Arithmetic aspects of mirror symmetry

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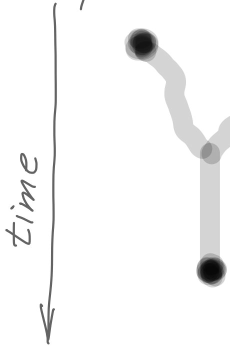
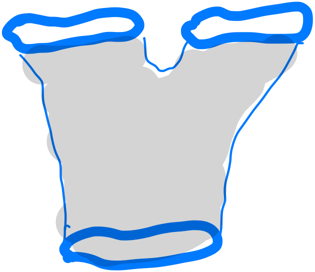
STRING THEORY

point-like particles ⇝ one-dimensional strings



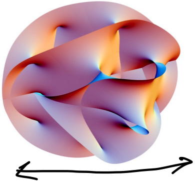


spacetime

4 macroscopic open dimensions

+

at least 6 extra compact dimensions shaped like a Calabi-Yau manifold



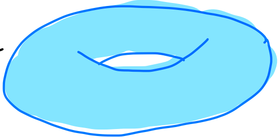
$\ell_P \approx 1.6 \times 10^{-35}$ m

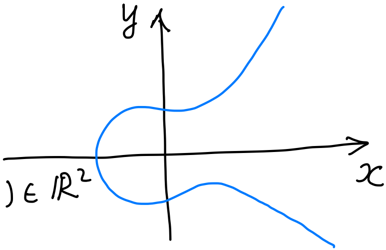
CALABI-YAU MANIFOLDS OF DIMENSION n

$n = 1$ elliptic curves

$$y^2 = x^3 + a x^2 + b x + c$$

$(x, y) \in \mathbb{C}^2$





$(x, y) \in \mathbb{R}^2$

$n = 2$ K3 surfaces

...

Calabi-Yau manifold

is a smooth projective complex manifold of dim n with a nowhere zero holomorphic differential n-form $\omega = \frac{dx}{y}$

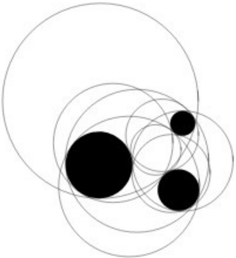
MIRROR SYMMETRY

n = 3

Two CY manifolds can be different geometrically but be equivalent when employed as extra dimen-sions in string theory

ENUMERATIVE GEOMETRY

Problem of Apollonius (262-190 BC)



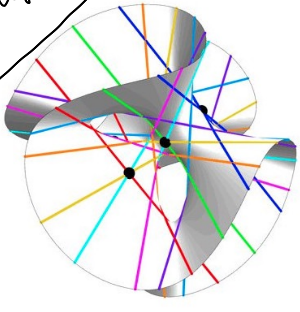
How many circles are there tangent to the three given circles?

8

How many lines are there on a smooth hypersurface of degree 3 in $\mathbb{P}^3$?

27

George Salmon
Arthur Cayley
1840s



$$\sum_{i+j+k+\ell=3} a_{ijk\ell}\, x^i y^j z^k w^\ell = 0$$

$\binom{3+3}{3} = 20$ terms

ENUMERATIVE GEOMETRY

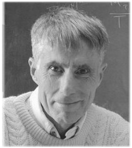
$X \subset \mathbb{P}^4$ hypersurface of degree 5

$$\sum_{i+j+k+l+m=5} a_{ijklm} \, x^i y^j z^k w^l v^m = 0$$

$\binom{5+4}{4} = 126$ terms

Calabi-Yau manifold of dim 3

Hermann Schubert, 1886: there are $n_1 = 2875$ lines on X



Conjecture (Herbert Clemens, 1984)

For $d \geq 1$ there are finitely many rational curves of degree d on X

Sheldon Katz 1986

$n_2 = 609250$

$n_d :=$ number of rat. curves $Y \subset X$ of degree d

BEGINNINGS OF MIRROR SYMMETRY

A-side X | X' B-side ~ differential equation for period integrals

1991
Philip Candelas
Xenia de la Ossa
Paul Green
Linda Parkes

$$Y(q) = 1 + \sum_{d \geq 1} N_d \, d^3 \frac{q^d}{1-q^d}$$

Yukawa coupling

instanton numbers

$N_1 = 2875 = n_1 \quad N_2 = 609250 = n_2 \quad N_3 = 317206375 \ldots$

1993 Geir Ellingsrud, Stein Arild Strømme: $n_3 = 317206375$

Physics wins!

PERIODS IN ALGEBRAIC GEOMETRY

(Alexander Grothendieck)

Periods of an algebraic variety $X/\mathbb{Q}$ are complex numbers which one obtains by integrating rational differential k-forms along topological k-cycles on $X(\mathbb{C})$.

$$\alpha = \int_\gamma \omega \qquad \omega \in H^k_{dR}(X, \mathbb{Q}) \qquad \gamma \in H_k(X(\mathbb{C}), \mathbb{Q})$$

E.g. $X = \mathbb{A}^1 \setminus \{0\}$

$X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$

$H^1_{dR}(X, \mathbb{Q}) = \langle \frac{dx}{x} \rangle$

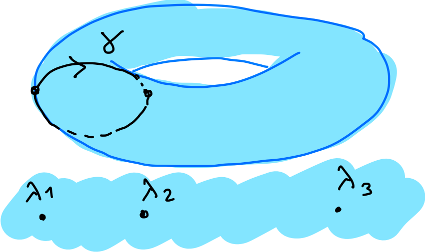
$\alpha = \oint \frac{dx}{x} = 2\pi i$

$$\underbrace{\mathbb{Q} \subset \overline{\mathbb{Q}}}_{\text{algebraic numbers}} \subset \underbrace{\{\text{periods}\}}_{\text{geometric numbers}} \subsetneq \mathbb{C}$$

PERIODS AND DIFFERENTIAL EQUATIONS

$$y^2 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

elliptic curve X



$$(x,y) \mapsto x$$

$$H^1_{dR}(X) = \left\langle \frac{dx}{y}, x\frac{dx}{y} \right\rangle$$

periods = elliptic integrals

$$\alpha = \int_\gamma \frac{dx}{y} = 2\int_{\lambda_1}^{\lambda_2} \frac{dx}{\sqrt{(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)}}$$

$$F(t) = \int_0^1 \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

satisfies differential equation

$$t(1-t)F'' + (1-2t)F' - \frac{1}{4}F = 0$$

PERIODS AND DIFFERENTIAL EQUATIONS

$$y^2 = x(x-1)(x-\underset{\uparrow}{t})$$

parameter

Legendre's family of elliptic curves

All period functions

$$F(t) = \int_{-\infty}^{0} \text{ or } \int_{0}^{1} \text{ or } \int_{1}^{t} \text{ or } \int_{t}^{\infty} \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

satisfy the same diff. equation

$$t(1-t)F'' + (1-2t)F' - \frac{1}{4}F = 0$$

Differential equations originating from algebraic geometry are called Picard-Fuchs or Gauss-Manin equations.

BACK TO MIRROR SYMMETRY

Picard-Fuchs differential operator

$$\mathcal{L} = \delta^4 - 5^5\,\delta\left(\delta+\frac{1}{5}\right)\left(\delta+\frac{2}{5}\right)\left(\delta+\frac{3}{5}\right)\left(\delta+\frac{4}{5}\right)$$

where $\delta = t\frac{d}{dt}$

$t=0$ regular singularity

Solutions to $\mathcal{L}(F(t)) = 0$ near $t=0$:

$$F_0(t) = \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5}\, t^n$$

$$F_1(t) = F_0(t)\log t + G_1(t)$$

$$G_1 = \sum_{n=1}^{\infty} \frac{(5n)!}{n!^5}\left(\sum_{k=1}^{5n} \frac{5}{k}\right) t^n$$

$$F_2(t) = F_0(t)\frac{(\log t)^2}{2!} + G_1(t)\log t + G_2(t)$$

$$F_3(t) = F_0(t)\frac{(\log t)^3}{3!} + \ldots$$

DIFFERENTIAL EQUATION AND ARITHMETIC

$$L(F(t)) = 0 \qquad F_i(t) = G_0(t)\frac{(\log t)^i}{i!} + G_1(t)\frac{(\log t)^{i-1}}{(i-1)!} + \dots + G_i(t)$$

G_i power series with coefficients in $\mathbb{Q}$

Observations of physicists (1991):

$$q(t) := \exp\left(\frac{F_1(t)}{F_0(t)}\right) = t\exp\left(\frac{G_1(t)}{G_0(t)}\right) = t + 770t^2 + \dots$$

canonical coordinate $\in \mathbb{Z}[[t]]$

This is true!
B.-H. Lian
S.T. Yau
1996

$$\frac{F_0}{F_0} = 1 \qquad \frac{F_1}{F_0} = \log q \qquad \frac{F_2}{F_0} = \frac{1}{2}(\log q)^2 + 575q + \frac{975375}{4}q^2 + \dots$$

$$Y(q) := \left(q\frac{d}{dq}\right)^2 \frac{F_2}{F_0} = 1 + 575q + 975375q^2 + \dots$$

Yukawa coupling $\rightsquigarrow$ instanton numbers $N_d \in \mathbb{Z} \;\forall d$

Conjecture

MIRROR THEOREM: $n_d = N_d \quad \forall d \geq 1$

(A. Givental, B. Lian - K. Liu - S.-T. Yau, circa 1995)

enumerative geometry

n_d = "number" of rational of degree d on X

Gromov-Witten invariants

$n_d \in \mathbb{Q}$

differential equation for period functions

$$Y(q) = 1 + \sum_{d=1}^{\infty} N_d \, d^3 \frac{q^d}{1-q^d}$$

instanton numbers

$N_d \in \mathbb{Q}$

INTEGRALITY OF INSTANTON NUMBERS

$N_d \in \mathbb{Z}$?

Theorem (Frits Beukers - MV, 2020)

YES for $L = \delta^4 - 5^5 t(\delta+\frac{1}{5})(\delta+\frac{2}{5})(\delta+\frac{3}{5})(\delta+\frac{4}{5})$

and some other examples

* In 2018 E.N. Ionel and T.H. Parker proved integrality of BPS-invariants using methods of symplectic topology. Combined with the Mirror Theorem, this should also imply integrality of instanton numbers.

** The advantage of our proof is that it is happening entirely on the B-side. We gave an explict proof using the p-adic Frobenius structure for L (after J. Stienstra, M. Kontsevich - A. Schwarz - V. Vologodsky)

P-ADIC FROBENIUS STRUCTURE

X algebraic variety / $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

$\exists\ \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_k \in \overline{\mathbb{Q}}$

such that

$$\# X(\mathbb{F}_{p^s}) = \sum_{i=1}^{m} \alpha_i^s - \sum_{j=1}^{k} \beta_j^s \quad \forall s \geq 1$$

Frobenius eigenvalues

Frobenius matrix: a matrix Φ with entries in $\mathbb{Q}_p$ whose eigenvalues are a subset of α_i's and β_j's

Frobenius str-re: $\Phi(t) \longleftrightarrow X_t$

THE QUINTIC EXAMPLE

$$X_t : \quad t\left(x_1+x_2+x_3+x_4+\frac{1}{x_1x_2x_3x_4}\right)=1$$

$$\Phi(t) = \mathcal{U}(t)\begin{pmatrix} 1 & 0 & 0 & -\frac{8}{25}p^3\zeta_p(3) \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^3 \end{pmatrix}\mathcal{U}(t^p)^{-1}$$

where $\mathcal{U}(t) = \left(\delta^i F_j(t)\right)_{0\le i,j\le 3}$

is the Wronskian matrix

for $L = \delta^4 - 5^5 t^5(\delta+1)(\delta+2)(\delta+3)(\delta+4)$

$\approx$ our quintic case $t \longleftrightarrow t^5$

MORE P-ADIC ZETA VALUES

Theorem (Beukers-V, arxiv: 2302.09603)

Family $1 - t\left(x_1 + \ldots + x_n + \frac{1}{x_1 \cdot \ldots \cdot x_n}\right) = 0$ $\quad n \geq 2$

for each $p > n+1$ has a p-adic Frobenius str-re given by

$$\Phi(t) = \mathcal{U}(t) \begin{pmatrix} 1 & p\alpha_1 & p^2\alpha_2 & \cdots & p^{n-1}\alpha_{n-1} \\ 0 & p & p^2\alpha_1 & \cdots & p^{n-1}\alpha_{n-2} \\ & & & \cdots & \end{pmatrix} \mathcal{U}(t^p)^{-1}$$

with

α_i = coeff. of x^i in $\left(\frac{\Gamma_p(x)}{\Gamma_p\left(\frac{x}{n+1}\right)}\right)^{n+1}$

$i = 1, \ldots, n-1$

$$\log \Gamma_p(x) = \Gamma_p'(0)x - \sum_{m \geq 2} \frac{\zeta_p(m)}{m} x^m$$

THANK YOU!

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