Quantum and deformed algebras in quantum physics: from hadronic to galactic scales

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Plan of the talk:

Quantum algebras & applications –

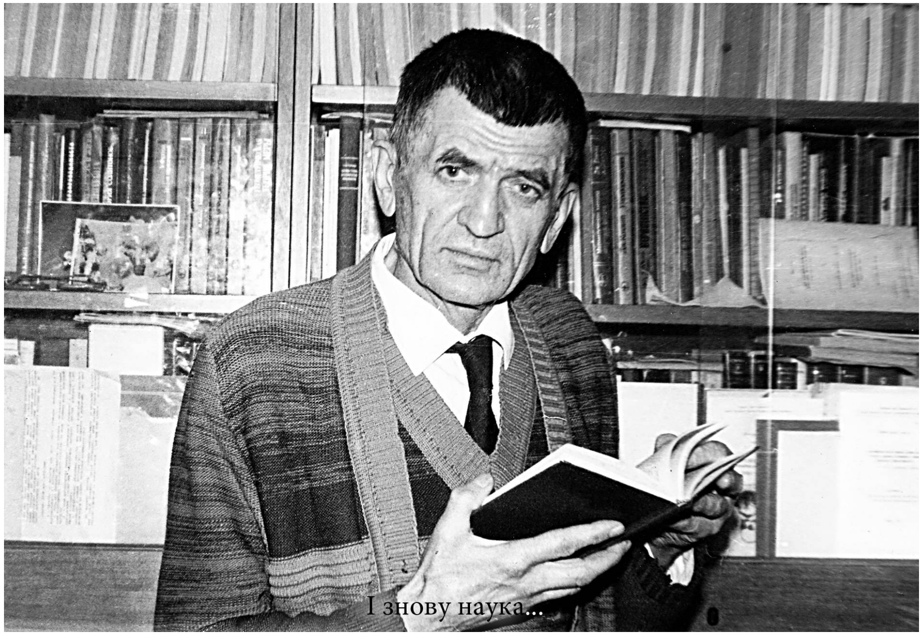
- $U_q(su_n)$ → **hadron mass** sum rules
- $U_q(so_n)$ → (2+1)-dim. Q-gravity; *n*-dim. Q-gravity,

Deformed algebras & applications –

at **hadronic scale** → correl. function intercepts of pions produced at RHICs

at **galactic scales** → properties of DM (dark matter) of dwarf galaxies, including galaxy rotation curves

Ostap Parasiuk -- the founder and the first head of department of Mathematical Methods in Theoretical Physics at the BITP



(21.12.1921 - 22.11.2007)

Prof. acad., Ostap Parasiuk, famous by his joint with acad. Nikolai Bogolyubov **BP theorem** and **BP R-operation** in quant. field theory. His favorite place at BITP was the library, and his passion, of course besides books, was most “fresh” preprints. Those often gave him an inspiration. Once, he was very impre-ssed by the work of Dirk Kreimer.

In the mid-sixties of 20th century, of great importance has become the role of unitary symmetries and group-theoetical methods in general. That produced very successful classification of hadrons and resonances, appearance of quark model, introducing the concept of color, and growing role of gauge theories.

In the staff of department, Anatoli Klimyk was “responsible” for symmetry theory.

Prof. Anatoli Klimyk initiated in the Department, which he headed after Prof. Parasiuk, the research on quantum* groups & algebras, their representations (and applications), as well as properties of deformed oscillator algebras. Involved: Anatoli Klimyk, Ivan Burban, Alexandre Gavrilik, Ivan Kachurik, Valentyna Groza, Mykola Iorgov, Yuri Mishchenko, Anastasiia Rebesh. Later - reserch on deformed models of thermostatistics (theor. and applied aspects).

*A.Klimyk & K.Schmuedgen, *Quantum groups and their representations*, Springer (1997), > 1560 citations

Quantum algebras $U_q(su_n)$ and $U_q(u_{n,1})$ in hadron phenomenology

1. Use q-Algebras $U_q(u_{n,1})$, their representations in GZ basis.
2. Both finite- and infinite-dimensional irreps are used.
3. $U_q(su_n)$ for flavor, $U_q(u_{n+1})$ or $U_q(u_{n,1})$ as dynamical symmetries
4. Application to vector meson mass sum rules (MSRs).
5. Application to baryon octet/decuplet MSRs
6. Several implications:

- "Nonperturbative" treatment → account of all-order SU(3)-breaking effects, *beyond 1st & 2nd orders in hyperchargeY (in usual scheme)*.
- Relation with knots (via Alexander polynomials of torus knots)
- Role of q-Serre relations. Use of the Hopf-algebra structure
- Use of anyonic realization in case of decuplet MSRs
- "Best" value q_7 linked with Cabbibo angle → then, $\theta_C = \pi/14$
- Relation to quark/diquark model of baryons

QUANTUM ALGEBRA $U_q(gl_n)$ AND ITS REAL FORMS

We will use the denotion $[B]_q \equiv [B] \equiv (q^B - q^{-B})/(q - q^{-1})$ where B is either a number or an operator. The elements $\mathbf{1}$, A_{jj+1}, A_{j+1j}, A_{jj}, $j = 1, 2, ..., n-1$, A_{nn} that generate the q-deformed (universal enveloping) algebra $U_q(gl_n)$, satisfy the relations [10]

$$[A_{ii}, A_{jj}] = 0, \qquad [A_{ii}, A_{jj+1}] = \delta_{ij} A_{ij+1} - \delta_{ij+1} A_{ji},$$

$$[A_{ii}, A_{j+1j}] = \delta_{ij+1} A_{ij} - \delta_{ij} A_{j+1i},$$

$$[A_{ii+1}, A_{j+1j}] = \delta_{ij} [A_{ii} - A_{i+1i+1}]_q,$$

$$[A_{ii+1}, A_{jj+1}] = [A_{i+1i}, A_{j+1j}] = 0 \qquad \text{for} \qquad |i-j| \geq 2,$$

$$(A_{i\mp1i})^2 A_{ii\pm1} - [2]_q A_{i\mp1i} A_{ii\pm1} A_{i\mp1i} + A_{ii\pm1} (A_{i\mp1i})^2 = 0,$$
$$(A_{ii\pm1})^2 A_{i\mp1i} - [2]_q A_{ii\pm1} A_{i\mp1i} A_{ii\pm1} + A_{i\mp1i} (A_{ii\pm1})^2 = 0.$$
(q-Serre relations)

Endowed with comultiplication, counit and antipode (which we do not reproduce here), the q-deformed algebra $U_q(gl_n)$ becomes a quantum (Hopf) algebra.

'compact' quantum algebra $U_q(u_n)$ is singled out by means of the $*$-operation

$$(A_{jj})^* = A_{jj}, \qquad (A_{j+1j})^* = A_{jj+1}, \qquad (A_{jj+1})^* = A_{j+1j}$$

The *'noncompact' quantum algebra* $U_q(u_{n,1})$, *in addition, needs*

$$(A_{n+1n})^* = -A_{nn+1}, \quad (A_{nn+1})^* = -A_{n+1n}, \quad (A_{n+1\,n+1})^* = A_{n+1\,n+1}$$

A sketch of the irreps of $U_q(su_n)$ and $U_q(u_{n,1})$

Our approach is insensitive to the substitution $U_q(su_n) \to U_q(u_n)$, and also for $U_q(u_{n,1})$.
Finite-dimensional irreps of $U_q(u_n)$ are given by sets of

ordered integers $\mathbf{m_n} = (\mathrm{m_{1n}}, \mathrm{m_{2n}}, ..., \mathrm{m_{nn}})$

Note that standard branching rules survive through q-deformation, so we use q-analog of Gel'fand-Zetlin formalism (GZ basis and formulas).

The representations of the algebra $U_q(u_{n,1})$ are characterized by their signatures χ, that is, by the sets of $n+1$ numbers: $\chi \equiv (l_1, l_2, ..., l_{n-1}; c_1, c_2)$. Here c_1, c_2 are complex numbers such that $c_1 + c_2 \in \mathbf{Z}$, and all the l_i, $i = 1, ...n-1$, are integers related with the components $m_1, m_2, .., m_{n-1} \equiv \mathbf{m}$ of the highest weight $\mathbf{m}$ of irrep of the subalgebra $U_q(u_{n-1})$, namely, $l_i = m_i - i - 1$. The condition on the components of highest weight in terms of l_i reads: $l_1 > l_2 > ... > l_{n-1}$. Under restriction to the 'compact' subalgebra $U_q(u_n)$, the representation T_χ decomposes into direct sum of all those irreps $T_{\mathbf{l}_n}$ $(\mathbf{l}_n \equiv (l_{1n}, l_{2n}, ..., l_{nn}), l_{jn} = m_{jn} - j,$ $l_{1n} > l_1 \geq l_{2n} > l_2 \geq ... \geq l_{n-1n} > l_{n-1} \geq l_{nn}$ (each enters with multiplicity1).

(The list of infinite-dim. irreps of $U_q(u_{n,1})$ can be found e.g. in **[A.G., I.Kachurik, A.Tertychny, hep-ph/9504233]**) In the case of baryons, some ∞-dim. irreps of $U_q(u_{n,1})$ were applied

Quantum algebras $U_q(su_n)$ and $U_q(u_{n,1})$ in hadron phenomenology

1. q-Algebras $U_q(u_{n,1})$, their representations in GZ basis

2. To derive baryons (octet/decuplet) MSRs, we use infinite-dimensional irreps of "non-compact" dynamical quant.group, within which calculation is performed. Mass operator is constructed from "non-compact" generators.

3. At the end, we'll see that our treatment is in a sense "Nonperturbative" treatment → account of all-order SU(3)-breaking effects (*beyond 1st & 2nd orders in hyperch. Y of the usual scheme*)

Baryon octet mass q-relation

$$[2]m_N + \frac{[2]}{[2]-1}m_\Xi = [3]m_\Lambda + \left(\frac{[2]^2}{[2]-1} - [3]\right)m_\Sigma$$

Additional structure does arise →

$$+ \frac{A_q}{B_q}\left([2]m_N + m_\Xi - m_\Lambda - [2]m_\Sigma\right)$$

where $A_q = [2]^4 + [2]^3([5]-[4]) + [2]^2([6]-[5]) - [2]([6]+[4]^2) + [4]^2$

$$B_q = \left([2]^3 - [2]^2[4] + 3[5] - [3]\right)([2]-1).$$

$$A_q = ([2]-2)[2]^3([4]-[2])$$

In factorized form →

$$= ([2]-2)[2]^4([2]^2-3)$$

zero if q=e$^{i\pi/6}$

Corrections are not small ("nonperturbative"):

$$(1-\Delta_N)m_N + (1+\Delta_\Xi)m_\Xi = \left(\frac{3}{2} - \Delta_\Lambda\right)m_\Lambda + \left(\frac{1}{2} + \Delta_\Sigma\right)m_\Sigma$$

$$\Delta_N \equiv \cancel{\frac{A_q}{B_q}} \qquad \Delta_\Xi \equiv -\cancel{\frac{A_q}{[2]B_q}} + \frac{1}{[2]-1} - 1 \qquad \Delta_\Lambda \equiv \cancel{\frac{A_q}{[2]B_q}} + \frac{3}{2} - \frac{[3]}{[2]}$$

$$\Delta_\Sigma \equiv -\cancel{\frac{A_q}{B_q}} + \frac{[2]}{[2]-1} - \frac{[3]}{[2]} - \frac{1}{2} \quad \to\to \quad \Delta_\Sigma = \frac{\sqrt{3}}{\sqrt{3}-1} - \frac{2}{\sqrt{3}} - \frac{1}{2} \approx 0.71$$

Gavrilik & Iorgov → confirmed more explicitly, via the terms like q^{Y}

Vector meson mass *q*-relations and MSRs

In case of VM, adjoint (finite-dim.) irreps. are used. Using dynamical *q*-algebras, we derive VM mass *q*-relations for singlet, triplet and doublet

n=3 (flavors) $$\frac{[3]_q}{[2]_q} m_{\omega_8} + \left(2 - \frac{[3]_q}{[2]_q}\right) m_\rho = 2 m_{K^*}$$

If q=1, we get famous Gell-Mann – Okubo MSR: $3m_{\omega_8} + m_\rho = 4m_{K^*}$

which requires ω-ϕ mixing with fitted angle. But,

if $q = e^{i\pi/5}$ (and $[3]_q = [2]_q$) → Okubo's nonet MSR: $m_{\omega_8} + m_\rho = 2m_{K^*}$

This holds ideally with mass m_ϕ (1020 *MeV*) put in place of m_{ω_8} (no mixing is needed!).

Likewise, for more flavors, e.g.

n=4 (flavors) $$\frac{[4]}{[3]} m_{\omega_{15}} + \left(2\frac{[4]^2}{[3]^2} - \frac{8}{[2]}\frac{[4]}{[3]} - \frac{[4]}{[3]} + 4\right) m_\rho =$$

$$= 2\, m_{D^*} + \left(2\frac{[4]^2}{[3]^2} - \frac{8}{[2]}\frac{[4]}{[3]} + 2\right) m_{K^*},$$

setting $[4]_q = [3]_q$ (and $[n]_q = [n-1]_q$, $n = 5, 6$) over-simplifies the relations and yields higher analogs of Okubo's nonet sum rule (isodoublets in r.h.s):

Vector mesons (vect.quarkonia) VS torus knots

$$m_{\omega_{15}} + (5 - 8/[2]_{q_4}) m_\rho =$$
$$= 2\, m_{D^*} + (4 - 8/[2]_{q_4}) m_{K^*},$$
$$m_{\omega_{24}} + (9 - 16/[2]_{q_5}) m_\rho =$$
$$= 2\, m_{D_b{}^*} + (4 - 8/[2]_{q_5})(m_{D^*} + m_{K^*})$$

A.M.Gavrilik *J. Phys. A: Math. Gen.* Vol.**27** No.3, L91 (1994)

The latter MSR, with (mass of) Υ in place of ω_{24}, holds within 0.7%

The Alexander polynomials (AP) naturally appear:

$\Delta_q\{(2n-1)_1\}$ of $(2n-1)_1$-torus knots. E.g., $[3]_q - [2]_q = q^2 + q^{-2} - q - q^{-1} + 1 \equiv \Delta_q\{5_1\}$, $[4]_q - [3]_q = q^3 + q^{-3} - q^2 - q^{-2} + q + q^{-1} - 1 \equiv \Delta_q\{7_1\}$ correspond to the 5_1- and 7_1-knots. Since the ex

Zeros of "senior" polynomials do matter:

- $$\frac{[3]_q}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{\Delta\{3_1\}+1}$$
- $$\frac{[n]_q}{[n-1]_q} = 1 + \frac{[n]_q - [n-1]_q}{[n-1]_q} = 1 + \frac{\Delta_{\{(2n-1)_1\}}}{1 + \sum_{r=2}^{n-1} \Delta_{\{(2r-1)_1\}}}$$

Thus, "senior" AP, by its root, fixes param. *q* rigidly

Short summary:

- ❖ ~~Mass sum rules for mesons & baryons of very high precizion are gained.~~
- ❖ All-order effects (corrections) in flavor SU(3) breaking are taken into account.
- ❖ A relation found: flavors (= vect. quarkonia) ↔ torus knots (Alexander pol.).
- ❖ Relation: def.param. $q \leftrightarrow \theta_C$ (Cabibbo angle) found, with exact $\theta_C = \pi/14$.
- ❖ Hopf algebra structure can be used → without new insights.
- ❖ If anyonic realization (and its dual) of $U_q(u_n)$ is used, then $<\tilde{\ldots}| M_{Di\ \ldots}>$.
- ❖ A relation with diquark-quark model of baryons is found.
- ❖ Use of quantum groups → to be extended to other parts of the ***standard model***. *(Some results exist for **el.weak** sector, e.g. P. Watts, D.Finkelshtein & others)*

1. **Vector meson MSRs ↔ torus knot invariants (Alexander polynomials)**

- **torus knots**

  5_1, 7_1, 9_1, 11_1 are put into correspondence with vector quarkonia $s\bar{s}$, $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ res

2. **Baryon MSRs: θ_C & the value of q (deformation parameter).**

3. **Highest precision**

- a new formula giving quark mass ratio in terms (very precisely known) octet baryon masses:

$$\frac{m_s}{m_d} = \frac{3M_\Sigma - M_\Lambda - 3M_N + M_\Xi}{M_\Sigma + M_\Lambda - M_N - M_\Xi} = 18.63 \pm 0.16$$

which agrees well with value $\frac{m_s}{m_d} = 18.9 \pm 0.8$

H. Leutwyler, hep-ph/0011049

Nonstandard q-algebras $U'_q(so_n)$

Advantages:

A. Gavrilik, A. Klimyk,
***Lett. Math. Phys.* - 1991**

1. Obey canonical embeddings
2. Representations in GZ basis
3. Admit all the noncompact forms and inhomogen. extensions (Euclid.,Poincare)
4. In quant. geometry -- construction of quantum spheres $S_q^{(n)}$ for any n, & also other q-coset spaces (Grassm., Stiefel)
5. Appear in (2+1) Anti-de Sitter gravity
6. Applications to n-dim. quantum gravity
7. Many other aspects

Nonstandard *q*-algebras $U'_q(so_n)$ as an alternative to Drinfeld-Jimbo (standard) quantization of B_r, D_r

In the DJ quantization of U(so_n) → it is impossible to construct irreps using Gel'fand-Zetlin (GZ) formalism, as the series *n=2r+1* and *n=2r* are quantized disjointly, while GZ requires canonical embeddings like $so_n > so_{n-1} > so_{n-2}$ …

We use *q*-numbers, i.e. $[x] \equiv (q^x - q^{-x})/(q - q^{-1})$ When $q \to 1$, $[x] \to x$.

$$I^2_{k,k-1}I_{k-1,k-2} + I_{k-1,k-2}I^2_{k,k-1} - [2]_q\, I_{k,k-1}I_{k-1,k-2}I_{k,k-1} = -I_{k-1,k-2},$$
$$I^2_{k-1,k-2}I_{k,k-1} + I_{k,k-1}I^2_{k-1,k-2} - [2]_q\, I_{k-1,k-2}I_{k,k-1}I_{k-1,k-2} = -I_{k,k-1},$$

$$[I_{i,i-1}, I_{k,k-1}] = 0 \qquad \text{if} \quad |\, i-k \,| > 1, \qquad i,k = 2,3,...,n.$$

real forms - compact $U_q(\mathrm{so}_n)$ and noncompact $U_q(\mathrm{so}_{n-1,1})$ - are singled out from $U_q(\mathrm{so}(n,\mathbf{C}))$ by imposing the $*$-structures

$$I^*_{k,k-1} = -I_{k,k-1}, \qquad k = 2,...,n,$$ -- "compact"

$$I^*_{k,k-1} = -I_{k,k-1}, \quad k = 2,...,n-1, \qquad I^*_{n,n-1} = I_{n,n-1}$$ -- "non-compact"

A. Gavrilik, A. Klimyk, *Lett. Math. Phys.* (1991)

Remark: besides trilinear, also the **bilinear** formulation is possible for $U'_q(so_n)$

Bilinear formulation of $U'_q(so_n)$

$$[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \quad [I^+_{kl}, I^+_{km}]_q = I^+_{lm},$$

$$[I^+_{km}, I^+_{lm}]_q = I^+_{kl} \ \text{ if } k > l > m,$$

$$[I^+_{kl}, I^+_{mp}] = 0 \ \text{ if } k>l>m>p \quad \text{or} \quad k>m>p>l; \qquad (2$$

I_{21}

$$[I^+_{kl}, I^+_{mp}] = (q-q^{-1})(I^+_{lp}I^+_{km} - I^+_{kp}I^+_{ml}) \ \text{ if } k>m>l>p.$$

[A.G., N.Iorgov, arxiv:9911201]

If $n=3$, to the set I_{21}, I_{32} we add $I_{31} \equiv [I_{21}, I_{32}]_q = q^{1/2} I_{21} I_{32} - q^{-1/2} I_{32} I_{21}$, and the other two relations $[I_{31}, I_{21}]_q = I_{32},$ $[I_{32}, I_{31}]_q = I_{21}$.The result is known as cyclically symmetric q-algebra [D.Fairlie, *J.Ph.A* (1990), A.Odesskii, Func.An. Apl.(1986)]

Signatures and basis

$$\{\xi_n\} \equiv \left\{ \begin{array}{l} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{array} \right\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\}$$

sets $\mathbf{m}_n$ consisting of $[\frac{n}{2}]$ components $m_{1,n}, m_{2,n}, ..., m_{[\frac{n}{2}],n}$

$m_{1,2p+1} \geq m_{2,2p+1} \geq ... \geq m_{p,2p+1} \geq 0,$ for $n = 2p+1$

$m_{1,2p} \geq m_{2,2p} \geq ... \geq m_{p-1,2p} \geq |m_{p,2p}|.$ for $n = 2p$

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq ... \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1},$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq ... \geq m_{p-1,2p-1} \geq |m_{p,2p}|.$$

introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j.$$

Basis element defined by scheme $\{\xi_n\}$ is denoted as $|\{\xi_n\}\rangle$ or

Infinitesimal operator $I_{2p+1,2p}$ of the representation, given by $\mathbf{m}_{2p+1}$, of $U_q(\mathrm{so}_{2p+1})$ on the GT basis elements acts as

$$I_{2p+1,2p}|\mathbf{m}_{2p+1},\mathbf{m}_{2p},\beta\rangle = \sum_{j=1}^{p} A_{2p}^{j}(\mathbf{m}_{2p})|\mathbf{m}_{2p+1},\mathbf{m}_{2p}^{+j},\beta\rangle - \sum_{j=1}^{p} A_{2p}^{j}(\mathbf{m}_{2p}^{-j})|\mathbf{m}_{2p+1},\mathbf{m}_{2p}^{-j},\beta\rangle$$

A.G., N.Iorgov, q-alg/9709036 (1997)

where $A_{2p}^{j}(\xi_{2p+1}) = d(l_{j,2p})\left|\frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}]}\right|$

At q=1 this is ½ $\quad d(l_{j,2p}) \equiv \left(\frac{[l_{j,2p}][l_{j,2p}+1]}{[2l_{j,2p}][2l_{j,2p}+2]}\right)^{\frac{1}{2}}$ is the q-deformation of ½ !

Most nontrivial point: how to deform the "classical" coeff. ½ ? Indeed, q-numbers [½] or 1/[2] don't work. But the **function** $d(l_{j,2p})$ does work!

Likewise the operator $I_{2p,2p-1}$ of the representation, given by $\mathbf{m}_{2p}$, of $U_q(\mathrm{so}_{2p})$

acts as $I_{2p,2p-1}|\mathbf{m}_{2p},\mathbf{m}_{2p-1},\beta\rangle = \sum_{j=1}^{p-1} B_{2p-1}^{j}(\mathbf{m}_{2p-1})|\mathbf{m}_{2p},\mathbf{m}_{2p-1}^{+j},\beta\rangle$

with

B_{2p-1}, C_{2p-1} obtained by… $\quad - \sum_{j=1}^{p-1} B_{2p-1}^{j}(\mathbf{m}_{2p-1}^{-j})|\mathbf{m}_{2p},\mathbf{m}_{2p-1}^{-j},\beta\rangle + i\, C_{2p-1}(\mathbf{m}_{2p-1})|\mathbf{m}_{2p},\mathbf{m}_{2p-1},\beta\rangle.$

Example: *q*-Euclidean algebra U$'_q$(iso_n), *n*=3

Bilinear formulation of U$'_q$(iso_n):

A.G., N. Iorgov, *Symmetry in Nonlin. Math. Phys.* (1997)

$$[I_{21}, I_{32}]_q = I_{31}^+ \qquad [I_{21}, P_3] = 0$$
$$[I_{32}, I_{31}^+]_q = I_{21} \qquad [I_{32}, P_1^+] = 0$$
$$[I_{31}^+, I_{21}]_q = I_{32} \qquad [I_{31}^+, P_2^+] = (q - q^{-1})(P_1^+ I_{32} - P_3 I_{21}$$

$$[I_{32}, P_3]_q = P_2^+ \qquad [I_{21}, P_2^+]_q = P_1^+ \qquad [I_{31}^+, P_3]_q = P_1^+$$
$$[P_2^+, I_{32}]_q = P_3 \qquad [P_1^+, I_{21}]_q = P_2^+ \qquad [P_1^+, I_{31}^+]_q = P_3$$
$$\boxed{[P_3, P_2^+]_q = 0 \qquad [P_2^+, P_1^+]_q = 0 \qquad [P_3, P_1^+]_q = 0}$$

Here -- non-commutativity of translation generators

Similarly constructed *q*-Poincare algebra contains the *q*-deformed Lorentz (sub)algebra and *q*-commutative subalgebra of momenta

That basically differs from well-known *κ*-Poincare algebra of *J. Lukiersky* et al. 1993 (***non-deformed*** *Lorentz, momenta subalgs.*)

2+1 quant.gravity, algebra of Nelson & Regge

Bilinear formulation of $U'_q(so_n)$

A.Gavrilik, *UJP* (2002), arxiv:gr-qc-0401067

$$[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \quad [I^+_{kl}, I^+_{km}]_q = I^+_{lm},$$

$$[I^+_{km}, I^+_{lm}]_q = I^+_{kl} \text{ if } k > l > m,$$

$$[I^+_{kl}, I^+_{mp}] = 0 \text{ if } k>l>m>p \quad \text{or} \quad k>m>p>l; \qquad (2$$

$$[I^+_{kl}, I^+_{mp}] = (q-q^{-1})(I^+_{lp}I^+_{km} - I^+_{kp}I^+_{ml}) \text{ if } k>m>l>p.$$

commutator algebra $A(n)$ specific for $2+1$ quantum gravity with negative Λ. For each quadruple of indices $\{j,l,k,m\}$, $j,l,k,m = 1,\ldots,n$, such that

J. Nelson & T, Regge, Phys .Lett. 1991)

$$[a_{mk}, a_{jl}] = [a_{mj}, a_{kl}] = 0,$$

$$[a_{jk}, a_{kl}] = (1 - \tfrac{1}{K})(a_{jl} - a_{kl}a_{jk}),$$

$$[a_{jk}, a_{km}] = (\tfrac{1}{K} - 1)(a_{jm} - a_{jk}a_{km}),$$

$$[a_{jk}, a_{lm}] = (K - \tfrac{1}{K})(a_{jl}a_{km} - a_{kl}a_{jm}).$$

$$K = \frac{4\alpha - ih}{4\alpha + ih}, \quad \alpha^2 = -\frac{1}{3\Lambda}, \quad \Lambda < 0.$$

Isomorphism of the Algebras $A(n)$ and $U'_q(\mathrm{so}_n)$

$$\text{Redefine:} \qquad \{K^{1/2}(K-1)^{-1}\}a_{ik} \longrightarrow A_{ik},$$

$$\text{Identify:} \qquad A_{ik} \longrightarrow I_{ik}, \qquad K \longrightarrow q.$$

Then, the Nelson—Regge algebra $A(n)$

these two deformed algebras are isomorphic to each other (of course, for $K \neq 1$). Recall that n is linked to the genus g as $n = 2g + 2$, while $K = (4\alpha - ih)/(4\alpha + ih)$ with $\alpha^2 = -\frac{1}{\Lambda}$.

A.Gavrilik, *UJP (*2002),
arxiv:gr-qc-0401067

Using the q-algebras $U'_q(so_n)$ for n-dimensions

A sketch of G=SO(n) Spin Networks

A generalized spin network associated with a Lie group G is defined as a triple (Γ, ρ, I) where Γ is an oriented graph (=directed edges and vertices), ρ is a labeling of each edge *e by an irrep* ρ_e of G; I is a labeling of each vertex v of Γ by an intertwinner I_v mapping tensor product of irreps incoming at v to the product of irreps outgoing from v.

Simple $G = \mathrm{SO}(n)$ Spin Networks

consider restricted case of $G = \mathrm{SO}(n)$ *simple* spin networks. Simple spin networks associated with $G = \mathrm{SO}(n)$ are evaluated as Feynman integrals over the coset space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, i.e. over the sphere S^{n-1}. Simplicity means that only the $\mathrm{SO}(n)$ representations of class 1 (with respect to $\mathrm{SO}(n-1)$) labeled by single nonnegative integer l, are employed.

To such irreps → zonal spherical functions and thus Gegenbauer polynomials.

Basic ingredient is the 'propagator' expressed in terms of zonal spherical functions $t^{nl}_{00}(y)$, $y = \cos\theta$, or in view of the equality [8]

$$t^{Nl}_{00}(\cos\theta) = \frac{\Gamma(2p)l!}{\Gamma(2p+l)} C^{p}_{l}(\cos\theta)\ , \qquad p = (N-2)/2\ , \quad (1)$$

directly through the Gegenbauer polynomials:

$$G^{(N)}_{m}(x,y) = \frac{N+2m-2}{N-2} C^{(N-2)/2}_{m}(x\cdot y)\ . \qquad (2)$$

Used linearization, recursion relations

→ calculated Θ-graph (9)

A. Freidel, K. Krasnov,

J. Math. Phys. (2001)

q-ultraspherical Polynomials

obey the recursion relation:

$$(1-q^n)C_n(x;\beta|q) = 2x(1-\beta q^{n-1})C_{n-1}(x;\beta|q)$$

$$-(1-\beta^2 q^{n-2})C_{n-2}(x;\beta|q), \qquad (n \geq 2),$$

along with special values

$$C_0(x;\beta|q) = 1, \quad C_1(x;\beta|q) = 2(1-\beta)x/(1-q).$$

With $\beta = q^\lambda$, the "classical" limit $q \to 1$ yields

$$C_n(x;\beta|q) \xrightarrow{q\to 1} C_n^\lambda(x).$$

T. Sugitani, *Compositio Math*. (1995)

The explicit expression for the q-ultraspherical polynomials is [9] as follows:

$$C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} \mathrm{e}^{\mathrm{i}(n-2k)\theta}$$

$$= \frac{(\beta;q)_n}{(q;q)_n} \mathrm{e}^{in\theta} {}_2\Phi_1(q^{-n},\beta;\beta^{-1}q^{1-n};q,q\beta^{-1}\mathrm{e}^{-2i\theta}).$$

q-ultraspherical Polynomials

where:

$$(a;q)_n = \begin{cases} 1, & n=0 \\ (1-a)(1-qa)\ldots(1-q^{n-1}a), & n \geq 1. \end{cases}$$

The orthogonality relation

$$\int_0^\pi C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q)W_\beta(\cos\theta|q)\mathrm{d}\theta = \frac{\delta_{mn}}{h_n(\beta|q)},$$

where the weight function and normalization factor are:

$$W_\beta(\cos\theta|q) = \frac{(\mathrm{e}^{2i\theta}, \mathrm{e}^{-2i\theta}; q)_\infty}{(\beta\mathrm{e}^{2i\theta}, \beta\mathrm{e}^{-2i\theta}; q)_\infty}, \qquad (a_1, a_2; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty$$

$$h_n(\beta|q) = \frac{(q, \beta^2; q)_\infty (q;q)_n (1-\beta q^n)}{2\pi(\beta, \beta q; q)_\infty (\beta^2; q)_n (1-\beta)} \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

Then, the q-analog of Θ-graph (9) is calculated, along with its q-deformed recursion relation

Linearization (Rogers) formula is to be used

A.Gavrilik, *UJP (*2002), arxiv:gr-qc-0401067

Clearly, other more complicated graphs are to be calculated.

From quant. algebras to deformed ones: gen.remarks:

So, various quantum or *q*-deformed algebras show their efficiency in diverse problems of quantum physics.

Related with these, deformed oscillators (deformed bosons) as well play important role in modern physics:

1). If instead of treating particles as point-like structureless objects, one tends to take into account either nonzero proper volume or composite nature of particles, then it is natural to modify or deform the standard commutation relations.

2). Yet another reason to deal with deforrned models is the com-plication due to nonlinearities and/or (self)interactions. And, there are other reasons to deal wiith deformed oscillators or models.

quantum algebras ←→ deformed oscillators

As known, Lie algebra *su*(2) with relns $[J_0\,,\,J_\pm]=\pm J_\pm\,,\quad [J_+\,,\,J_-]=2J_0$
realizes by 2 copies of harmonic oscillator: $\{a_1,a_1^\dagger,N_1\}$ and $\{a_2,a_2^\dagger,N_2\}$

$$[a\,,\,a^\dagger]=1\,,\quad [N\,,\,a^\dagger]=a^\dagger$$

REALIZATION (Jordan-Schwinger):

$$J_0=\frac{1}{2}(N_1-N_2)\,,\quad J_+=a_1^\dagger a_2\,,\quad J_-=a_2^\dagger a_1$$

How to realize quantum algebra U_q(*su*(2)) with relations:

$$[\mathcal{J}_0\,,\,\mathcal{J}_\pm]=\pm\mathcal{J}_\pm\,,\quad [\mathcal{J}_+\,,\,\mathcal{J}_-]=[\![2J_0]\!]_q$$

? We have to take two copies of *special* deformed oscillator $\{A_1,A_1^\dagger,\mathcal{N}_1\}$ and $\{A_2,A_2^\dagger,\mathcal{N}_2\}$ **such that**

$$\boxed{AA^\dagger-qA^\dagger A=q^{-\mathcal{N}}\,,\quad [\mathcal{N}\,,\,A^\dagger]=A^\dagger}$$

$$[\![n]\!]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$$

i.e. BM *q*-oscillator (Biedenharn-Macfarlane)

***Another (but equival.) presentation* – using structure function of deformation** $\boxed{\varphi(N)\equiv A^\dagger A}$**. For BM *q*-oscillator struct. functions [[*N*]]$_q$. Then** $AA^\dagger - A^\dagger A$ = φ(*N*+1)- φ(*N*) =[[*N*+1]]$_q$ - [[*N*]]$_q$

Importance of structure function of deformation (DSF)

$$a^\dagger a = \varphi(N), \qquad aa^\dagger = \varphi(N+1).$$

For the ordinary quantum oscillator: $a^\dagger a = N$, $aa^\dagger = N+1$.

Commutation relation for operators $a^\dagger$, a:

$$aa^\dagger - a^\dagger a = \varphi(N+1) - \varphi(N).$$

In the q-analog of Fock space:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{\varphi(N)!}}|0\rangle, \quad N|n\rangle = n|n\rangle, \quad \varphi(N)|n\rangle = \varphi(n)|n\rangle$$

where $\varphi(N)! = \varphi(N)\cdot\varphi(N-1)\cdot\ldots\cdot\varphi(1), \quad \varphi(0)! = 1.$

Structure functions (SF) → deformed oscillators

Diverse types of deformation:

- *q*- and *q,p*-oscillators – SFs of exponential type:

q,p-SF (or *q,p*-bracket):

$$\boxed{\varphi(N) = \frac{p^N - q^N}{p - q}}$$

AC if *p=1*;
BM if *p=1/q*;
TD if *p=q*; plehtora if *p=f(q)*;

- So-called *μ*-oscillator (Janussis) – SF of rational type:

μ-SF (or *μ*-bracket):

$$\boxed{\varphi(N) = \frac{N}{1+\mu N}}$$

- There exist deformed oscillators of polynomial type:

μ̃-SF (or *μ̃*-bracket):

$$\boxed{\varphi_{\tilde{\mu}}(\hat{N}) = (1+\tilde{\mu})\hat{N} - \tilde{\mu}\hat{N}^2}$$

- There exist a plenty of deformed oscillators of hybrid type :

q,μ̃-SF (or *q,μ̃* -bracket):

$$\boxed{\varphi_{\tilde{\mu},q}(n) = (1+\tilde{\mu})[n]_q - \tilde{\mu}\left([n]_q\right)^2 \equiv [n]_{\tilde{\mu},q}}$$

Below we give applications of deformed bosons of 4-th and 2-nd types to some *micro-* and *macro-*systems, respectively

Some cases of DOs i.e. their deformation structure function

--- *q,p-oscillator:*

$$A_i A_j^\dagger - q^{\delta_{ij}} A_j^\dagger A_i = \delta_{ij} \cdot p^{N_i} \qquad [N_i, A_j^\dagger] = \delta_{ij}\ A_j^\dagger$$

$$A_i^\dagger A_i = [\![N_i]\!]_{qp} \qquad [\![X]\!]_{qp} \equiv \frac{q^X - p^X}{q-p}$$ ***(q,p-bracket)***

--- *q-oscillators:* 1) *if* p=1 - **AC** (Arik-Coon) type,

2) *if* $p=q^{-1}$ - **BM** (Bied.-Macfarlane) type,

3) *if* $p=q$ - **TD** (Tamm-Dancoff) type

– **"plethora of 1-parameter" DOs (*G., R.*, MPLA 2008)** .

(μ-bracket)

$$\varphi_\mu(N) = \frac{N}{1+\mu N}$$

μ-oscillator of Jannusis: ***structure f-n***

From deform. Oscillator (DO) to def. Bose gas model (DBGM)

-- by deforming thermodynamics

sample of hybrid DSF in deforming thermodynamics:

$$\varphi_{\tilde{\mu},q}\left(z\frac{d}{dz}\right) \equiv \varphi_{\tilde{\mu}}(D_q) = (1+\tilde{\mu})D_q - \tilde{\mu}D_q^2, \quad D_q \equiv \left[z\frac{d}{dz}\right]_q$$

-- by deforming distributions & correlations

the same hybrid DSF in deforming distributions and correl. functions:

$$\varphi_{\tilde{\mu},q}(n) = (1+\tilde{\mu})[n]_q - \tilde{\mu}\left([n]_q\right)^2 \equiv [n]_{\tilde{\mu},q}$$

***lintersept of correlation function*:**

$$\lambda^{(r)}(\mathbf{k}) = \frac{\langle (a_{\mathbf{k}}^{\dagger})^r (a_{\mathbf{k}})^r \rangle}{\langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \rangle^r} - 1$$

1- & 2-particle distributions in *q,p*-Bose gas model

Ideal gas of def. bosons: thermal averages, one-particle distribution:

$$\langle A\rangle = \frac{\mathrm{Sp}(A\cdot e^{-\beta H})}{\mathrm{Sp}(e^{-\beta H})}, \qquad H=\sum_i \omega_i\ N_i^{(qp)}, \qquad \omega_i=\sqrt{m^2+\mathbf{k}_i^2}$$

$$\langle A^\dagger A\rangle = \frac{(e^{\beta\omega}-1)}{(e^{\beta\omega}-p)(e^{\beta\omega}-q)} \xrightarrow{p\to 1} \frac{1}{e^{\beta\omega}-q} \xrightarrow{q\to 1} \frac{1}{e^{\beta\omega}-1},$$

(*q,p*-Bose) **(AC type *q*-Bose)** **(Bose)**

	1-particle distribution	2-particle distribution
AC type *q*-Bose	$\langle a^\dagger a\rangle = \frac{1}{e^{\beta\omega}-q}$	$\langle a^{\dagger 2} a^2\rangle = \frac{(1+q)}{(e^{\beta\omega}-q)(e^{\beta\omega}-q^2)}$
BM *q*-Bose	$\langle b^\dagger b\rangle = \frac{e^{\beta\omega}-1}{e^{2\beta\omega}-(q+q^{-1})e^{\beta\omega}+1}$	$\langle b^{\dagger 2} b^2\rangle = \frac{(q+q^{-1})}{(e^{\beta\omega}-q^2)(e^{\beta\omega}-q^{-2})}$
***q,p*-Bose** $\longrightarrow$	$\langle A^\dagger A\rangle = \frac{(e^{\beta\omega}-1)}{(e^{\beta\omega}-p)(e^{\beta\omega}-q)}$	$\langle A^{\dagger 2} A^2\rangle = \frac{(p+q)\ (e^{\beta\omega}-1)}{(e^{\beta\omega}-q^2)(e^{\beta\omega}-pq)(e^{\beta\omega}-p^2)}$

Two-particle momentum correlation function:

$$C^{(2)}(k_1,k_2) = \gamma \frac{P_2(k_1,k_2)}{P_1(k_1)P_1(k_2)},$$

can be rewritten in variables $Q = k_1 - k_2$, $K = (k_1+k_2)/2$:

$$C^{(2)}(Q,K) \xrightarrow{k_1=k_2} C^{(2)}(Q=0,K) = 1 + \lambda^{(2)}(m,\mathbf{K}),$$

$\lambda^{(2)}$ - intercept of two-particle correlation function.

If assume that the particle are bosons then $\lambda^{(2)} = 1$.

Chapman & Heinz, Phys. Lett. B (1994)

$$\lambda^{2)} \equiv \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^\dagger a \rangle^2} - 1$$

For **true** bosons $\lambda^{(2)} = 1$, unlike deformed analogs of BGM (or DBGM): in the latter, **one- and *n*-particle distributions depend on the deformation parameters**

Combined account of two factors

Compositeness:

$$\varphi_{\tilde{\mu}}(\hat{N}) = (1+\tilde{\mu})\hat{N} - \tilde{\mu}\hat{N}^2 \qquad \tilde{\mu} = 1/m$$

Avancini,… A.G & Yu.Mishchenko

Particle-particle interactions:

q-deformation of Arik–Coon type $\qquad [N]_q = \frac{q^N-1}{q-1}$

Narayana Swami,… A.G & Yu.Mishchenko

Hybrid (combined) deformation:

$$\varphi_{\tilde{\mu},q}\left(z\frac{d}{dz}\right) \equiv \varphi_{\tilde{\mu}}(D_q) = (1+\tilde{\mu})D_q - \tilde{\mu}D_q^2, \quad D_q \equiv \left[z\frac{d}{dz}\right]_q$$

for this hybrid model, intercept of 2-particle correlations was obtained, it reads:

Model with joint account of the two factors

For this 2-param. deformed model, $\lambda^{(2)}$ was found in [**A.G., Yu.Mishchenko, Nuc.Phys.B 891,466 (2015)**]

$$\lambda^{(2)} = -1 + \frac{\varphi_{\tilde{\mu},q}(2)(z-q)(z-q^2)}{(z-[3]_q+\varphi_{\tilde{\mu},q}(2))^2(z-q^3)(z-q^4)}$$

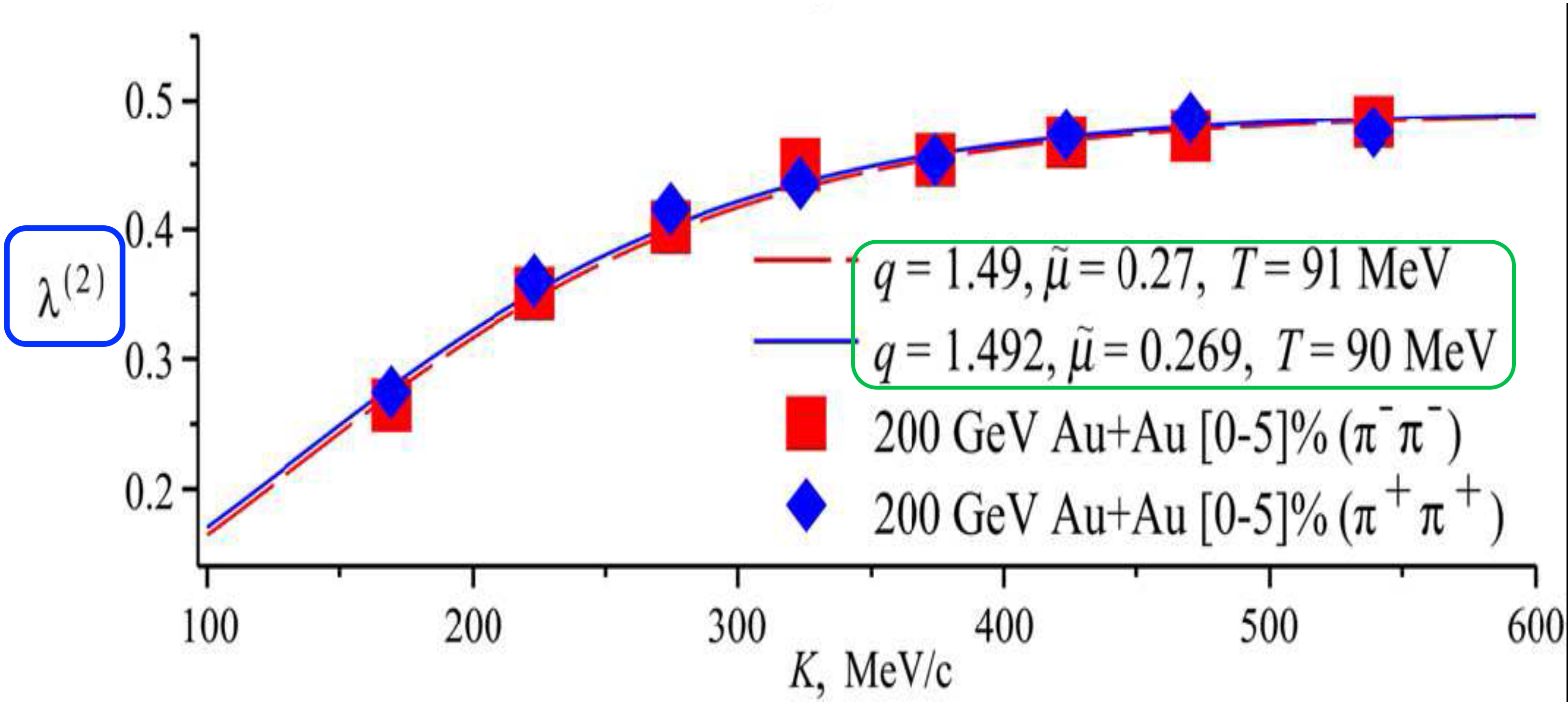
$$\cdot \left\{ (z-q^3)(z-q^4) + \left(\varphi_{\tilde{\mu},q}(3)-[3]_q\right)\left(z+q^2-\frac{q^2[4]_q}{\varphi_{\tilde{\mu},q}(2)}\right)\right\}$$

where $z = e^x,\ x = \beta\hbar\omega,\ \beta = (k_B T)^{-1}$

for this hybrid model, intercept of 2-particle correlations was obtained, namely:

RHIC/STAR data on two-pion correlations

Phys. Rev. C 71 (2005) 044906 Phys. Rev. C 80 (2009) 024905



Very nice agreement with data achieved in:

[**A.G., Yu.Mishchenko, Nuc.Phys.B 891,466 (2015)**]

Hence, the (not small) values of *q* and *μ* witness that <u>both compositeness and interactions do matter</u>!

μ-Bose gas model, *EXACT* results:

$$a^+a^- = \varphi_\mu(N),$$

Structure function

$$\langle a^+a\rangle = \langle [N]_\mu\rangle = \langle \frac{N}{1+\mu N}\rangle$$

$$\langle a^+a^+aa\rangle = \langle a^+[N]_\mu a\rangle = \langle a^+a[N-1]_\mu\rangle = \langle [N]_\mu[N-1]_\mu\rangle.$$

Intercept $\lambda_\mu^{(2)}$:

$$\lambda^{(2)}(K) = \frac{\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle^2} - 1 = \frac{\langle [N]_\mu[N-1]_\mu\rangle}{\langle [N]_\mu\rangle^2} - 1,$$

$$\lambda_\mu^{(2)} = \left\{X^{-1} - \left(\frac{1}{\mu}+\frac{1}{\mu^2}\right)\Phi(e^{-\beta},1,\mu^{-1}) - \left(\frac{1}{\mu}-\frac{1}{\mu^2}\right)\Phi(e^{-\beta},1,\mu^{-1}-1)\right\}\times$$

$$\times\left(X^{-1} - \mu^{-1}\Phi(e^{-\beta},1,\mu^{-1})\right)^{-2}X^{-1} - 1, \quad (1-e^{-\beta}) = X \qquad \textbf{(1)}$$

Here Φ is Lerch transcedent: $\Phi = \sum_{n=0}^{\infty} z^n/(n+\alpha)^s$

A.G., Yu.Mishchenko, *Phys.Lett. A* (2012)

μ-Bose gas model,

Intercept λ_μ(3):

Intercept of three-particle correlation function: $\lambda^{(3)}(K) = \frac{\langle a^\dagger a^\dagger a^\dagger aaa \rangle}{\langle a^\dagger a \rangle^3} - 1$

$$\lambda_\mu^{(3)} = X^{-2}\Big\{X^{-1} - \Big(\frac{1}{\mu} + \frac{3}{2\mu^2} + \frac{1}{2\mu^3}\Big)\underline{\Phi(e^{-\beta}, 1, \mu^{-1}})- \Big(\frac{1}{\mu} - \frac{1}{\mu^3}\Big)\underline{\Phi(e^{-\beta}, 1, \mu^{-1} - 1})-$$
$$- \Big(\frac{1}{\mu} - \frac{3}{2\mu^2} + \frac{1}{2\mu^3}\Big)\underline{\Phi(e^{-\beta}, 1, \mu^{-1} - 2})\Big\} \cdot \Big(X^{-1} - \mu^{-1}\underline{\Phi(e^{-\beta}, 1, \mu^{-1}})\Big)^{-3} - 1.$$

Here Φ is Lerch transcedent: $\Phi = \sum_{n=0}^{\infty} z^n/(n+\alpha)^s$

Expression for *r*-th order intercepts:

$$\underline{\lambda_\mu^{(r)}(k)} = \Big(1 + \mu^{-1}(1 - e^{-\beta\varepsilon}) \sum_{l=0}^{r-1} A_l^{(r)}(\mu)\underline{\Phi(e^{-\beta\varepsilon}, 1, \mu^{-1} - }l)\Big)$$
$$\cdot \Big(1 + \mu^{-1}(1 - e^{-\beta\varepsilon}) A_0^{(1)}(\mu)\underline{\Phi(e^{-\beta\varepsilon}, 1, \mu^{-1}})\Big)^{-r} - 1, \quad r = 2, 3, \ldots$$

A. G., Yu. Mishchenko, *Phys. Lett. A* (2012)

Thermodynamics of μ-Bose gas

(deformed) total number of particles:

$$N \equiv N^{(\mu)} = z\mathcal{D}_z^{(\mu)} \ln Z = -z\mathcal{D}_z^{(\mu)} \sum_i \ln(1 - ze^{-\beta\varepsilon_i}),$$

$$\mathcal{D}_x^{(\mu)} x^n = [n]_\mu x^{n-1},$$

$$\boxed{[n]_\mu = \frac{n}{1+\mu n}}$$

Deformed partition function:

$$\boxed{\ln Z^{(\mu)} = \left(z\frac{d}{dz}\right)^{-1} N^{(\mu)}.}$$

By this, all other thermodyn. functions can be obtained

$$\varphi_\mu(N) = \frac{N}{1+\mu N}$$

(correln.) $N \leftrightarrow z\frac{d}{dz}$ **(thermodyn.)**

[A.G., Kachurik, Rebesh' 2013-2014]

$\tilde{\mu}$-calculus, $\tilde{\mu}$ thermodyn. of

(deformed) total number of particles:

$$N \equiv N^{(\mu)} = z\mathcal{D}_z^{(\mu)} \ln Z = -z\mathcal{D}_z^{(\mu)} \sum_i \ln(1 - ze^{-\beta\varepsilon_i}),$$

$$= \frac{V}{\lambda^3} g_{\frac{3}{2}}^{(\mu)}(z) + g_0^{(\mu)}(z)$$

deformed partition function:

$$\ln Z^{(\mu)} = \left(z\frac{d}{dz}\right)^{-1} N^{(\mu)} = \frac{V}{\lambda^3} g_{\frac{5}{2}}^{(\mu)}(z) + g_1^{(\mu)}(z)$$

Here the thermal wavelength and μ-polylogarithm are:

$$\boxed{\lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}}$$

$$g_l^{(\mu)}(z) = \sum_{n=1}^{\infty} \frac{z^n}{(1+\mu n)n^l}$$

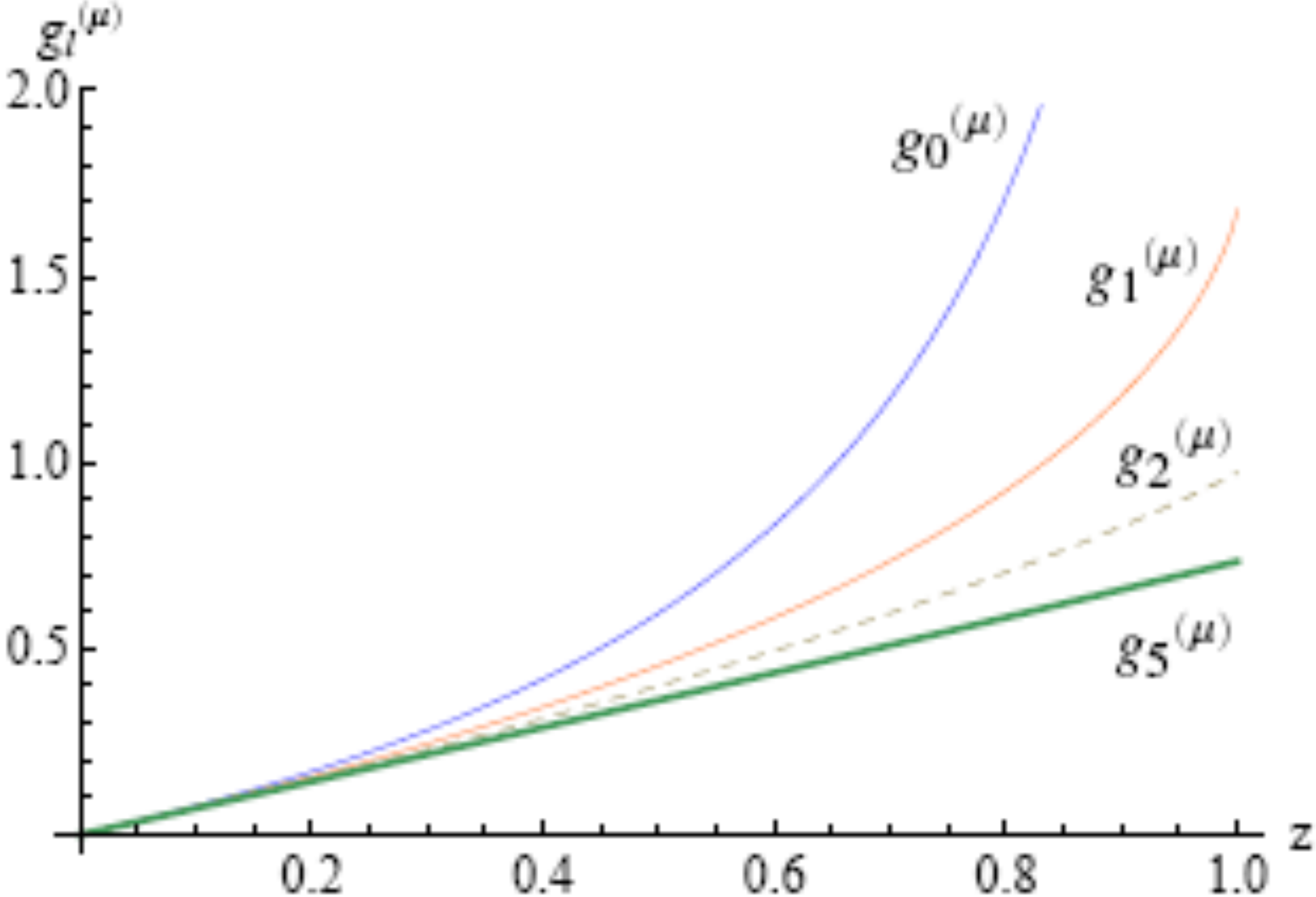


Fig. 4. Dependence of the functions g_0^{μ}, g_1^{μ}, g_2^{μ}, and g_5^{μ} on the $\boxed{\text{fugacity } z = \exp(\beta\tilde{\mu})}$ at $\mu = 0.4$

Thermodynamics: μ-Bose gas

- **Partition function (μ-deformed):** $Z^{(\mu)}(z,T,V) = \exp\left(\frac{V}{\lambda^3} g_{5/2}^{(\mu)}(z) + g_1^{(\mu)}(z)\right)$
- **Critical temperature (μ-depend.):** $$\frac{T_c^{(\mu)}}{T_c} = \left(\frac{2.61}{g_{3/2}^{(\mu)}(1)}\right)^{2/3}.$$

Entropy-per-volume versus deform. parameter μ

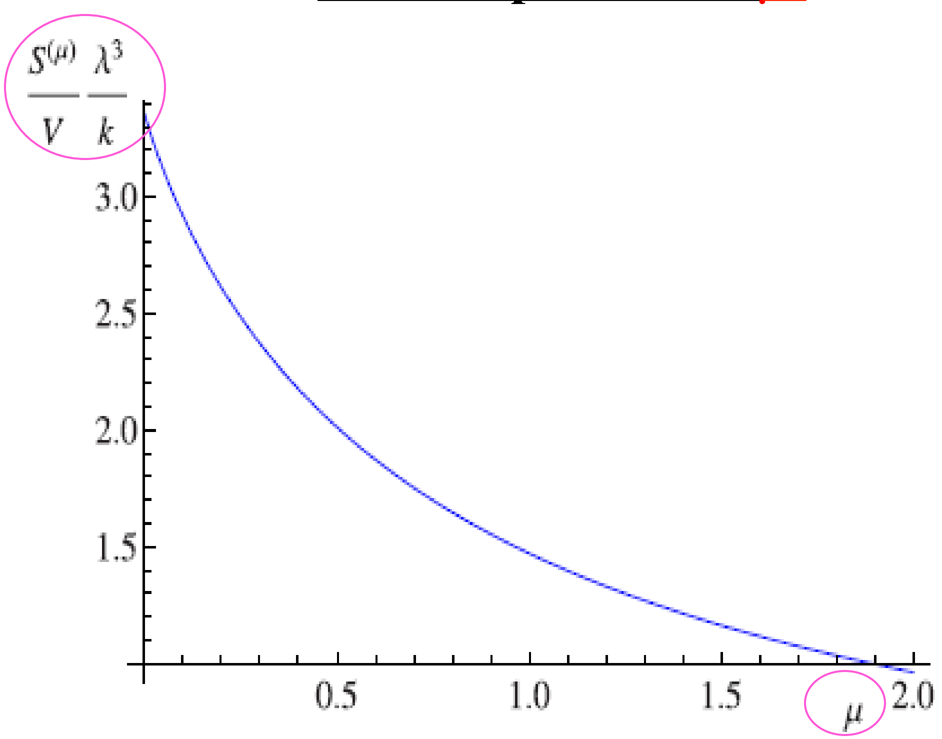
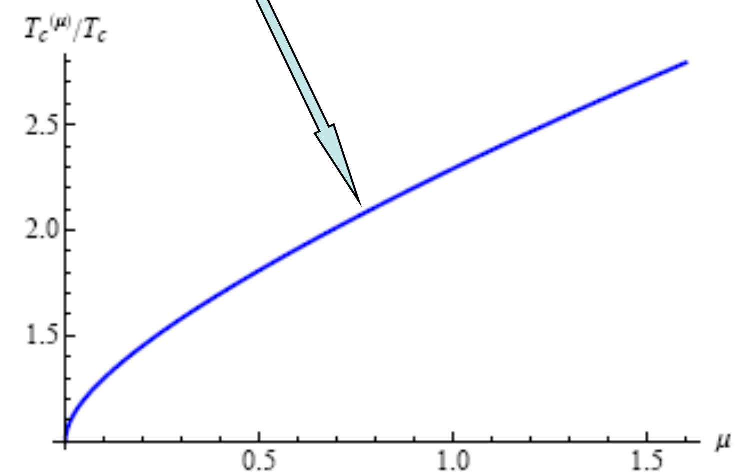


Fig. 2: Ratio $\frac{S^{(\mu)}}{V}$ (times $\frac{\lambda^3}{k}$) versus deformation parameter μ.



Dependence of the ratio $T_c^{(\mu)}/T_c$ on the μ-parameter

NB: For greater μ (stronger deform.)

→ **higher T_c, and lesser entropy!**

<u>Thermodynamical geometry</u>

In 2013, "***<u>Infinite statistics condensate as a model of dark matter</u>***" **<u>haz been proposed. But, infinite statistics is only one, rather exotic example of nonstandard statistics.</u>**

<u>Diverse DBGMs (earlier known, or developed by us) may, in principle, as well serve for such a modeling and thus are worth of being studied.</u>

[AG., Kachurik, Khelashvili and Nazarenko] in: arXiv:1709.05931, arXiv:1805.02504, and Physica A 506 (2018) explored TG of μ-Bose gas model and confirmed Bose-like condensation. This, and other results on thermodynamics of the μ-BGM *allowed to propose the model for effective modeling of dark matter*.

Thermodynamical geometry of μ-Bose gas model in 2-dim. space with coordinates $\boxed{\beta,\ \gamma = -\mu\beta}$

$$G_{\beta\beta} = -\left(\frac{\partial U}{\partial \beta}\right)_{\gamma} = \frac{15}{4}\frac{V}{\lambda^3\beta^2} g_{\frac{5}{2}}^{(\mu)}(z)$$

$$G_{\beta\gamma} = -\left(\frac{\partial N}{\partial \beta}\right)_{\gamma} = \frac{3}{2}\frac{V}{\lambda^3\beta} g_{\frac{3}{2}}^{(\mu)}(z)$$

$$G_{\gamma\gamma} = -\left(\frac{\partial N}{\partial \gamma}\right)_{\beta} = \frac{V}{\lambda^3} g_{\frac{1}{2}}^{(\mu)}(z) + g_{-1}^{(\mu)}(z)$$

$$\boxed{\Gamma_{\lambda\mu\nu} = \frac{1}{2}(\ln Z)_{,\lambda\mu\nu}}$$

$$\Gamma_{\gamma\beta\beta} = -\frac{15}{8}\frac{V}{\lambda^3\beta^2} g_{\frac{3}{2}}^{(\mu)}(z)$$

$$det|G_{ij}| \equiv g = \frac{3}{4}\frac{V}{\lambda^3\beta^2}\left(5g_{\frac{5}{2}}^{(\mu)}(z)g_{-1}^{(\mu)}(z) + \frac{V}{\lambda^3}\left(5g_{\frac{5}{2}}^{(\mu)}(z)\, g_{\frac{1}{2}}^{(\mu)}(z) - 3g_{\frac{3}{2}}^{(\mu)}(z)\, g_{\frac{3}{2}}^{(\mu)}(z)\right)\right)$$

Recall that μ-polylogarithm is:

$$\boxed{g_l^{(\mu)}(z) = \sum_{n=1}^{\infty} \frac{z^n}{(1+\mu n)n^l}}$$

BEC models of Dark Matter (DM)

S.J. Sin, Phys. Rev. D **50** (1994) 3650

C.G. Boehmer and T. Harko, JCAP **0706** (2007) 025.

core–cusp problem, settled in:

T. Harko, JCAP **05** (2011) 022.

A review on BEC DM models:

A. Suarez, V.H. Robles and T. Matos, Astrophysics and Space Science Proceedings, vol. 38, Springer, 2013, p. 107.

S.F. Guzman et al., JCAP **09** (2013) 034.

(gravit. collapse problem)

In BEC models of Dark Matter:

$$R = \pi\sqrt{\frac{\hbar^2 a}{Gm^3}}, \qquad M = \frac{4}{\pi}R^3\rho^{(c)};$$

In our μ–Bose gas model, at:

$\mu_0 \simeq 1.895$, we obtain the relation

$$g_{3/2}(1) = 3.3535\ g_{3/2}^{(\mu=\mu_0)}(1)$$

Due to this,

the (critical) volume-per-particle in the case of μ-deformed thermodynamics: $v = \lambda^3/g^{(\mu)}_{3/2}(1)$. It means that $\rho^{(c)}_{(\mu)} = \Big(g^{(\mu)}_{3/2}(1)/g^{(0)}_{3/2}(1)\Big)\rho^{(c)}$ and therefore

$M^{(\mu)} = \Big(g^{(\mu)}_{3/2}(1)/g^{(0)}_{3/2}(1)\Big)M$ will play the role of new corrected characteristics. Since $g^{(\mu)}_{3/2}(1) < g^{(0)}_{3/2}(1)$ at $\mu > 0$, these predictions can give instead of $M^{BEC} \equiv M^{(0)}$ a better agreement with the observational data.

Rotational curves in μ-deformed approach

Use the μ-derivative (via μ-bracket)

$$\mathcal{D}_x^{(\mu)} \equiv \boxed{\left[\frac{d}{dx}\right]_\mu = \frac{\frac{d}{dx}}{1+\mu\frac{d}{dx}}} = \frac{d}{dx}\left(1 - \mu\frac{d}{dx} + \mu^2\frac{d}{dx}\frac{d}{dx} - \dots\right)$$

to construct μ-deformed Lane-Emden equation:

$$\boxed{\frac{1}{r^2}\mathcal{D}_r^\mu\left(r^2\mathcal{D}_r^\mu\rho(r)\right) + k^2\rho(r) = 0} \qquad \rho(0) = \rho_c, \ \rho'(0) = 0$$

Then its solution is:

$$\rho(kr) = \rho_c\,[1]_\mu\,\frac{\sin_\mu(kr)}{kr}.$$

[Note that ~~while~~ $\frac{d}{dx}\sin_\mu(x) \neq \cos_\mu(x)$

$$\mathcal{D}_x^\mu \sin_\mu(x) = \cos_\mu(x)$$

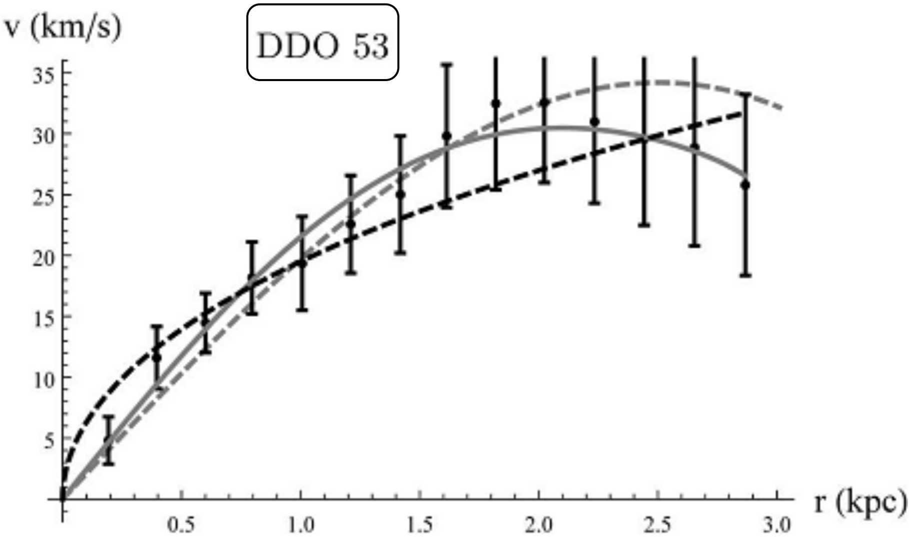
Here

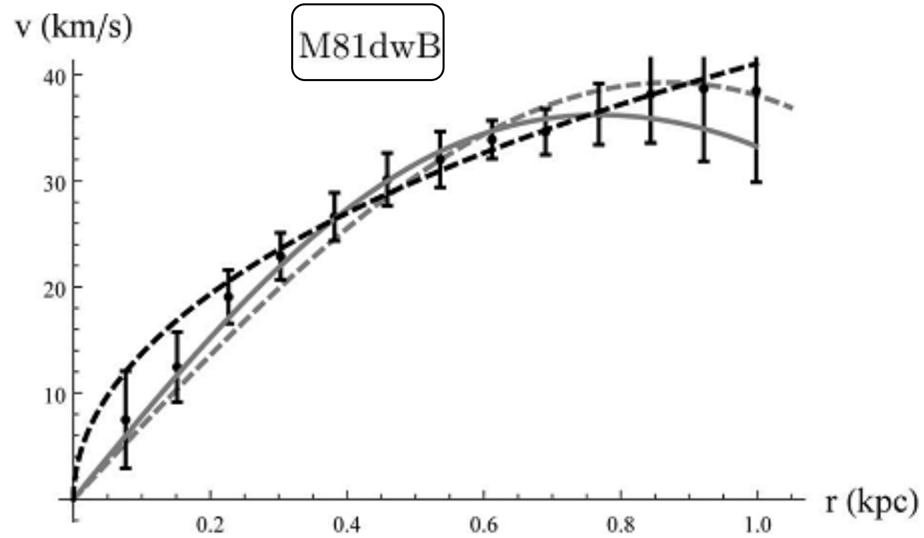
$$\boxed{\sin_\mu x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{[2n]_\mu!}}.$$

$x = kr$

From density profile $\rho(kr)$ we obtain rotation curves

Rotational curves in μ-deformed approach:





	A, km/s	k, kpc^{-1}	μ	χ^2
DDO 53	32.27	0.97	0.180	1.9
M81dwB	38.35	2.64	0.180	2.5

deform. param.

Our curves (solid) agree with data better than NFW, and BECDM (Harko)

A.G., I.Kachurik. M.Khelashvili,
Ukr. J. Phys. Vol.64 (11) (2019),
arXiv: 1910.10796

New deformed Heisenberg algebra

- In the usual BEC model (Harko), the two shapes of LEE (Lane-Emden equation): based on Laplace-Beltrami, or ($2^{nd}+1^{st}$)-order derivatives, are equivalent ⟹ i.e. possess the same solution
- If μ-deformed analogs of Lane-Emden equation are built, the situation is different: to get equivalence, coefficients in the latter form should be replaced by certain functions of r and def. parameter μ
- From their equivalence – derive new μ-deformed analog of Heisenberg algebra

(**main implications – both minimal and maximal uncertainties of position and momentum**)

Two versions of μ-deformed Lane-Emden eqn.

- $$\frac{1}{r^2}\mathcal{D}_r^{\mu}\left(r^2\mathcal{D}_r^{\mu}\rho(r)\right)+k^2\rho(r)=0$$
- $$\left(D_r^{(\mu)}D_r^{(\mu)}+\boxed{g_\mu(r)}\frac{2}{r}D_r^{(\mu)}+\boxed{h_\mu(r)}k^2\right)\rho(r)=0$$

The two eqns.are *equivalent* -- *common* solution is *sin*$_\mu$(kr)/kr

$$\boxed{g_\mu(r)}=\frac{1}{1-2\mu}\left(1-\frac{1-\mu}{1+\mu}\mu^2k^2r^2\right),$$

$$\boxed{h_\mu(r)}=\frac{1+2\mu}{1-2\mu}-2\mu^2\frac{1-\mu^2k^2r^2}{(1+\mu)(1-2\mu)}$$

$g_\mu(r)\longrightarrow 1$, $h_\mu(r)\longrightarrow 1$ if $\mu\rightarrow 0$

From equivalence → new μ-deformed Heisenberg algebra

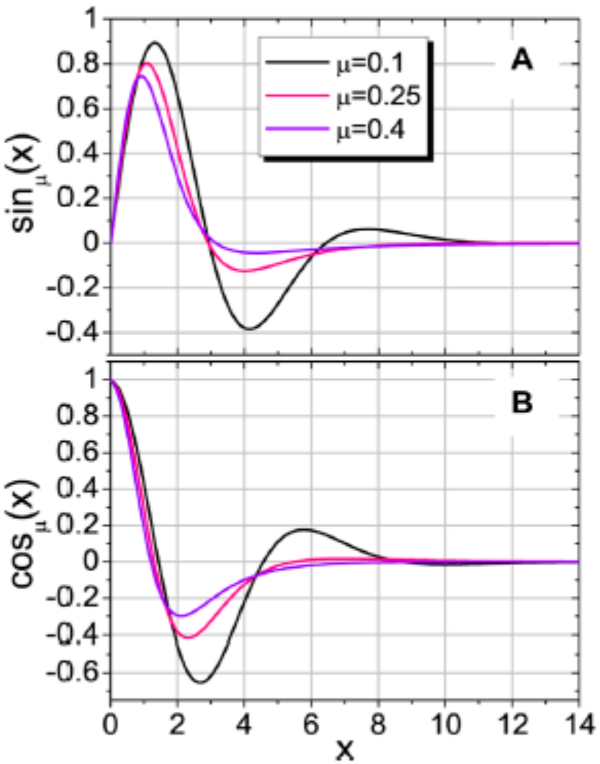
$$\sigma(x)\,x\,D_x^{(\mu)}-D_x^{(\mu)}\,x=-\lambda(x),$$

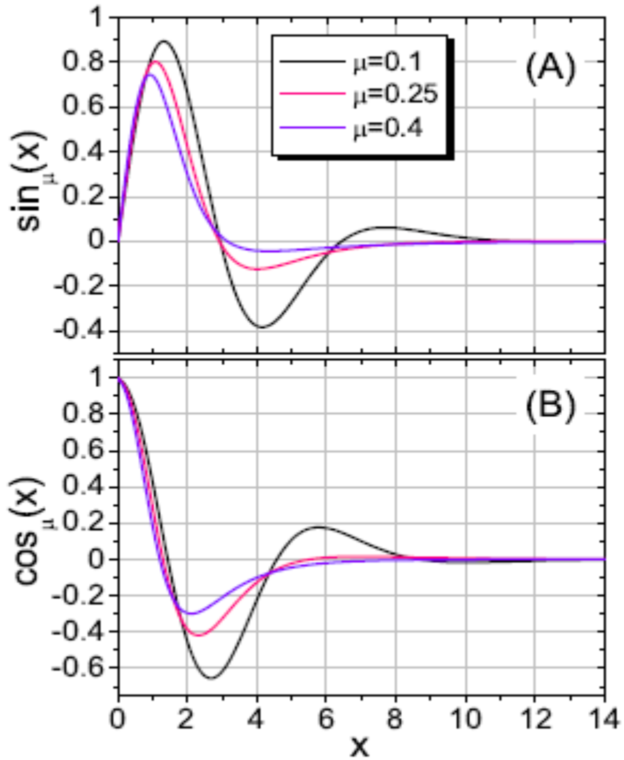
$$\sigma(x)=\frac{1}{\sqrt{\boxed{h_\mu}}}=\left[\frac{(1-2\mu)(1+\mu)}{1+\mu(3+2\mu^3x^2)}\right]^{1/2},$$

$$\lambda(x)=\frac{2\boxed{g_\mu}}{(1+\sigma)\boxed{h_\mu}}=\frac{1+\mu-(1-\mu)\mu^2x^2}{\mu[2+\mu(1+\mu^2x^2)]}(1-\sigma)$$

Deformed sine, cosine, and *μ*-spherical functions

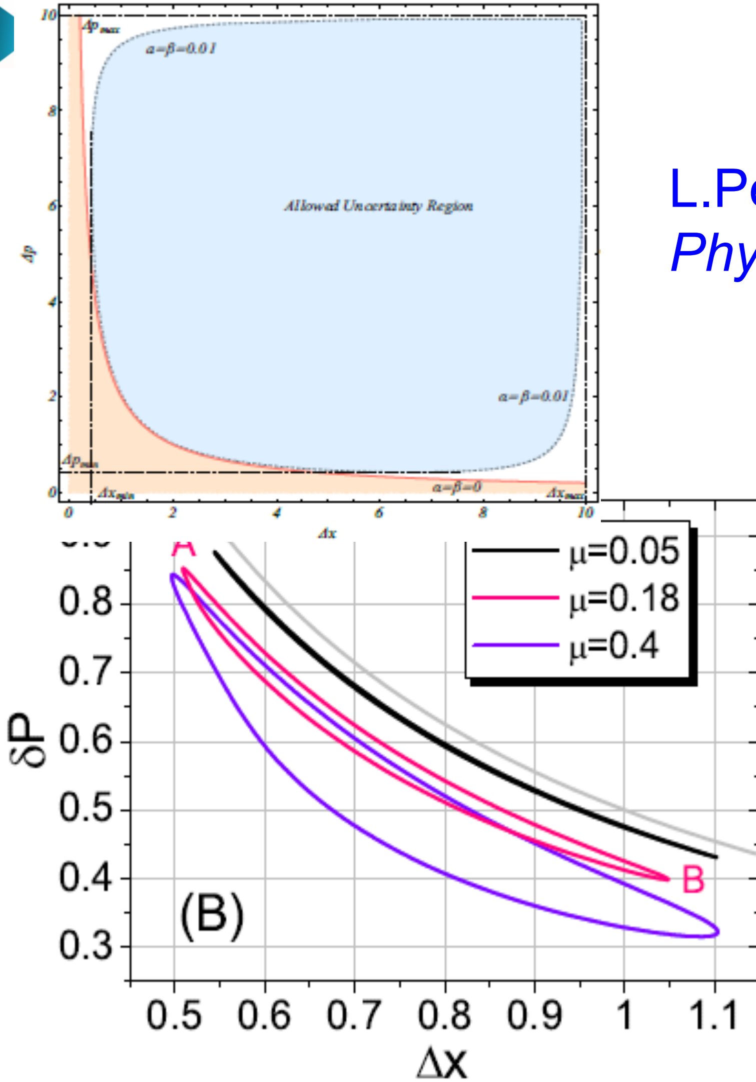
$$\sin_\mu x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]_\mu!}, \quad \cos_\mu x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]_\mu!}$$



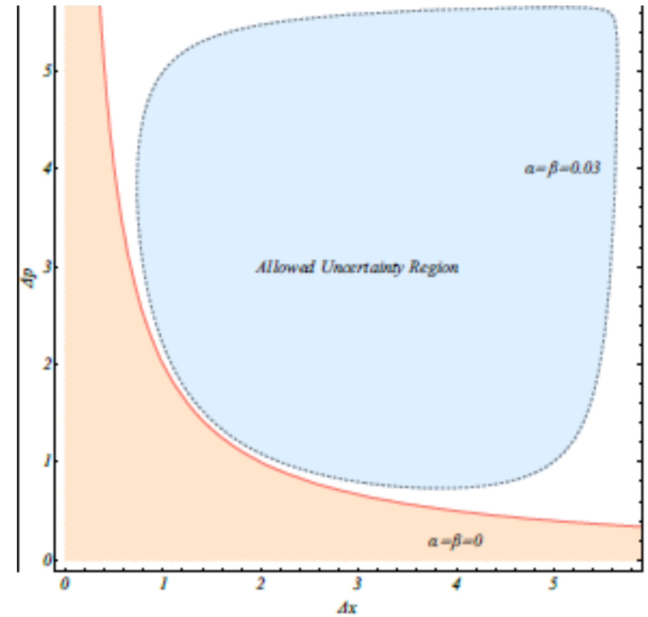


A.G., I. Kachurik, A.Nazarenko,
***Frontiers in Astronomy and Space Sciences* (2023)**

$$\boxed{\sigma(x)\, x\, \hat{P} - \hat{P}\, x = \mathrm{i}\lambda(x)}$$



L.Perivolaropoulos,
Phys. Rev. D (2017)



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Frontiers … (2023)

Minimal and maximal position/momentum uncertainties from μ-HA

TABLE 2 The parameters for the dark matter halos of dwarf galaxies.

Galaxy	μ	k, kpc^{-1}	$(\Delta r)_{max}$, kpc	$(\Delta r)_{min}$, kpc	$(\Delta P_r)_{max}$, $10^{-27} eV/c$	$(\Delta P_r)_{min}$, $10^{-27} eV/c$
M81dwB	0.18	2.64	0.398	0.193	14.38	6.75
DDO 53	0.18	0.97	1.082	0.526	5.28	2.48
IC 2574	0.179	0.17	6.18	3.0	0.926	0.435
NGC 2366	0.178	0.37	2.84	1.38	2.02	0.946
HO I	0.151	1.27	0.830	0.402	6.98	3.33

While Perivolaropoulos relates *max.position uncertainty* with *cosmological horizon*, in our case $(\Delta r)_{max}$, as seen from table, *refers to galactic scales*

Remark: we dealt with various q- or μ-deformed functions, e.g.

- q-Polynomials (related to Alexander polynomials (= torus knot invariants)
- q-deformed Gegenbauer polynomials
- Lerch transcendents (in correlation functions of μ-bosons, and in thermodynamical geometry of infinite statistics gas)
- μ-deformations, e.g., μ-sin, μ-cos, μ-Bessel, μ-(poly)logarythms

for which $\frac{d}{dx}\sin_\mu(x) \neq \cos_\mu(x) \Longrightarrow \mathcal{D}_x^\mu \sin_\mu(x) = \cos_\mu(x)$

where $$\mathcal{D}_x^{(\mu)} \equiv \left[\frac{d}{dx}\right]_\mu = \frac{\frac{d}{dx}}{1+\mu\frac{d}{dx}} = \frac{d}{dx}\left(1-\mu\frac{d}{dx}+\mu^2\frac{d}{dx}\frac{d}{dx}-\ldots\right)$$

Thanks for your attention!