Algebraic structures in the theory of integrable systems and their physical applications

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Plan of the talk

1. Algebraic structures in the theory of classical integrable systems: two types of structures
2. Algebraic structures in the theory of quantum integrable systems: two types of structures
3. Physical applications: integrable models with nearest-neighbours interactions
4. Physical applications: integrable spin models with long-range spin-spin interactions
5. Physical applications: integrable fermion models of nuclear physics
6. Physical applications: integrable spin-boson models of quantum optics

I. Algebraic structures in the classical theory

I.A. Hamiltonian systems and Lax representation

Assume that Hamiltonian equations on a Poisson manyfold $(\mathcal{P}, \{\ ,\ \})$ with a Hamiltonian H are written in the Lax form:

$$\frac{dL(u)}{dt} = [L(u), M_H(u)], \qquad (1)$$

where $L(u)$ and $M_H(u)$ are some matrices depending on the initial dynamical variables — local coordinates on the space $\mathcal{P}$ — and the auxiliary complex parameter u. The Lax matrix $L(u)$ takes values in a finite-dimensional Lie algebra $\mathfrak{g}$.
The Lax representation (1) provides generating functions of the first integrals of the corresponding Hamiltonian equations, which may be chosen to be the traces of its powers

$$I_k(u) = \mathrm{tr}\, L(u)^k, k \in \overline{1, n}. \qquad (2)$$

I.B. Classical r-matrices and linear tensor structure

In order to guarantee the Poisson-commutativity of the above generating functions needed for the Liouville integrability of the corresponding hamiltonian system we will assume that the initial Poisson brackets are re-written on the level of the Lax matrices in the form of the so-called linear tensor brackets:

$$\{L_1(u), L_2(v)\}_1 = [r_{12}(u,v), L_1(u)] - [r_{21}(v,u), L_2(v)], \quad (3)$$

where $L_1(u) = L(u) \otimes 1$, $L_2(v) = 1 \otimes L(v)$ and the function of two complex variables

$$r(u,v) = \sum_{a,b=1}^{\dim\mathfrak{g}} r_{ab}(u,v) X_a \otimes X_b \quad (4)$$

with values in $\mathfrak{g} \otimes \mathfrak{g}$ is called classical r-matrix.

The Poisson bracket (3) guarantee the commutativity of (2):

$$\{I_k(u), I_l(v)\}_1 = 0. \quad (5)$$

The generalized classical Yang-Baxter equation:

$$[r_{12}(u,v), r_{13}(u,w)] - [r_{23}(v,w), r_{12}(u,v)] + \\ + [r_{32}(w,v), r_{13}(u,w)] = 0 \quad (6)$$

provides the Jacobi condition for the brackets (3).
We will consider r-matrices possessing the decomposition:

$$r(u,v) = \frac{\Omega}{u-v} + r_0(u,v), \quad (7)$$

where $\Omega = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} X_\alpha \otimes X_\beta$ and $r_0(u,v)$ is a regular on the diagonal $u = v$ function with values in $\mathfrak{g} \otimes \mathfrak{g}$.
Example. The simplest possible r-matrix has the form:

$$r(u,v) = \frac{\Omega}{u-v}. \quad (8)$$

I.C. Classical r-matrices and quadratic tensor structures

In the case of skew-symmetric r-matrices, i.e. when

$$r_{12}(u_1, u_2) = -r_{21}(u_2, u_1),$$

the generalized classical Yang-Baxter equation reduces to the usual classical Yang-Baxter equation [Sklyanin 1979]:

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2) + r_{13}(u_1, u_3)], \tag{9}$$

solutions of which have been classified [Belavin, Drinfeld 1982]. In this case are defined also quadratic brackets [Sklyanin 1979]:

$$\{L_1(u), L_2(v)\}_2 = [r_{12}(u, v), L_1(u)L_2(v)], \tag{10}$$

which also guaranty the Poisson-commutativity of (2):

$$\{I_k(u), I_l(v)\}_2 = 0. \tag{11}$$

Let $r(u-v)$ be a skew-symmetric classical r-matrix. Let σ be an automorphism of $\mathfrak{g}$ of second order, such that

$$(\sigma \otimes \sigma) \cdot r_{12}(u-v) = r_{12}(u-v).$$

In this case one can define the following quadratic brackets (Sklyanin 1988):

$$\begin{aligned} \{L_1(u_1), L_2(u_2)\}_2 = [r_{12}(u_1-u_2), L_1(u_1)L_2(u_2)] + \\ + L_1(u_1)s_{12}(u_1+u_2)L_2(u_2) - L_2(u_2)s_{12}(u_1+u_2)L_(u_1), \end{aligned} \quad (12)$$

where

$$s_{12}(u_1+u_2) \equiv 1 \otimes \sigma \cdot r_{12}(u_1+u_2).$$

It also guaranties the Poisson-commutativity of (2):

$$\{I_k(u), I_l(v)\}_2 = 0 \quad (13)$$

and possess one more generalization.

Let us consider four tensor a, b, c, d satisfying equations:

$$[a_{12}(u_1, u_2), a_{13}(u_1, u_3)] = [a_{23}(u_2, u_3), a_{12}(u_1, u_2) + a_{13}(u_1, u_3)],$$

$$[d_{12}(u_1, u_2), d_{13}(u_1, u_3)] = [d_{23}(u_2, u_3), d_{12}(u_1, u_2) + d_{13}(u_1, u_3)],$$

$$[a_{12}(u_1, u_2), c_{13}(u_1, u_3)] = [c_{23}(u_2, u_3), a_{12}(u_1, u_2) + c_{13}(u_1, u_3)],$$

$$[d_{12}(u_1, u_2), b_{13}(u_1, u_3)] = [b_{23}(u_2, u_3), d_{12}(u_1, u_2) + b_{13}(u_1, u_3)].$$

where tensors a, d are skew-symmetric,

$$c_{12}(u_1, u_2) = b_{21}(u_2, u_1).$$

Then it is possible to define the Poisson brackets (Maillet):

$$\{L_1(u_1), L_2(u_2)\} = a_{12}(u_1, u_2) L_1(u_1) L_2(u_2) + L_1(u_1) b_{12}(u_1, u_2) \times L_2(u_2) - L_2(u_2) c_{12}(u_1, u_2) L_1(u_1) - L_1(u_1) L_2(u_2) d_{12}(u_1, u_2), \tag{14}$$

This is the most general form of quadratic brackets satisfying (13) under condition:

$$a_{12}(u_1, u_2) + b_{12}(u_1, u_2) = c_{12}(u_1, u_2) + d_{12}(u_1, u_2). \tag{15}$$

II. Algebraic structures in quantum theory

II.A. Classical r-matrices and linear quantum structures

In quantum case the Lax matrix $L(u)$ is replaced by the quantum Lax matrix: $L(u) \to \hat{L}(u)$ which reflects the fact that the basic dynamical variables — coordinates on $\mathcal{P}$ — are now quantum operators.

The quantisation of the linear brackets (3) is achieved by substitution of the Poisson brackets by commutator:

$$[\hat{L}_1(u), \hat{L}_2(v)] = i\hbar\big([r_{12}(u,v), \hat{L}_1(u)] - [r_{21}(v,u), \hat{L}_2(v)]\big). \tag{16}$$

Finding commutative quantum analogs of all $\operatorname{tr} L(u)^k, k \in \overline{1,n}$ is an open problem. Nevertheless it is possible to show that for small grades we indeed have the needed commutativity:

$$[\operatorname{tr}(\hat{L}(u)^k), \operatorname{tr}(\hat{L}(v)^l)] = 0, \qquad k, l \in 1, 2. \tag{17}$$

II.B. Quantum R-matrices and quadratic algebras

In the case of the quadratic algebras the quantisation problem is more complicated. After replacement of the Lax matrix $L(u)$ by the quantum Lax matrix $L(u) \to \hat{L}(u)$ the quantum analog of the Poisson relations (10) are the following (Faddeev et all):

$$R_{12}(u,v)\hat{L}_1(u)\hat{L}_2(v) = \hat{L}_2(v)\hat{L}_1(u)R_{12}(u,v), \tag{18}$$

where the quantum R-matrix $R^{12}(u,v)$ satisfies quantum Yang-Baxter equation:

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2) \tag{19}$$

and possess the following quasi-classical expansion:

$$R_{12}(u_1,u_2) = 1 \otimes 1 + i\hbar r_{12}(u_1,u_2) + o(\hbar^2), \tag{20}$$

The algebra (18) provides commutativity of the quantum analogs of $\mathrm{tr}\, L(u)^k, k \in \overline{1,n}$.

After replacement of the Lax matrix $L(u)$ by the quantum Lax matrix $L(u) \to \hat{L}(u)$ the quantum analogs of the relations (12) are the following ones (Sklyanin 1989):

$$R_{12}(u,v)\hat{L}_1(u)S_{12}(u,v)\hat{L}_2(v) = \hat{L}_2(v)S_{12}(u,v)\hat{L}_1(u)R_{12}(u,v), \tag{21}$$

where the quantum R-S matrices satisfy the equations:

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2),$$

$$R_{12}(u_1,u_2)S_{13}(u_1,u_3)S_{23}(u_2,u_3) = S_{23}(u_2,u_3)S_{13}(u_1,u_3)R_{12}(u_1,u_2),$$

and possess the following quasi-classical expansions:

$$R_{12}(u_1,u_2) = 1 \otimes 1 + i\hbar r^{12}(u_1,u_2) + o(\hbar^2), \tag{22}$$

$$S_{12}(u_1,u_2) = 1 \otimes 1 + i\hbar s^{12}(u_1,u_2) + o(\hbar^2), \tag{23}$$

The algebra (21) also provides commutativity of the quantum analogs of $\mathrm{tr}\, L(u)^k, k \in \overline{1,n}$.

After replacement of the Lax matrix $L(u)$ by the quantum Lax matrix $L(u) \to \hat{L}(u)$ the quantum analogs of the relations (14) are the following ones (Maillet 1991):

$$A_{12}(u,v)\hat{L}_1(u)B_{12}(u,v)\hat{L}_2(v) = \hat{L}_2(v)C_{12}(u,v)\hat{L}_1(u)D_{12}(u,v), \tag{24}$$

where the quantum A,B, C, D matrices satisfy the equations:

$$A_{12}(u_1,u_2)A_{13}(u_1,u_3)A_{23}(u_2,u_3) = A_{23}(u_2,u_3)A_{13}(u_1,u_3)A_{12}(u_1,u_2),$$

$$A_{12}(u_1,u_2)C_{13}(u_1,u_3)C_{23}(u_2,u_3) = C_{23}(u_2,u_3)C_{13}(u_1,u_3)A_{12}(u_1,u_2),$$

$$D_{12}(u_1,u_2)D_{13}(u_1,u_3)D_{23}(u_2,u_3) = D_{23}(u_2,u_3)D_{13}(u_1,u_3)D_{12}(u_1,u_2)$$

$$D_{12}(u_1,u_2)B_{13}(u_1,u_3)B_{23}(u_2,u_3) = B_{23}(u_2,u_3)B_{13}(u_1,u_3)D_{12}(u_1,u_2),$$

possess the following quasi-classical expansions:

$$A_{12}(u_1,u_2) = 1\otimes 1 + i\hbar a_{12}(u_1,u_2) + o(\hbar^2),$$

$$B_{12}(u_1,u_2) = 1\otimes 1 + i\hbar b_{12}(u_1,u_2) + o(\hbar^2),$$

$$C_{12}(u_1,u_2) = 1\otimes 1 + i\hbar c_{12}(u_1,u_2) + o(\hbar^2),$$

$$D_{12}(u_1,u_2) = 1\otimes 1 + i\hbar d_{12}(u_1,u_2) + o(\hbar^2).$$

and the following properties:

$$B_{12}(u_1,u_2) = C_{21}(u_2,u_1).$$

It is also assumed that the unit matrix satisfies (24), i.e. *ABCD*-matrices are connected among themselves as follows:

$$A_{12}(u_1,u_2)B_{12}(u_1,u_2) = C_{12}(u_1,u_2)D_{12}(u_1,u_2).$$

The algebra (24) also provides commutativity of the quantum analogs of $\mathrm{tr}\, L(u)^k, k\in\overline{1,n}$.

Quadratic structures and Heisenberg models

The famous Heisenberg Hamiltonian is the Hamiltonian with the nearest neighbours interaction:

$$\hat{H} = \sum_{\alpha=1}^{3} \sum_{k=1}^{N} J_\alpha \hat{S}_\alpha^{(k)} \hat{S}_\alpha^{(k+1)}, \qquad (25)$$

The Hamiltonian (25) is integrable if all quantum spin operators $\hat{S}_\alpha^{(l)}, \alpha \in \overline{1,3},\ l \in \overline{1,N}$ act in a representation of $so(3)^{\oplus N}$ with all spins being equal to one-half.

The integrability of the Hamiltonian (25) is based on the theory of quantum algebras. In particular the Hamiltonian (25) is connected with the algebra (18) and quantum elliptic R-matrix of Baxter and J_α are expressed via the branching points of the elliptic curve.

Other quadratic quantum structures e.g. (21) lead to the additional boundary terms in the hamiltonian (25).

Linear structures and Gaudin-type models

Let $\hat{S}_a^{(l)}, a \in \overline{1, \dim\mathfrak{g}},\ l \in \overline{1, N}$ be quantum operators that constitute a representation of the Lie algebra $\mathfrak{g}^{\oplus N}$, i.e.:

$$[\hat{S}_a^{(l)}, \hat{S}_b^{(k)}] = \delta^{kl} \sum_{c=1}^{\dim\mathfrak{g}} C_{ab}^c \hat{S}_c^{(k)}.$$

Let $\nu_k,\ \nu_k \neq \nu_l,\ k, l \in 1, ..., N$ be some fixed points in the complex plane belonging to the open region $\mathcal{U}$ in which the r-matrix $r(u, v)$ possesses the decomposition (7).

Let $c(u)$ be a "constant Lax matrix" solving the equation:

$$[r_{12}(u, v), c_1(u)] - [r_{21}(v, u), c_2(v)] = 0. \qquad (26)$$

Let us define the following Lax matrix:

$$\hat{L}(u) = \sum_{k=1}^{N} \sum_{a,b=1}^{\dim\mathfrak{g}} r^{ab}(\nu_k, u) \hat{S}_a^{(k)} X_b + \sum_{a=1}^{\dim\mathfrak{g}} c^a(u) X_a.$$

Then the operators $\hat{H}_l$

$$\hat{H}_l = \frac{1}{2} res_{u=\nu_l} \mathrm{tr}(\hat{L}(u)^2).$$

of the following explicit form:

$$\hat{H}_l = \sum_{a,b=1}^{\dim\mathfrak{g}} \sum_{k=1,k\neq l}^{N} r^{ab}(\nu_k, \nu_l) \hat{S}_a^{(k)} \hat{S}_b^{(l)} +$$

$$+ \sum_{a,b=1}^{\dim\mathfrak{g}} \frac{r_0^{ab}(\nu_l, \nu_l)}{2} (\hat{S}_a^{(l)} \hat{S}_b^{(l)} + \hat{S}_b^{(l)} \hat{S}_a^{(l)}) + \sum_{a=1}^{\dim\mathfrak{g}} c^a(\nu_l) \hat{S}_a^{(l)}. \quad (27)$$

constitute an abelian (commutative) algebra (Skrypnyk 2006). In the skew-symmetric case they coincide with the usual Gaudin hamiltonians in an external field (Gaudin 1976):

$$\hat{H}_l = \sum_{a,b=1}^{\dim\mathfrak{g}} \sum_{k=1,k\neq l}^{N} r^{ab}(\nu_k, \nu_l) \hat{S}_a^{(k)} \hat{S}_b^{(l)} + \sum_{a=1}^{\dim\mathfrak{g}} c^a(\nu_l) \hat{S}_a^{(l)}. \quad (28)$$

Example 1: Standard rational Gaudin hamiltonians

The rational Gaudin hamiltonians in an external magnetic field are obtained by a specification of the formula (28) for the case of the rational r-matrix:

$$\hat{H}_l = \sum_{k=1, k\neq l}^{N} \frac{\sum_{a,b=1}^{\dim\mathfrak{g}} g^{ab}\hat{S}_a^{(k)}\hat{S}_b^{(l)}}{(\nu_l - \nu_k)} + \sum_{a=1}^{\dim\mathfrak{g}} k^a \hat{S}_a^{(l)}, \qquad (29)$$

here k^a are the components of an external field $K \in \mathfrak{g}$ and g^{ab} are the components of the invariant bilinear form.

Let us assume that K belongs to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. It has a reductive centralizer $\mathfrak{g}_0^K$ generated by the elements

$$\hat{M}_a = \sum_{k=1}^{N} \hat{S}_a^{(k)}, \qquad (30)$$

where $X_a \in \mathfrak{g}_0^K$. The Casimir element $\hat{C}_{\mathfrak{g}_0^K}$ of $\mathfrak{g}_0^K$ commute with all $\hat{H}_l$. Besides $\hat{C}_{\mathfrak{g}_0^K}$ and $\hat{H}_l$ commute with any integral belonging to the "global" Cartan subalgebra, in particular with

$$\hat{h}_{\delta_K} = \sum_{l=1}^{N} \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \delta_K(H_i)\hat{S}_i^{(l)}, \text{ where } \delta_K \equiv \sum_{\alpha \in (\Delta/\Delta_K)_+} \alpha.$$

The linear combination of these hamiltonians:

$$\hat{H}^s_{gBCS} = \sum_{l=1}^{N} \nu_l \hat{H}_l + \frac{1}{2}\sum_{l=1}^{N} \hat{C}_l - \frac{1}{2}\hat{C}_{\mathfrak{g}_0^K} - \frac{1}{2}\hat{h}_{\delta_K}, \qquad (31)$$

in terms of the root basis it is written as follows:

$$\hat{H}^s_{gBCS} = \sum_{l=1}^{N} \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \nu_l k_i \hat{S}_i^{(l)} + \sum_{k,l=1}^{N} \sum_{\alpha \in (\Delta/\Delta_K)_+} \hat{S}^{(l)}_{-\alpha}\hat{S}^{(k)}_{\alpha}. \qquad (32)$$

This is spin Hamiltonian that will be used for the construction of the integrable fermion models of the s-type.

Example 2: Z_2-graded r-matrices

Let σ be an automorphism of $\mathfrak{g}$ of a second order. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ be the corresponding Z_2-grading of $\mathfrak{g}$, such that

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0^K,\ \mathfrak{g}_{\bar{1}} = \mathfrak{g}_1^K + \mathfrak{g}_{-1}^K,$$

where subalgebra $\mathfrak{g}_0^K$ is reductive, subalgebras $\mathfrak{g}_0^K$ are abelian. The corresponding Z_2-graded r-matrix has the form:

$$r_{12}(u,v) = \frac{2v}{u^2 - v^2}\Big(\sum_{i=1}^{\mathrm{rank}\mathfrak{g}} H_i \otimes H_i + \sum_{\alpha\in(\Delta_K)_+} (X_\alpha \otimes X_{-\alpha} + X_{-\alpha} \otimes X_\alpha)\Big) + \frac{2u}{u^2 - v^2} \sum_{\alpha\in(\Delta/\Delta_K)_+} (X_\alpha \otimes X_{-\alpha} + X_{-\alpha} \otimes X_\alpha), \quad (33)$$

where Δ is a system of roots of the algebra $\mathfrak{g}$ and Δ_K is a subsystem of roots of the subalgebra $\mathfrak{g}_0^K$.

Example 2: Z_2-graded Gaudin hamiltonians

In the case of Z_2-graded r-matrices the generalized Gaudin hamiltonians in magnetic field (28) have the following form:

$$\hat{H}_l = \sum_{k=1, k\neq l}^{N} \Big(\frac{2\nu_l}{(\nu_k^2 - \nu_l^2)} \Big(\sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \hat{S}_i^{(k)} \hat{S}_i^{(l)} + \sum_{\alpha\in(\Delta_K)_+} (\hat{S}_\alpha^{(k)} \hat{S}_{-\alpha}^{(l)} + \hat{S}_{-\alpha}^{(k)} \hat{S}_\alpha^{(l)}) \Big)$$

$$+ \frac{2\nu_k}{(\nu_k^2 - \nu_l^2)} \sum_{\alpha\in(\Delta/\Delta_K)_+} (\hat{S}_\alpha^{(k)} \hat{S}_{-\alpha}^{(l)} + \hat{S}_{-\alpha}^{(k)} \hat{S}_\alpha^{(l)}) \Big)$$

$$- \frac{1}{2\nu_l} \Big(\sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \hat{S}_i^{(l)} \hat{S}_i^{(l)} + \sum_{\alpha\in(\Delta_K)_+} (\hat{S}_\alpha^{(l)} \hat{S}_{-\alpha}^{(l)} + \hat{S}_{-\alpha}^{(l)} \hat{S}_\alpha^{(l)}) \Big) +$$

$$+ \frac{1}{2\nu_l} \sum_{\alpha\in(\Delta/\Delta_K)_+} \big(\hat{S}_\alpha^{(l)} \hat{S}_{-\alpha}^{(l)} + \hat{S}_{-\alpha}^{(l)} \hat{S}_\alpha^{(l)} \big) + \frac{c}{\nu_l} \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} k_i \hat{S}_i^{(l)}, \quad (34)$$

where $K = \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} k_i H_i$ is the element of the Cartan subalgebra centralized by $\mathfrak{g}_0^K$.

Let us consider the following combination of the integrals (34):

$$\hat{H}^{p_x+ip_y}_{gBCS} = \sum_{l=1}^{N} \nu_l^{-1}\hat{H}_l + \frac{1}{2}\nu_l^{-2}\sum_{l=1}^{N}\hat{C}_l, \tag{35}$$

where $\hat{C}_l$ are quadratic Casimir operators of l-th copy of $\mathfrak{g}$:

$$\hat{C}_l = \sum_{a,b=1}^{\dim\mathfrak{g}} g^{ab}\hat{S}_a^{(l)}\hat{S}_b^{(l)}.$$

More explicitly:

$$\hat{H}^{p_x+ip_y}_{gBCS} = \sum_{k=1}^{N}\nu_l^{-2}\sum_{i=1}^{\mathrm{rank}\mathfrak{g}}(k_i+\delta_K(H_i))\hat{S}_i^{(l)} + \\ + 2\sum_{k,l=1}^{N}\nu_k^{-1}\nu_l^{-1}\sum_{\alpha\in(\Delta/\Delta_K)_+}\hat{S}_{-\alpha}^{(k)}\hat{S}_{\alpha}^{(l)}, \tag{36}$$

where $\delta_K \equiv \sum\limits_{\alpha\in(\Delta/\Delta_K)_+}\alpha$. The hamiltonian (36) is our general integrable p_x+ip_y hamiltonian written in the spin form.

Integrable BCS-type hamiltonians

Using the fermionization procedure, i.e. expressing the spin operators via fermion creation-anihilation operators we define integrable pairing hamiltonian containing m types of fermions:

$$\hat{H}_{gBCS} = \sum_{i=1}^{m}(\sum_{l=1}^{N}\sum_{\sigma\in\pm}\epsilon_{l,i}c^{\dagger}_{l,i,\sigma}c_{l,i,\sigma} - 2\sum_{k,l=1}^{N}G^{T=1}_{ii,kl}c^{\dagger}_{l,i,+}c^{\dagger}_{l,i,-}c_{k,i,-}c_{k,i,+})$$

$$-\sum_{i,j=1,i<j}^{m}\sum_{k,l=1}^{N}G^{T=1}_{ij,kl}(c^{\dagger}_{l,j,+}c^{\dagger}_{l,i,-}+c^{\dagger}_{l,i,+}c^{\dagger}_{l,j,-})(c_{k,i,-}c_{k,j,+}+c_{k,j,-}c_{k,i,+})-$$

$$-\sum_{i,j=1,i<j}^{m}\sum_{k,l=1}^{N}G^{T=0}_{ij,kl}(c^{\dagger}_{l,j,+}c^{\dagger}_{l,i,-}-c^{\dagger}_{l,i,+}c^{\dagger}_{l,j,-})(c_{k,i,-}c_{k,j,+}-c_{k,j,-}c_{k,i,+}),$$

where $c_{k,i,\sigma}, c^{\dagger}_{l,j,\sigma}$, $k,l \in \overline{1,N}$, $i,j \in \overline{1,n}$ are fermion operators:

$$\{c^{\dagger}_{k,i,\sigma}, c_{l,j,\sigma'}\} = \delta_{kl}\delta_{ij}\delta_{\sigma\sigma'},\ \{c^{\dagger}_{k,i,\sigma}, c^{\dagger}_{l,j,\sigma'}\} = 0,\ \{c_{k,i,\sigma}, c_{l,j,\sigma'}\} = 0.$$

In $m = 2$ case they are $N = Z$ proton-neutron Hamiltonians.

The integrability requirements for the free energies are:

$$\epsilon_{l,i} = \epsilon_{l,j} = \epsilon_l, \quad \forall i, j \in \overline{1, m}$$

The integrability requirements for s-type couplings are:

$$1) G^{T=0}_{ij,kl} = G^{T=1}_{ij,kl} = \frac{1}{2} g, \quad 2) G^{T=0}_{ij,kl} = 0, G^{T=1}_{ij,kl} = g,$$
$$3) G^{T=1}_{ij,kl} = 0, G^{T=0}_{ij,kl} = g.$$

The integrability requirements for $p_x + ip_y$-type couplings are:

$$1) G^{T=0}_{ij,kl} = G^{T=1}_{ij,kl} = \frac{1}{2} g \sqrt{\epsilon_k}\sqrt{\epsilon_l}, \quad 2) G^{T=0}_{ij,kl} = 0, \ G^{T=1}_{ij,kl} = g \sqrt{\epsilon_k}\sqrt{\epsilon_l},$$
$$3) G^{T=1}_{ij,kl} = 0, \ G^{T=0}_{ij,kl} = g \sqrt{\epsilon_k}\sqrt{\epsilon_l}.$$

The three cases above are connected with the generalized Gaudin Hamiltonians based on Lie algebras $gl(2m)$, $sp(2m)$ and $so(2m)$, respectively.

All of the above integrable Hamiltonians are diagonalizable by means of the nested Bethe anzats (T. Skrypnyk 2012).

Remark. The above s-type integrable fermion Hamiltonians in the cases of $gl(2)$ and $sp(4)$ were found by Richardson in 1967.

The generalized Jaynes-Cummings-Dicke hamiltonians

For the classical r-matrices possessing special point ν_0 , it is possible to define spin-boson hamiltonians (T. Skrypnyk 2015):

$$\hat{H}_l^{JCD} = \hat{H}_l^G + \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \sum_{b=1}^{\dim\mathfrak{g}} \frac{1}{2} k_i \hat{S}_b^{(l)} \partial^2_{\nu_0} r^{ib}(\nu_0, \nu_l) +$$

$$+ \sum_{a=1}^{\dim\mathfrak{g}^0_{\nu_0}} \sum_{b=1}^{\dim\mathfrak{g}} \hat{l}_a^{(0)}(\hat{b}^-_\beta, \hat{b}^+_\alpha) \hat{S}_b^{(l)} r^{ab}(\nu_0, \nu_l)$$

$$+ \sum_{\alpha\in(\Delta/\Delta_K)_+} \sum_{b=1}^{\dim\mathfrak{g}} \Big(\sqrt{\alpha(K)} \hat{S}_b^{(l)} (\hat{b}^+_\alpha \partial_{\nu_0} r^{\alpha b}(\nu_0, \nu_l) + \hat{b}^-_\alpha \partial_{\nu_0} r^{-\alpha b}(\nu_0, \nu_l) \Big),$$

where $\hat{H}_l^G$ is Gaudin-type hamiltonian, $\hat{b}^+_\alpha$, $\hat{b}^-_\alpha$ Bose operators:

$$[\hat{b}^+_\alpha, \hat{b}^-_\beta] = \delta_{\alpha,\beta}\hat{1},\ [\hat{b}^+_\alpha, \hat{b}^+_\beta] = [\hat{b}^-_\alpha, \hat{b}^-_\beta] = 0,\ \text{where } \alpha, \beta \in (\Delta/\Delta_K)_+. \quad (37)$$

and $\hat{l}_a^{(0)}$ is expressed via $\hat{b}^-_\beta$, $\hat{b}^+_\alpha$ with the help of the generalized Jordan-Schwinger formulae (Skrypnyk 2015).

The hamiltonians $\hat{H}_l^{JCD}$ mutually commute (Skrypnyk 2015):

$$[\hat{H}_k^{JCD}, \hat{H}_l^{JCD}] = 0$$

and one can define the following quantum Hamiltonian:

$$\hat{H}_{JCD} = \sum_{l=1}^{N} \hat{H}_l^{JCD} = \sum_{l=1}^{N} \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \sum_{b=1}^{\mathrm{dim}\mathfrak{g}} \frac{1}{2} k_i \hat{S}_b^{(l)} \partial^2_{\nu_0} r^{ib}(\nu_0, \nu_l) +$$

$$+ \sum_{l=1}^{N} \sum_{\alpha \in (\Delta/\Delta_K)_+} \sum_{b=1}^{\mathrm{dim}\mathfrak{g}} \sqrt{\alpha(K)} (\hat{b}^+_\alpha \partial_{\nu_0} r^{\alpha b}(\nu_0, \nu_l) + \hat{b}^-_\alpha \partial_{\nu_0} r^{-\alpha b}(\nu_0, \nu_l)) \hat{S}_b^{(l)}$$

$$+ \sum_{l=1}^{N} \sum_{a=1}^{\mathrm{dim}\mathfrak{g}^0_{\nu_0}} \sum_{b=1}^{\mathrm{dim}\mathfrak{g}} \hat{I}_a^{(0)}(\hat{b}^-_\beta, \hat{b}^+_\alpha) \hat{S}_b^{(l)} r^{ab}(\nu_0, \nu_l) + \sum_{l=1}^{N} \hat{H}_l^G, \quad (38)$$

which is an r-matrix generalization of JCD hamiltonian.
Remark. In the case of Cartan-invariant r-matrices one can add to $\hat{H}_{JCD}$ also any combination of linear integrals $\hat{M}^b_{H_i}$.

The rational Jaynes-Cummings-Dicke hamiltonian

In the case of the rational r-matrix we obtain (T. Skrypnyk 2008):

$$\hat{H}_l^{JCD} = \hat{H}_l^G + \sum_{\alpha\in(\Delta/\Delta_K)_+} \sqrt{\alpha(K)}(\hat{b}_\alpha^+ \hat{S}_{-\alpha}^{(l)} + \hat{b}_\alpha^- \hat{S}_\alpha^{(l)}) + \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \nu_l k_i \hat{S}_i^{(l)}.$$

Adding to it linear integrals $\hat{M}_{H_i}^b$ we will have:

$$\hat{H}_{JCD} = \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} w_i \hat{M}_{H_i}^b + g \sum_{l=1}^{N} \hat{H}_l. \quad (39)$$

More explicitly:

$$\hat{H}_{JCD} = -\sum_{i=1}^{\mathrm{rank}\mathfrak{g}} w_i \sum_{\alpha\in(\Delta/\Delta_K)+} \alpha(H_i)\hat{b}_\alpha^- \hat{b}_\alpha^+ + \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \sum_{k=1}^{N} (w_i + g\nu_l k_i)\hat{S}_i^{(k)}$$

$$+ g \sum_{\alpha\in(\Delta/\Delta_K)+} \sqrt{\alpha(-K)} \sum_{l=1}^{N} (\hat{b}_\alpha^+ \hat{S}_{-\alpha}^{(l)} + \hat{b}_\alpha^- \hat{S}_\alpha^{(l)}). \quad (40)$$

The generalized $N = 1$ Jaynes-Cummings hamiltonian. The $\mathfrak{g} = gl(n)$ case

In this case generalized n-level JC hamiltonian has the form:

$$\hat{H} = -\sum_{i,j=1,i<j}^{n} (w_i - w_j)\hat{b}_{ij}^{-}\hat{b}_{ij}^{+} + \sum_{i=1}^{n}(w_i + gc_i)\hat{S}_{ii} + g\sum_{i,j=1,i<j}^{n} \sqrt{k_j - k_i}(\hat{b}_{ij}^{+}\hat{S}_{ji} + \hat{b}_{ij}^{-}\hat{S}_{ij}). \quad (41)$$

The first term in this hamiltonian is an energy of $n(n-1)/2$ modes of the electromagnetic field; the second term correspond to a free energy of n-level atom; the last term is an atom-field interaction corresponding to the passages from the level j to the level i and vice verse with the simultaneous creation/anihilation of photon.

The Hamiltonian (41) is diagonalizable by means of the nested Bethe anzats (T. Skrypnyk 2008).

Remark. In the $n = 2$ case the Hamiltonian (41) yields the famous two-level, one-mode Jaynes-Cummings hamiltonian.

Conclusion and Discussion

In the present talk we have reviewed the theory of algebraic structures in the theory of classical and quantum integrable systems and shown that

- Not only quadratic tensor structures are important for the theory of quantum integrable systems. There are many physically interesting quantum integrable models associated with linear tensor structures and classical (non-skew-symmetric in general) r-matrices

The main open problems in this context are the following:

- To classify all non-skew-symmetric classical r-matrices and related classical and quantum integrable models
- To develop the method of solution (separation of variables, Bethe ansatz — both "off-shell" and "on-shell") for the corresponding classical and quantum models

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Thank you for the attention!