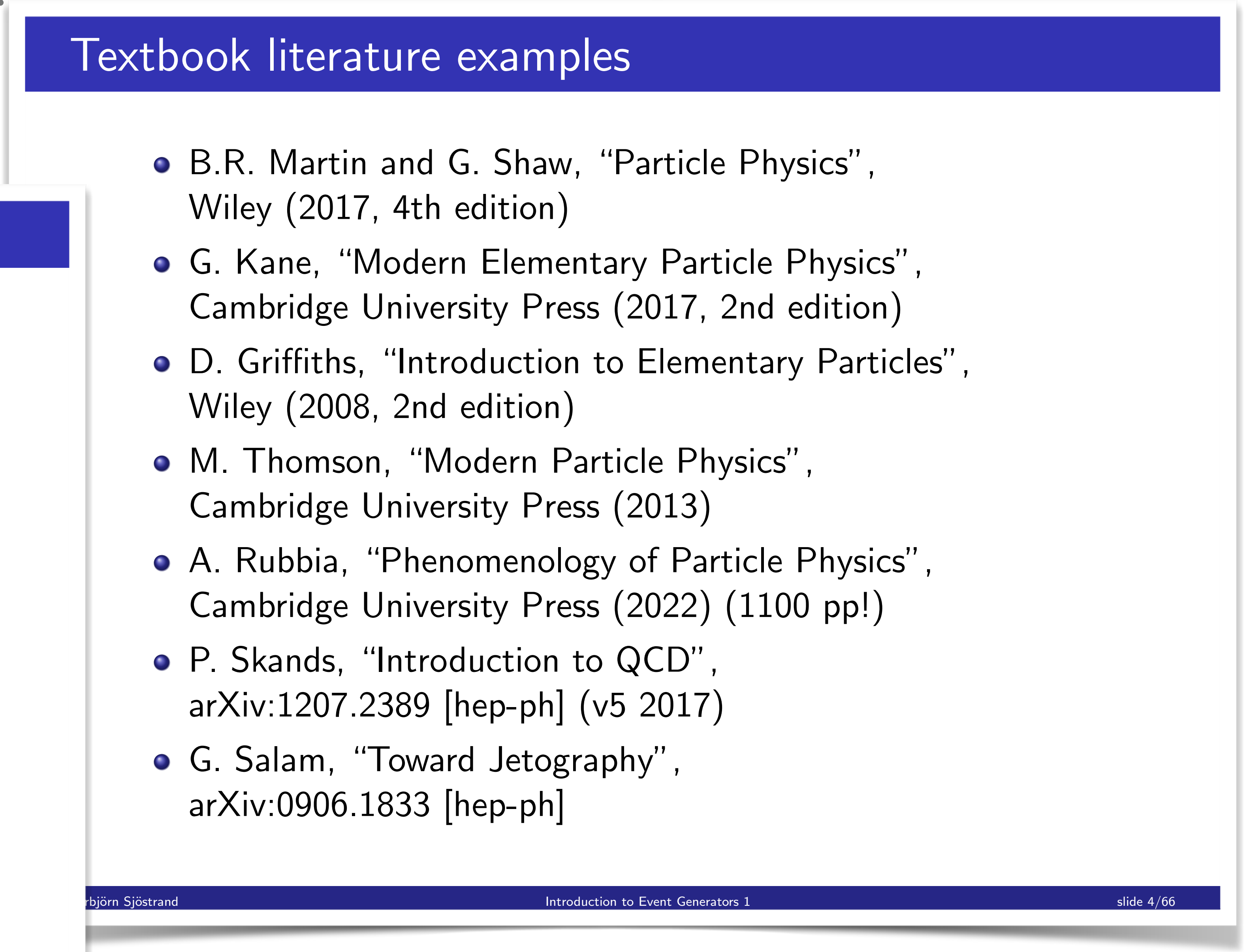
Textbook literature examples

- B.R. Martin and G. Shaw, "Particle Physics", Wiley (2017, 4th edition)
- G. Kane, "Modern Elementary Particle Physics", Cambridge University Press (2017, 2nd edition)
- D. Griffiths, "Introduction to Elementary Particles", Wiley (2008, 2nd edition)
- M. Thomson, "Modern Particle Physics", Cambridge University Press (2013)
- A. Rubbia, "Phenomenology of Particle Physics", Cambridge University Press (2022) (1100 pp!)
- P. Skands, "Introduction to QCD", arXiv:1207.2389 [hep-ph] (v5 2017)
- G. Salam, "Toward Jetography", arXiv:0906.1833 [hep-ph]

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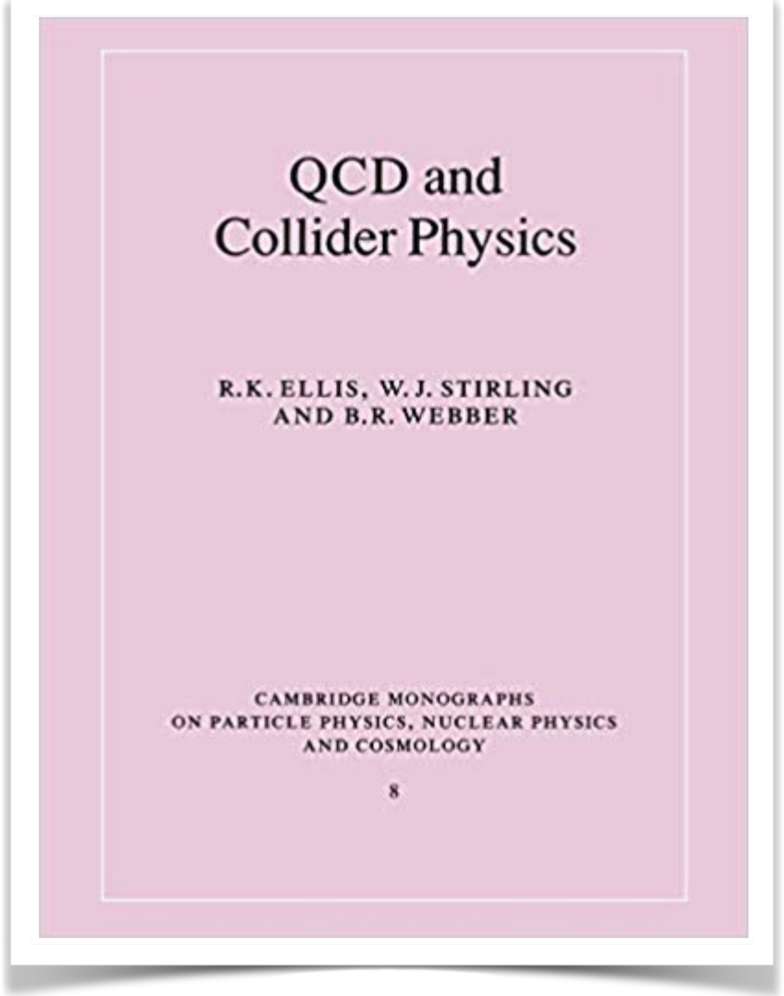
- "QCD and Collider Physics"



- "Introduction to Parton Shower Event Generators" TASI lecture by Stefan Höche

Recap: Unitary Parton Shower Algorithms

- Start with state (i.e. ingoing and outgoing partons with specified momentum, flavour and charges) determined by hard process

- Given some probability to split a parton, assume unitarity, i.e.

$$P(\text{no splitting}) = 1 - P(\text{any splittings}) = 1 - \sum_n P(\text{n ordered splitting})$$

$$= 1 - \sum_n \frac{1}{n!} P^n(\text{one splitting}) = \exp\left(-P\left(\text{one splitting}\right)\right)$$

- Select new scale and kinematics according to this, produce splitting

- Repeat until we reach some cutoff scale

Recap: Unitary Parton Shower Algorithms

- Start with state (i.e. ingoing an flavour and charges) determine
- Given some probability to split

$$P(\text{no splitting}) = 1 - P(\text{any split}$$

$$= 1 - \sum_n \frac{1}{n!} P^n(\text{o}$$

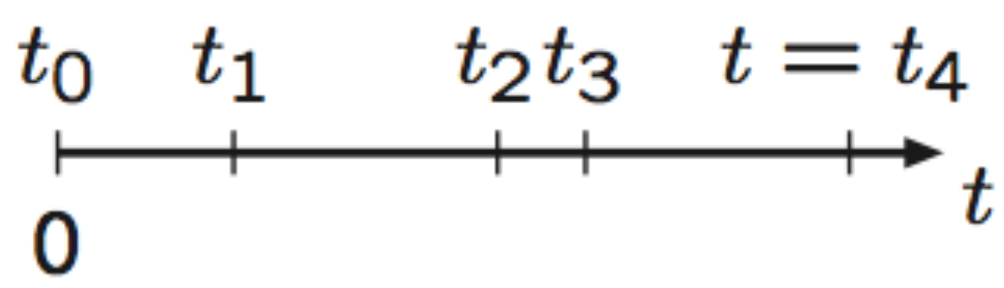
- Select new scale and kinemati
- Repeat until we reach some cu

Reminder: Algorithmically solved by veto algorithm, see lecture by Torbjörn!

The veto algorithm: solution

The veto algorithm

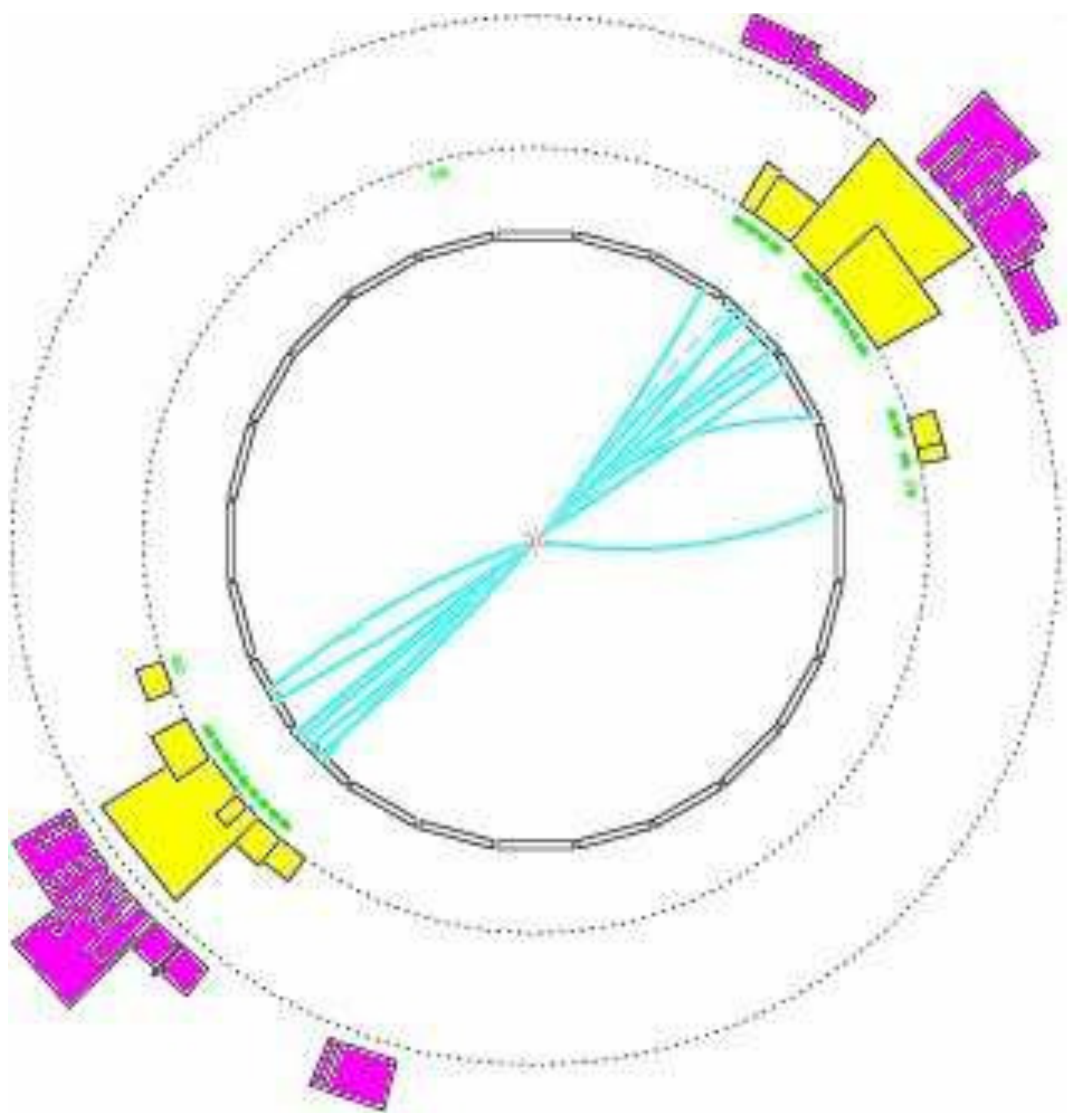
1. start with $i = 0$ and $t_0 = 0$
2. $i = i + 1$
3. $t = t_i = G^{-1}(G(t_{i-1}) - \ln R)$, i.e $t_i > t_{i-1}$
4. $y = R\, g(t)$
5. while $y > f(t)$ cycle to 2

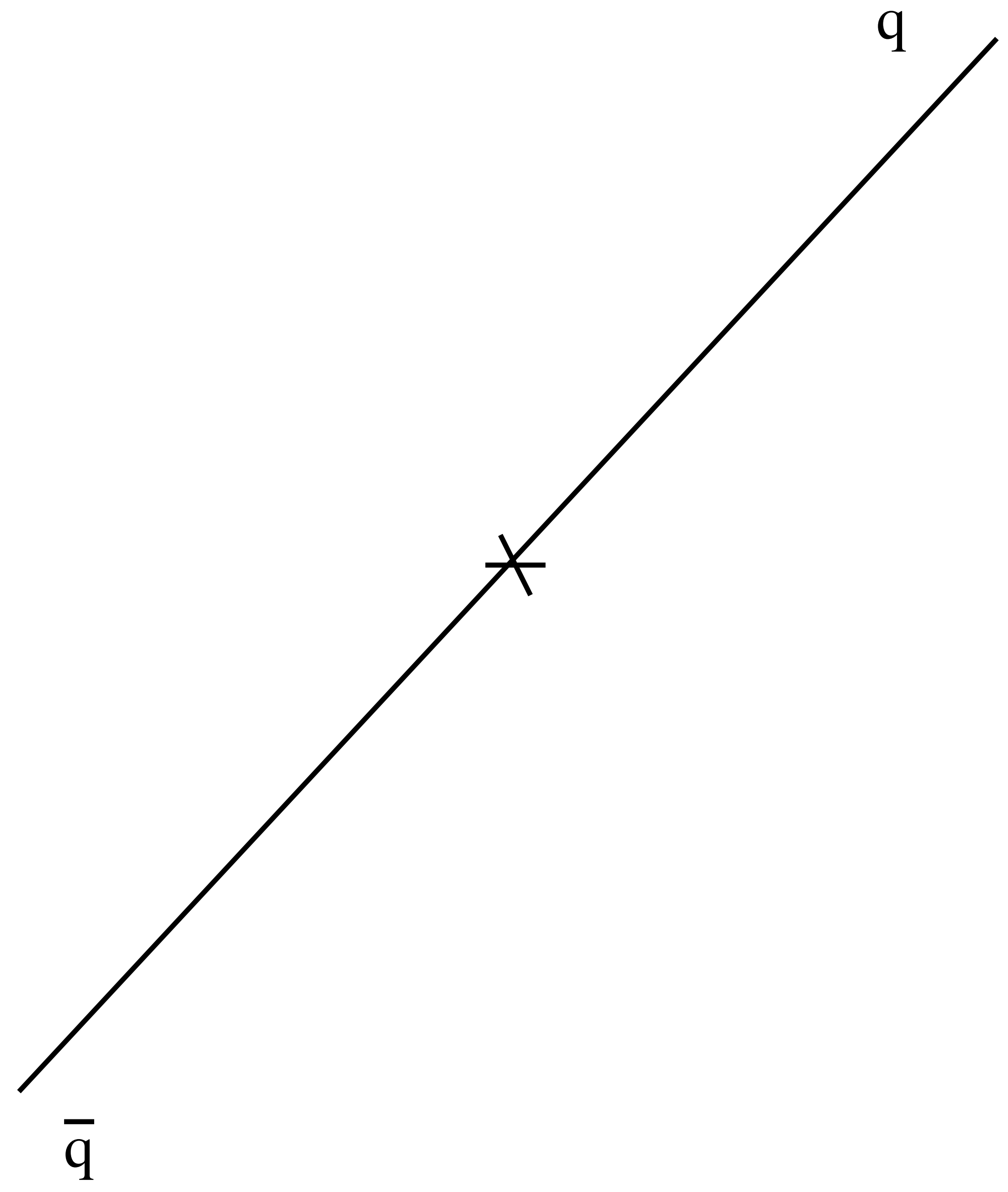


That is, when you fail, you keep on going from the time when you failed, and *do not* restart at time $t = 0$. (Memory!)

jet evolution - qualitative picture

- Start from simple partonic state

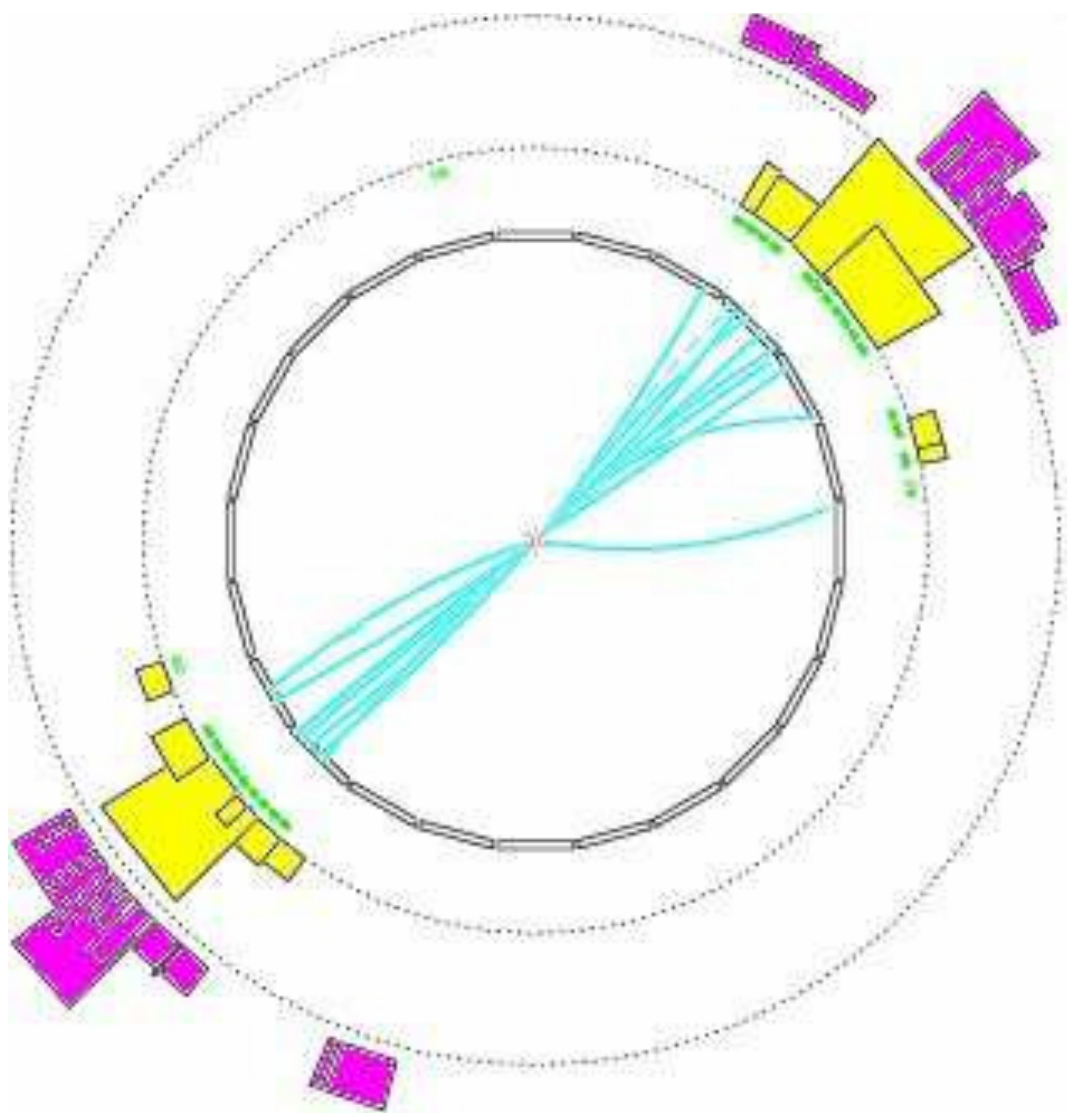


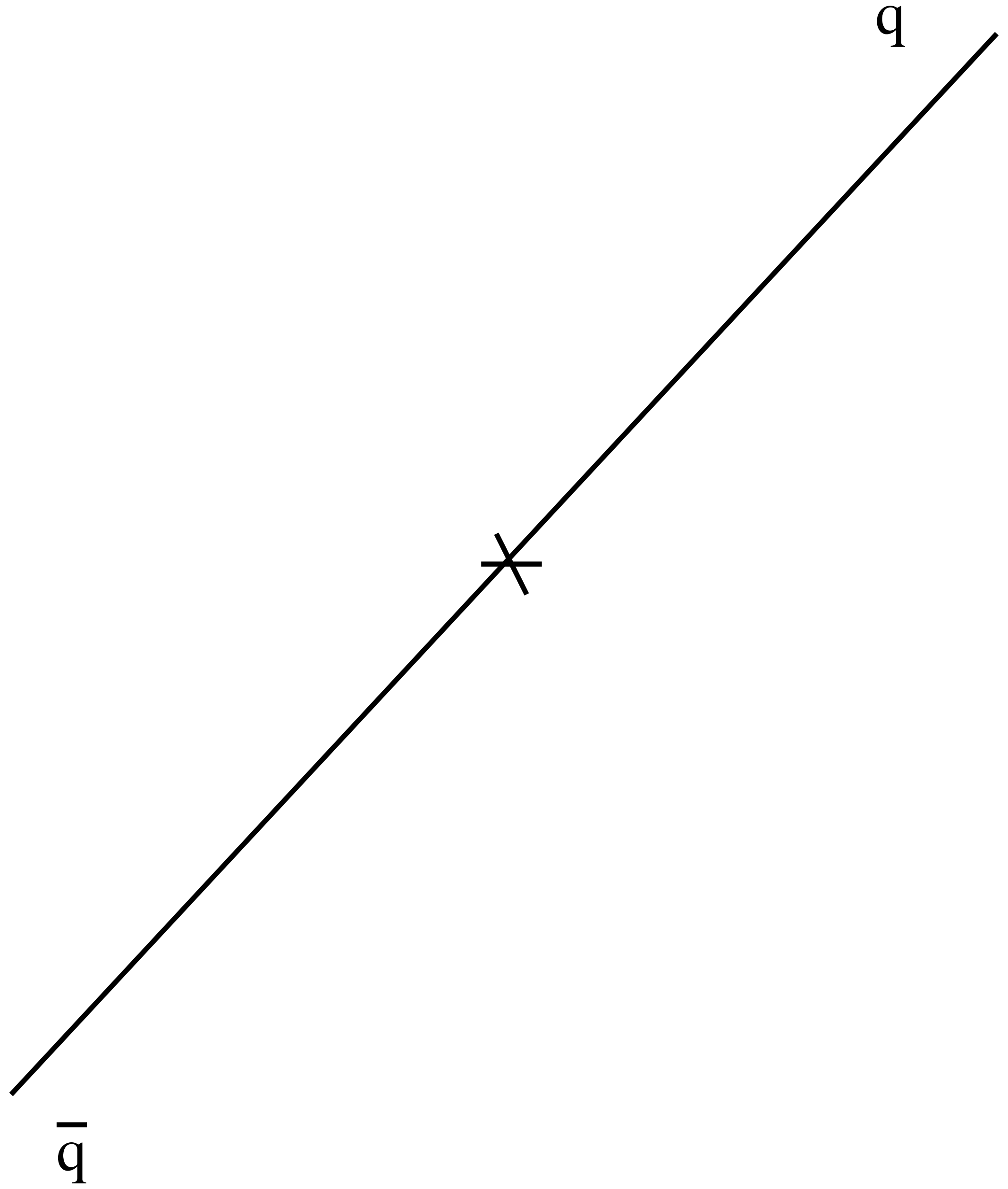


jet evolution - qualitative picture

- Start from simple partonic state

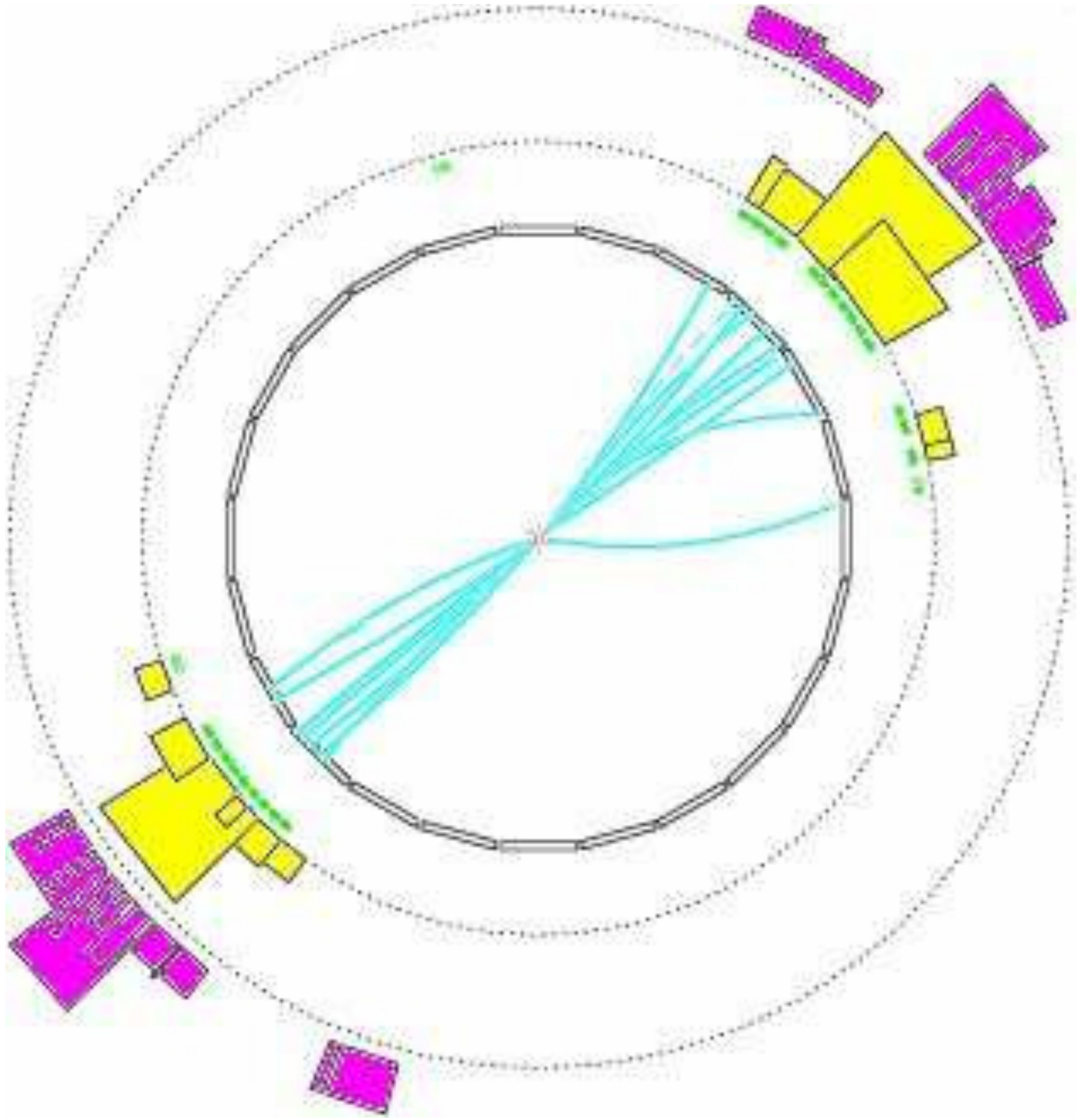
- determine new splitting scale and kinematics according to “no-emission probability”

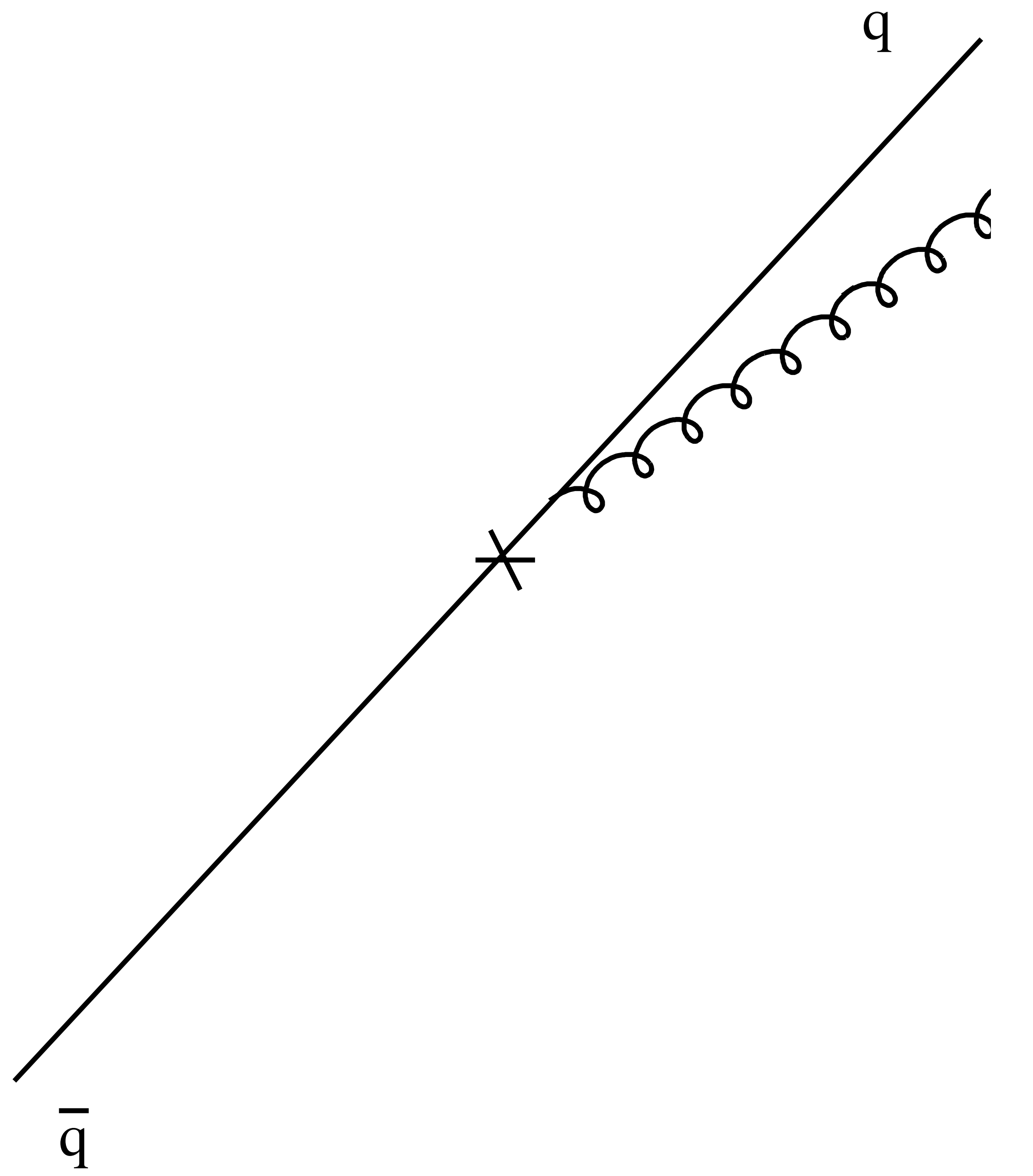




jet evolution - qualitative picture

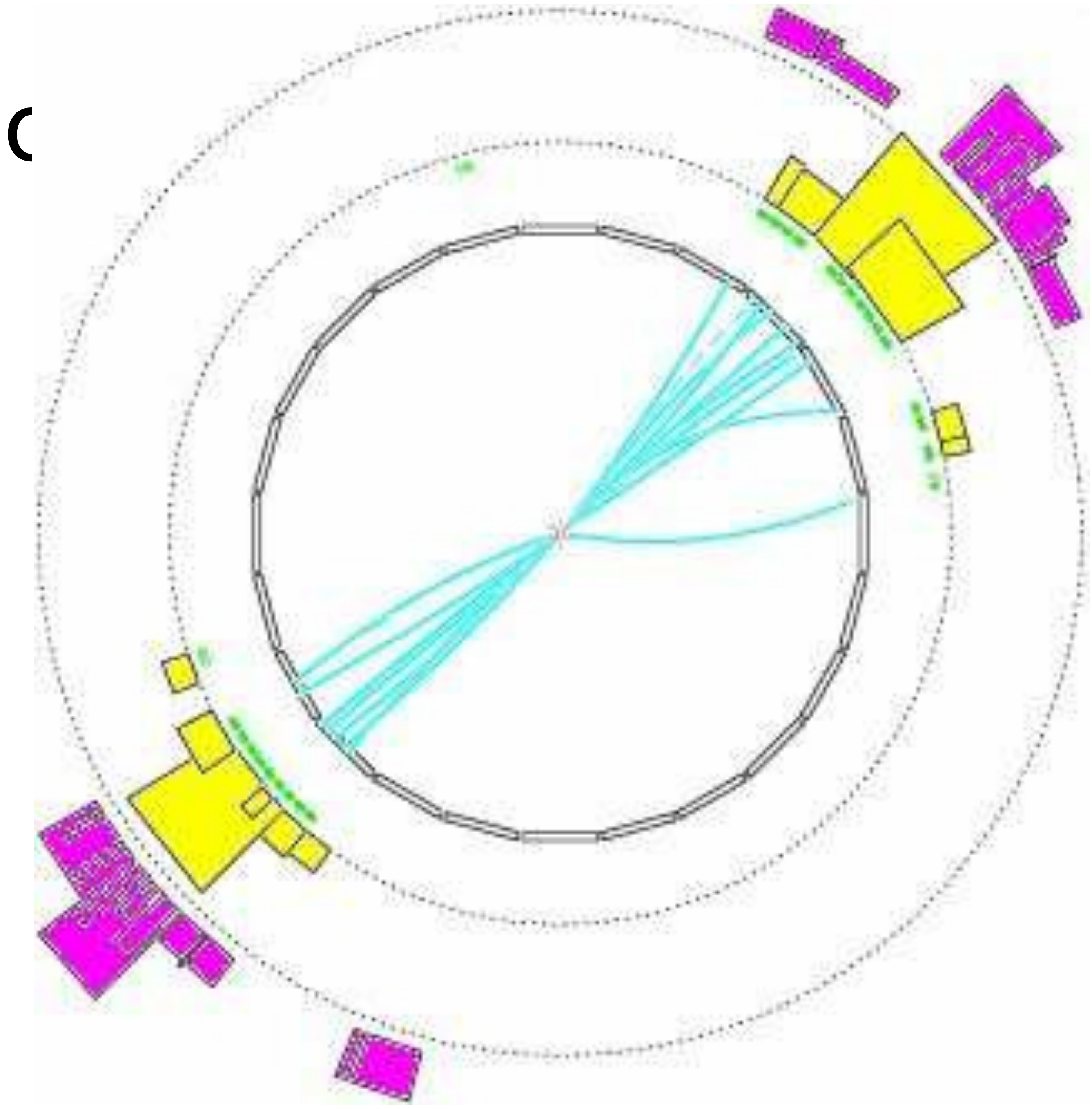
- Start from simple partonic state
- determine new splitting scale and kinematics according to “no-emission probability”
- produce new state with additional gluon

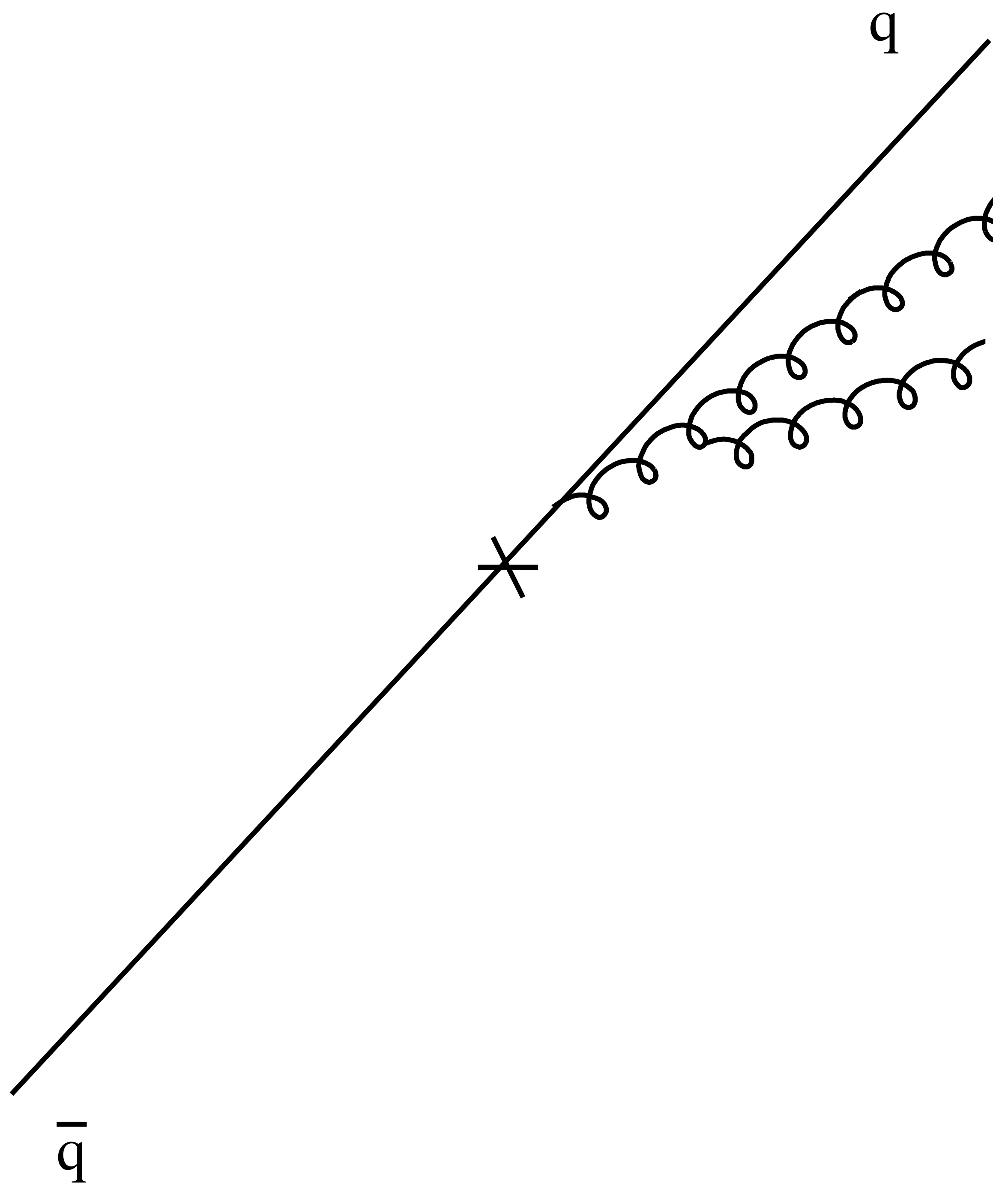




jet evolution - qualitative picture

- Start from simple partonic state
- determine new splitting scale and kinematics according to “no-emission probability”
- produce new state with additional gluon
- iterate!





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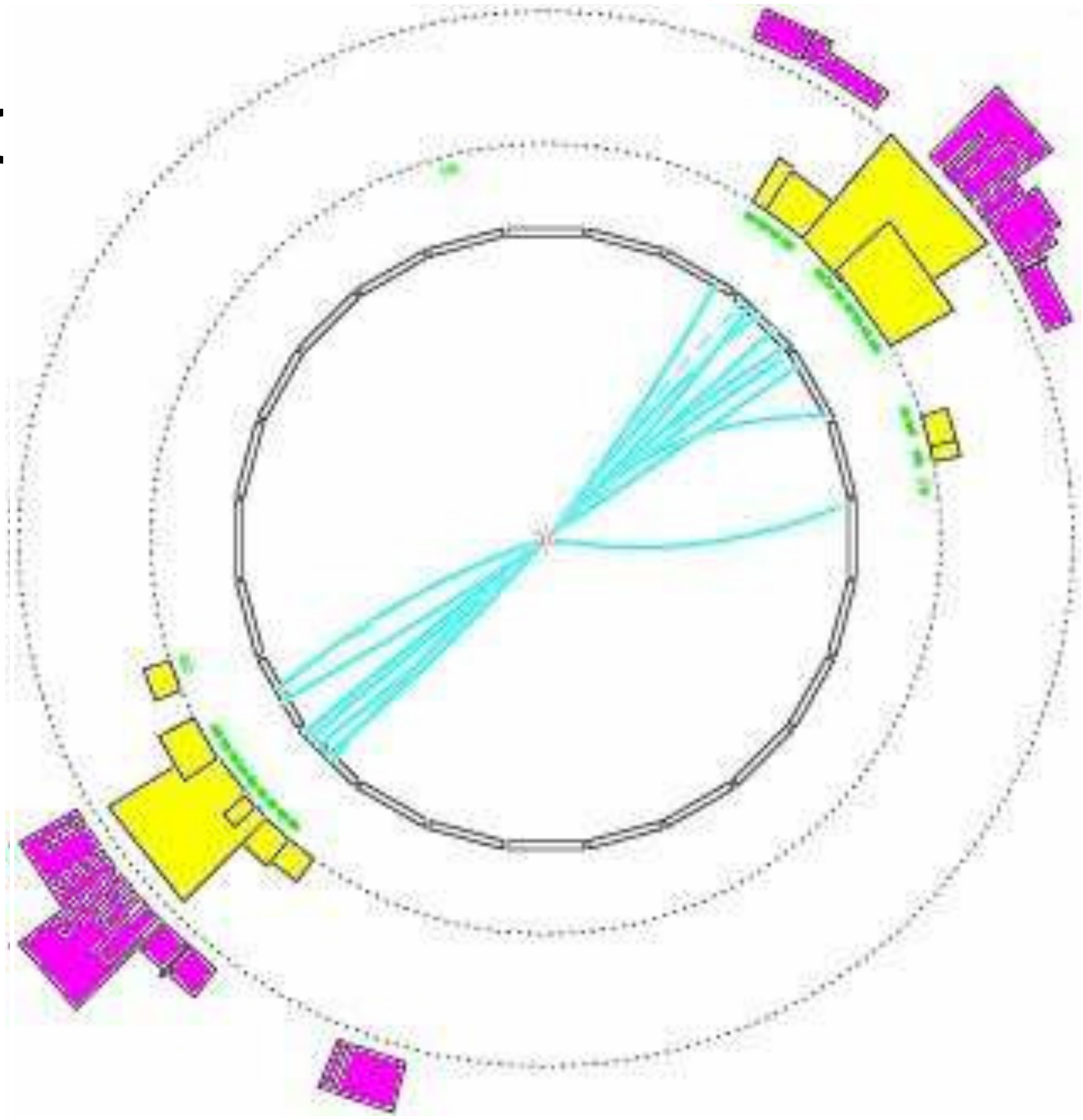

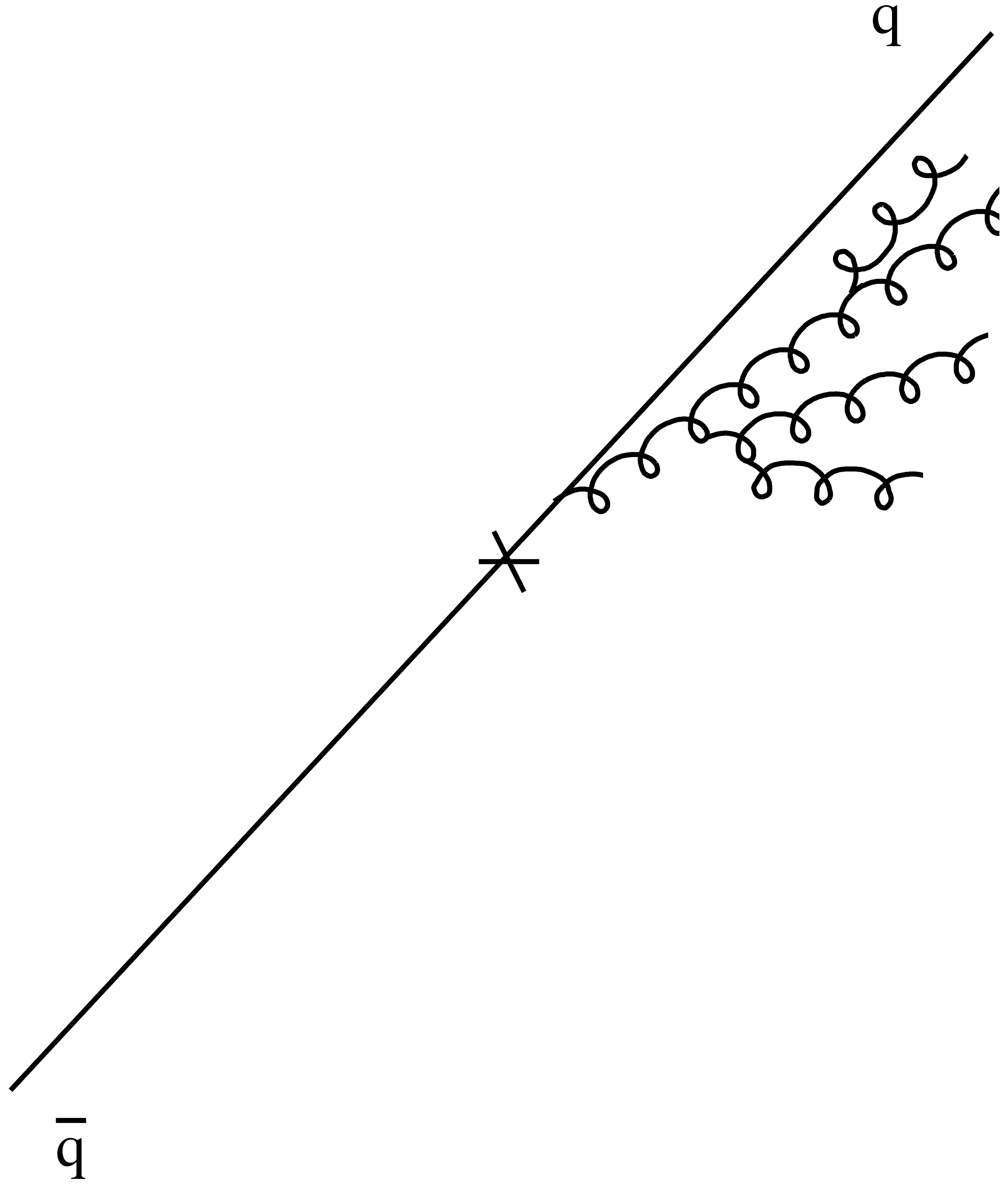

jet evolution - qualitative picture

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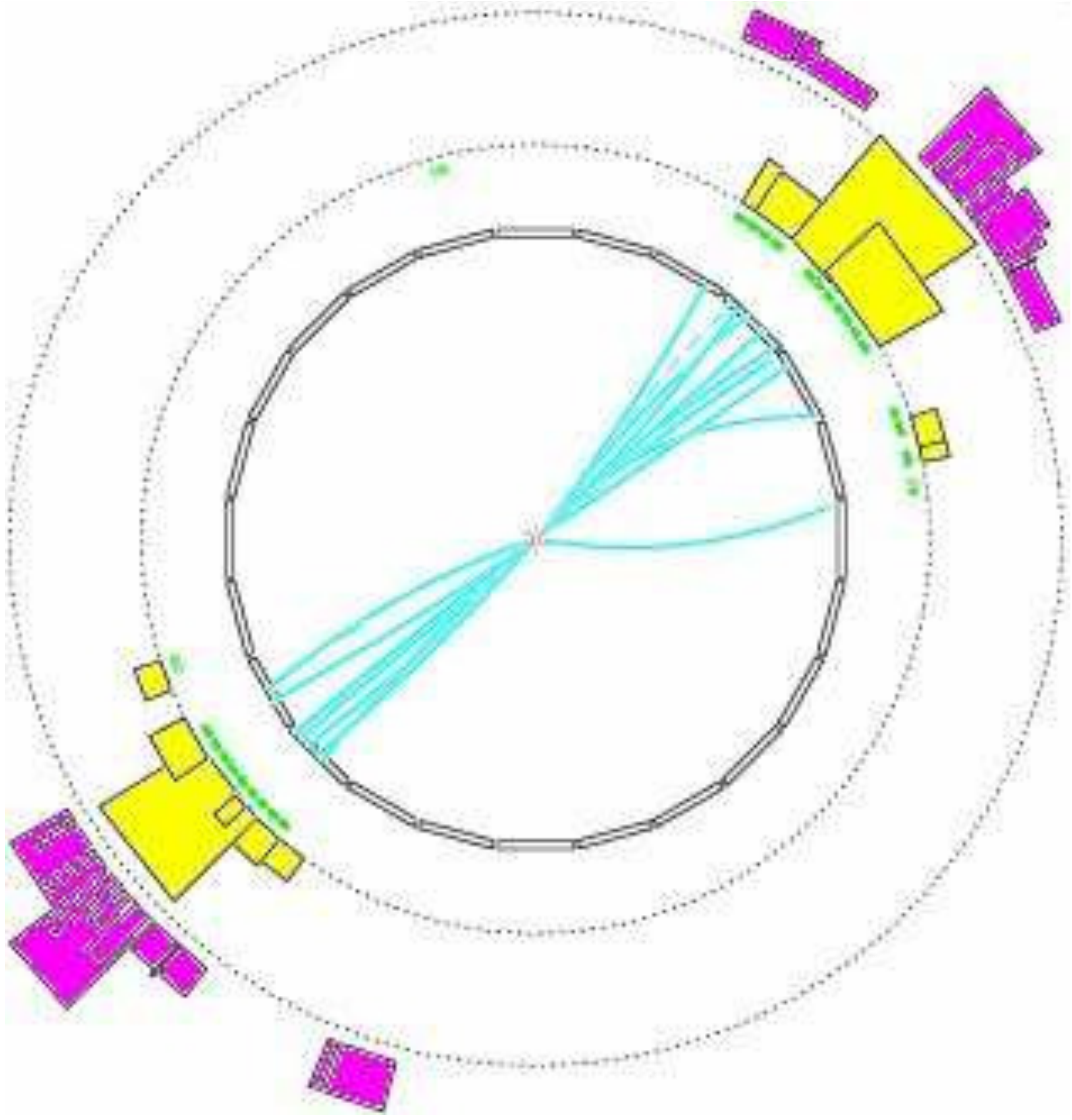

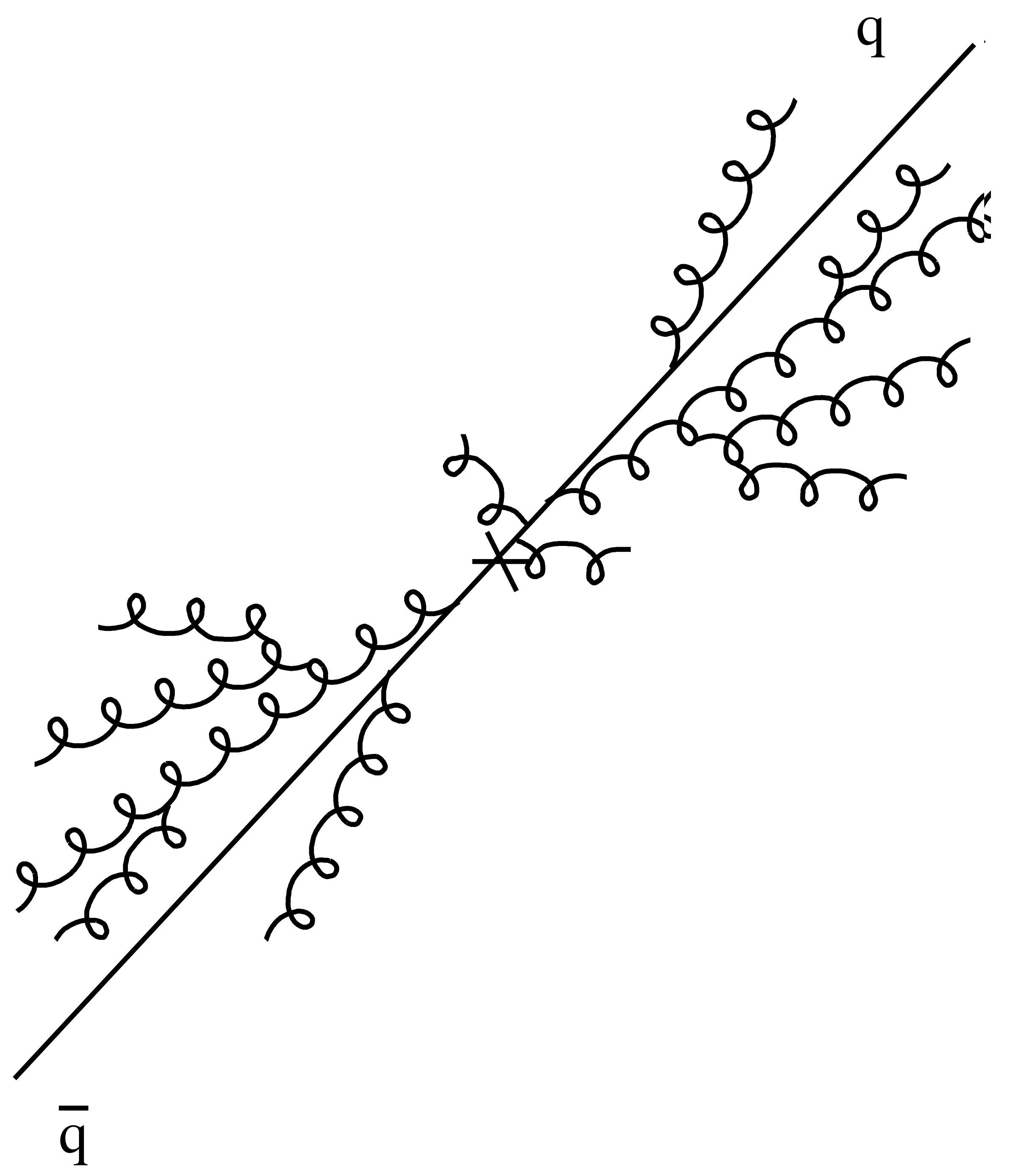

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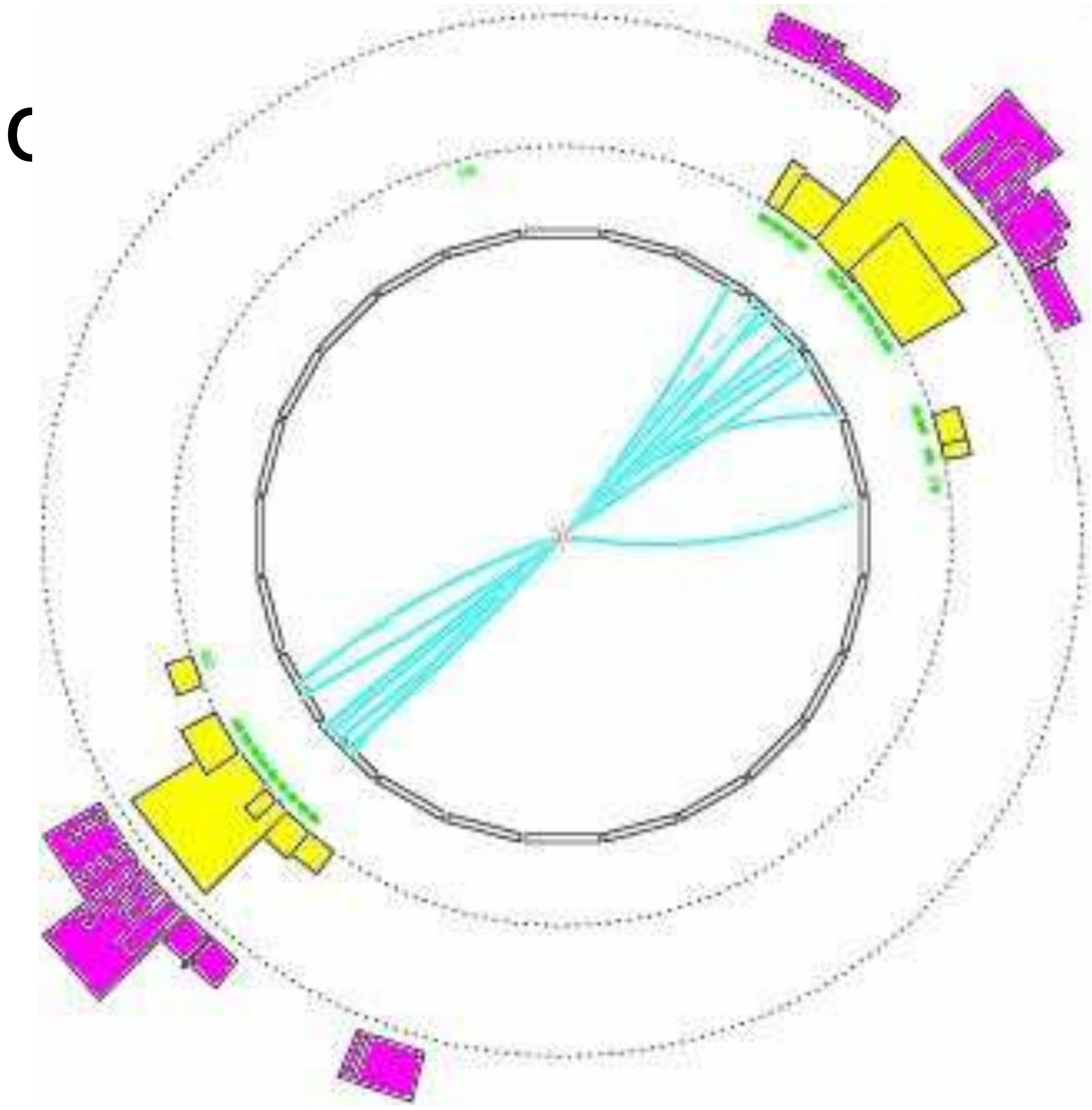
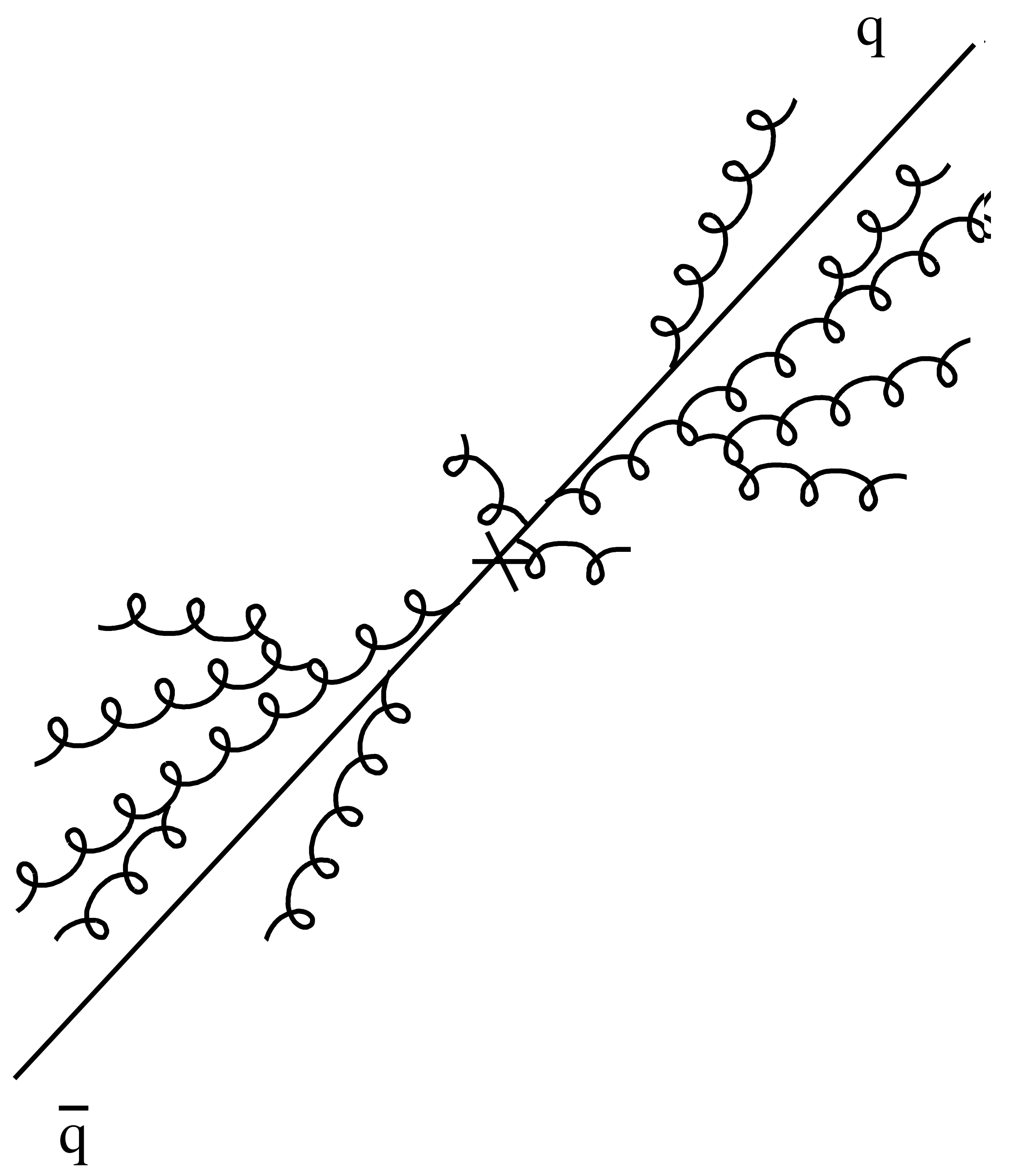




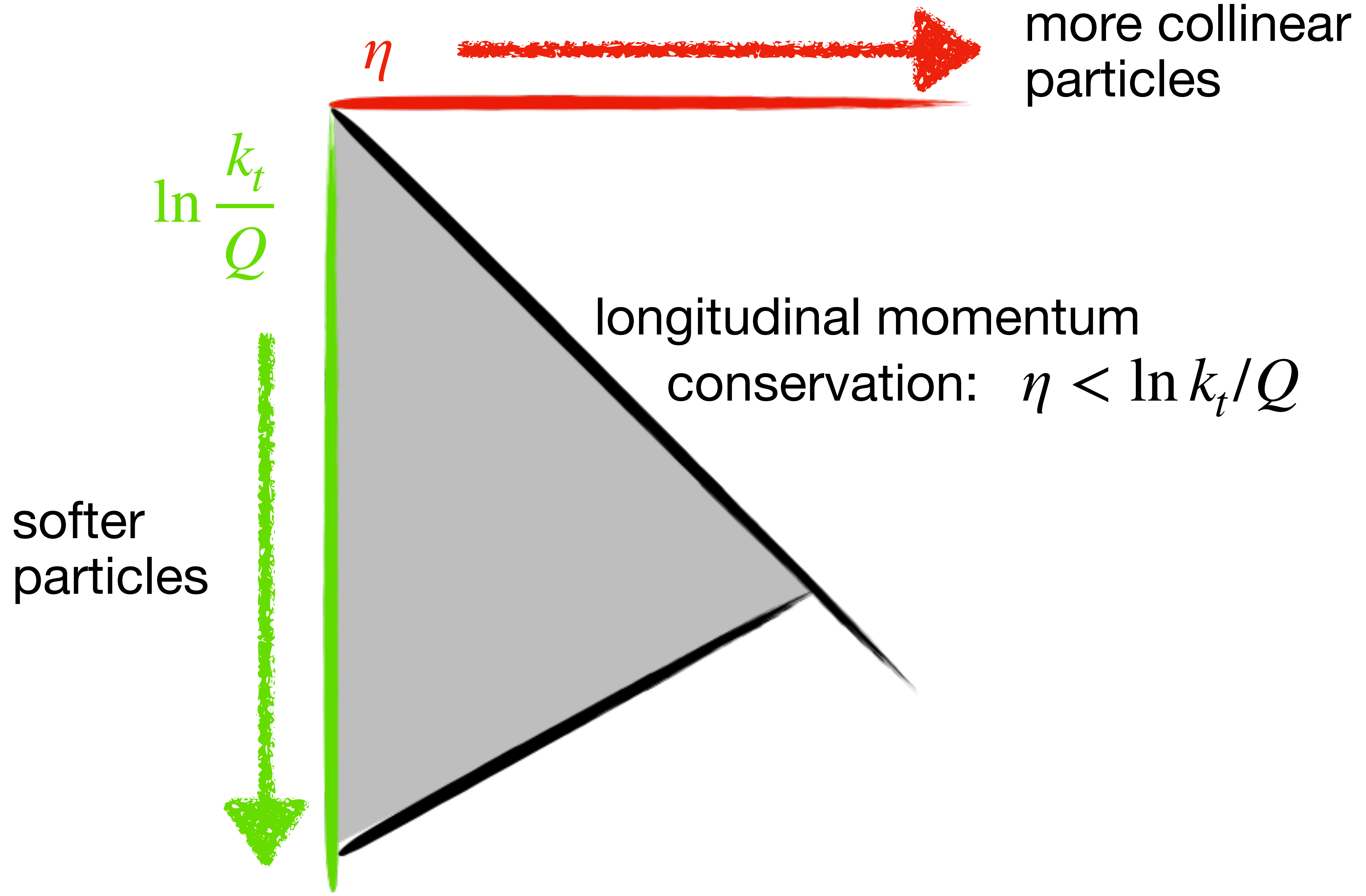
Recap: Unitary Parton shower Algorithms

- Main ingredients to a shower:

    1. splitting kernels $P(z)$ captures soft and collinear limits of matrix elements

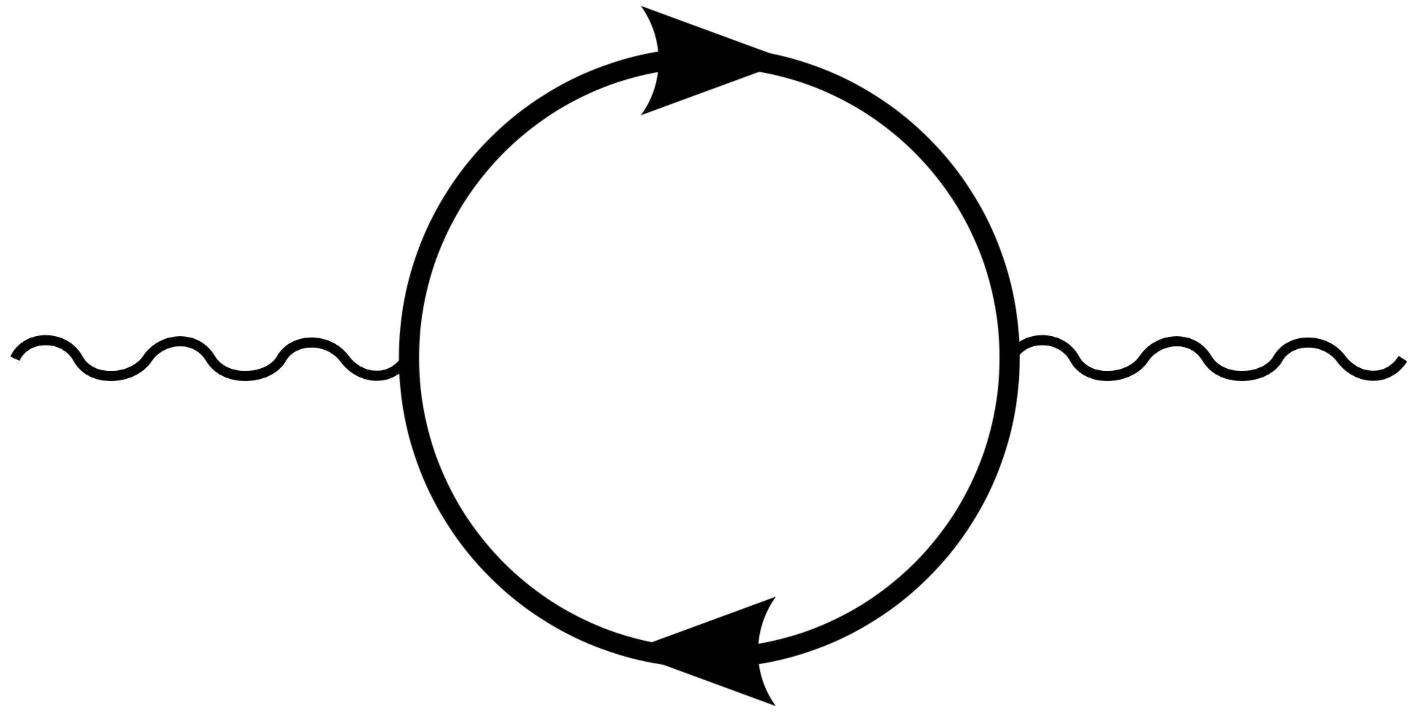
    2. fill phase space ordered in evolution variable $(k_t, \theta, q^2, \dots)$

    3. generate new final state after emission according to recoil scheme



Collinear Splitting Functions

- Calculate for example $g \to q\bar{q}$



assume Sudakov decomposition like

$$p_i^\mu = z_i \hat{p}_{ij}^\mu + \frac{-k_t^2}{z_i\, 2p_{ij}\bar{n}} \bar{n}^\mu + k_t^\mu \,,$$

$$p_j^\mu = z_j \hat{p}_{ij}^\mu + \frac{-k_t^2}{z_j\, 2p_{ij}\bar{n}} \bar{n}^\mu - k_t^\mu$$

polarisation tensor:

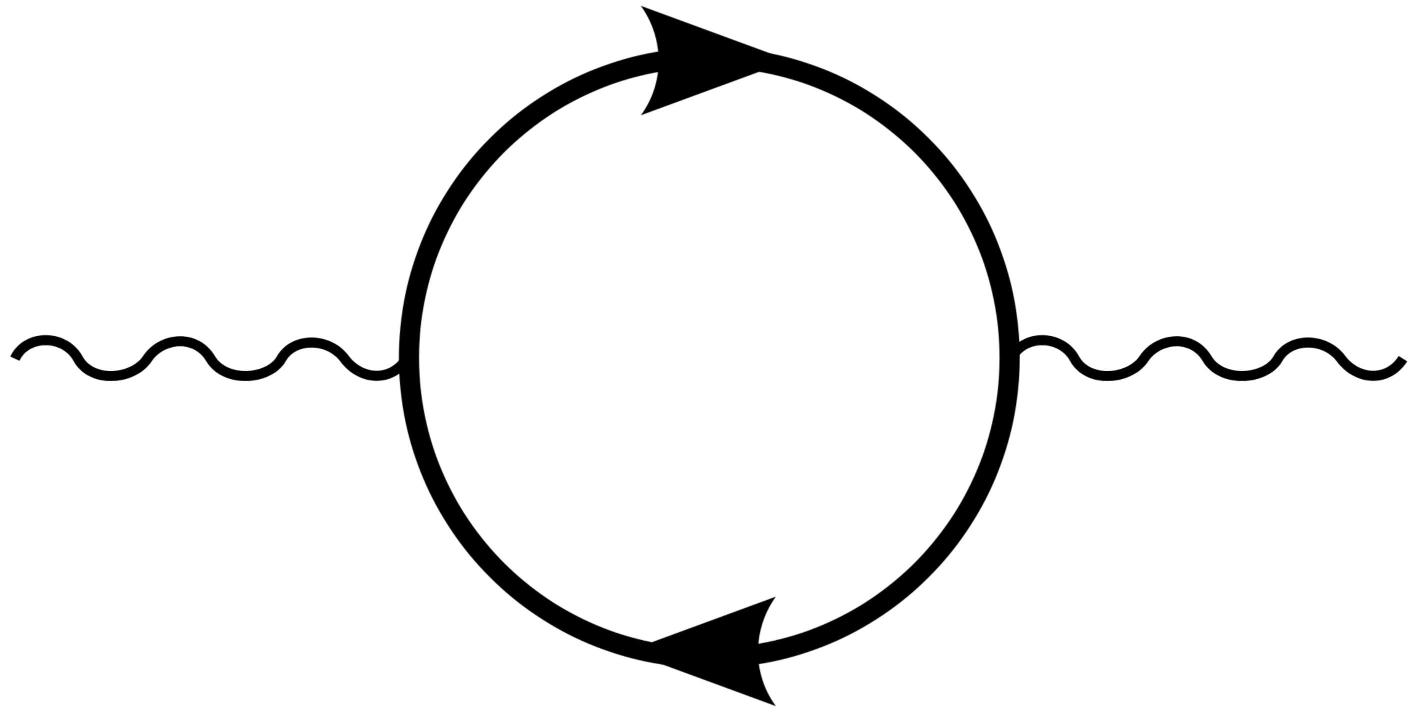
$$d^{\mu\nu}(p,n) = -g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{pn}$$

$$P^{\mu\nu}_{g\to q}(p_i,p_j) = \frac{T_R}{2p_{ij}^2}\, d^\mu{}_\rho(p_{ij},\bar{n}) \mathrm{Tr}[\not{p}_i \gamma^\rho \not{p}_j \gamma^\sigma] d^\nu{}_\sigma(p_{ij},\bar{n})$$

- evaluate in collinear limit: $\to T_R \left[-g^{\mu\nu} + 4 z_i z_j \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right]$
  - spin average: $\to T_R \left[1 - 2 z_i z_j \right] = \left[1 - 2z(1-z) \right]$

Collinear Splitting Functions

- Calculate for example $g \to q\bar{q}$



$$P^{\mu\nu}_{g\to q}(p_i,p_j) = \frac{T_R}{2p_{ij}^2}\, d^{\mu}{}_{\rho}(p_{ij},\bar{n})\mathrm{Tr}[$$

- evaluate in collinear limit: $\to T$

  - spin average: $\to T$

Full set of DGLAP equations from introduction lectures:

The DGLAP equations

Generalizes to

DGLAP (Dokshitzer–Gribov–Lipatov–Altarelli–Parisi)

$$\mathrm{d}\mathcal{P}_{a\to bc} = \frac{\alpha_s}{2\pi}\frac{\mathrm{d}Q^2}{Q^2}P_{a\to bc}(z)\,\mathrm{d}z$$

$$P_{q\to qg} = \frac{4}{3}\frac{1+z^2}{1-z}$$

$$P_{g\to gg} = 3\frac{(1-z(1-z))^2}{z(1-z)}$$

$$P_{g\to q\bar{q}} = \frac{n_f}{2}(z^2+(1-z)^2) \quad (n_f = \text{no. of quark flavours})$$

Universality: any matrix element reduces to DGLAP in collinear limit.

e.g. $$\frac{\mathrm{d}\sigma(H^0\to q\bar{q}g)}{\mathrm{d}\sigma(H^0\to q\bar{q})} = \frac{\mathrm{d}\sigma(Z^0\to q\bar{q}g)}{\mathrm{d}\sigma(Z^0\to q\bar{q})} \quad \text{in collinear limit}$$

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DGLAP parton showers

- DGLAP equation determines collinear evolution
  - of IS PDFs
  - or FS: fragmentation functions

$$t\frac{\partial}{\partial t}f(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)f(x/z,t)$$

- introduce Sudakov factor

$$\Delta(t) \equiv \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int dz\,\frac{\alpha_S}{2\pi}\hat{P}(z)\right]$$

- leads to integral equation

$$f(x,t) = \Delta(t)f(x,t_0) + \int_{t_0}^{t}\frac{dt'}{t'}\frac{\Delta(t)}{\Delta(t')}\int\frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)f(x/z,t')$$

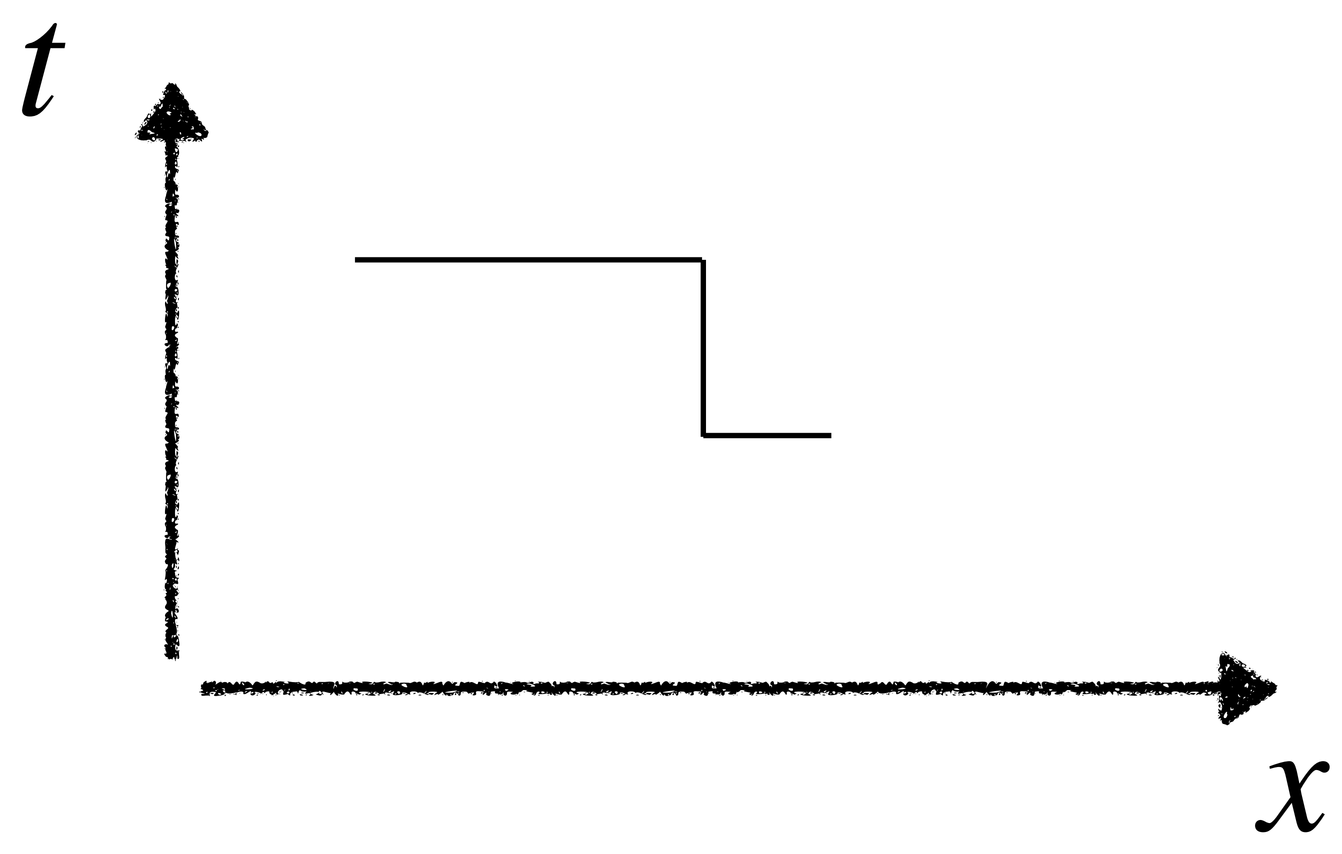
DGLAP parton showers

$$f(x,t) = \Delta(t) f(x,t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) f(x/z, t')$$

Evolution between t and t_0 without any splitting

Evolution between t and t' without any splitting

Splitting at t' with momentum fraction z



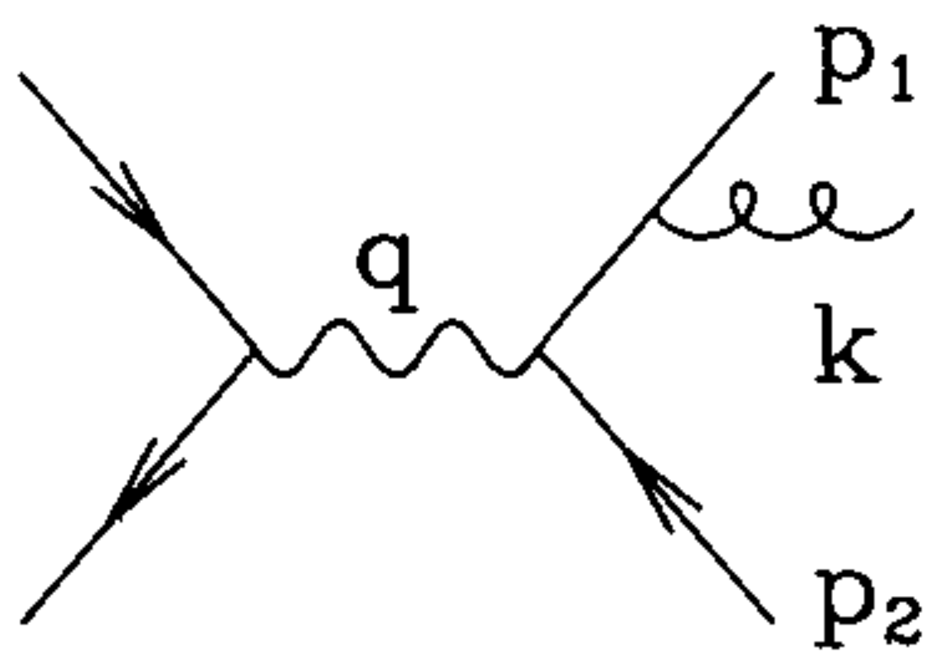
Select size of t step

Select size of x step

Towards Coherent Branching

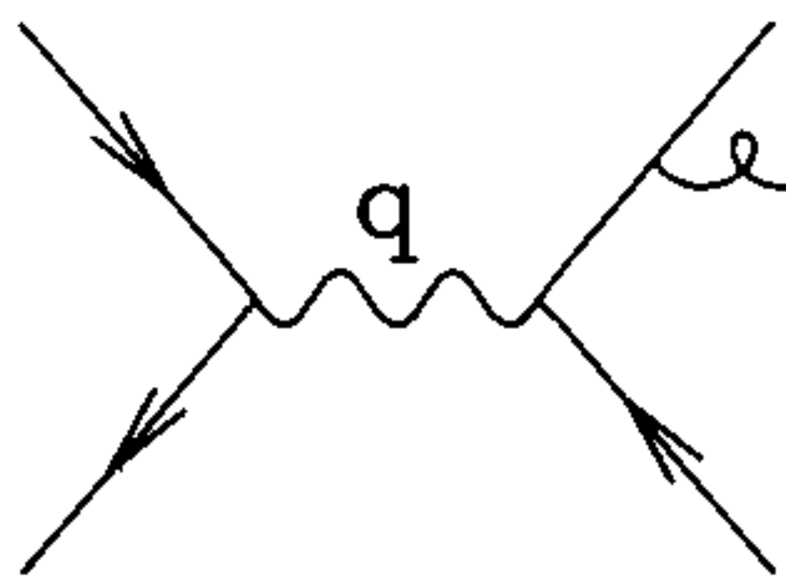
- Collinear enhancements under control at all orders
- What about soft gluons emissions?
- Limit of splitting functions enhanced by $\sim \frac{1}{1-z}$
- Is that all? Lets have a closer look!

QCD calculations — soft limit



$$\sim \bar{u}(p_1)(-ig_s)t^A\gamma^\alpha \frac{i(\not{p}_1 + \not{k})}{(p_1 + k)^2}(-ie)\gamma^\mu v(p_2)\epsilon_\alpha$$

QCD calculations — soft limit

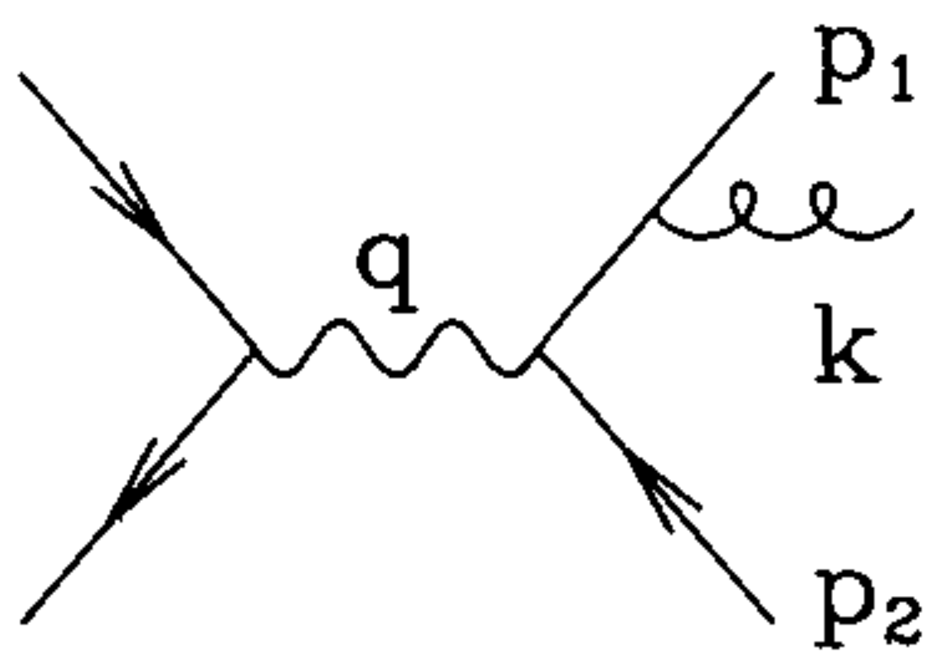


$$\sim \bar{u}(p_1)(-ig_s)t^A\gamma^\alpha \frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2}(-ie)\gamma^\mu v(p_2)\epsilon_\alpha$$

$$\sim \bar{u}(p_1)(-ig_s)t^A\not{\epsilon} \frac{i\not{p}_1}{2p_1\cdot k}(-ie)\gamma^\mu v(p_2)$$

assume massless partons, $p_1^2 = 0$, $k^2 = 0$ and analyse the soft gluon $k \to 0$ limit

QCD calculations – soft limit



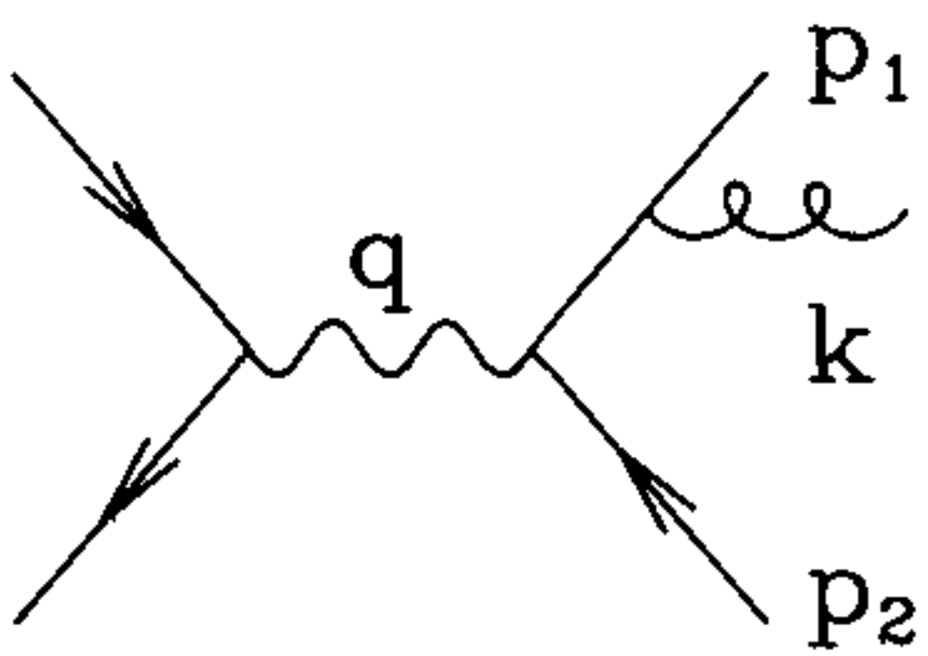
$$\sim \bar{u}(p_1)(-ig_s)t^A\gamma^\alpha \frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2}(-ie)\gamma^\mu v(p_2)\epsilon_\alpha$$

$$\sim \bar{u}(p_1)(-ig_s)t^A\not{\epsilon}\frac{i\not{p}_1}{2p_1\cdot k}(-ie)\gamma^\mu v(p_2)$$

$$\sim \bar{u}(p_1)(-ig_s)t^A\frac{ip_1\cdot\epsilon}{p_1\cdot k}(-ie)\gamma^\mu v(p_2)$$

use $\not{\epsilon}\not{p}_1 = 2\epsilon\cdot p_1 - \not{p}_1\not{\epsilon}$ and the Dirac equation $\bar{u}(p_1)\not{p}_1 = 0$

QCD calculations – soft limit

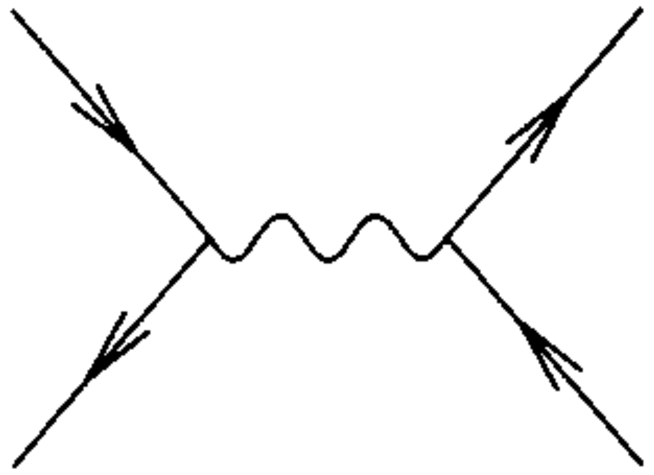


$$\sim \bar{u}(p_1)(-ig_s)t^A\gamma^\alpha \frac{i(\not{p}_1+\not{k})}{(p_1+k)^2}(-ie)\gamma^\mu v(p_2)\epsilon_\alpha$$

$$\sim \bar{u}(p_1)(-ig_s)t^A\not{\epsilon}\frac{i\not{p}_1}{2p_1\cdot k}(-ie)\gamma^\mu v(p_2)$$

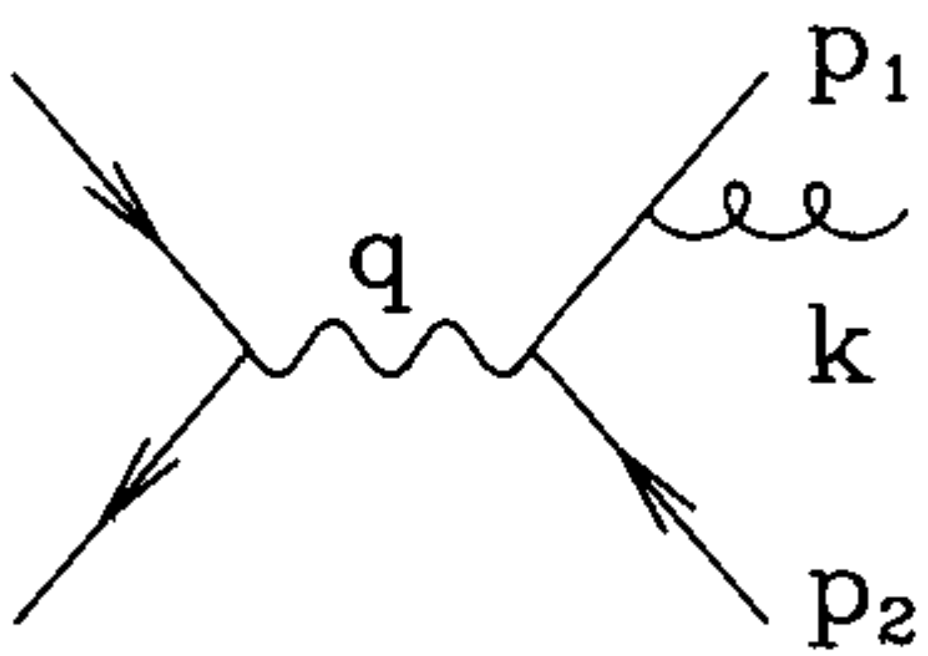
$$\sim \bar{u}(p_1)(-ig_s)t^A\frac{ip_1\cdot\epsilon}{p_1\cdot k}(-ie)\gamma^\mu v(p_2)$$

$$\sim g_s t^A \frac{p_1\cdot\epsilon}{p_1\cdot k}\times$$

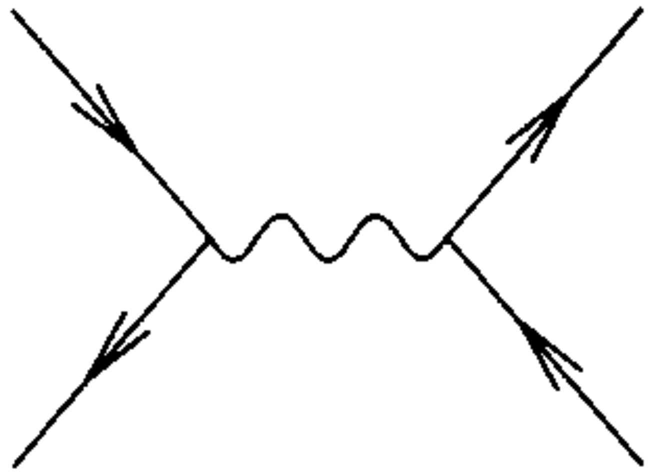


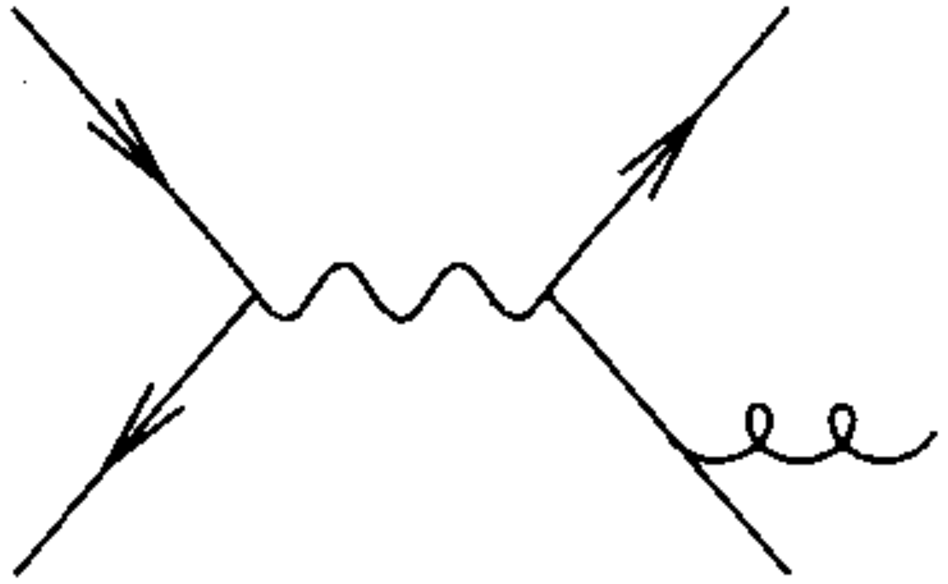
soft gluon emissions factorise!

QCD calculations — soft limit

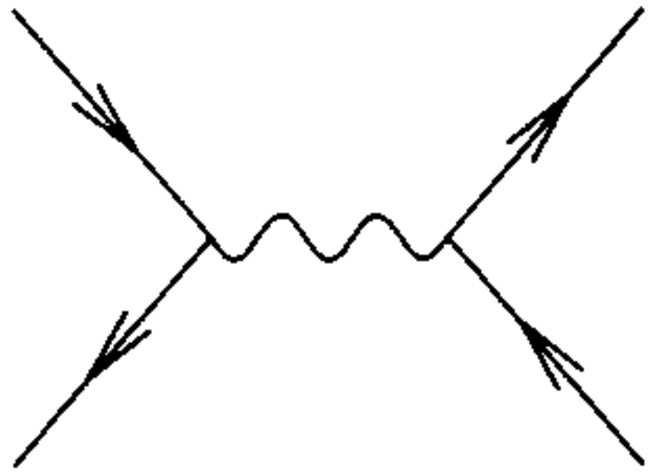


$$\sim g_s t^A \frac{p_1 \cdot \epsilon}{p_1 \cdot k} \times$$





$$\sim g_s t^A \frac{p_2 \cdot \epsilon}{p_2 \cdot k} \times$$



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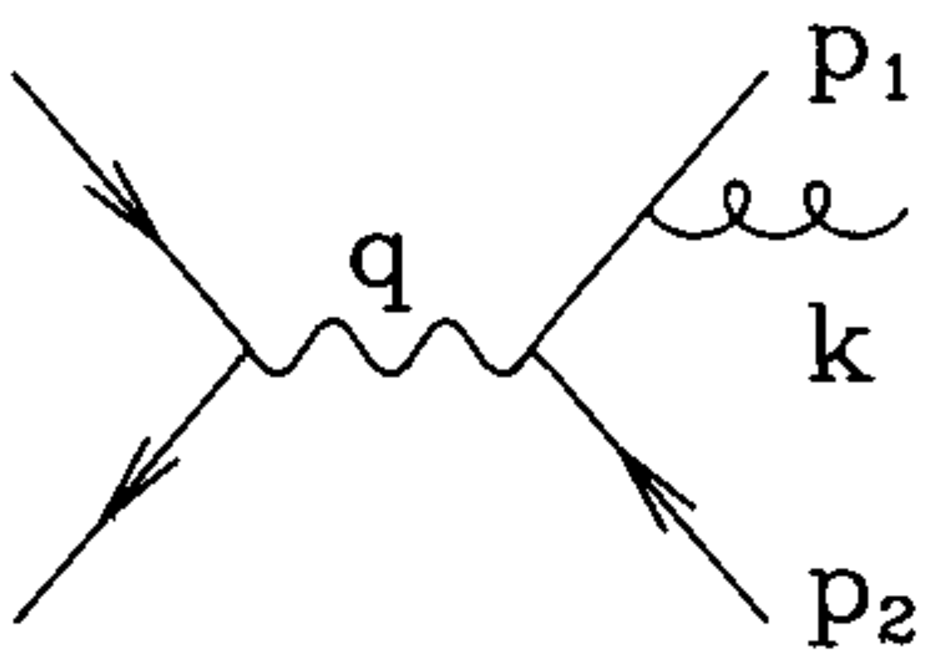
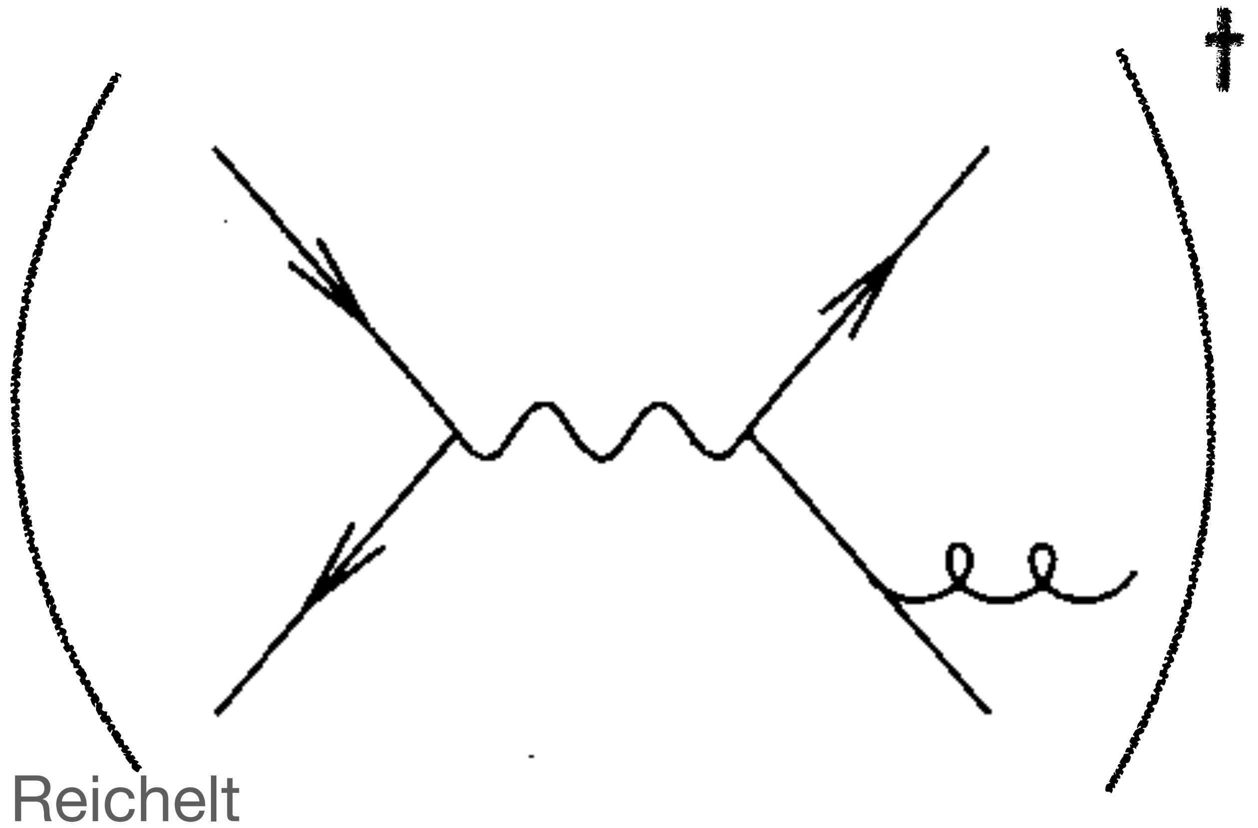
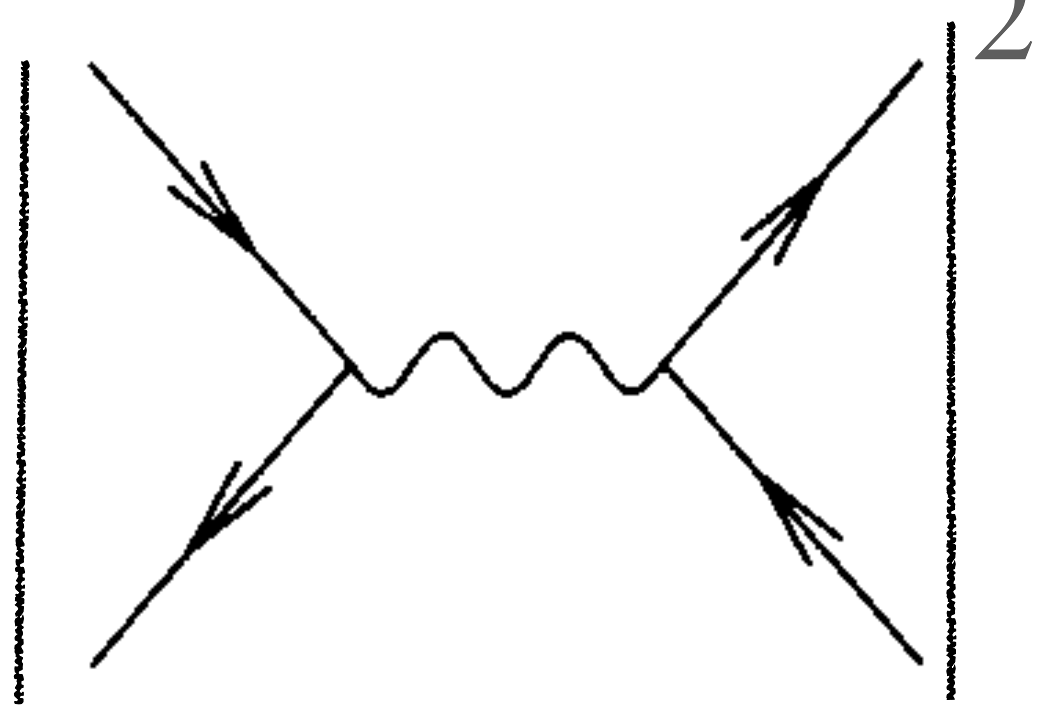
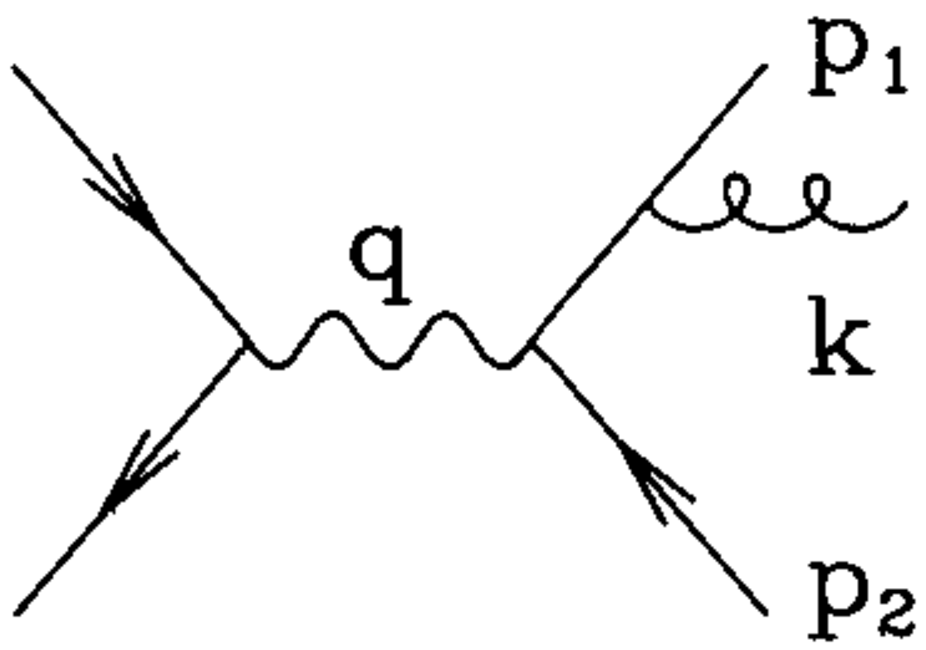
2 

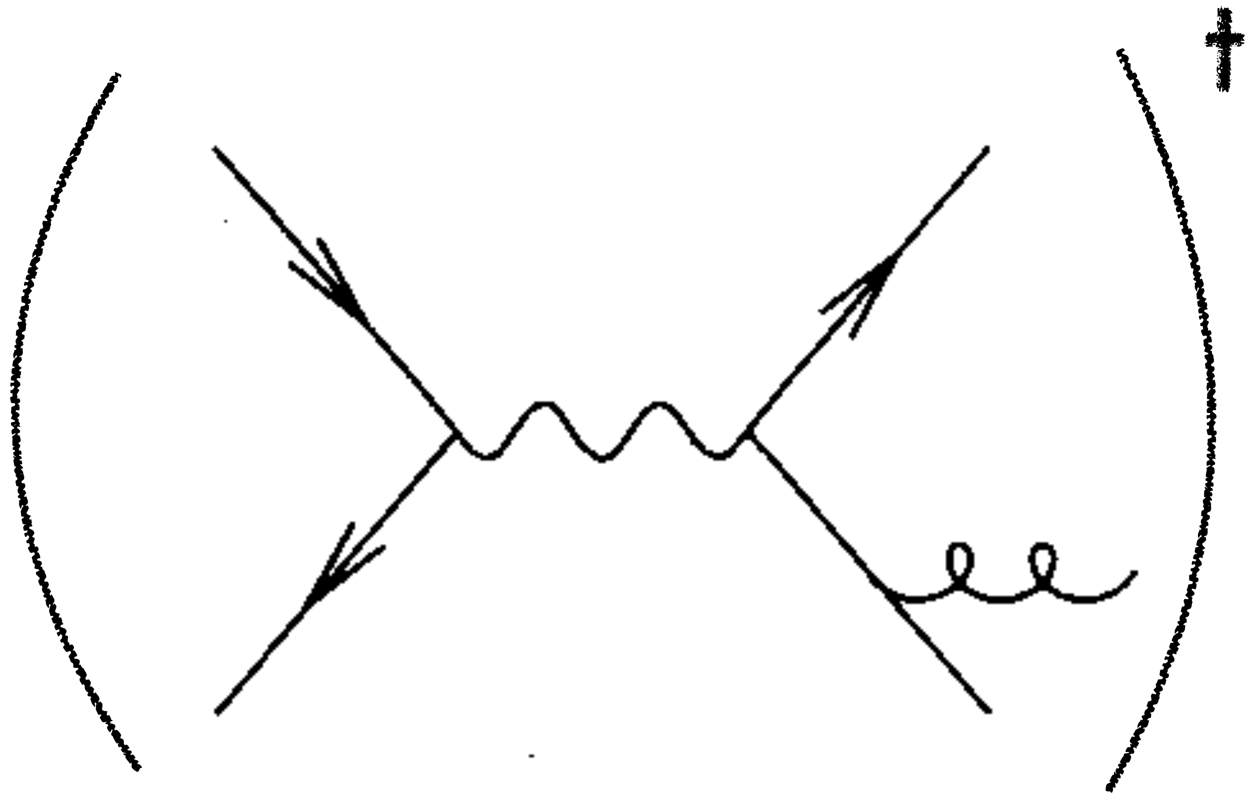


$$\sim 2 g_s^2 t^A t^B \frac{p_1 \cdot \epsilon}{p_1 \cdot k} \frac{p_2 \cdot \epsilon}{p_2 \cdot k}$$



QCD calculations – soft limit

2 



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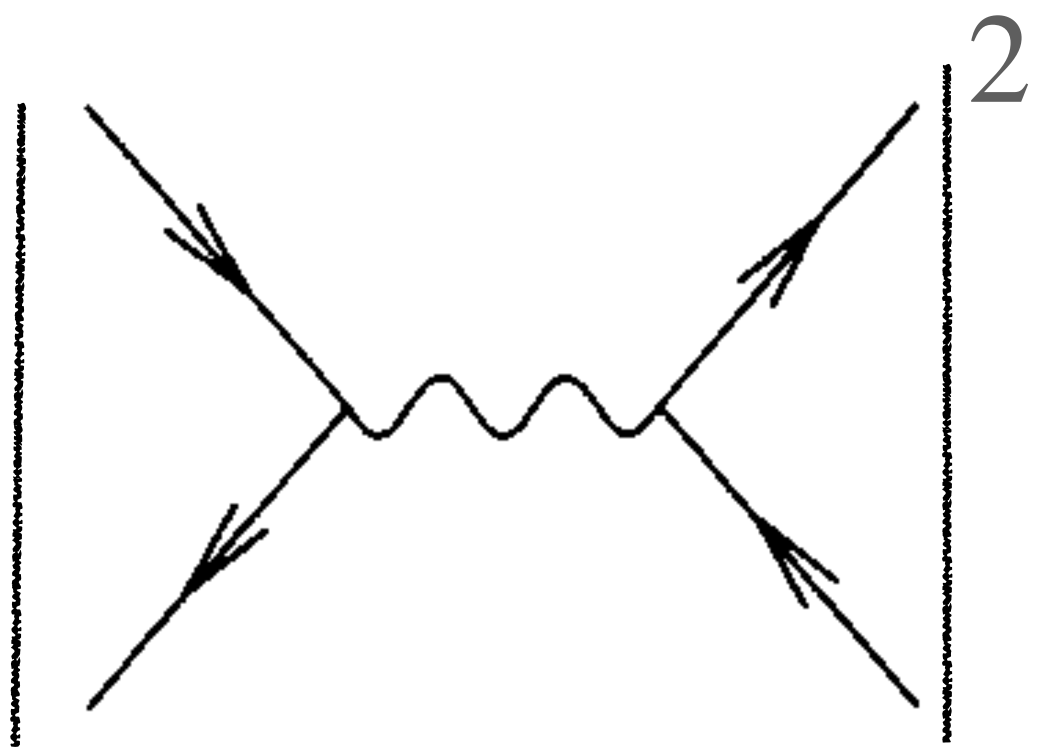
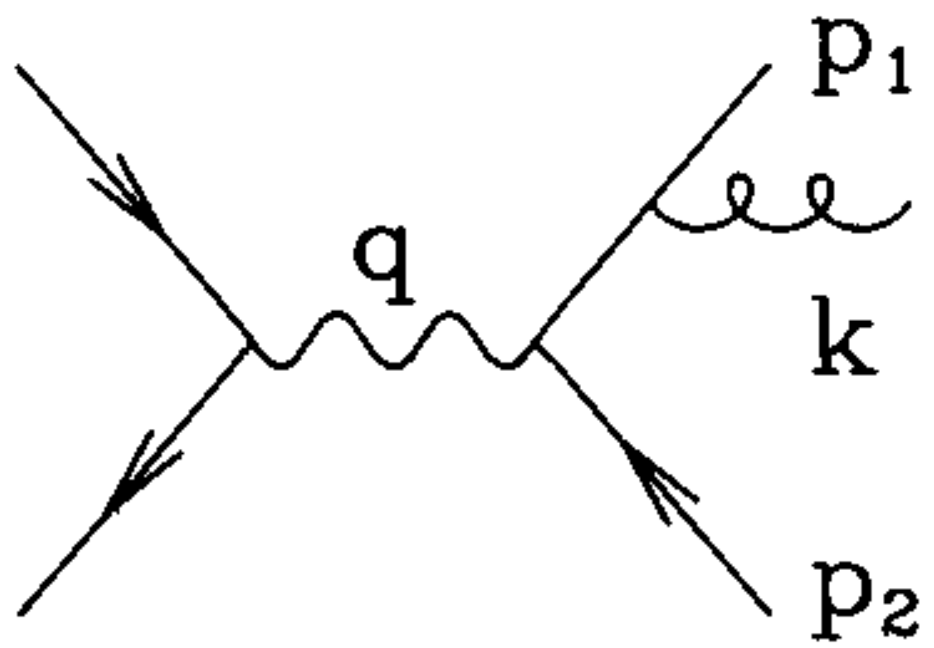
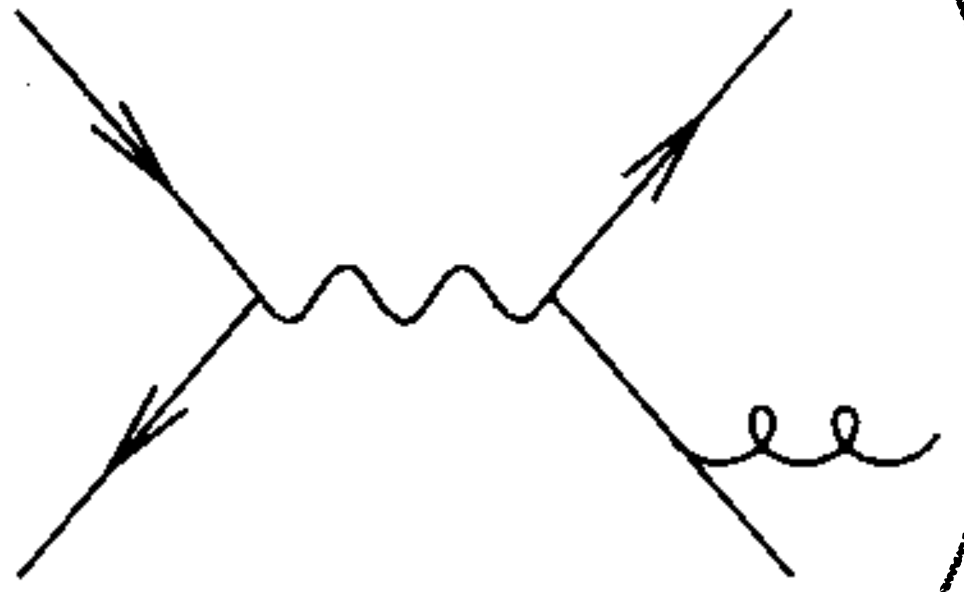
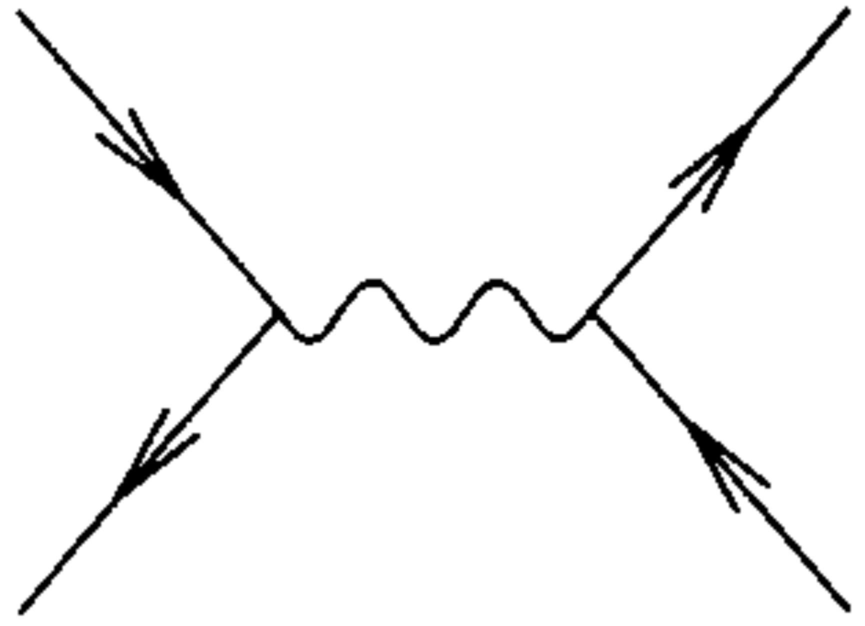
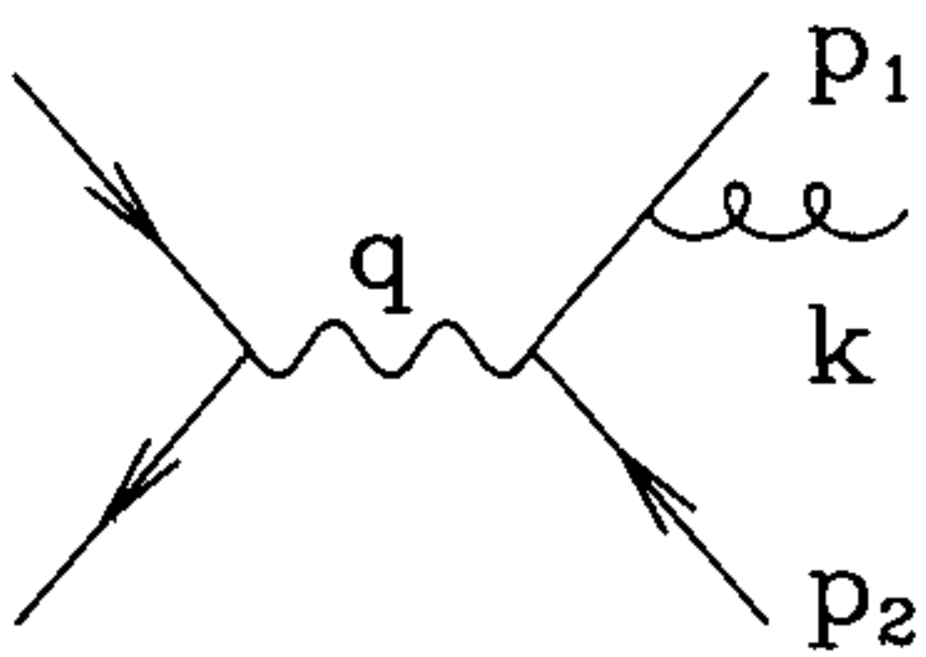


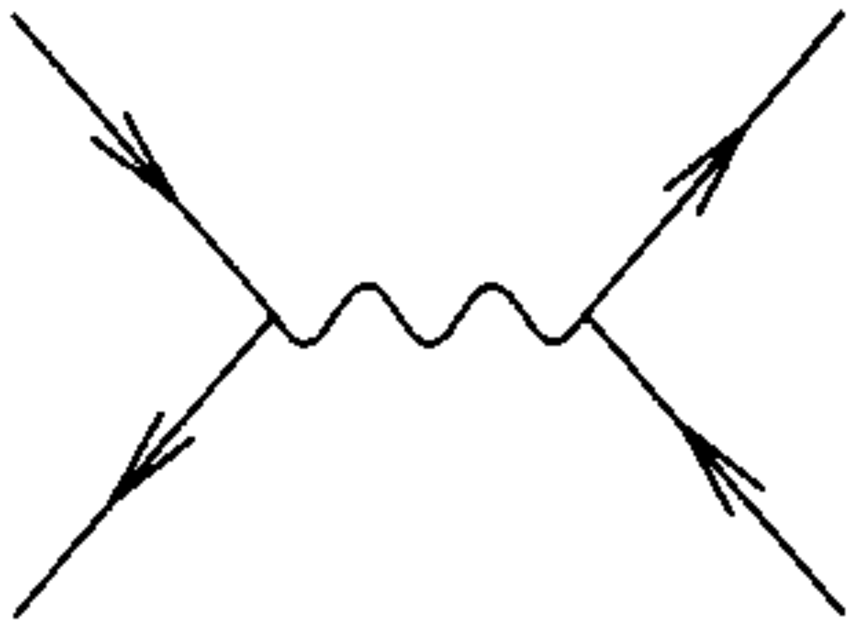
QCD calculations — soft limit

2 

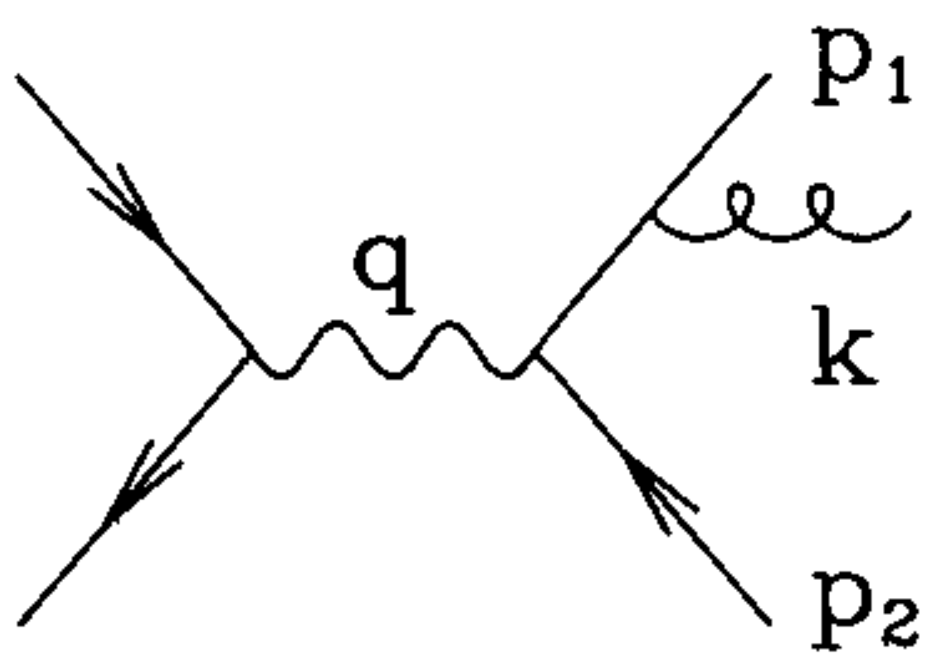
$\Big($  $\Big)^{\dagger}$ $\sim 2g_s^2 t^A t^B \frac{p_1 \cdot \epsilon}{p_1 \cdot k} \frac{p_2 \cdot \epsilon}{p_2 \cdot k}$ $\Big|$  $\Big|^2$

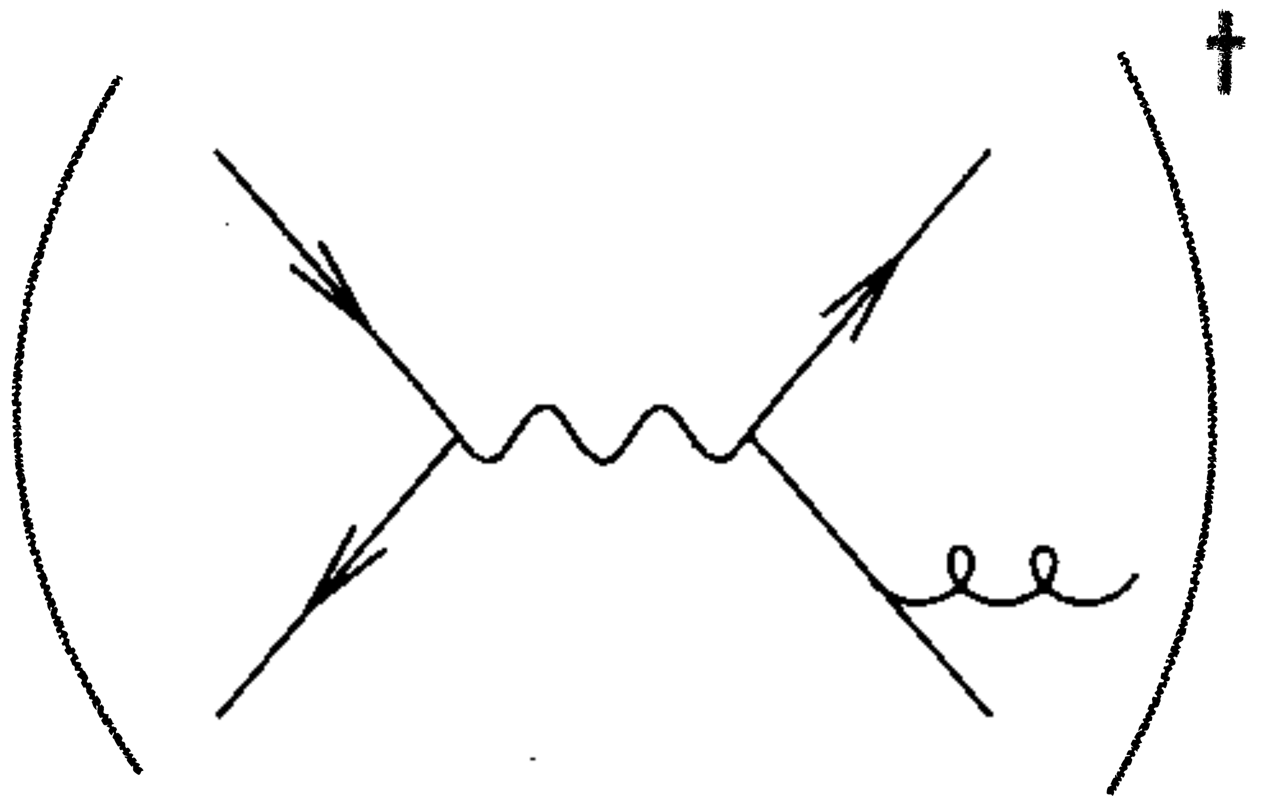
perform sum over gluon polarisations $\epsilon_\mu \epsilon_\nu \to -g_{\mu\nu}$, and colours $t^A t^B \to C_F$

$\Big|$  $\Big|^2$

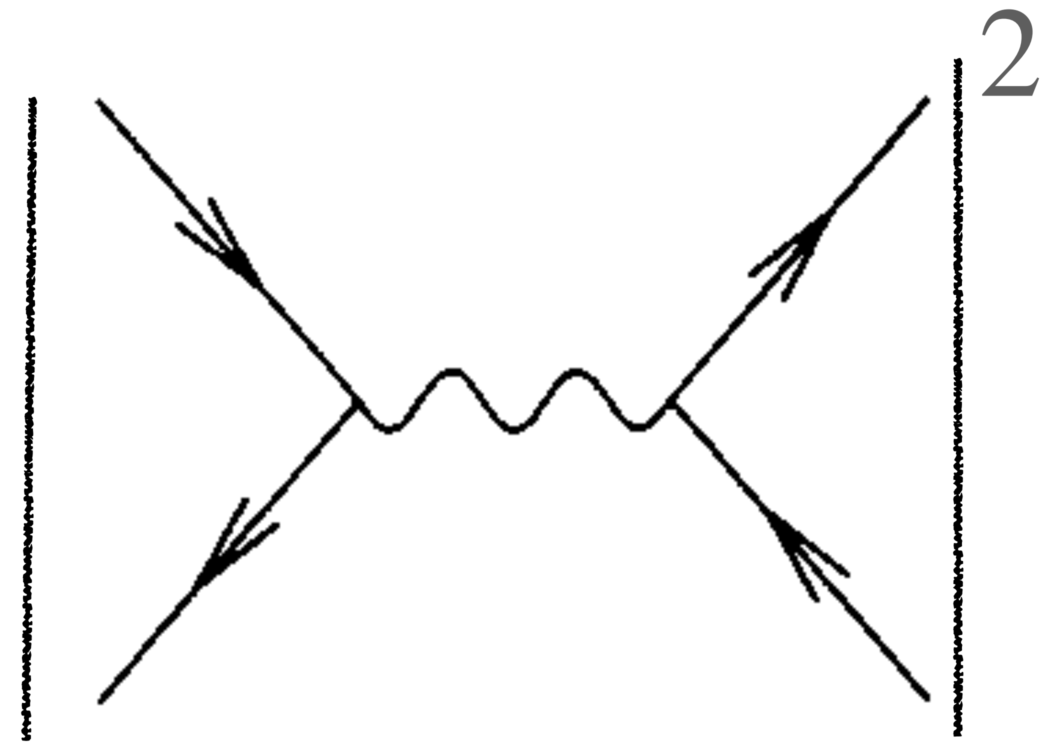
$\sim g_s^2 C_F \frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$ $\Big|$  $\Big|^2$

QCD calculations – soft limit

2 



$$\sim 2g_s^2 t^A t^B \frac{p_1 \cdot \epsilon}{p_1 \cdot k} \frac{p_2 \cdot \epsilon}{p_2 \cdot k}$$



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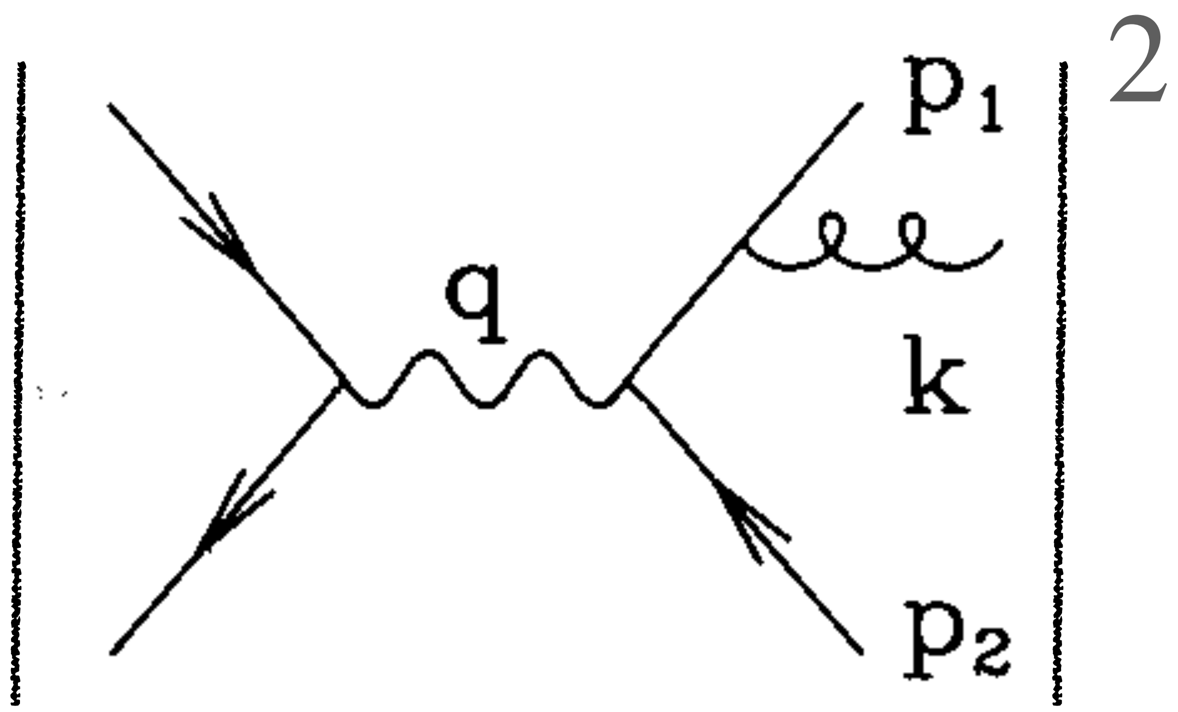
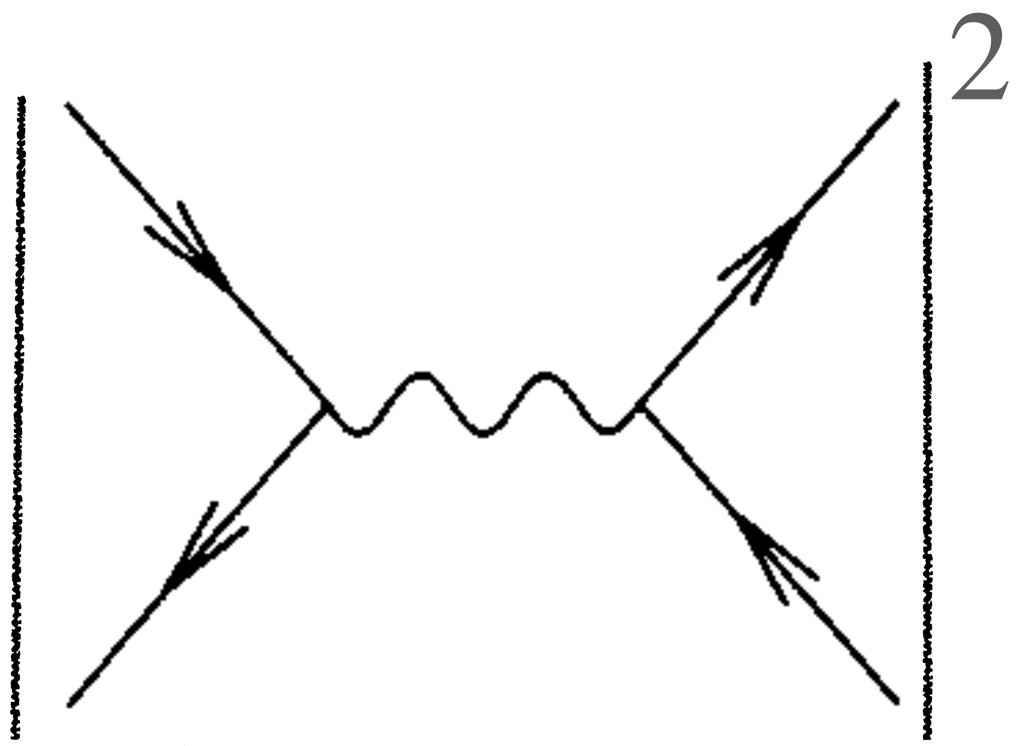


$$\sim g_s^2 C_F \frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$$



Note: phase space factorises as well $d\phi_{q\bar{q}g} = d\phi_{q\bar{q}} d\phi_{+1}$

Factorisation with “eikonal” factor!

Eikonal factor

$$\frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$$

observation: divergent if $k || p_1$ or $k || p_2$ or $k \to 0$

$\Rightarrow$ collinear and soft/infrared limits

Eikonal factor

$$\frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$$

observation: divergent if $k \parallel p_1$ or $k \parallel p_2$ or $k \to 0$

$\Rightarrow$ collinear and soft/infrared limits

Explicitly in some reference frame, use $p_i \cdot k = E_i E_k (1 - \cos\theta_{ik})$

$$\frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} = \frac{1}{E_k^2} \frac{1 - \cos\theta_{12}}{(1 - \cos\theta_{1k})(1 - \cos\theta_{2k})} \equiv \frac{W_{12,k}}{E_k^2}$$

$\Rightarrow$ divergencies visible for $\theta_{ik} \to 0$ (collinear) and $E_k \to 0$ (soft)

Coherent Branching - Angular Ordering

Eikonal derived in last slides:

$$W_{ik,j} = E_k^2 \frac{2p_i \cdot p_k}{(p_i \cdot p_j)(p_k \cdot p_j)} = \frac{1 - \cos\theta_{ik}}{(1 - \cos\theta_{ij})(1 - \cos\theta_{jk})}$$

We want to split this up, only divergent in one collinear region (by clever adding and subtracting divergent terms):

$$W_{ik,j} = \tilde{W}^i_{ik,j} + \tilde{W}^j_{ik,j}$$

$$\tilde{W}^i_{ik,j} = \frac{1}{2}\left(\frac{1 - \cos\theta_{ik}}{(1 - \cos\theta_{ij})(1 - \cos\theta_{jk})} + \frac{1}{1 - \cos\theta_{ij}} - \frac{1}{1 - \cos\theta_{jk}}\right)$$

Coherent Branching - Angular Ordering

- We have found

$$\tilde{W}^i_{ik,j} = \frac{1}{2}\left(\frac{1-\cos\theta_{ik}}{(1-\cos\theta_{ij})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{ij}} - \frac{1}{1-\cos\theta_{jk}}\right)$$

- Now perform the phase space integral, in particular angular integrals for emission j, choose coordinate system with i along z-axis $d\Omega_j = d\cos\theta_{ij}d\phi_j$

Coherent Branching - Angular Ordering

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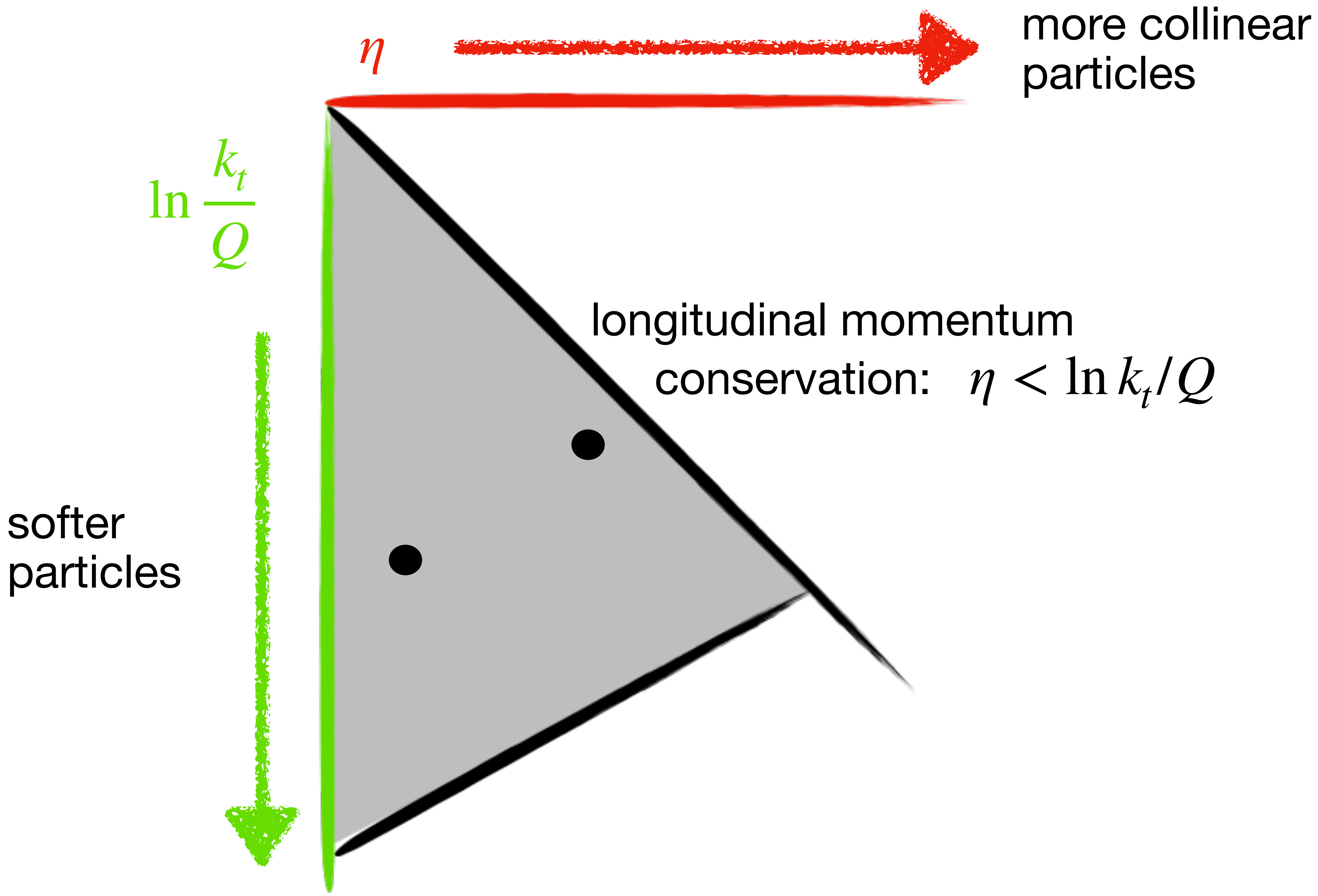
$$\tilde{W}^{i}_{ik,j} = \frac{1}{2}\left(\frac{1-\cos\theta_{ik}}{(1-\cos\theta_{ij})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{ij}} - \frac{1}{1-\cos\theta_{jk}}\right)$$

- Now perform the phase space integral, in particular angular integrals for emission j, choose coordinate system with i along z-axis $d\Omega_j = d\cos\theta_{ij} d\phi_j$:

$$\int d\phi_j \tilde{W}^{i}_{ik,j} = \begin{cases} \frac{1}{1-\cos\theta_{ij}} \text{ if } \theta_{ij} < \theta_{ik} \\ 0 \text{ else} \end{cases}$$

Coherent Branching - Angular Ordering

- What does this mean for parton shower?
  - We should first generate splitting at large angle
    - will be generated coherently from all more collinear partons, so generate them first while we only have original parent

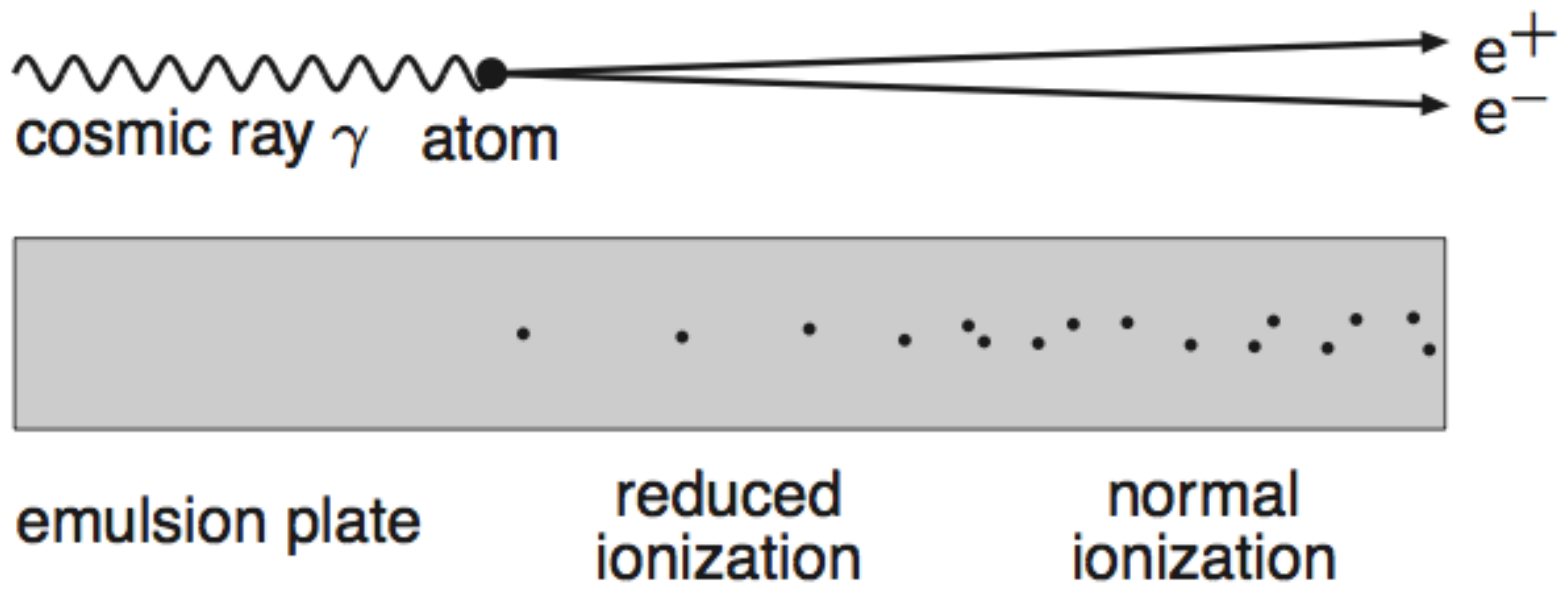


Coherent Branching - Angular Ordering

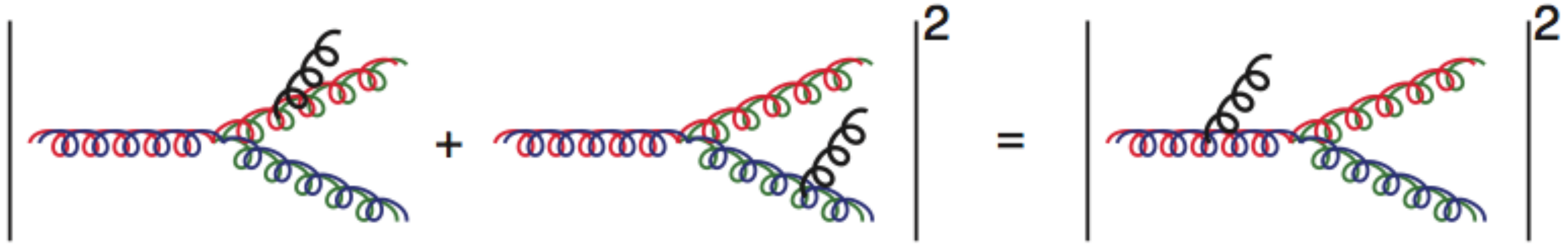
- Reminder: Consistent with expectations from QED

Coherence

QED: Chudakov effect (mid-fifties)



QCD: colour coherence for **soft** gluon emission



solved by • requiring emission angles to be decreasing
or • requiring transverse momenta to be decreasing

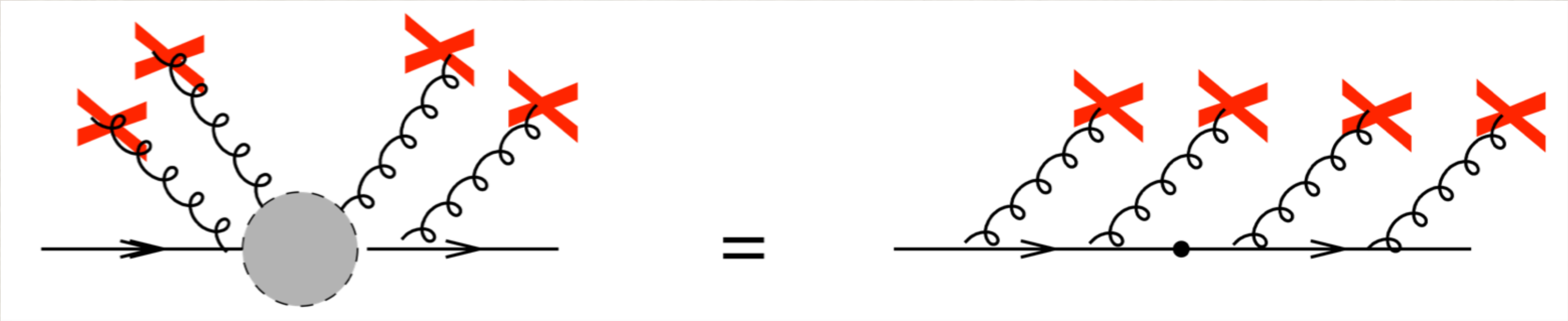
Torbjörn Sjöstrand Introduction to Event Generators 2 slide 21/59

Coherent Branching - Angular Ordering

- Reminder: Coherence in resummation of event shapes

MULTIPLE SOFT-COLLINEAR EMISSIONS

- We first consider an ensemble of soft-collinear emissions widely separated in angle (rapidity)
- Due to QCD coherence, the multi-gluon matrix element factorises into the product of single-emission matrix elements



- Contribution of multiple soft-collinear emissions to $\Sigma(v)$

$$\Sigma(v) = e^{-\int [dk] M^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i [dk_i] M^2(k_i)\, \Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_n))$$

virtual corrections, ensure that the inclusive sum over all emissions gives one

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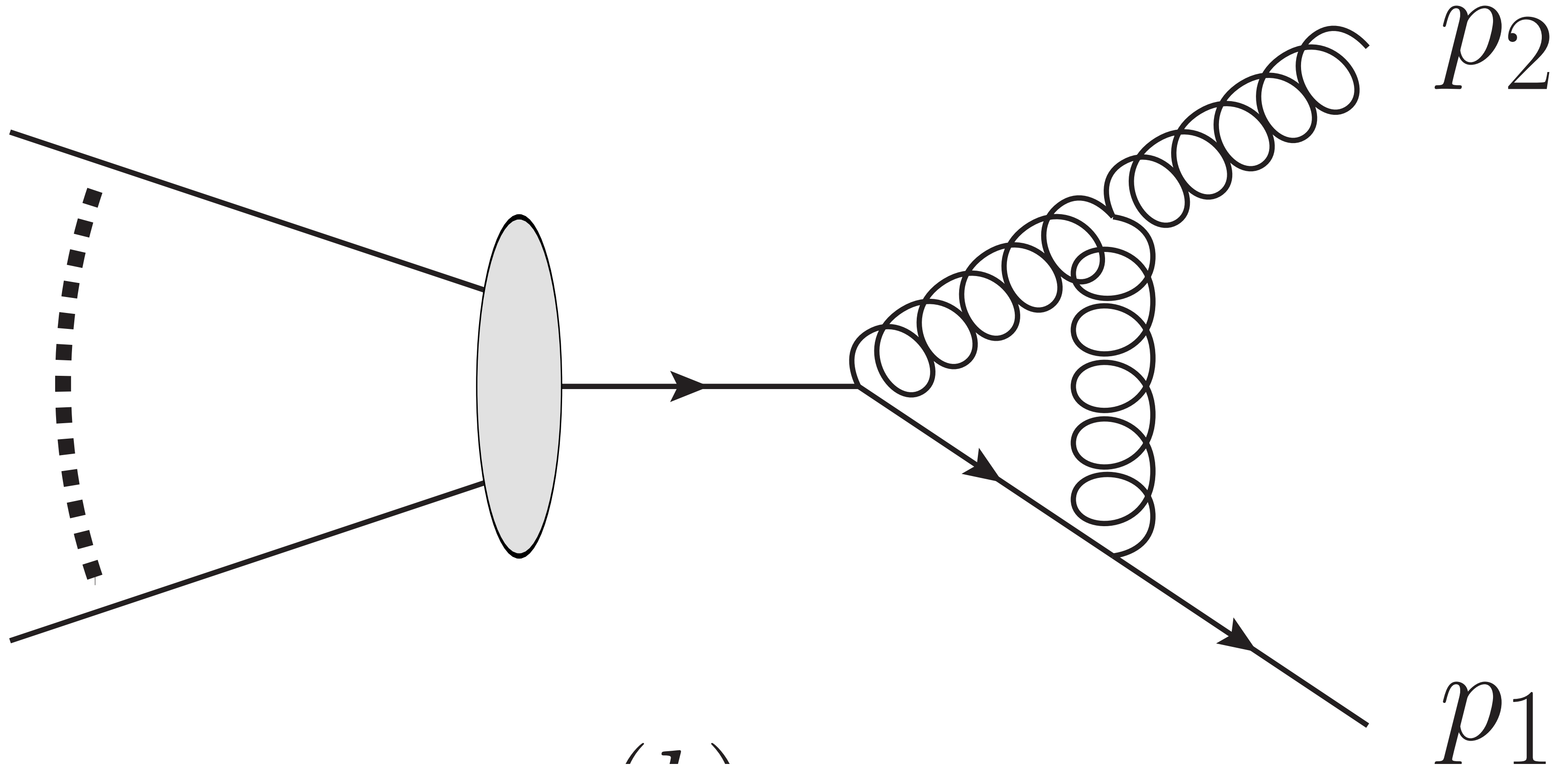
Towards NLL - argument of α_S

- Imagine we want to calculate loop corrections to splitting functions:
  - This will diverge in UV, need renormalization!
  - QCD is renormalized multiplicatively, effect: replace bare by renormalized coupling



$$\alpha_s^0 \to \alpha_s(\mu) \frac{e^{\epsilon \gamma_E}}{(4\pi)^\epsilon} \left(1 - \frac{1}{\epsilon} \frac{\alpha_s(\mu)}{2\pi} \beta_0 + \mathcal{O}(\alpha_s^2) \right)$$

  - On the other hand D-dim phase space w/ Sudakov parametrization

$$d^D p = \frac{2 p_1 \cdot p_2}{2} \, dz_1 \, dz_2 \, d\Omega(\epsilon) \, k_t^{D-3} dk_T \, \delta(p^2) \propto (k_t)^{-2\epsilon} \, dz_1 \, dz_2 \, d\Omega \, dk_t$$

$$\to \frac{\alpha_s}{2\pi} \beta_0 \ln \frac{\mu^2}{k_t^2}$$

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Essentially this term from lecture by Rene:

UV renormalization in QCD

$\overline{\text{MS}}$ Scheme (massless QCD, covariant gauge):

Wave functions:

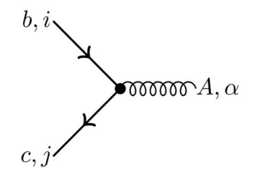
$$Z_q = 1 - \lambda C_F \frac{\alpha_s S_\varepsilon}{4\pi\varepsilon} + \mathcal{O}(\alpha_s^2)$$

$$Z_g = 1 - \frac{\alpha_s S_\varepsilon}{4\pi\varepsilon}\left[\left(\frac{\lambda}{2} - \frac{13}{6}\right) C_A + \frac{4}{3} T_F n_f\right] + \mathcal{O}(\alpha_s^2)$$

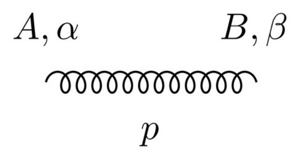
Coupling:

$$Z_{g_s} = 1 - \frac{\alpha_s S_\varepsilon}{4\pi\varepsilon}\frac{11 C_A - 4 n_f T_F}{6} + \ldots \qquad S_\varepsilon = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}$$

These introduce new diagrams in the perturbative expansions:
→ cancellation of all UV divergences



$\times (Z_{g_s} - 1)$



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18

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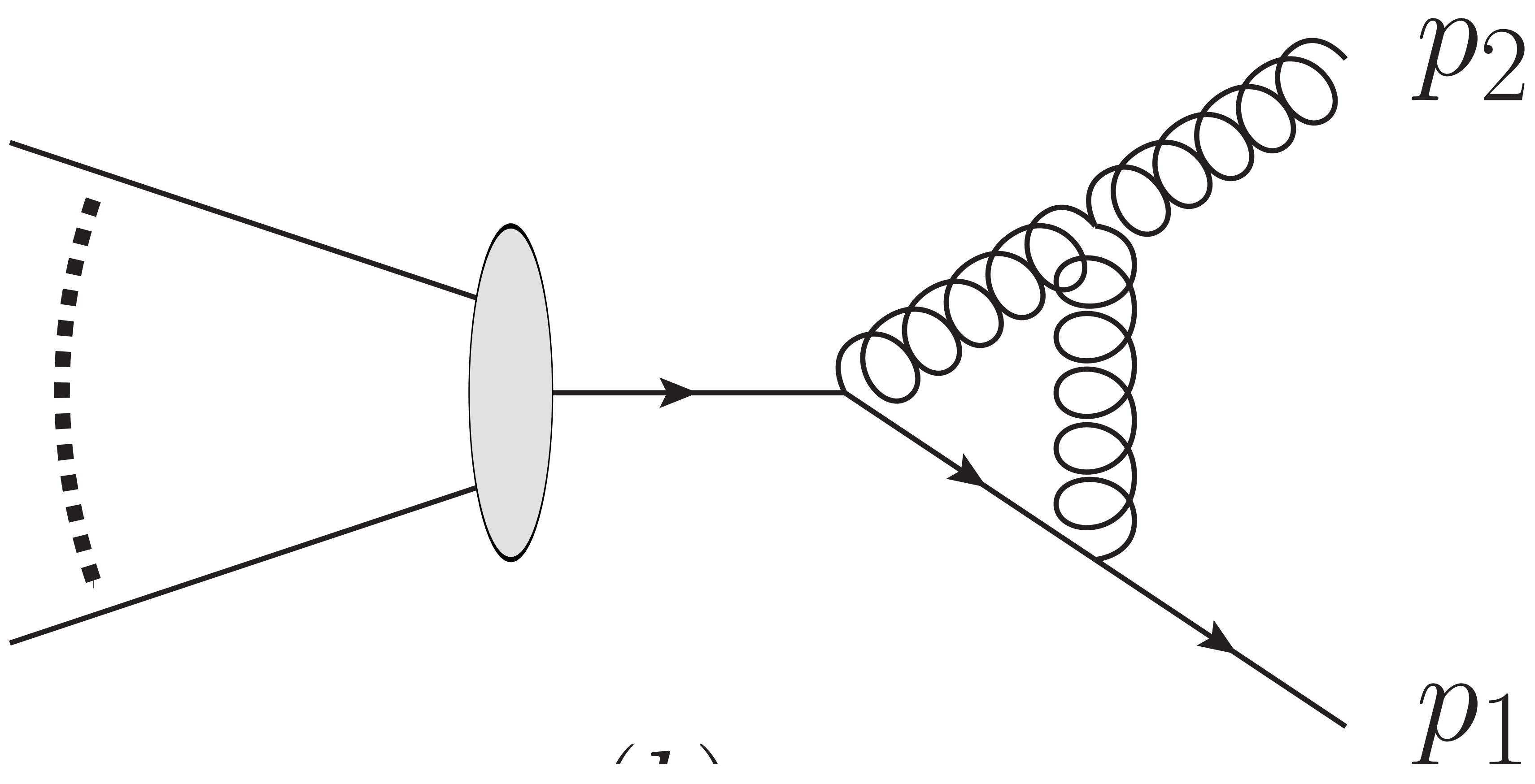

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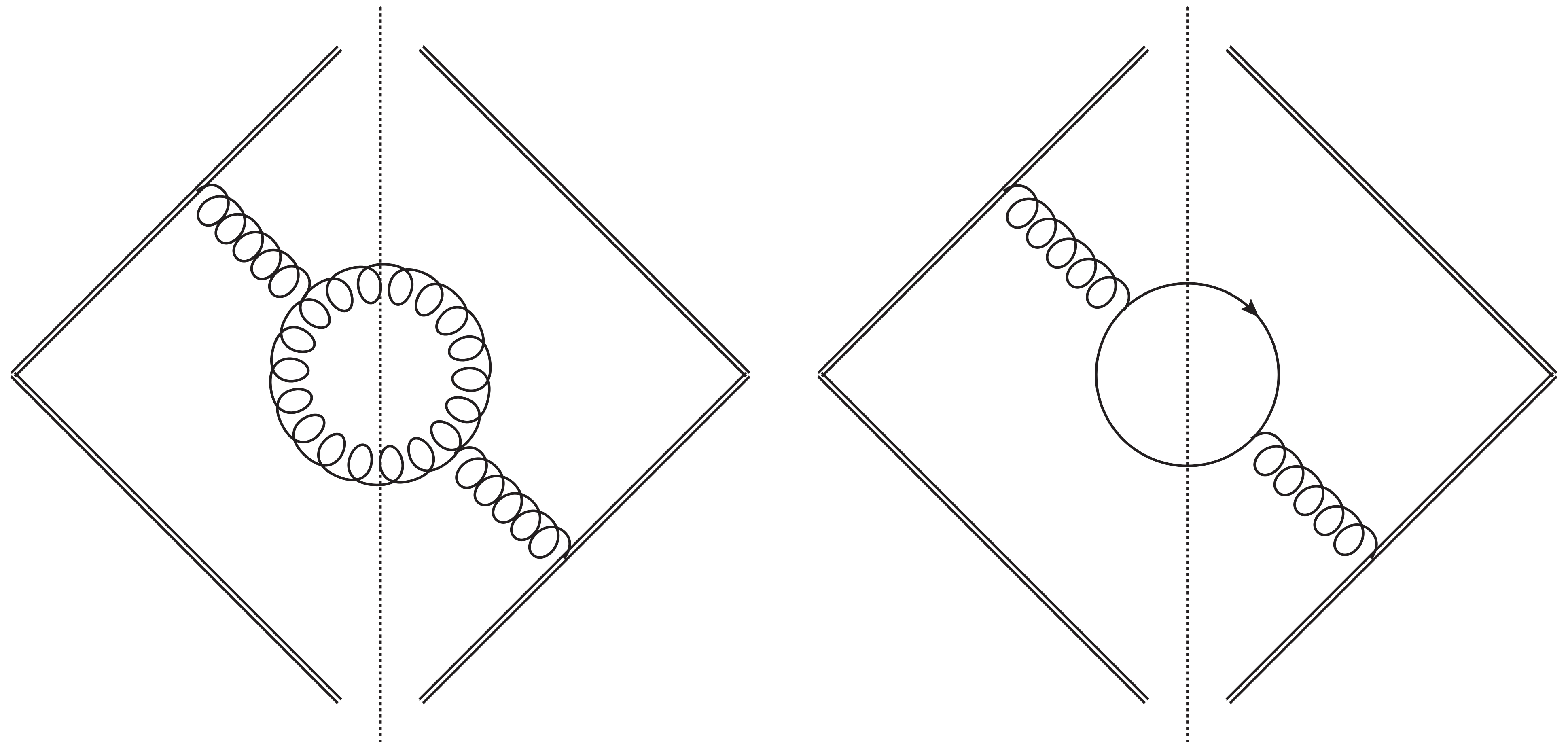
Towards NLL - the physical coupling scheme

- More higher order corrections
- Full calculation, again in D dimensions and and integrate inclusive over splitting variables
- In dim. reg. and $\overline{MS}$ scheme gives:
- Can be interpreted as local enhancement of gluon splitting, and absorbed into strong coupling (together with k_t log from last slide):



$$\left(\frac{67}{18} - \frac{\pi^2}{3}\right) C_A \qquad -\frac{10}{9} n_f T_R$$

$$\alpha_s(\mu^2) \to \alpha_s(\mu^2)\left[1 + \frac{\alpha_s(\mu^2)}{2\pi}\left(\beta_0 \ln \frac{k_t^2}{\mu^2} + K\right)\right]$$

Towards NLL - the physical coupling scheme

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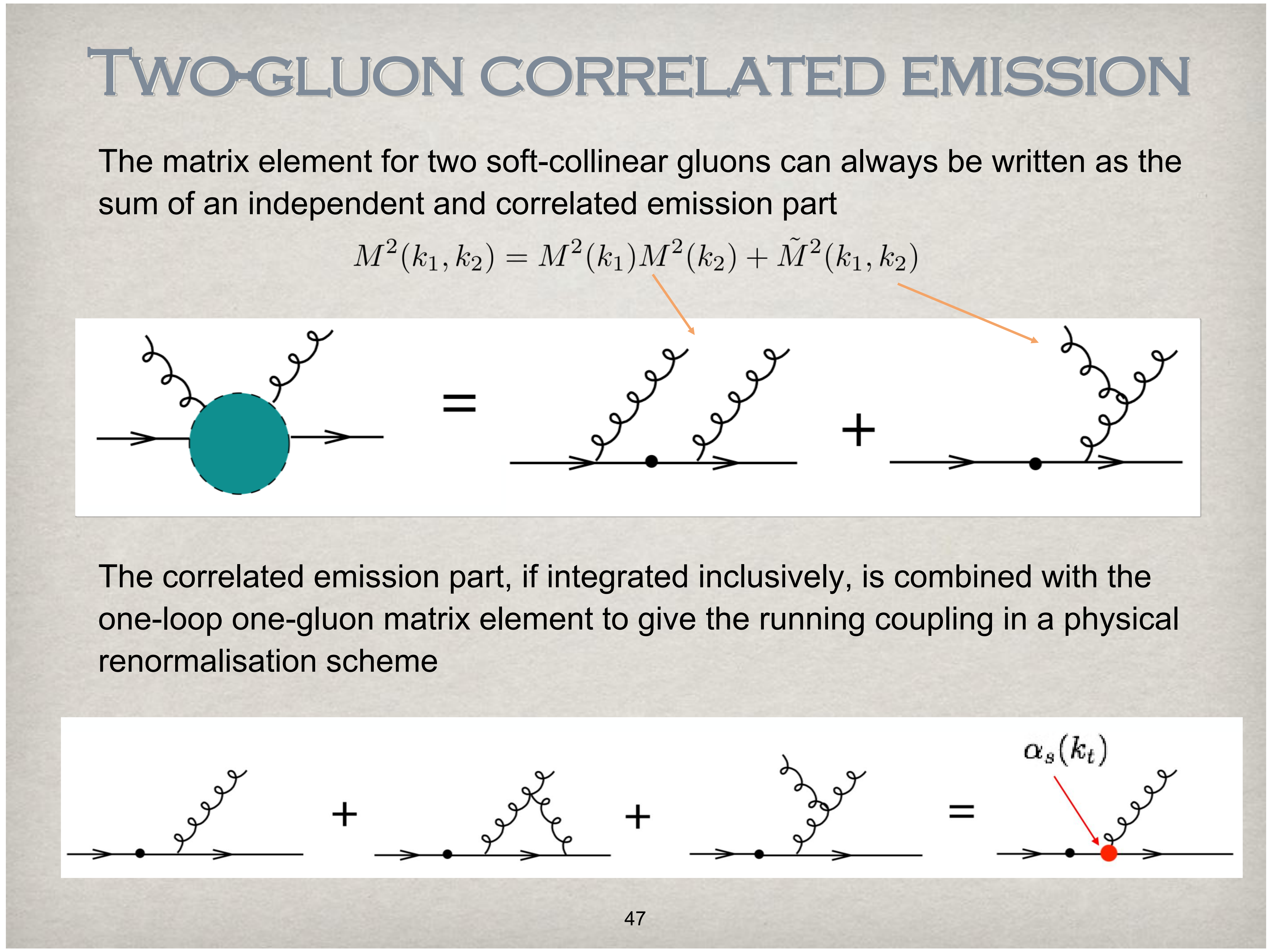
Compare to treatment in resummation in Lecture by Andrea:



Putting things together

- Consider an rIRC safe observable like thrust
- Parton shower (as discussed so far) will produce very similar expression to resummation (e.g. CAESAR master formula):

$$\Sigma_{\mathrm{PS}}(v) = \sum_{m=0}^{\infty} \prod_{i=0}^{m} \frac{1}{m!} \int \frac{d\xi_i}{\xi_i} \mathcal{P}(\xi_i, \xi_{i+1}) \Theta(v - V(\xi_1, \ldots, \xi_m))$$

$$\mathcal{P}(t', t) = \frac{d\Pi(t', t)}{d \ln t'} \qquad \Pi(t', t) = e^{-R(t,t')} \qquad R(t, t') = \int_{t'}^{t} \int_{z_-}^{z_+} dt dz \frac{\alpha_s}{2\pi} P(z)$$

- In practice V not evaluated in strict limit → leftover higher order corrections
- Apart from that, AO shower reproduces resummed Σ "by construction"

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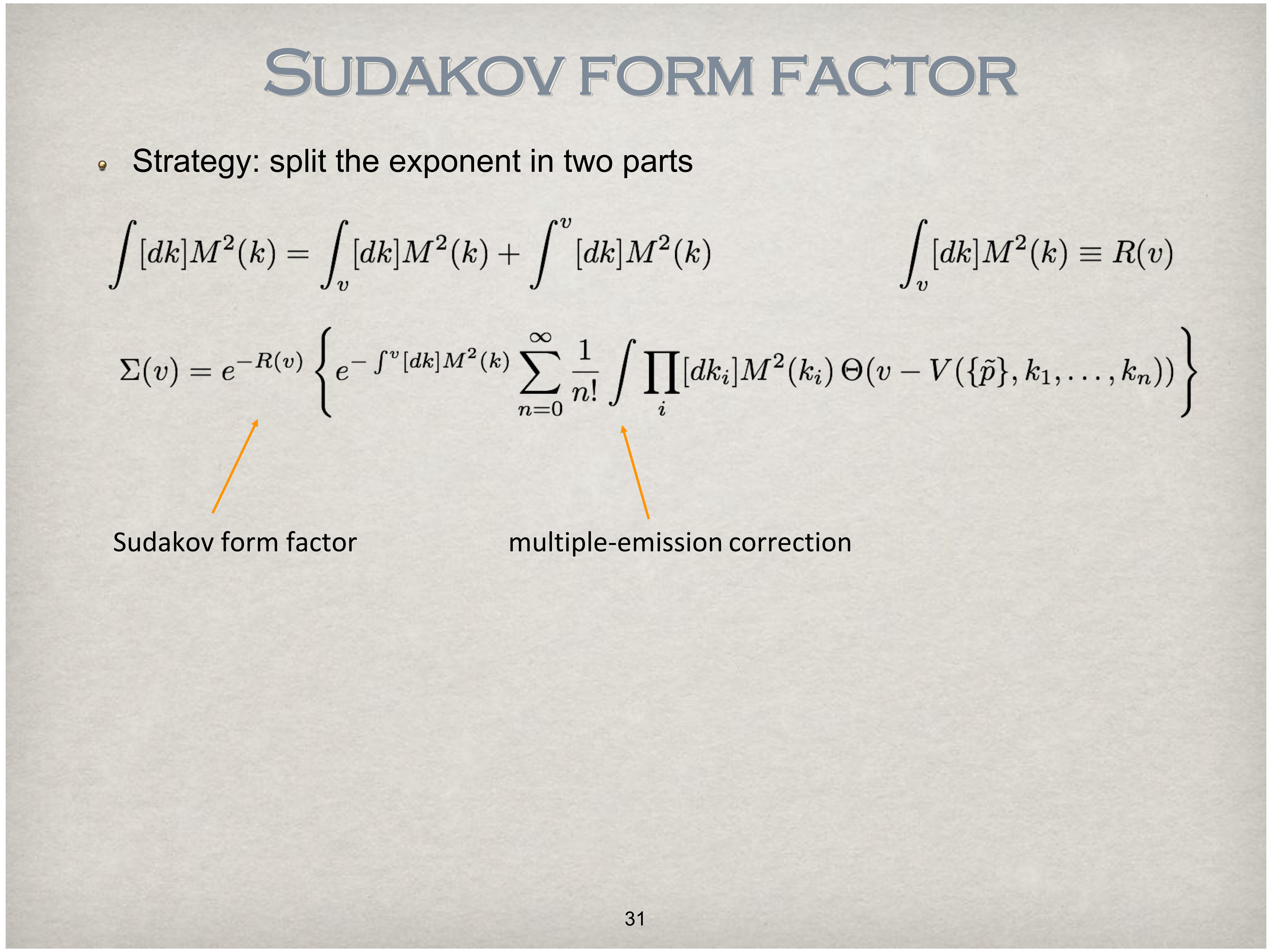
For example here:



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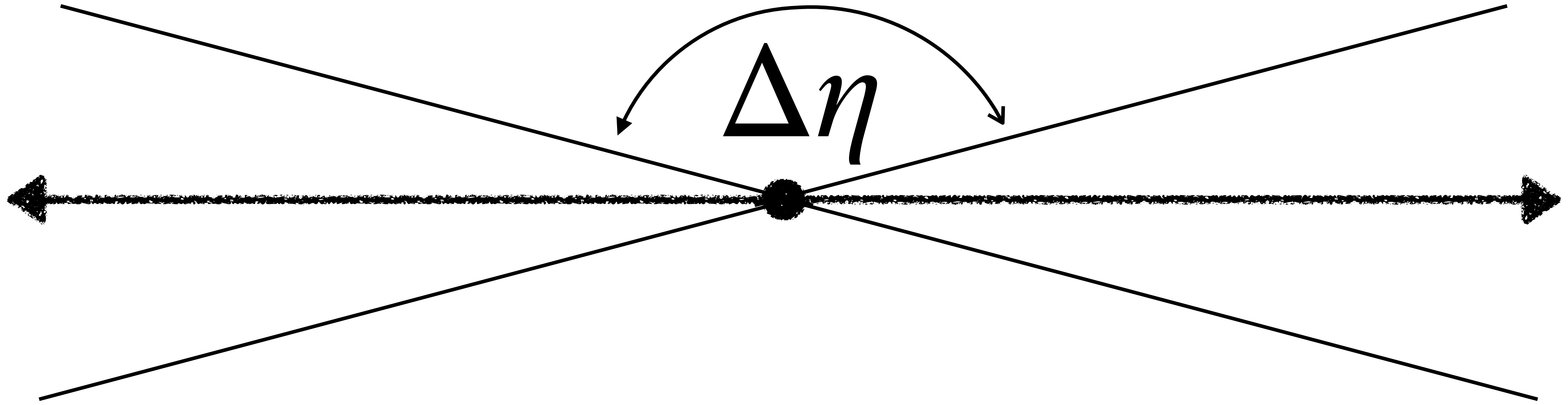
Non-global Observables - Dipole showers

- The argument that led us to coherent branching and angular ordering crucially depended on freely averaging over the azimuth ϕ_j !
- What if we are interested in observables that distinguish radiation at different ϕ ?
- Example: radiation into gap between jets ⇒ coherence argument fails!



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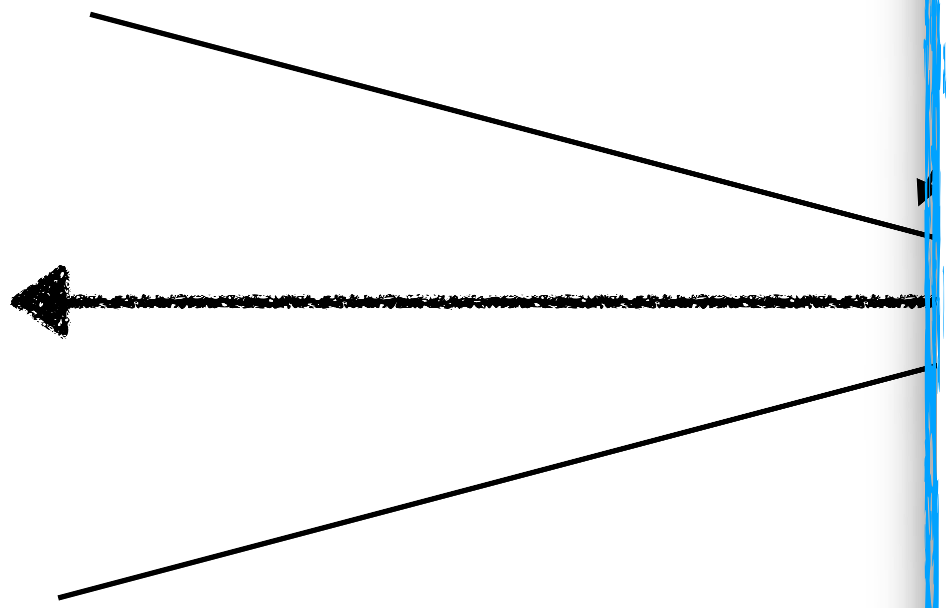
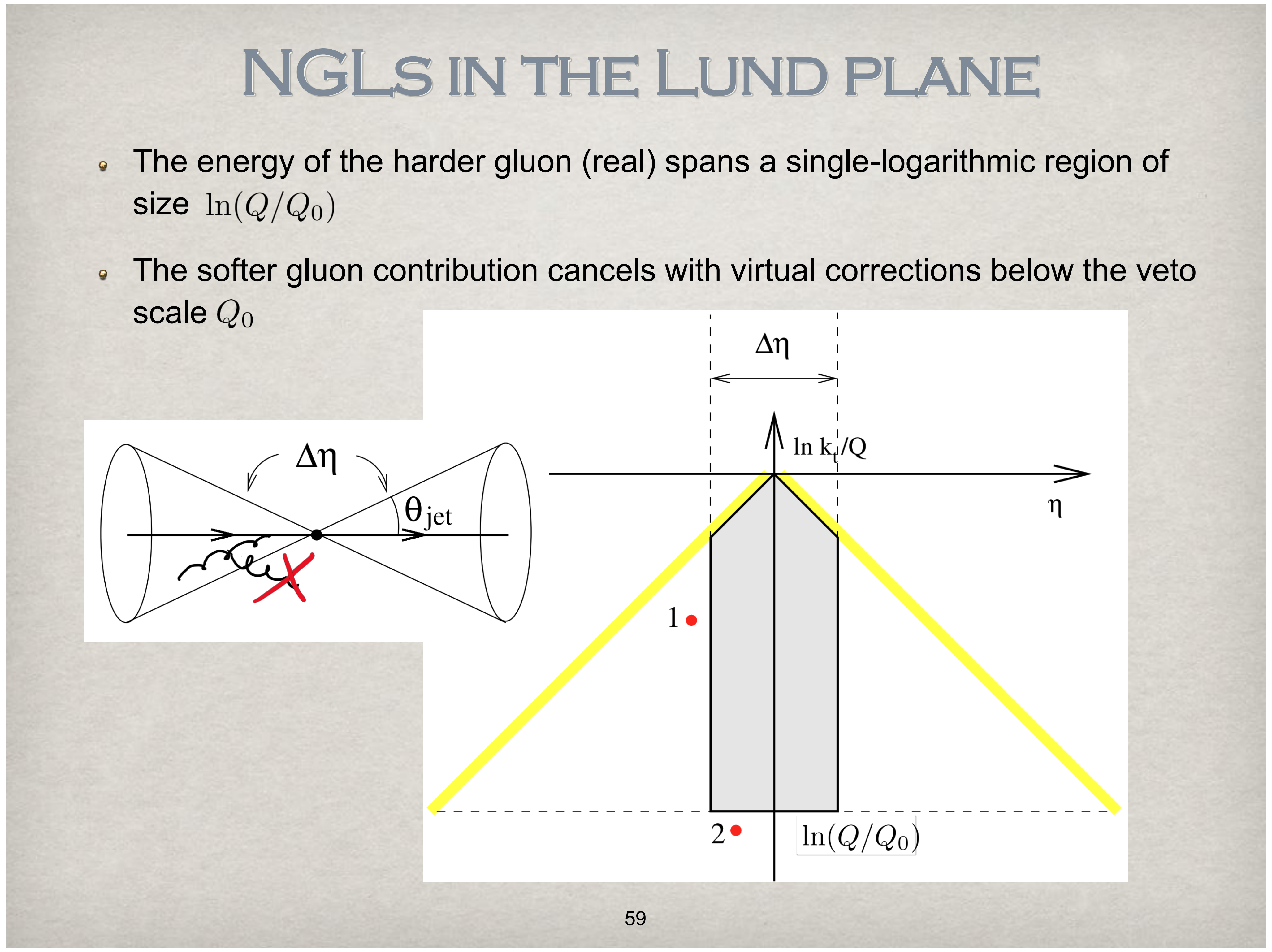


Those are exactly the NGLs:



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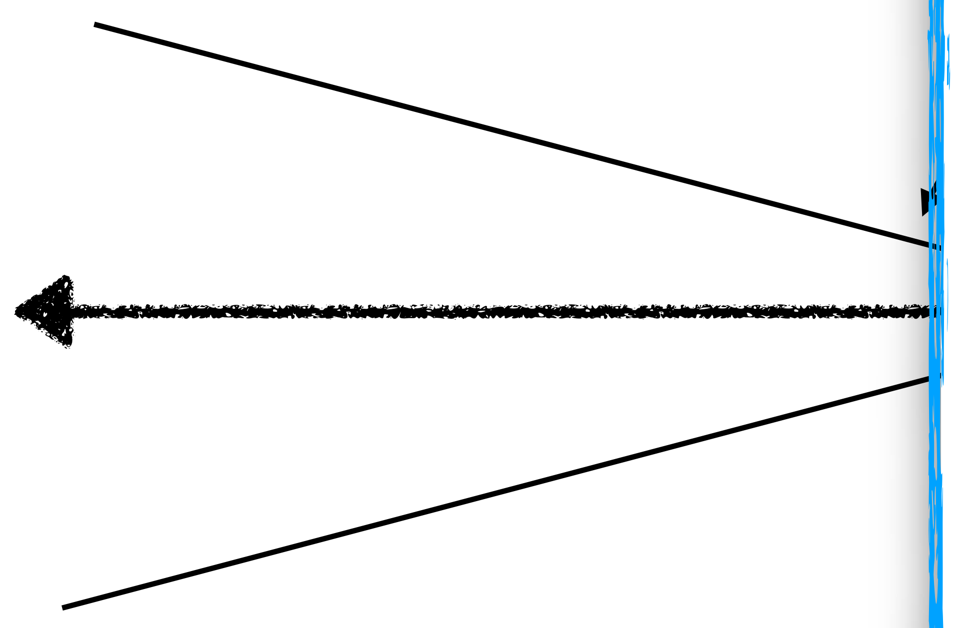
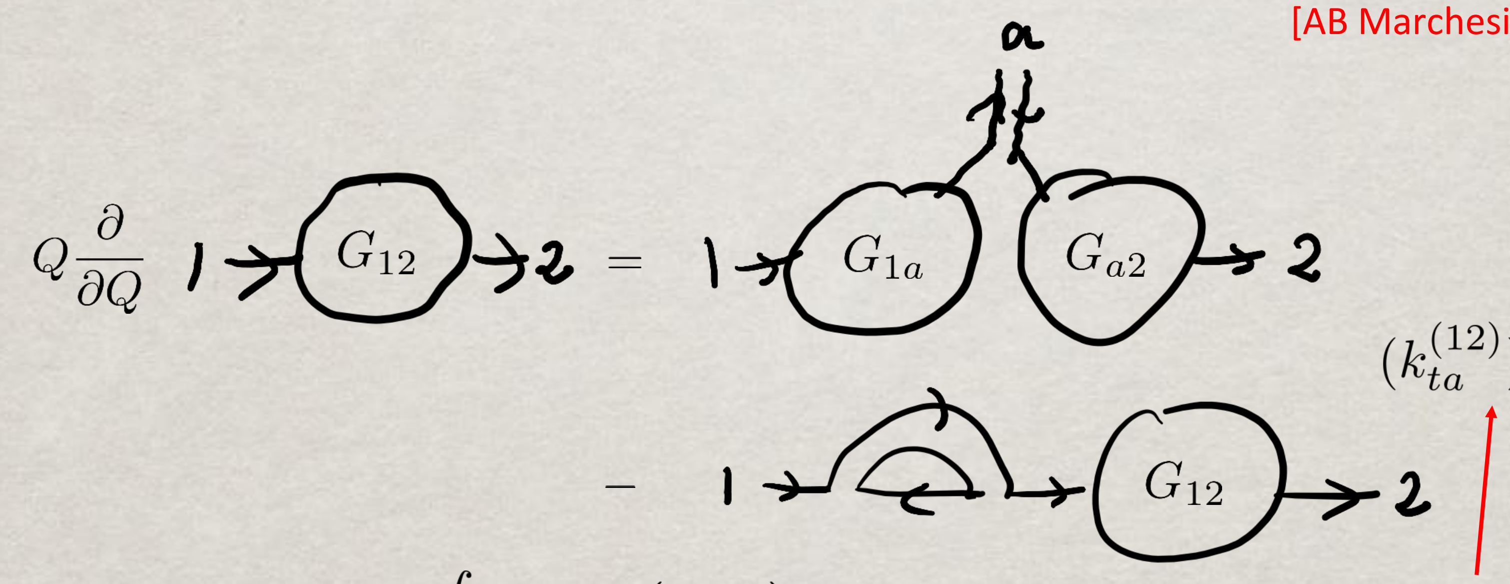


Correct Language are dipoles ordered in k_t:



THE BMS EQUATION

- The factorisation properties of the amplitude squared in the planar limit makes it possible to write closed differential equations for $G_{ij}[Q,u]$
- For $G_{ij}[Q;u]$ we obtain the Banfi-Marchesini-Smye (BMS) equation

[AB Marchesini Smye hep-ph/0206076]

$$(k_{ta}^{(12)})^2 \equiv 2\frac{(p_1 k_a)(k_a p_2)}{(p_1 p_2)}$$

$$k_{ta}^{(12)} \simeq E_a$$

ordering variable

$$Q\partial_Q G_{12}[Q,u] = \int [dk_a]\,\bar{\alpha}\left(k_{ta}^{(12)}\right) w_{12}(k_a)\times$$
$$\times\,\{u(k_a)G_{1a}[Q,u]G_{a2}[Q,u] - G_{12}[Q;u]\}\,Q\delta\left(Q - k_{ta}^{(12)}\right)$$

The solution of the BMS equation gives the LL resummation of NGLs

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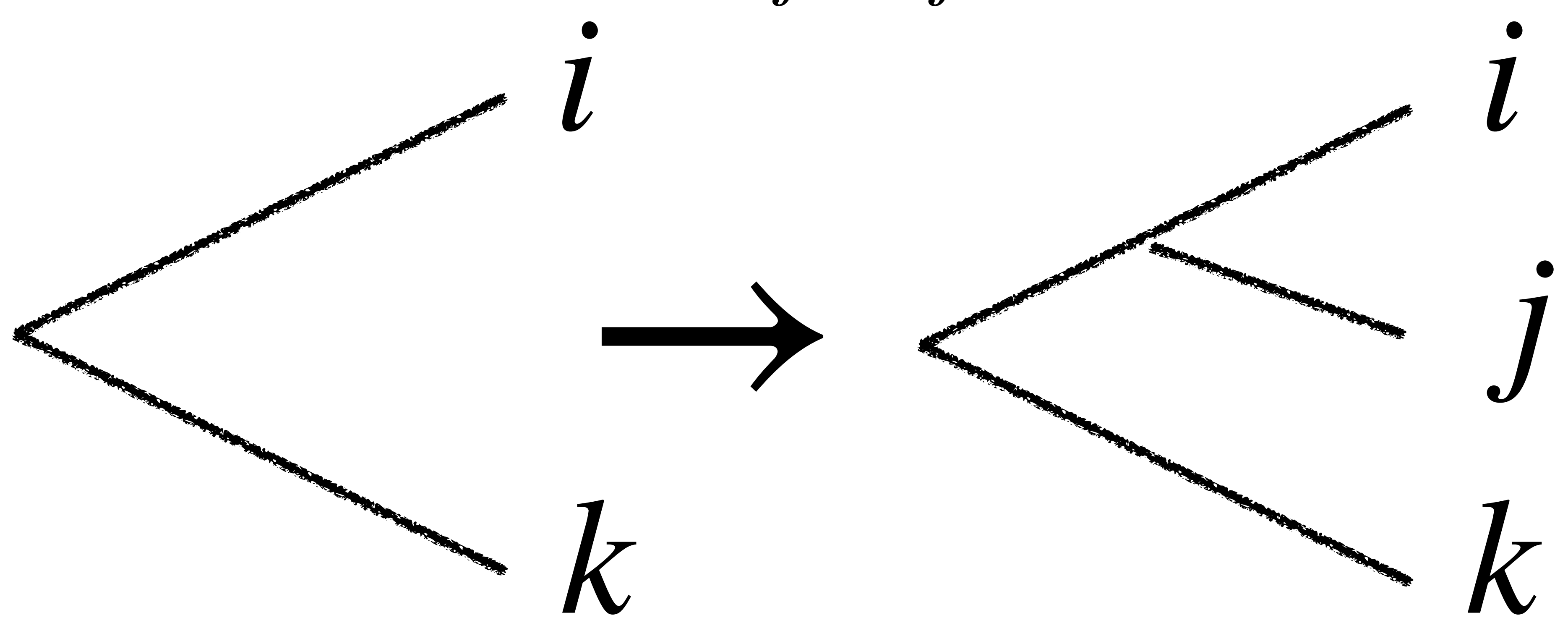
Dipole Showers!

- Correct language for NGLs are dipoles ordered in transverse momentum.
- Simple interpretation for shower:
  Probability for emission from dipole i, k to emit j
- Afterwards, we have new dipoles, $(i,k), (i,j), (j,k)$ all emitting with respective probabilities

$$\sim \frac{p_i p_k}{(p_i p_j)(p_j p_k)}$$



Addendum: dipole kinematics and matching

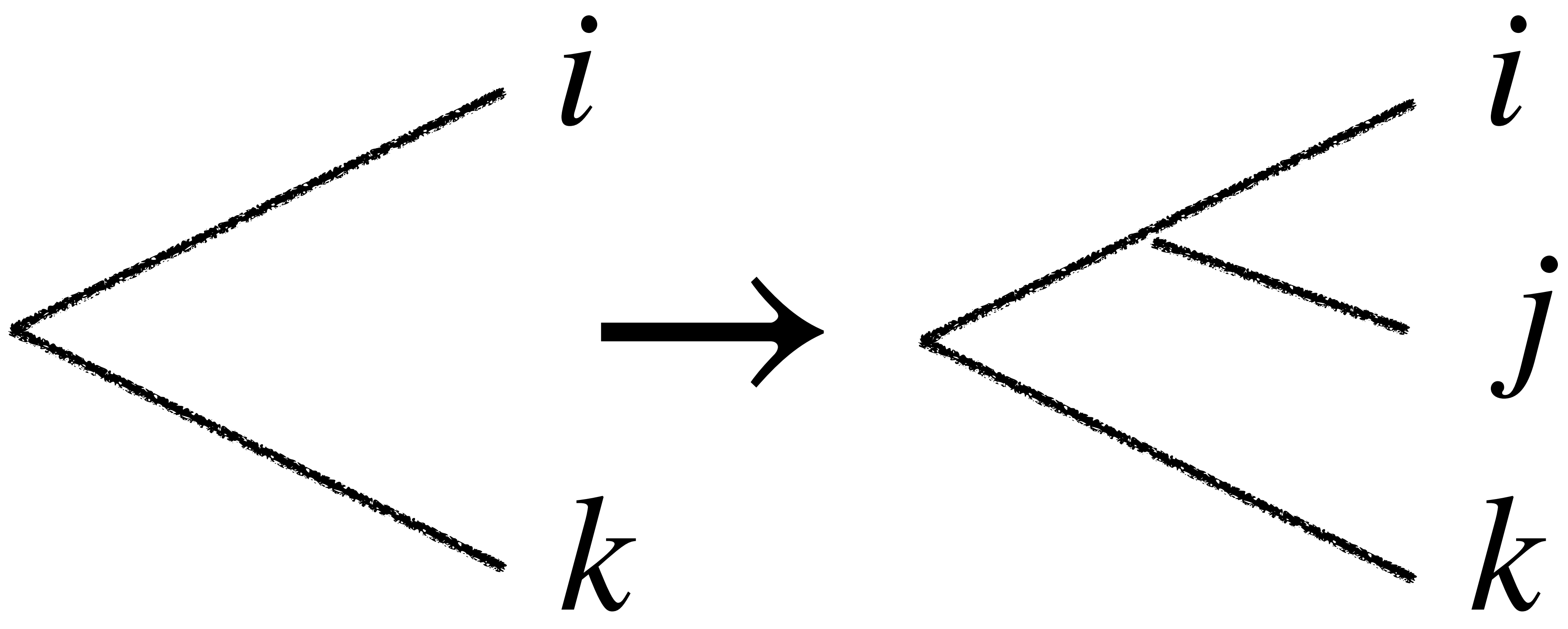
- Dipole showers are the correct language to discuss non-global logarithms
- But also:
  - close correspondence to NLO subtraction schemes, makes matching to fixed order significantly simpler
  - Probably the main reason for their rapid adoption concurrently with the wide availability of automated NLO calculations and PS matching schemes!

Dipole Recoil

- Preserve 4-momentum locally in both,
  - Scale t, i.e. after every emission project back to on-shell state
  - within the dipole, i.e.

$$\tilde{p}_{ij} + \tilde{p}_k = p_i + p_j + p_k$$

- See e.g. [Catani, Seymour '97]



$$p_i = z\tilde{p}_{ij} + (1-z)y\tilde{p}_k + k_\perp$$
$$p_j = (1-z)\tilde{p}_{ij} + zy\tilde{p}_k - k_\perp$$
$$p_k = (1-y)\tilde{p}_k \ .$$

Treatment of multiple emissions e.g. in CAESAR

- generalized rescaling

  [Banfi, Salam, Zanderighi ’05]:

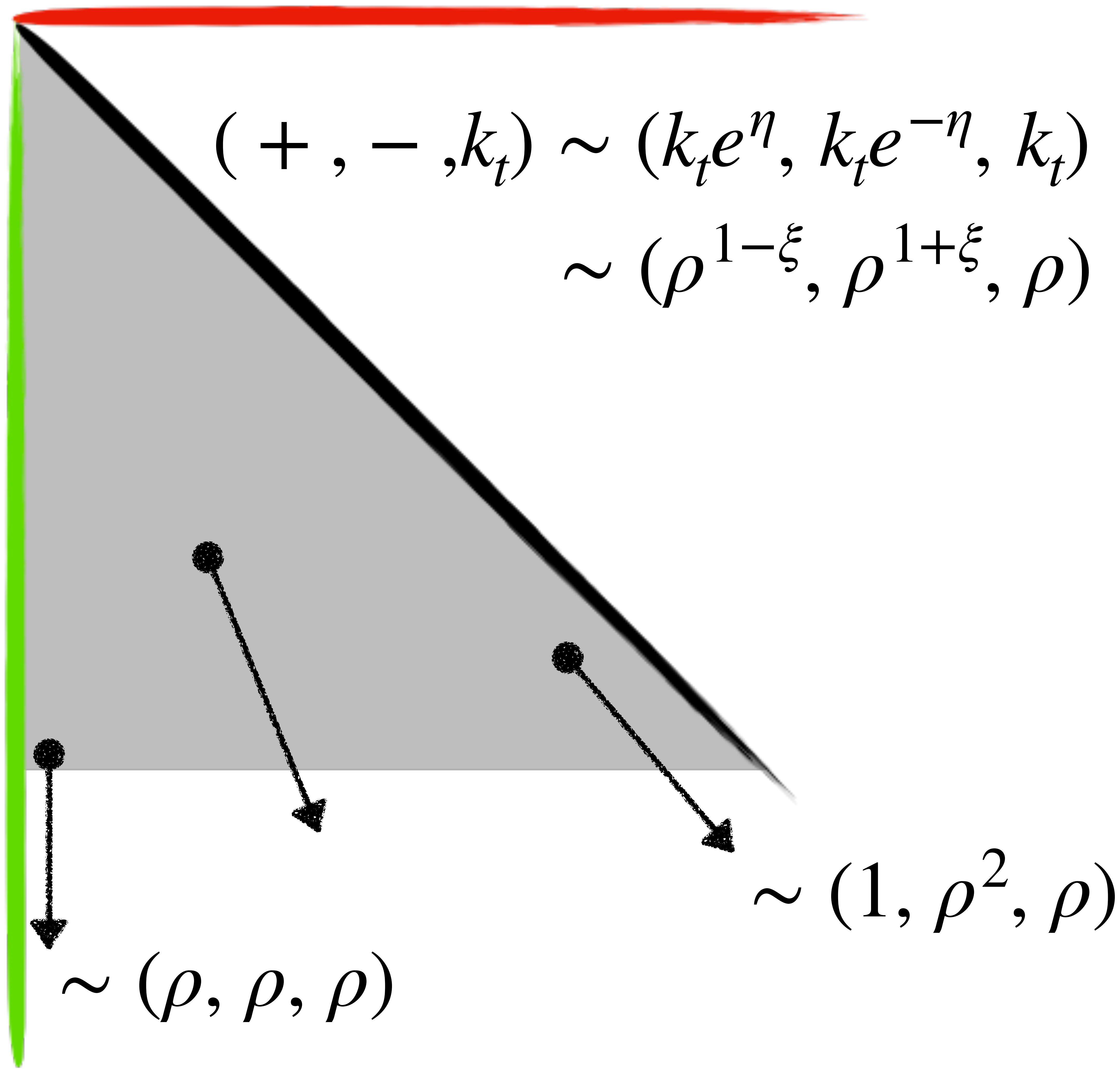
$$k_t^\rho = k_t \rho \qquad \xi = \frac{\eta}{\eta_{\text{max}}}$$

$$\eta^\rho = \eta - \xi \ln \rho$$

and assume

$$V(k_i^\rho) = \rho V(k_i)$$

→ numerically evaluate phase space integrals in this limit



* example assuming $V(k_t, \eta) \sim k_t/Q$ for brevity

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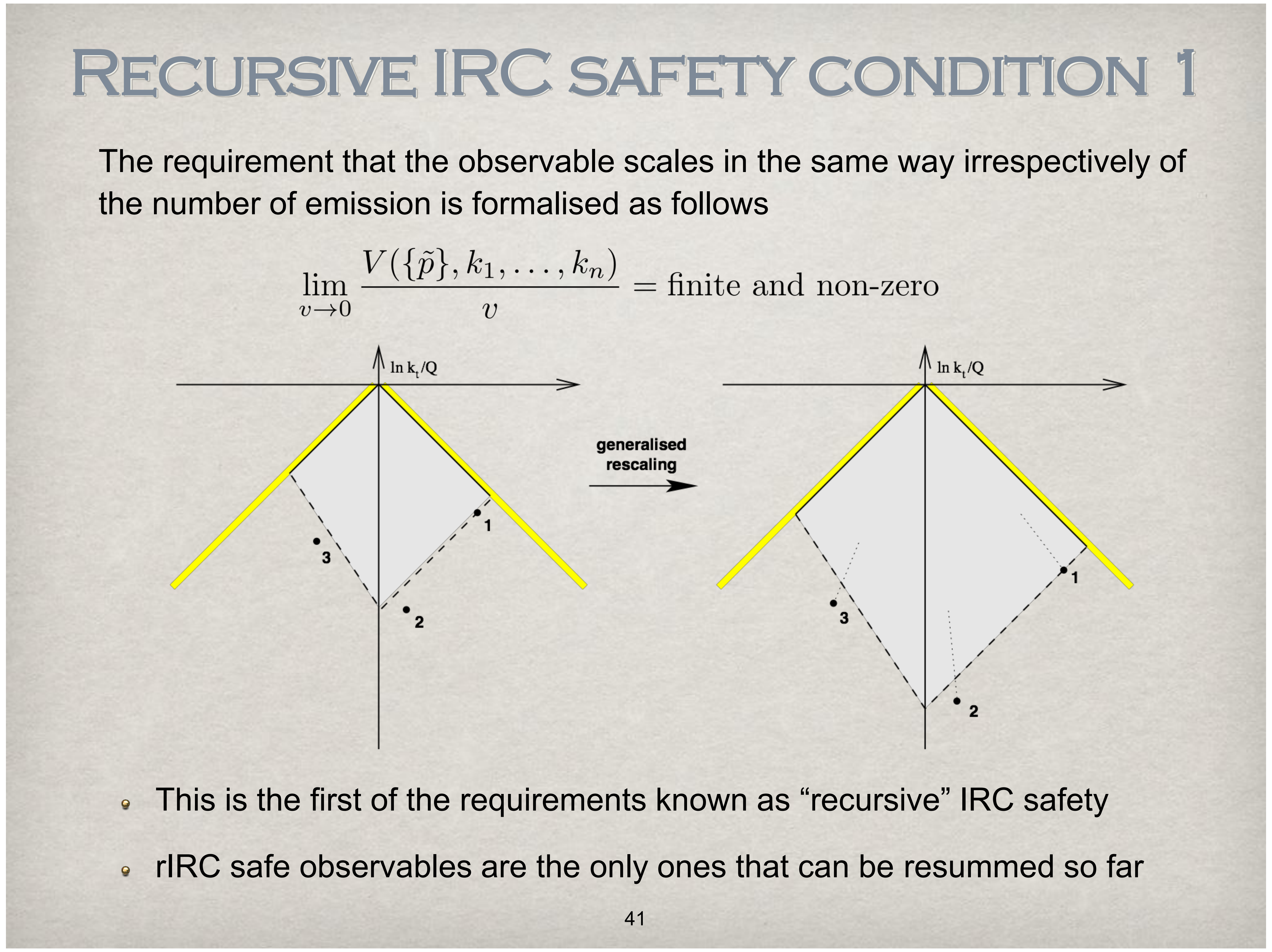
Generalized rescaling from Andrea's lecture:



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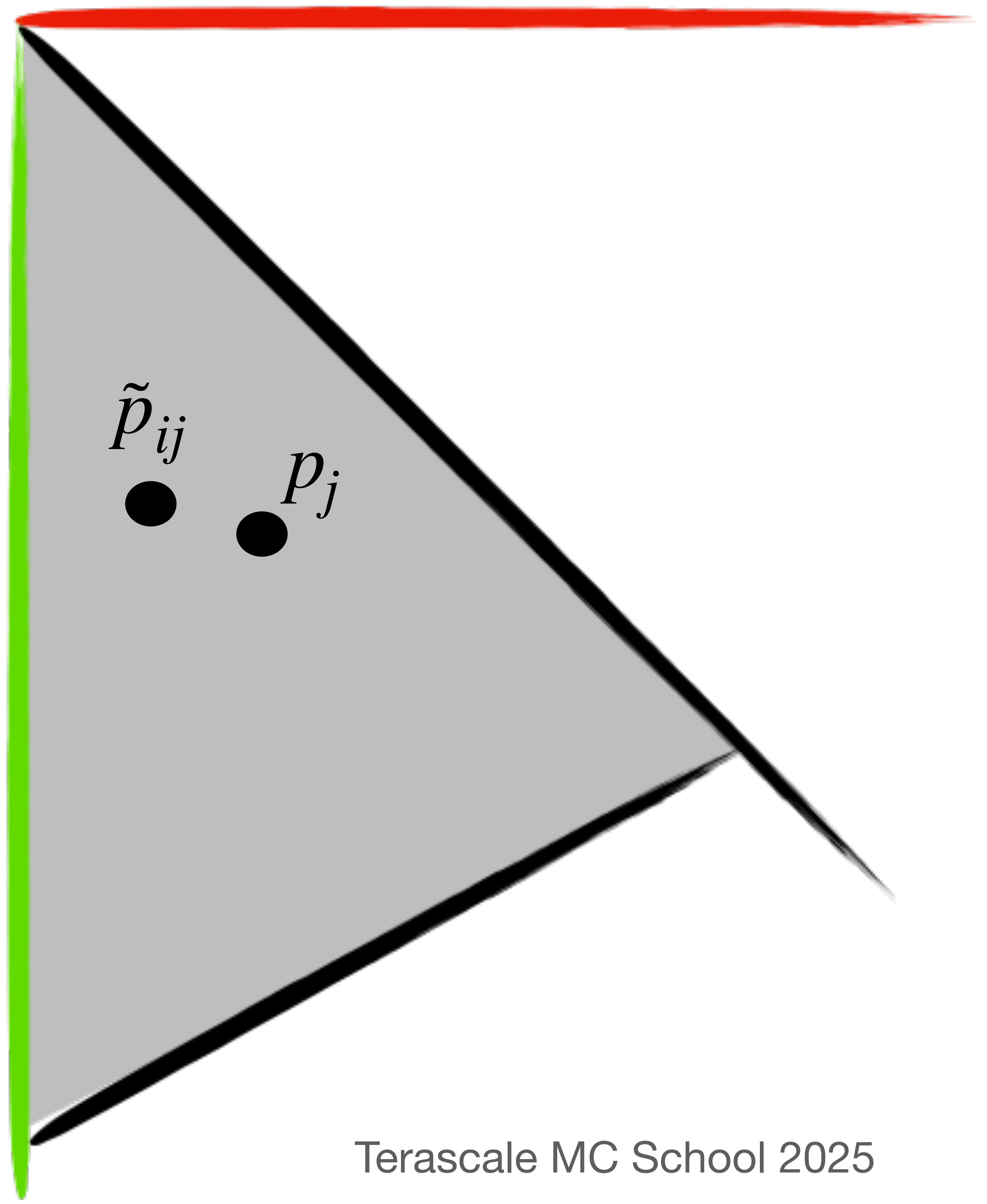
Effect of recoil on accuracy

- question: do recoil effects indeed vanish in soft limit (i.e. $\rho \to 0$)?*
  [Dasgupta,Dreyer,Hamilton,Monni,Salam '18]
- consider situation where we first emit $\tilde{p}_{ij}$ from p_a, p_b, then emit p_j,
  $\tilde{p}_{ij} \to p_i, p_j$
- transverse momentum of p_i will be
  $k_t^i \sim k_t^{ij} + k_t^j \to k_t^{ij}$ as $\frac{k_t^j}{k_t^i} \to 0$
- but, relevant limit is $\frac{\Delta k_t^i}{k_t^i} \to \frac{\rho k_t^j}{\rho k_t^i} = \mathcal{O}(1)$

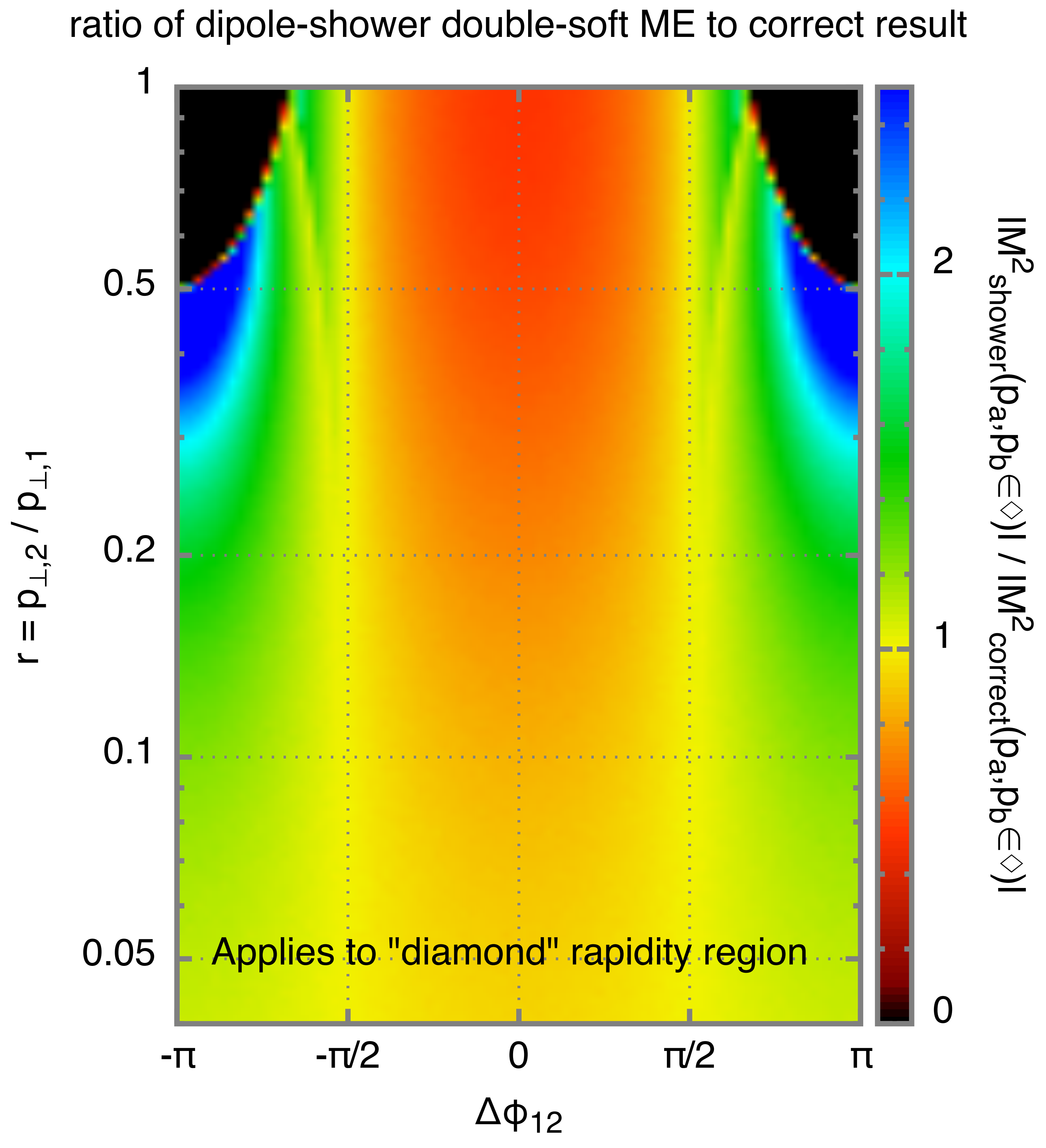
$$p_i = z\tilde{p}_{ij} + (1-z)y\tilde{p}_k + k_\perp$$
$$p_j = (1-z)\tilde{p}_{ij} + zy\tilde{p}_k - k_\perp$$
$$p_k = (1-y)\tilde{p}_k \ .$$



Effect of recoil on accuracy

[Dasgupta, Dreyer, Hamilton,Monni, Salam '18]

- With local dipole recoil, effects stay relevant even in the soft limit.
- Spoils NLL accuracy of most currently used (e.g. by ATLAS/CMS) dipole showers.



New Parton Showers - NLL accuracy

- Several solutions/re-evaluations of parton shower concepts:
- [Dasgupta, Dreyer, Hamilton,Monni, Salam, Soyez ’20], [vanBeekveld, Ferrario Ravasio, Hamilton, Salam, Soto-Ontoso,Soyez ’22] …
    - partitioning of splitting functions and appropriate choice of evolution variable can lead to NLL accurate shower for local and global recoil strategies
- [Forshaw, Holguin, Plätzer ’20]
    - Connections between angular ordered and dipole showers
- [Nagy, Soper ’11]
    - local transverse, global longitudinal recoil
- [Herren, Höche, Krauss, DR, Schönherr,’22], [Höche, Asse ’23], [Höche, Krauss, DR ‘24]
    - global recoil, enables analytic comparison to resummation and proof of NLL accuracy
- [Preuss ’24]
    - global recoil in antenna shower Vinca

Numerical accuracy tests

- First attempt: try to list all the differences between showers and resummation, look at the differences [Höche, DR, Siegert ’17]
- General expression: $\Sigma(v) = \exp\left\{-\int_v \frac{d\xi}{\xi} R'_{>v}(\xi) - \int_{v_{\min}}^{v} \frac{d\xi}{\xi} R'_{<v}(\xi)\right\}$

$$\times \sum_{m=0}^{\infty} \frac{1}{m!} \left(\prod_{i=1}^{m} \int_{v_{\min}} \frac{d\xi_i}{\xi_i} R'_{<v}(\xi_i)\right) \Theta\left(v - \sum_{j=1}^{m} V(\xi_i)\right)$$

$$R'_{\lessgtr v}(\xi) = \frac{\alpha_s^{\lessgtr v,\mathrm{soft}}(\mu^2_{\lessgtr})}{\pi} \int_{z_{\min}}^{z^{\max}_{\lessgtr v,\mathrm{soft}}} dz \, \frac{C_{\mathrm{F}}}{1-z} - \frac{\alpha_s^{\lessgtr v,\mathrm{coll}}(\mu^2_{\lessgtr v})}{\pi} \int_{z_{\min}}^{z^{\max}_{\lessgtr v,\mathrm{coll}}} dz \, C_{\mathrm{F}} \frac{1+z}{2}$$

- Can give either shower or resummed result, depending on choices

Numerical accuracy tests

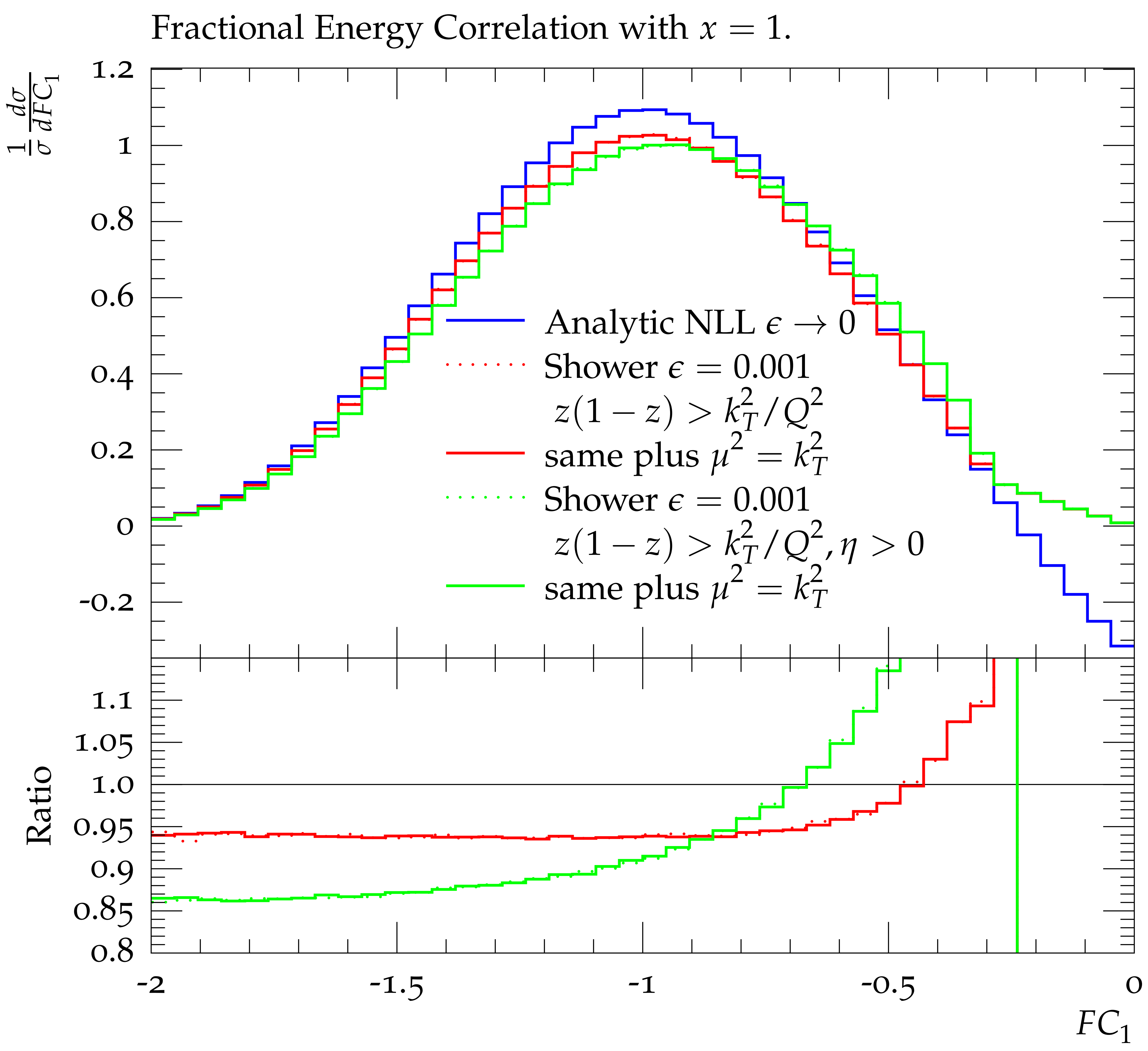
$$R'_{\lessgtr v}(\xi) = \frac{\alpha_s^{\lessgtr v,\mathrm{soft}}(\mu^2_{\lessgtr})}{\pi} \int_{z_{\min}}^{z^{\max}_{\lessgtr v,\mathrm{soft}}} dz\, \frac{C_\mathrm{F}}{1-z} - \frac{\alpha_s^{\lessgtr v,\mathrm{coll}}(\mu^2_{\lessgtr v})}{\pi} \int_{z_{\min}}^{z^{\max}_{\lessgtr v,\mathrm{coll}}} dz\, C_\mathrm{F}\,\frac{1+z}{2}$$

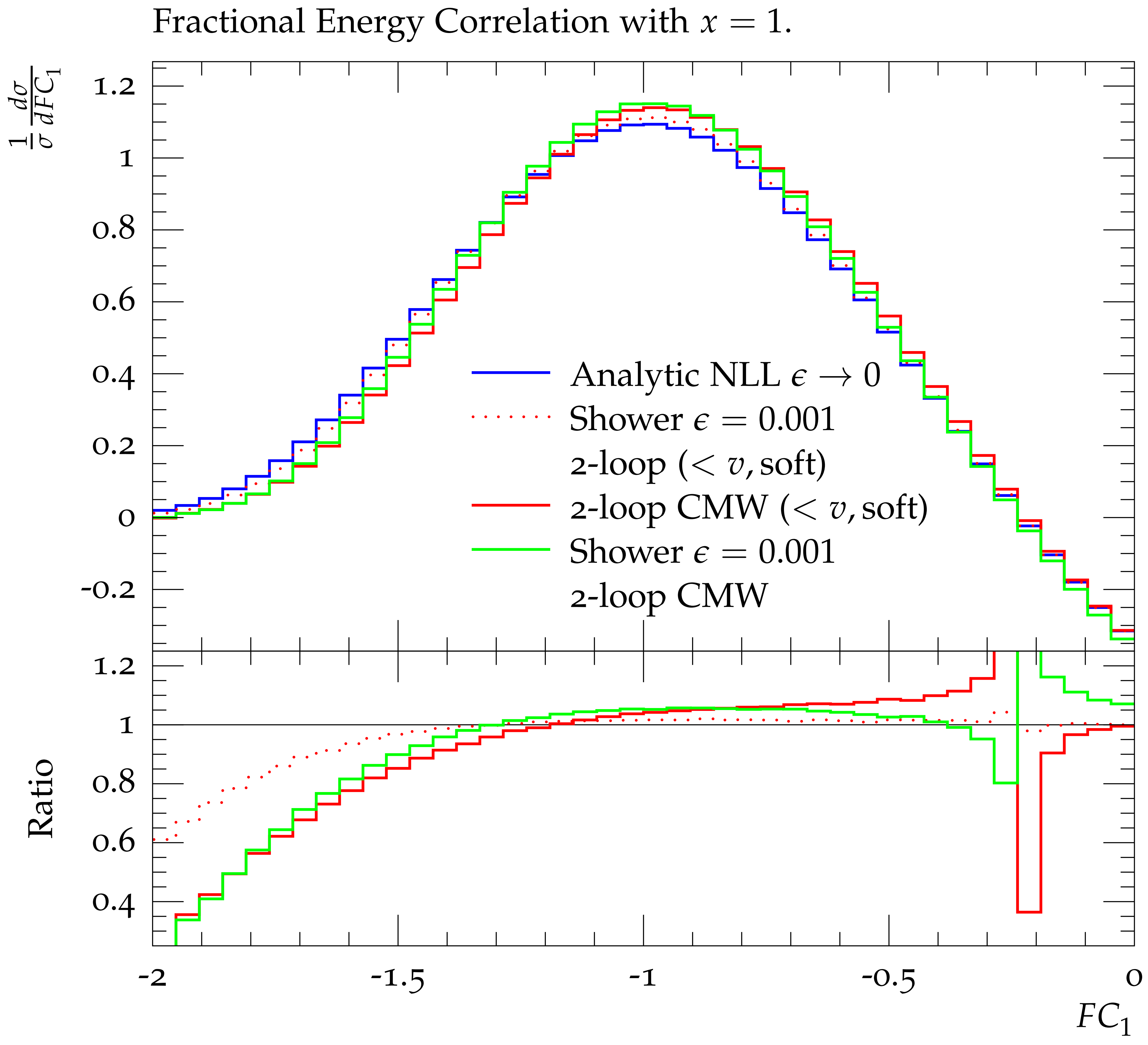
- Can give either shower or resummed result, depending on choices

	NLL	Parton Shower
$z^{\max}_{>v,\mathrm{soft}}$	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$	
$\mu^2_{>v,\mathrm{soft}}$	$\xi(1-z)^{\frac{2b}{a+b}}$	
$\alpha_s^{>v,\mathrm{soft}}$	2-loop CMW	
$z^{\max}_{<v,\mathrm{soft}}$	$1-v^{\frac{1}{a}}$	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{<v,\mathrm{soft}}$	$Q^2 v^{\frac{2}{a+b}}(1-z)^{\frac{2b}{a+b}}$	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{<v,\mathrm{soft}}$	1-loop	2-loop CMW

	NLL	Parton Shower
$z^{\max}_{>v,\mathrm{coll}}$	1	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{>v,\mathrm{coll}}$	ξ	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{>v,\mathrm{coll}}$	1-loop	2-loop CMW
$z^{\max}_{<v,\mathrm{coll}}$	0	$1-(\xi/Q^2)^{\frac{a+b}{2a}}$
$\mu^2_{<v,\mathrm{coll}}$	n.a.	$\xi(1-z)^{\frac{2b}{a+b}}$
$\alpha_s^{<v,\mathrm{coll}}$	n.a.	2-loop CMW

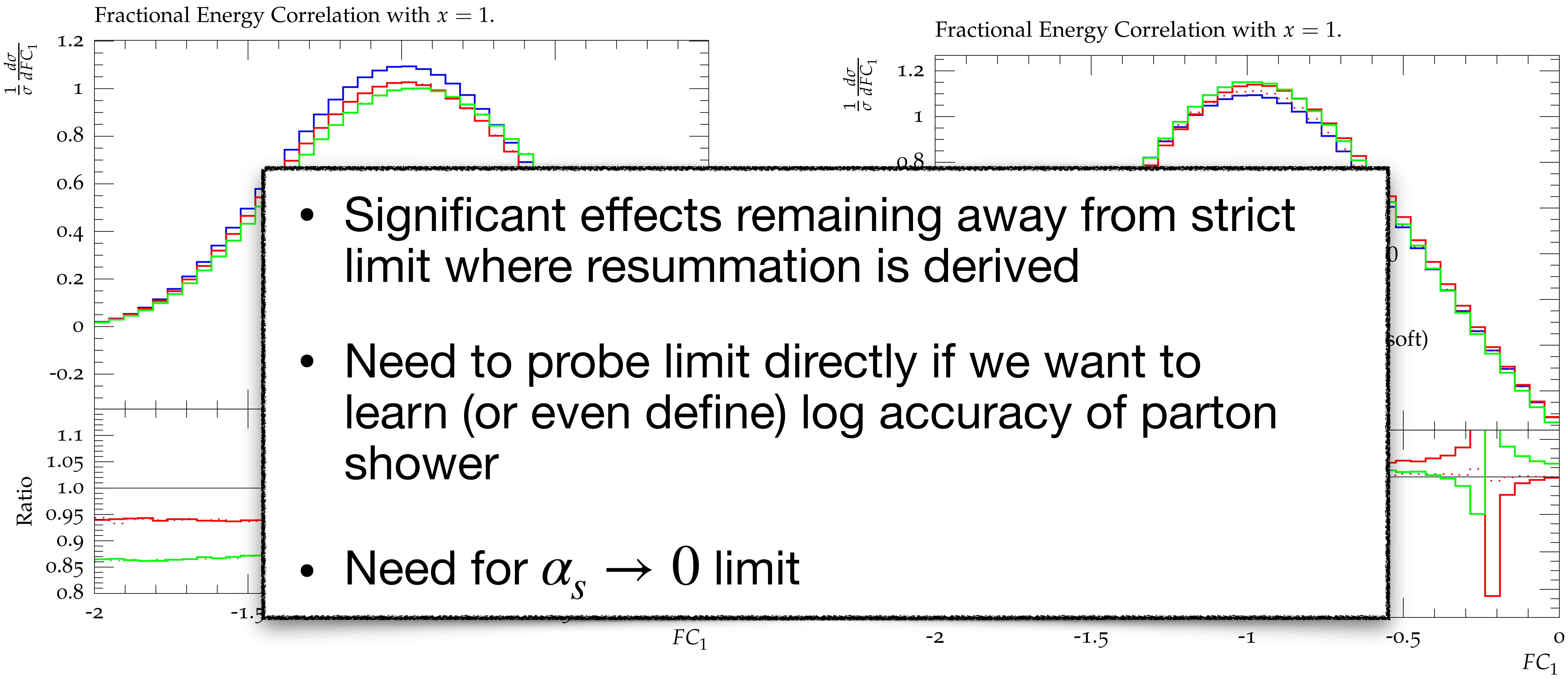
Numerical accuracy tests





[Höche, DR, Siegert '17]

Numerical accuracy tests



- Significant effects remaining away from strict limit where resummation is derived
- Need to probe limit directly if we want to learn (or even define) log accuracy of parton shower
- Need for $\alpha_s \to 0$ limit

[Höche, DR, Siegert '17]

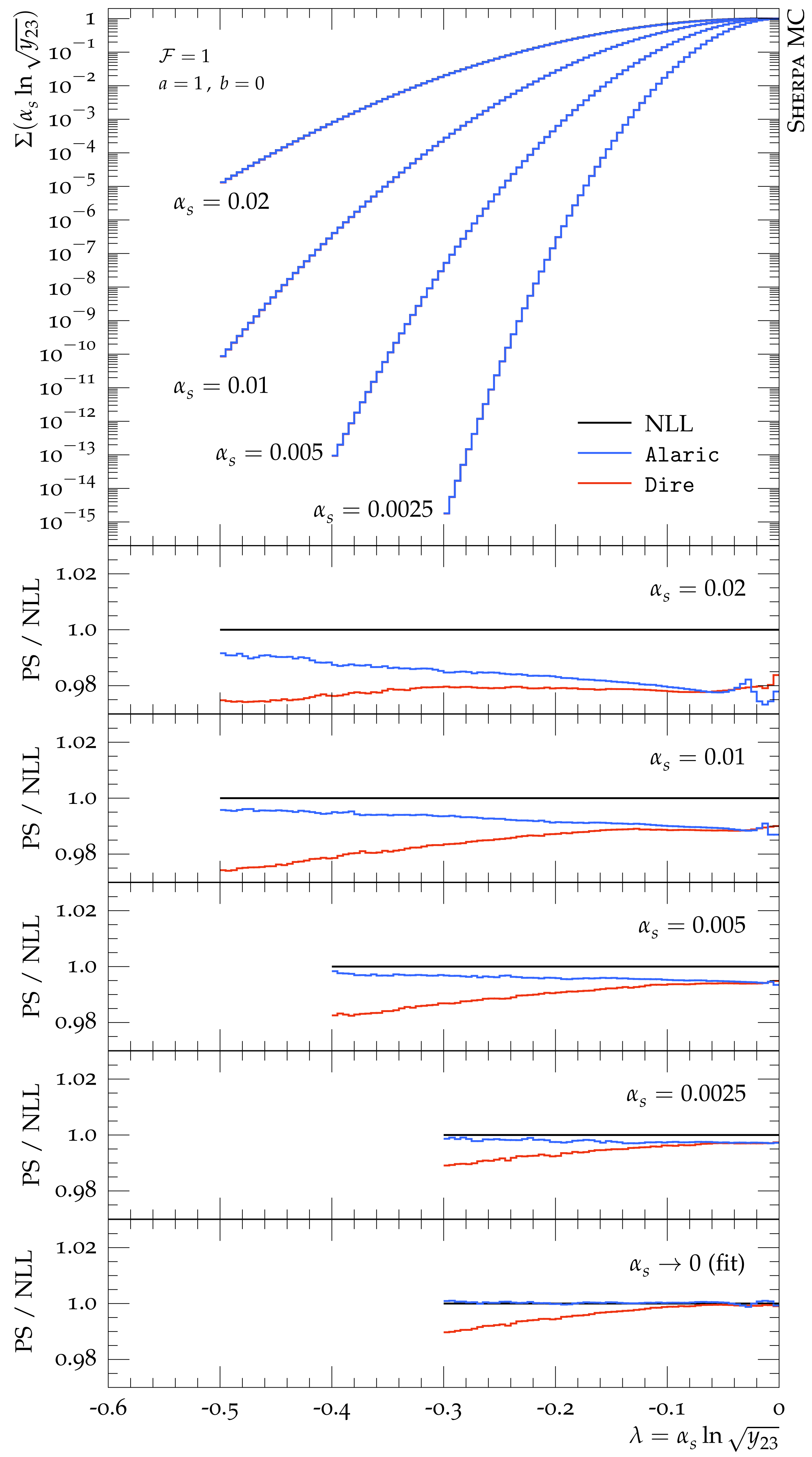
Numerical accuracy tests

- Limit $\alpha_s \to 0$ with $\lambda = \alpha_s L =$ const. (see [Dasgupta, Dreyer, Hamilton et. al. '20]) of

$$\frac{\Sigma^{\text{Shower}}}{\Sigma^{\text{NLL}}} \sim \exp\left(f^{LL}_{\text{Shower}} - Lg_1(\alpha_s^n L^n)\right) \times \exp\left(f^{NLL}_{\text{Shower}} - g_2(\alpha_s^n L^n)\right) \times \exp\left(\mathcal{O}(\alpha_s^{n+1} L^n)\right)$$

$\to 1$ if shower reproduces LL, NLL logs

- Observable: jet resolution y_{23} in Cambridge jet measure, $\mathcal{F} = 1 \to$ only largest emission matters, check that additional shower emissions vanish

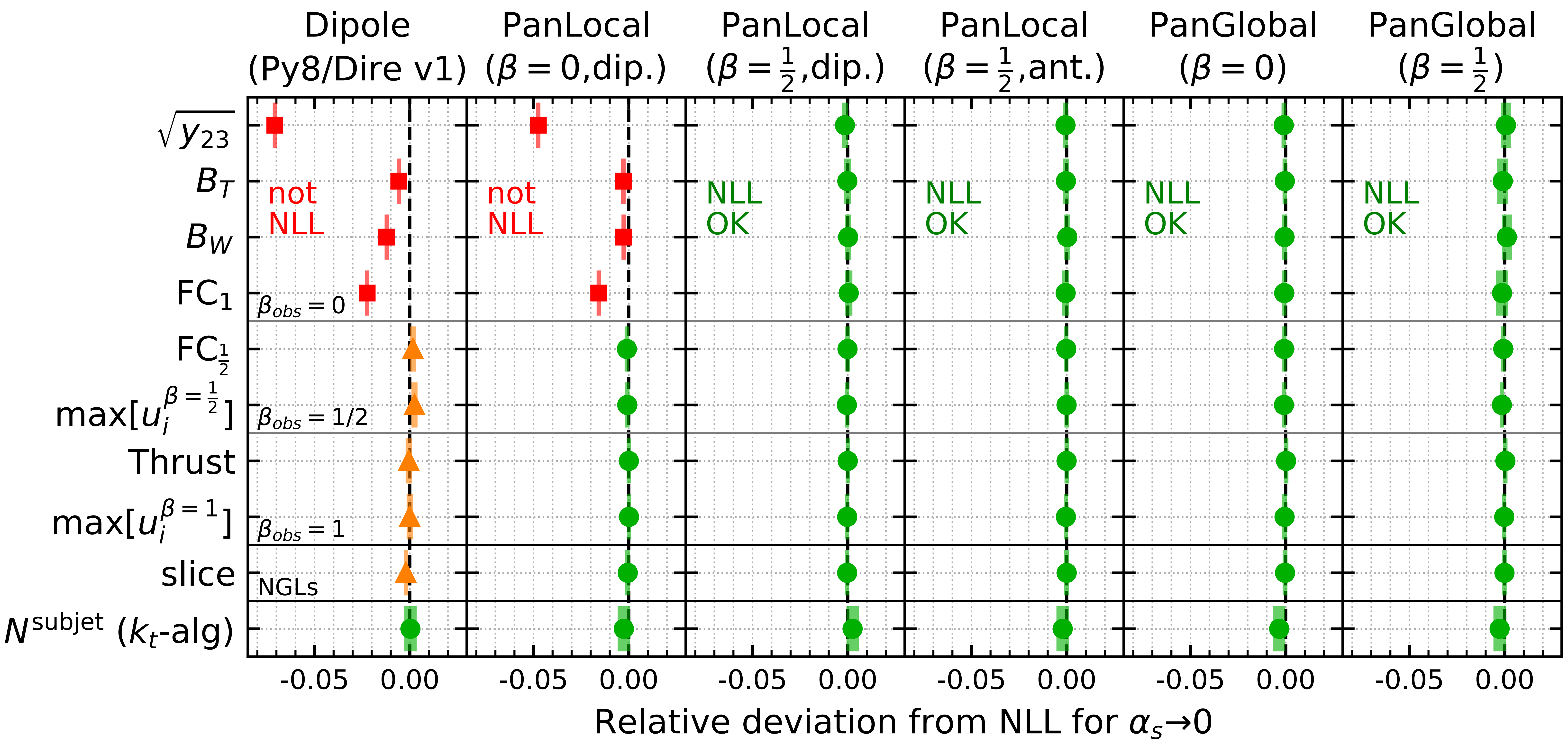


[Herren, Höche, Krauss, DR, Schönherr,'22]

Numerical accuracy tests

- Most extensive sets of tests by PanScales collaboration

[Dasgupta, Dreyer, Hamilton et. al. '20]

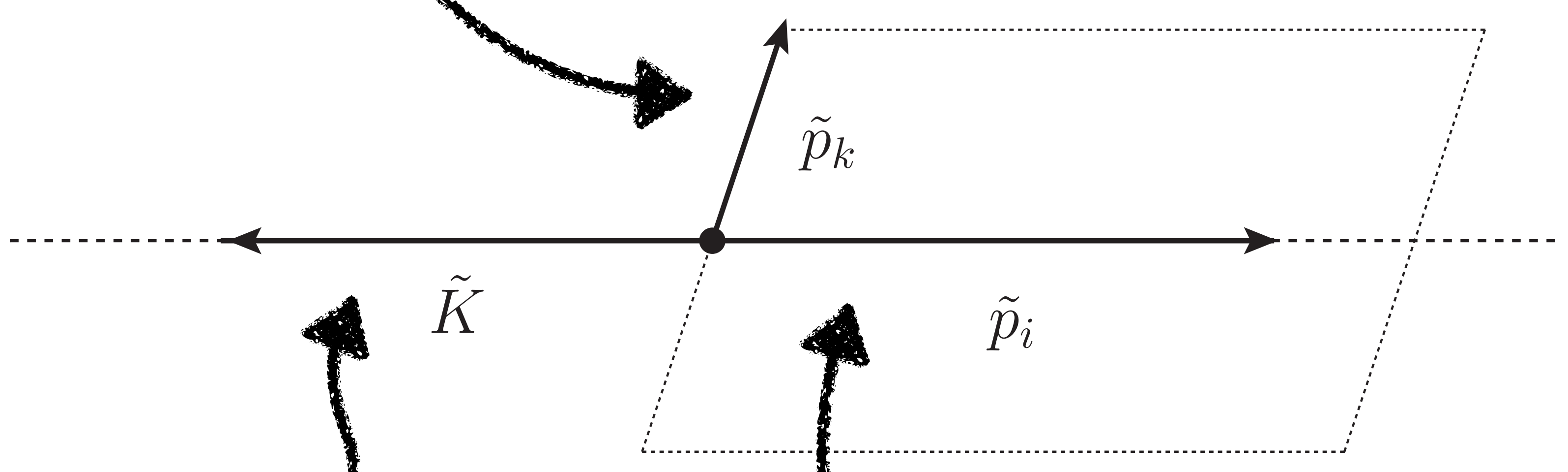


Kinematics - global recoil scheme (Alaric example)

[Herren, Höche, Krauss, DR, Schönherr,’22]

- Before splitting:

colour spectator

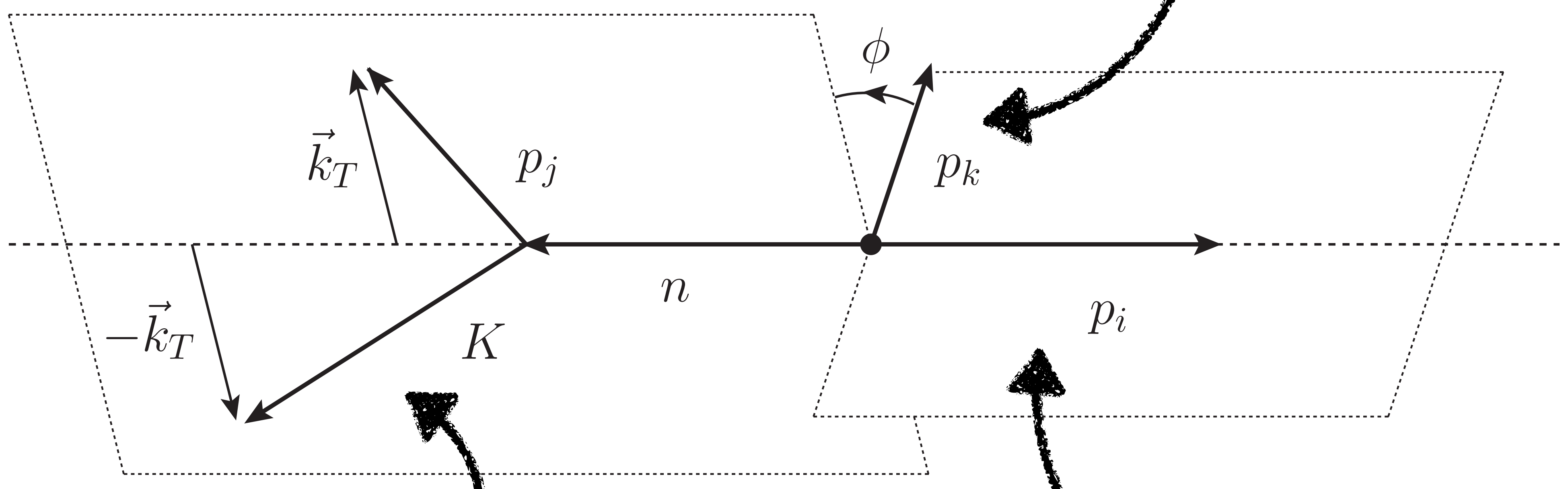


splitter

$\sum$ other momenta k_i

- After splitting:

$p_k = \tilde{p}_k$



$p_i = z\tilde{p}_i$

$K^2 = \tilde{K}^2$

$\tilde{K} + \tilde{p}_i = K + p_i + p_j$

$k_i^\mu \to \Lambda^\mu{}_\nu k_i^\nu$

[Catani, Seymour ’97]

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu - \frac{(K+\tilde{K})^\mu (K+\tilde{K})_\nu}{K\cdot\tilde{K} + \tilde{K}^2} + 2\frac{K^\mu \tilde{K}_\nu}{\tilde{K}^2} \to \Lambda^\mu{}_\nu \tilde{K}^\nu = K^\mu$$

What about angular ordered showers?

- Original recoil scheme:
  - keep momenta off-shell during shower
  - at the end, boost jets to globally preserve momentum
- But modern “local” schemes can be problematic: [Berwick, Ferrario-Ravasio, Richardson, Seymour ’19]
  - q_T preserving scheme favored by data in hard region, but not log-preserving
  - q^2 preserving scheme has problems describing data, but theoretically solid in the soft limit
  - Solution: new “dot product” preserving scheme as middle ground