Computing Feynman integrals, and hypergeometric functions

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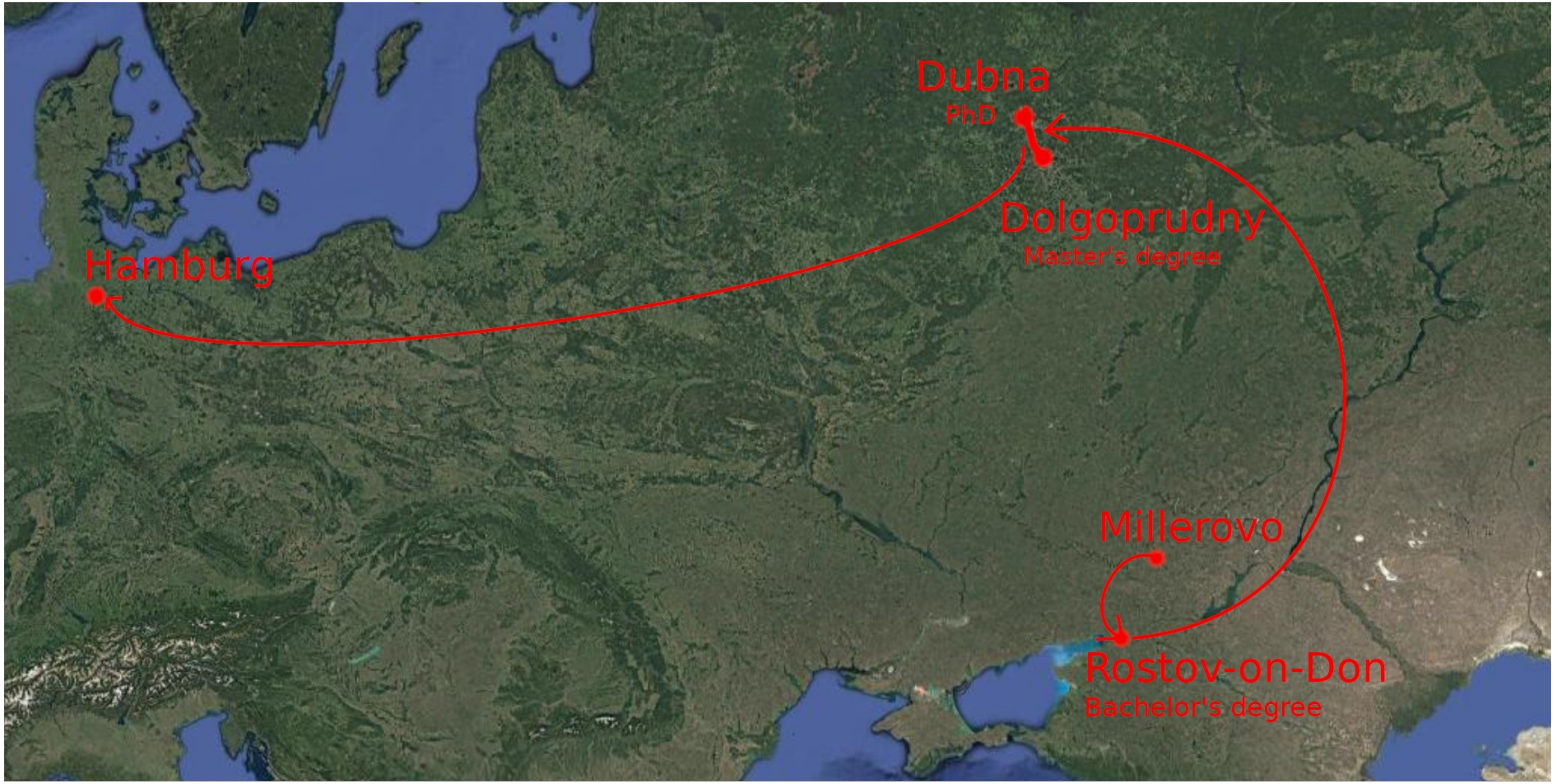

Southern Federal University

Bachelor's degree

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Cycling

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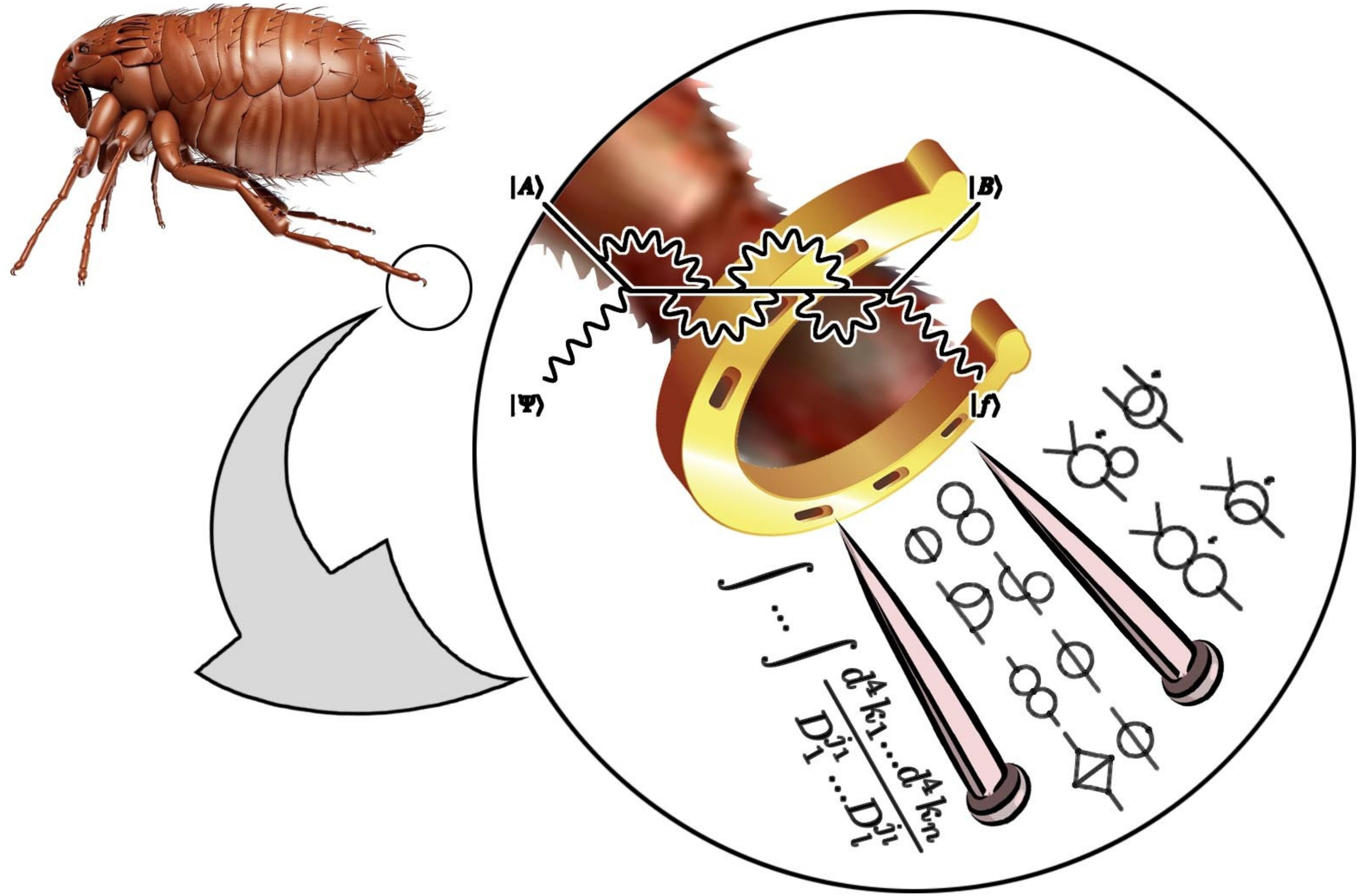
Cross-country skiing

Quantum field theory



Introduction

Feynman integral: $$\int \dots \int \frac{d^4k_1 \dots d^4k_n}{D_1^{j_1} \dots D_l^{j_l}}, \qquad D_r = \sum_{i \geq j \geq 1} A_r^{ij} p_i p_j - m_r^2$$

Integration by Parts (IBP) $$\int d^d k_1 d^d k_2 \dots \frac{\partial f}{\partial k_i^{\mu}} = 0$$ In dimensional regularization

F. V. Tkachov, Phys.Lett.B 100 (1981) 65-68

K.G. Chetyrkin, F.V. Tkachov, Nucl.Phys.B 192 (1981) 159-204

Any integral from a given family can be represented as a linear combination of some limited **basis** of integrals, elements of this basis are called **master integrals**.

Methods for calculating loop integrals

Solving a system of equations for the system of master integrals

- System of difference equations
- ***System of differential equations***

Kotikov, A. V., Phys.Lett.B 254 (1991) 158-164

Kotikov, A. V., Phys.Lett.B 267 (1991) 123-127

Kotikov, A. V., Phys.Lett.B 259 (1991) 314-322

Evaluating by direct integration using some parametric representation

- Feynman parametrisation
- Alpha parametrisation
- MB represention
- et al.

$$\frac{d}{dx}\begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_n \end{pmatrix} = A(x,\varepsilon)\begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_n \end{pmatrix},$$

«epsilon form»

$$A(x,\varepsilon) = \varepsilon\sum_i \frac{A_i}{x-c_i}, \qquad I_j = \sum_k I_j^{(k)}\varepsilon^k$$

J. M. Henn, Physical review letters, vol. 110, no. 25, p. 251601, 2013.

R. N. Lee, JHEP, vol. 04, p. 108, 2015.

Polylogarithms

$$G(a_1, ..., a_n; x) = \int_0^x \frac{G(a_2, ..., a_n; x')}{x' - a_1} dx', \qquad n > 0, \qquad G(; x) = 1, \qquad G(\vec{0}_n; x) = \frac{\log^n x}{n!}$$

A. B. Goncharov, Mathematical Research Letters 5, 497 (1998).
A. B. Goncharov, arXiv preprint math/0103059 (2001).

$$G(a; b) = \log\left(1 - \frac{b}{a}\right), \qquad a \neq 0$$

$$\mathrm{Li}_n(x) = -G\left(\vec{0}_{n-1}, \frac{1}{x}; 1\right) = \int_0^x \frac{dx'}{x'} \mathrm{Li}_{n-1}(x')$$

$$dG(a_1, ..., a_n; x) = \sum_{i=1}^{n} G(a_1, ..., a_{i-1}, a_{i+1}, ..., a_n; x) d\log\left(\frac{a_{i-1} - a_i}{a_{i+1} - a_i}\right)$$

Polylogarithms form Hopf algebra

Hypergeometric functions

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} c_n z^n, \qquad \frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\ldots(n+a_p)}{(n+b_1)\ldots(n+b_1)(n+1)}. $$

Appell functions

$$ F_1(\alpha, \beta_1, \beta_2, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n} m! n!} x^m y^n, \qquad |x| < 1,\ |y| < 1 $$

Lauricella functions

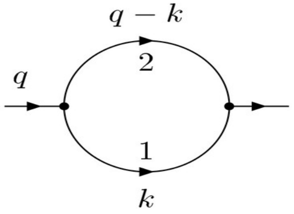
$$ F_A^{(n)}(\alpha; \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1} \ldots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \ldots (\gamma_n)_{m_n} m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n} $$

Methods for calculating Feynman integrals in terms of hypergeometry

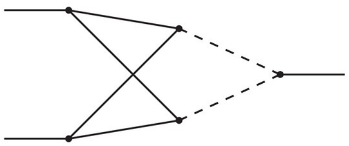
Mellin-Barnes

Dimensional reduction

Exact Frobenius



$$=\int \frac{d^{4-2\varepsilon}k}{i\pi^{2-\varepsilon}} \frac{1}{(k^2-m^2)((k-q)^2-m^2)} = m^{\varepsilon}\Gamma(\varepsilon)\; {}_2F_1\left(\begin{matrix}1,\varepsilon\\ 3/2\end{matrix}\,\middle|\,\frac{q^2}{4m^2}\right)$$



$$= -\frac{\pi^{3/2}(\varepsilon+6)\csc(\pi\varepsilon)\Gamma(2\varepsilon+2)}{2^{2\varepsilon+3}(\varepsilon+2)\Gamma(3-\varepsilon)\Gamma\left(\varepsilon+\frac{5}{2}\right)}\left\{F^{2:1:5}_{2:0:4}\left[\begin{matrix}2\;\; 2(1+\varepsilon)\\ \frac{5}{2}\;\; \frac{5}{2}+\varepsilon\end{matrix}\,\middle|\,\begin{matrix}1\\ -\end{matrix}\,\middle|\,\begin{matrix}1\;\; 2\;\; \frac{5}{2}\;\; 2+\varepsilon\;\; 3+\frac{\varepsilon}{3}\\ 3\;\; 3-\varepsilon\;\; 3+\varepsilon\;\; 2-\frac{\varepsilon}{3}\end{matrix};\frac{-s}{4};-s\right]\right.$$

$$\left.-\frac{\varepsilon(\varepsilon+1)(3\varepsilon+10)}{2(\varepsilon+6)}F^{2:1:6}_{2:0:5}\left[\begin{matrix}2\;\; 2(1+\varepsilon)\\ \frac{5}{2}\;\; \frac{5}{2}+\varepsilon\end{matrix}\,\middle|\,\begin{matrix}1\\ -\end{matrix}\,\middle|\,\begin{matrix}1\;\; 2\;\; \frac{5}{2}\;\; 2+\varepsilon\;\; 2+\varepsilon\;\; 3+\frac{3\varepsilon}{5}\\ 3\;\; 3-\varepsilon\;\; 3+\varepsilon\;\; 3\;\; 2+\frac{3\varepsilon}{5}\end{matrix};-\frac{s}{4};-s\right]\right\}+\dots$$

Generalized Kampé de Fériet function

$$F^{p:q:k}_{l:m:n}\left[\begin{matrix}(a_p)\\ (\alpha_l)\end{matrix}\,\middle|\,\begin{matrix}(b_q)\\ (\beta_m)\end{matrix}\,\middle|\,\begin{matrix}(c_k)\\ (\gamma_n)\end{matrix};x;y\right] = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}$$

Thank you for your attention!