Algebraic and asymptotic aspects of random matrices

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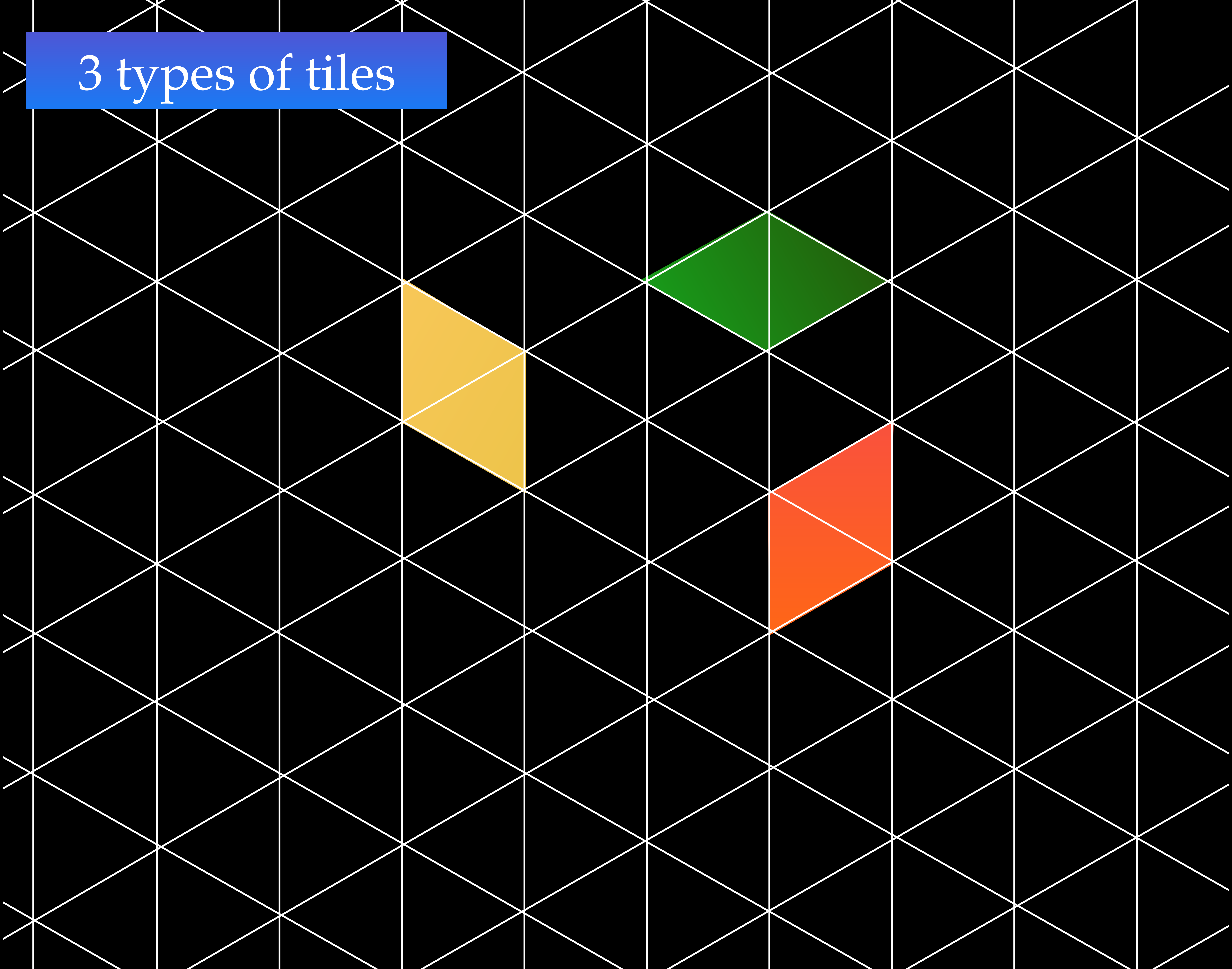
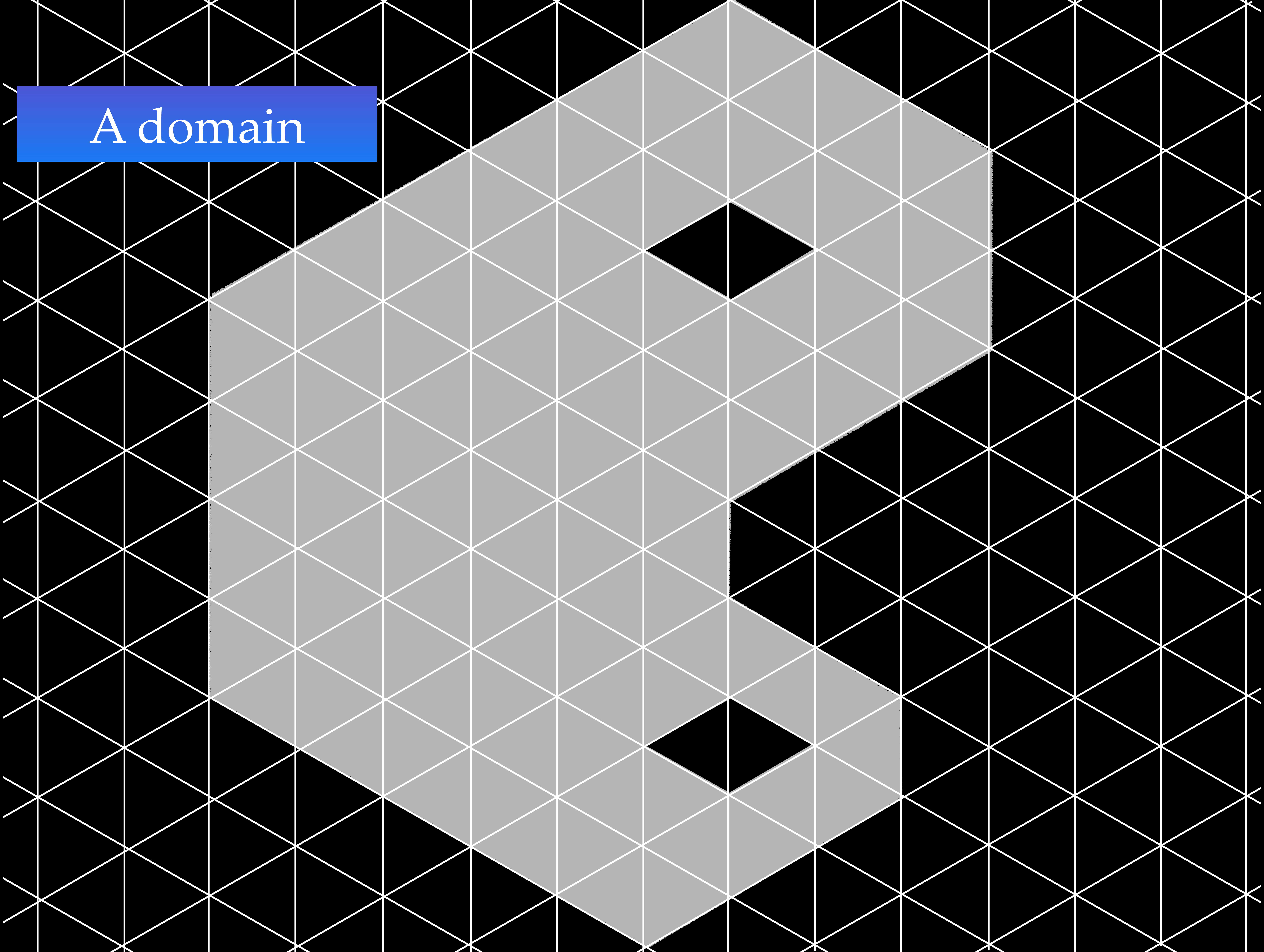
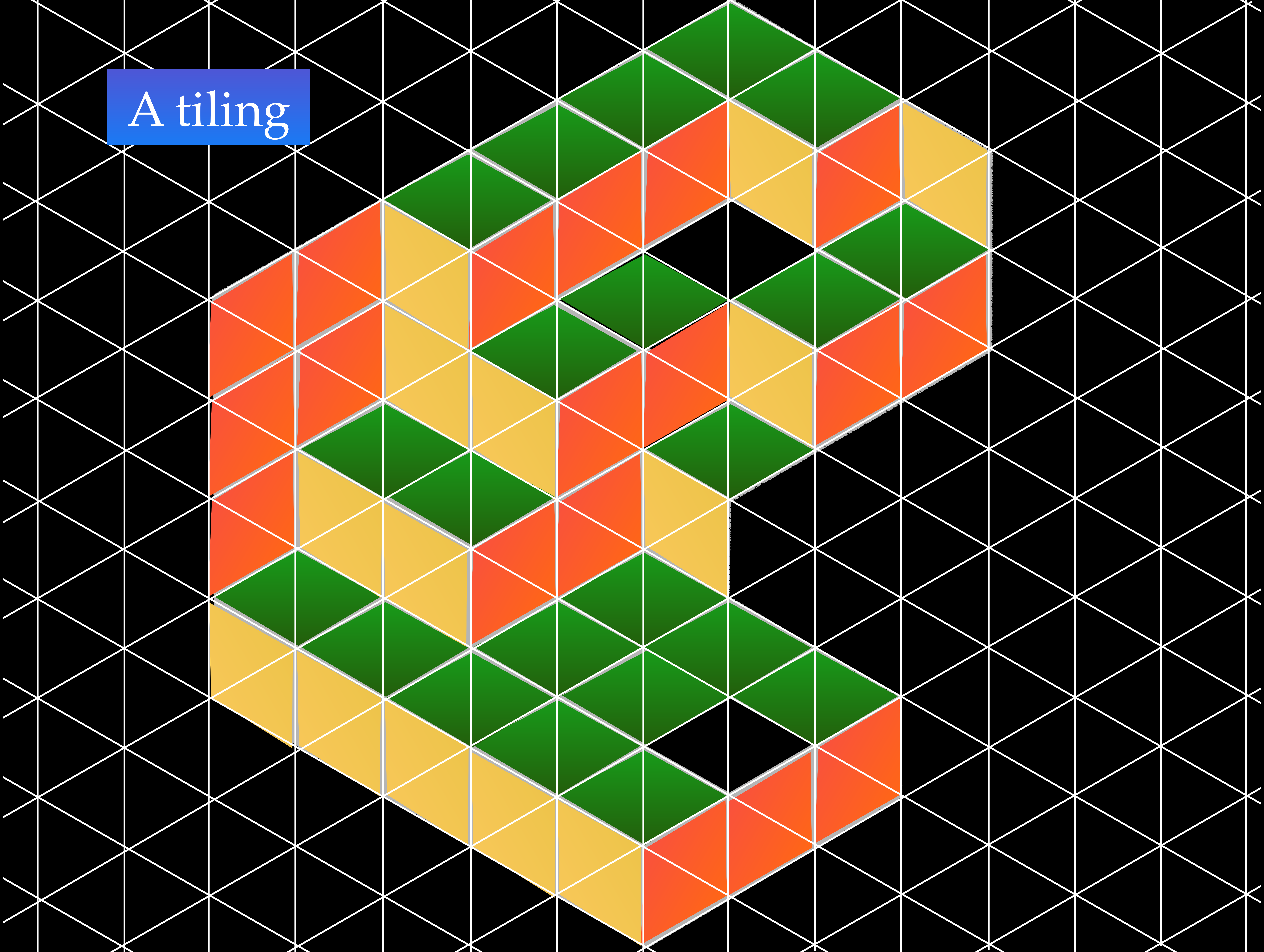
Introduction :
random lozenge tilings

3 types of tiles



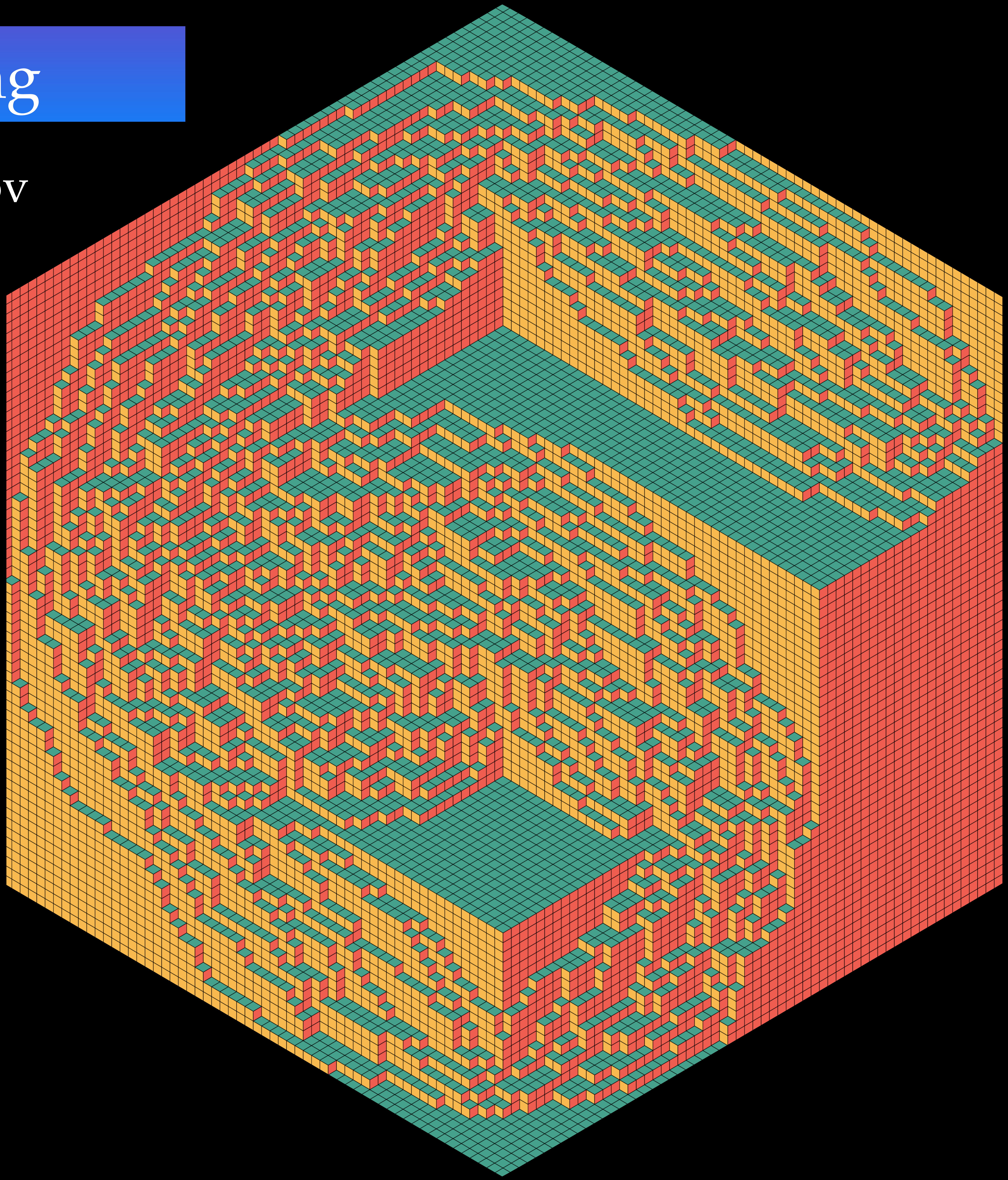
A domain

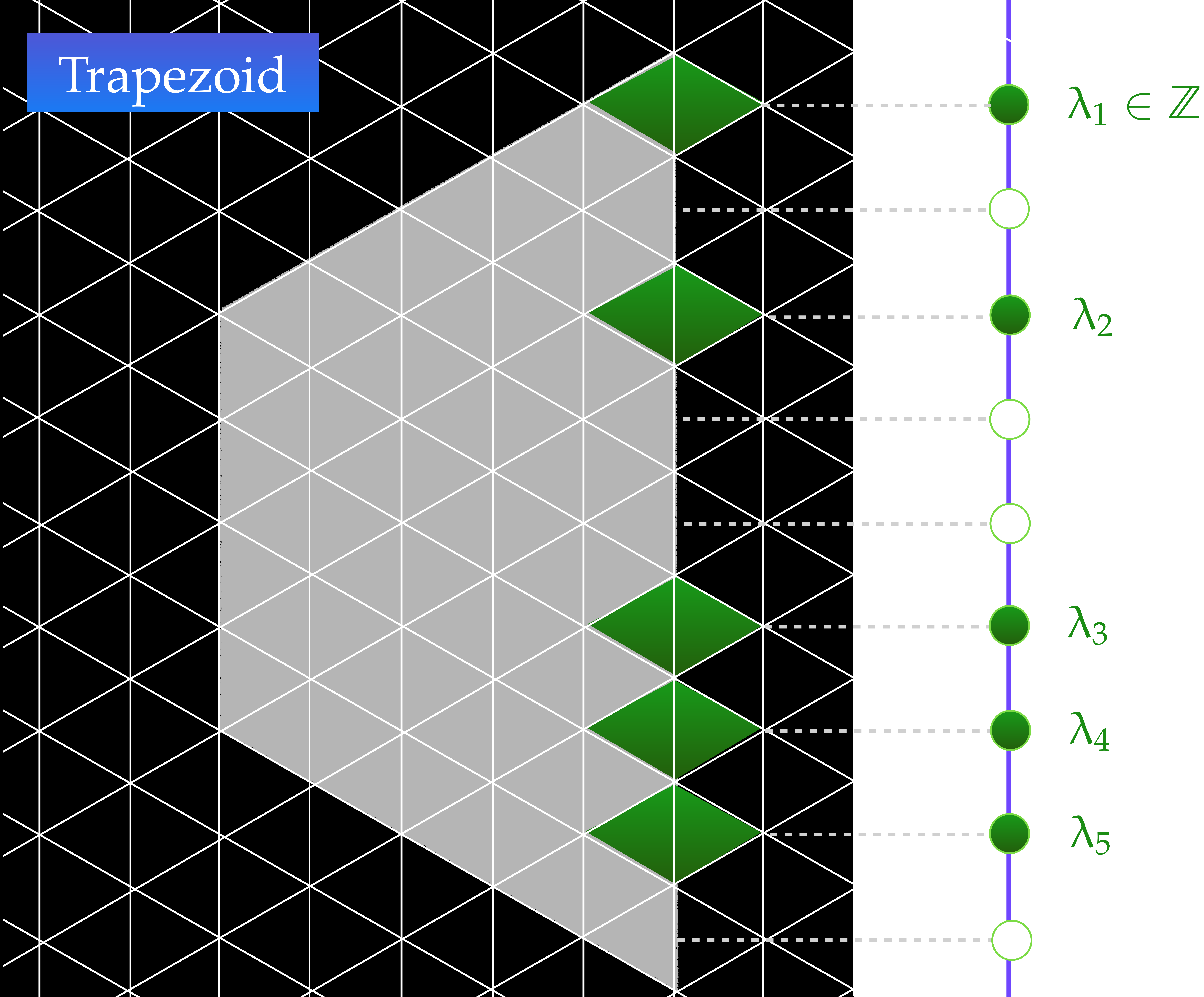



A tiling

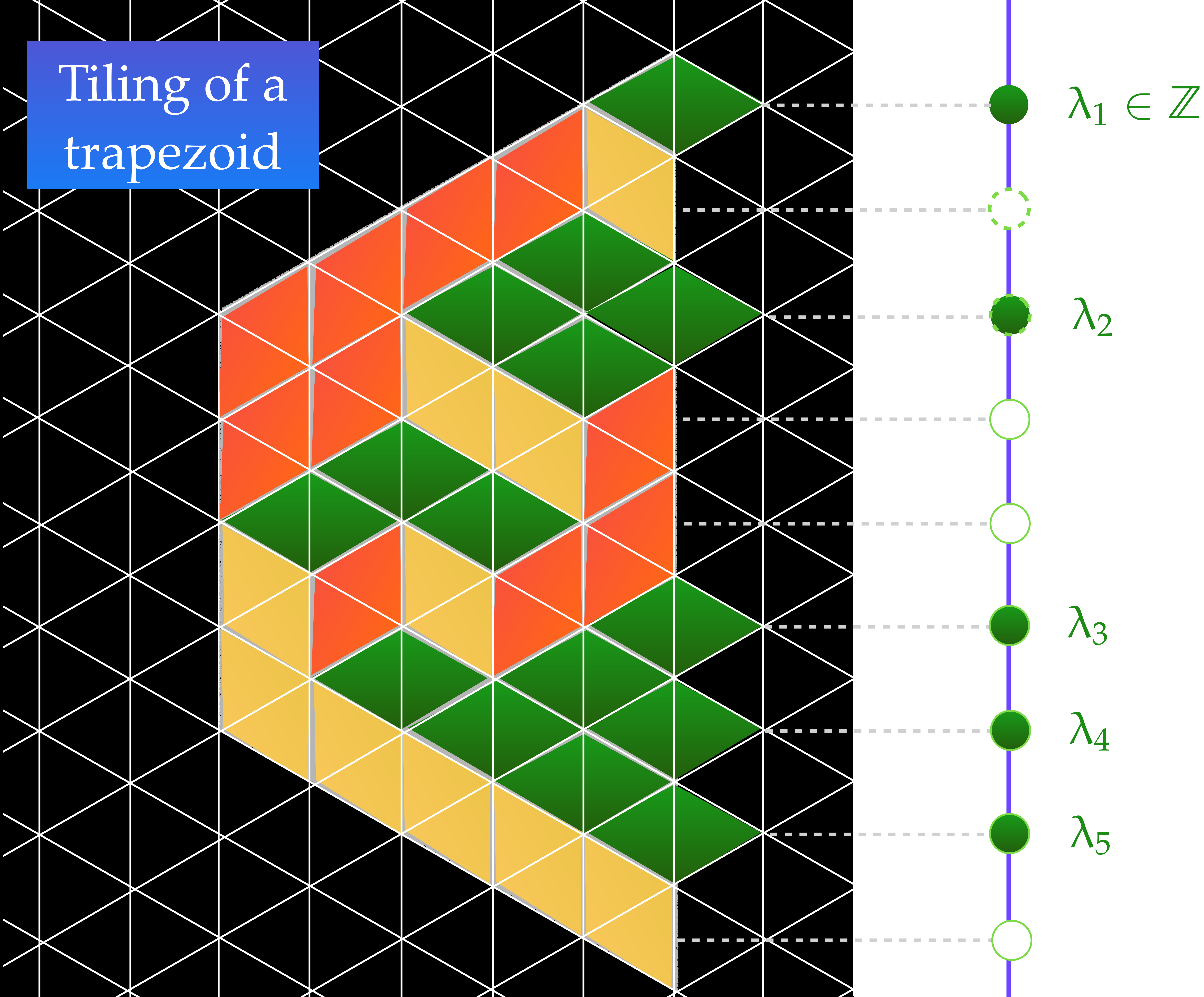
A large tiling

Simulation by L. Petrov





Trapezoid
$\lambda_1 \in \mathbb{Z}$
λ_2
λ_3
λ_4
λ_5


Tiling of a trapezoid
$\lambda_1 \in \mathbb{Z}$
λ_2
λ_3
λ_4
λ_5

Enumeration of tilings in trapezoids

A trapezoid of width w must have
w "horizontal" lozenges sticking out of its (large) base

Given their position $\ell_1 < \ldots < \ell_w$

there are $\prod_{i<j} \frac{\ell_j - \ell_i}{j - i}$ ways to complete into a tiling of the trapezoid

(Gelfand-Tsetlin patterns)

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The statistical mechanics models

II - Statistical mechanics models: setting

N particles $\lambda_1, \ldots, \lambda_N$, split in H groups

$$d\mathbb{P} \propto \prod_{i=1}^{N} d\lambda_i e^{-N V_{h(i)}(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^{2\theta_{h(i),h(j)}}$$

$$h(i) \in \{1, \ldots, H\} \qquad \lambda_i \in A_{h(i)} \subseteq \mathbb{R}$$

θ symmetric nonnegative HxH matrix
positive diagonal

V_h analytic function near $A \subseteq \mathbb{C}$ (e.g. polynomial)

$N_h =$ # particles in h-group, fixed

II - Statistical mechanics models: setting

We will be interested in the large N behavior of

- the partition function

$$Z_N = \int_{A^N} \prod_{i=1}^{N} \lambda_i e^{-NV_{h(i)}(\lambda_{h(i)})} \prod_{1 \leqslant i < j \leqslant n} |\lambda_i - \lambda_j|^{2\theta_{h(i),h(j)}}$$

- the k-point correlation functions

$$W_k\begin{pmatrix} x_1 & \cdots & x_k \\ h_1 & \cdots & h_k \end{pmatrix} = \mathbb{E}_N\Big[\sum_{\substack{1 \leqslant i_1 \leqslant N \\ h(i_1)=h_1}} \frac{1}{x_1 - \lambda_{i_1}} \cdots \sum_{\substack{1 \leqslant i_k \leqslant N \\ h(i_k)=h_k}} \frac{1}{x_k - \lambda_{i_k}} \Big]_c$$

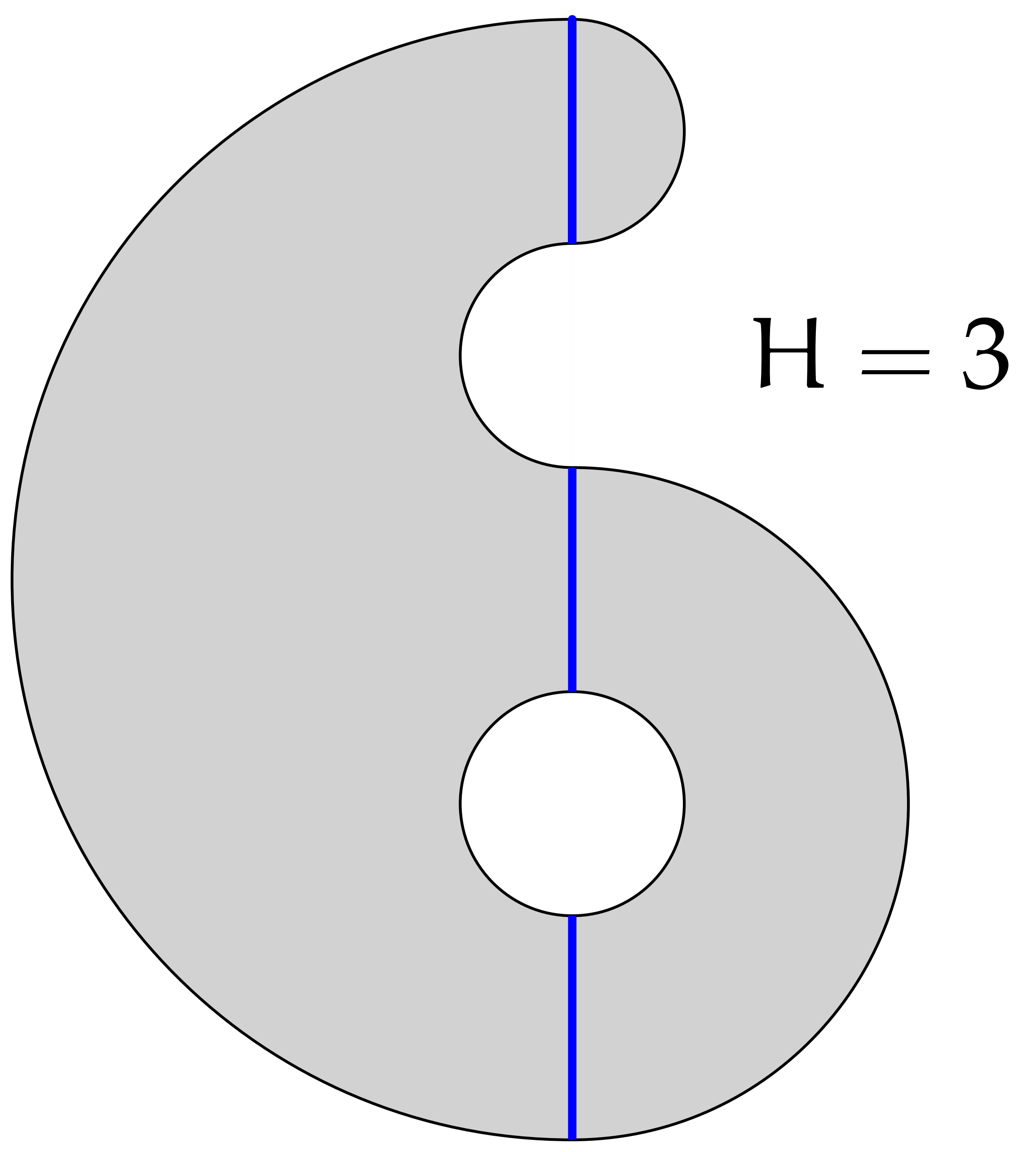
II - Statistical mechanics models: examples

Tiling models N particles $\lambda_1, \ldots, \lambda_N \in \mathbb{Z}$

$$\mathbb{P}(\lambda) \propto \prod_{i=1}^{N} e^{-N V_{h(i)}(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^{2\theta_{h(i),h(j)}}$$

$2\theta_{h,h'} = \#$ trapezoids common to h and h′

V_h determined by the geometry of the domain



$$\theta = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

Kenyon, Okounkov, …

Borodin, Guionnet, Gorin, …

Random matrix theory

$$d\mathbb{P}_N \propto \prod_{i=1}^{N} d\lambda_i e^{-NV(\lambda_i)} \prod_{1 \leqslant i < j \leqslant N} |\lambda_i - \lambda_j|^{2\theta}$$

$H = 1$ group

is the eigenvalue distribution of a random matrix $d\Lambda e^{-N\mathrm{Tr}V(\Lambda)}$

$\theta = 1/2$ Λ symmetric

$\theta = 1$ Λ hermitian

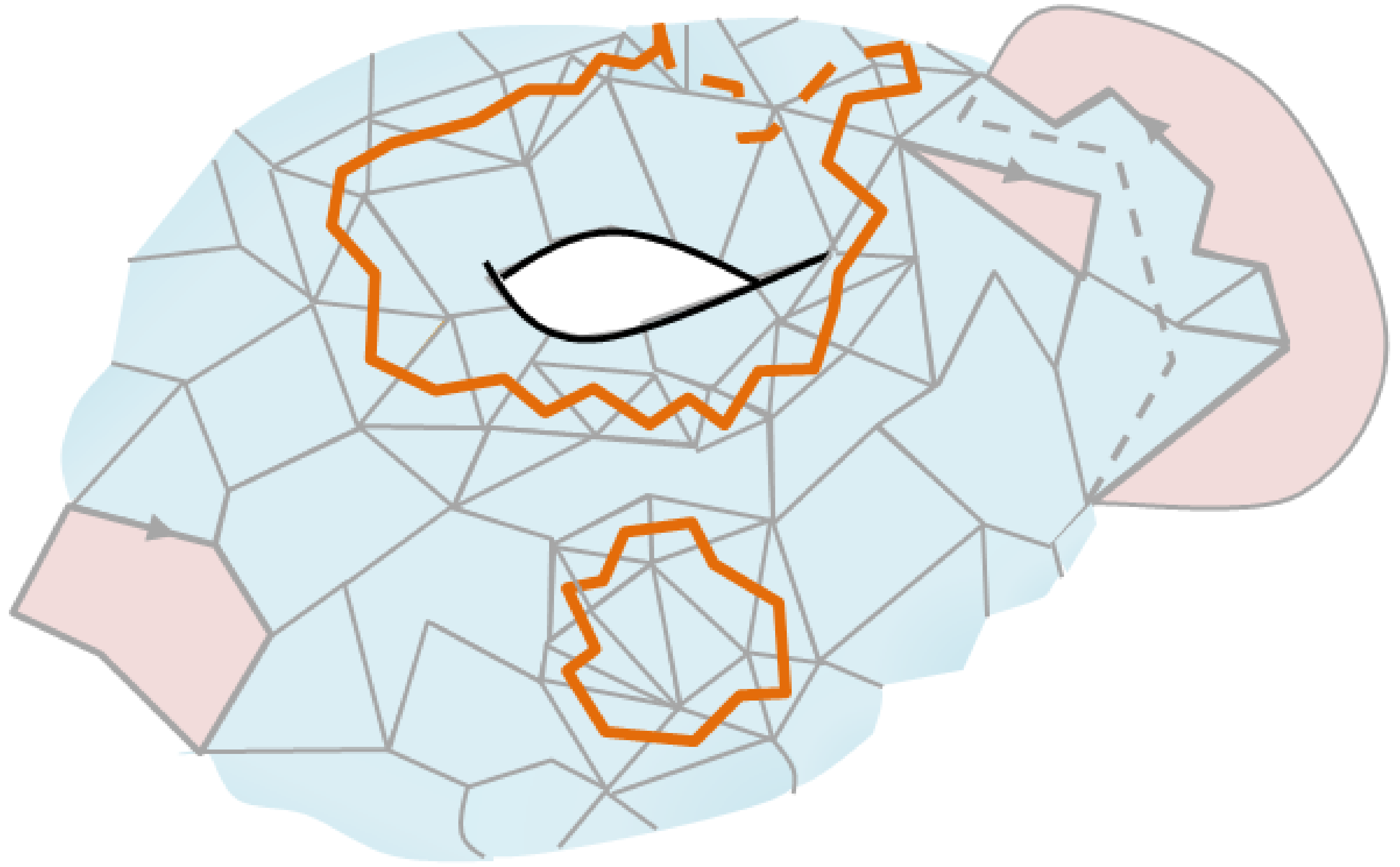
$\theta = 2$ Λ quaternionic self-dual

Wigner, Dyson, … (50s - …)

Mehta, Pastur, … (70s - …)

O(n) loop model on random surfaces

$$d\mathbb{P}_N = \prod_{i=1}^{N} d\lambda_i e^{-NV(\lambda_i)} \prod_{i<j} \left| \frac{(\lambda_i - \lambda_j)^2}{(\lambda_i + \lambda_j)^n} \right|$$



Gaudin, Kostov, Eynard, Staudacher, … (90s)

$$\ln Z_N \approx \sum_{g,p \geqslant 0} N^{2-2g} n^p \cdot \left\{ \begin{array}{c} \text{generating series of maps of genus } g \\ \text{carrying } p \text{ self-avoiding loops} \end{array} \right\}$$

$H = 2$ groups $\{1, \ldots, N\}, \{N+1, \ldots, 2N\}$

with symmetry $\lambda_i = -\lambda_{N+i}$

$$\theta = \begin{pmatrix} 1 & -n/2 \\ -n/2 & 1 \end{pmatrix}$$

II - Statistical mechanics models: examples

Chern-Simons theory on Seifert fibered spaces $S(a_1, \ldots, a_r)$

(perturbative contribution of trivial connection)

$(a_1, \ldots, a_r)$ order of exceptional fibers, $a = \mathrm{lcm}$

e.g. $S(a_1, a_2)$ = lens spaces

$S(2,3,5)$ = Poincare homology sphere

$$d\mathbb{P}_N = \prod_{i=1}^{N} d\lambda_i\, e^{-NV(\lambda_i)} \prod_{i<j} \left[|\lambda_i - \lambda_j|^{2-r} \prod_{m=1}^{r} \left|\lambda_i^{a/a_m} - \lambda_j^{a/a_m}\right| \right]$$

Mariño (00s)
Blau, Thompson, ...

$H = a$ groups of N particles with symmetries $\lambda_{Nq+i} = e^{2i\pi q/a}\lambda_i$

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The large N limit

III - The large N limit: leading order

The large N limit is governed by energy-minimizing configurations

$$d\mathbb{P} \propto \exp\big(-N^2\mathcal{E}[\mu^{(\lambda)}]\big)\prod_{i=1}^{N} d\lambda_i$$

empirical measure
in the h-th group

$$\mu_h^{(\lambda)} = \frac{1}{N}\sum_{\substack{1\leqslant i\leqslant N\\ h(i)=h}} \delta_{\lambda_i}$$

energy functional

$$\mathcal{E}[\mu] = -\sum_{1\leqslant h,h'\leqslant H}\iint_{x\neq y}\theta_{h,h'}\ln|x-y|d\mu_h(x)d\mu_{h'}(y) + \sum_{h=1}^{H}\int V_h(x)d\mu_h(x)$$

III - The large N limit: leading order

Assume A_h pairwise disjoint, $\epsilon_h = N_h/N$ fixed

Potential theory + large deviation theory shows

1. $\mathcal{E}$ is strictly convex and admits a unique minimiser μ^{eq}
   among positive measures μ_h on A_h with mass ϵ_h
2. $\mu_h^{(\lambda)} \underset{N\to\infty}{\longrightarrow} \mu_h^{eq}$ in probability (against nice test functions)
3. $\ln Z_N = -N^2\mathcal{E}[\mu^{eq}] + O(N\ln N)$

III - The large N limit: leading order

Characterisation of the minimiser

$$w_h(x) = \lim_{N\to\infty} \mathbb{E}_N \left[\frac{1}{N} \sum_{\substack{1\leqslant i\leqslant N \\ h(i)=h}} \frac{1}{x-\lambda_i} \right] = \int_{A_h} \frac{d\mu_h^{eq}(y)}{x-y}$$

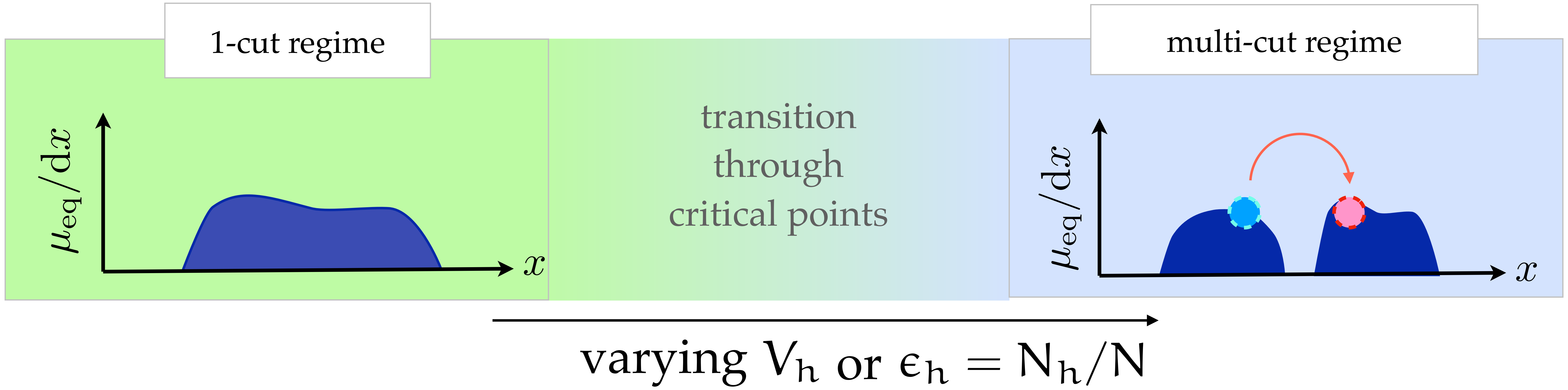
1. is holomorphic in $\mathbb{C} \setminus \mathrm{supp}(\mu_h^{eq})$
2. satisfies at infinity $w_h(x) \underset{x\to\infty}{\sim} \frac{\epsilon_h}{x}$
3. satisfies on the support

$$\theta_{h,h}\big(w_h(x+i0) + w_h(x-i0)\big) + \sum_{h'\neq h} 2\theta_{h,h'} w_{h'}(x) = \partial_x V_h(x)$$

III - The large N limit: support

$$V_h \text{ analytic} \implies \mathrm{supp}(\mu_h^{eq}) = \text{finite union of segments}$$

(proof via Dyson-Schwinger equations)



Multi-cut regime : oscillatory asymptotics from particle jumping understood in terms of theta functions

Bonnet, David, Eynard (heuristics)
Shcherbina 12, B., Guionnet 13 (rigorous)

III - The large N limit: finite-size corrections

Assume A_h pairwise disjoint, $\epsilon_h = N_h/N$ fixed

Assume the model is off-critical and $\mathrm{supp}(\mu_h^{\mathrm{eq}}) = [\alpha_h, \beta_h]$

1. We have asymptotic expansions

$$\ln Z_N = (c_1 N + c_0) \ln N + \sum_{g \geqslant 0} N^{2-2g} F_g + O(N^{-\infty})$$

$$W_k = \sum_{g \geqslant 0} N^{2-2g-k} W_{k;g} + O(N^{-\infty})$$

2. $W_{k;g}$ are computed by topological recursion

Albeverio, Pastur, Shcherbina 01

B., Guionnet, Kozlowski 13-15

Chekhov, Eynard, Orantin,
B., Marchal (05 - ...)

III - The large N limit: finite-size corrections

Leading covariance

$$B\begin{pmatrix} x_1 & x_2 \\ h_1 & h_2 \end{pmatrix} = W_{2;0}\begin{pmatrix} x_1 & x_2 \\ h_1 & h_2 \end{pmatrix} = \lim_{N\to\infty} \mathbb{E}_N\Bigg[\sum_{\substack{1\leqslant i_1\leqslant N \\ h(i)=h_1}} \frac{1}{x_1-\lambda_{i_1}} \cdot \sum_{\substack{1\leqslant i_2\leqslant N \\ h(i_2)=h_2}} \frac{1}{x_2-\lambda_{i_2}}\Bigg]_c$$

1. is holomorphic in $x_a \in \mathbb{C} \setminus \mathrm{supp}(\mu^{eq}_{h_a})$

2. is $O(x_1^{-2}) \cdot O(x_2^{-2})$ at infinity

3. satisfies for $x_1 \in \mathrm{supp}(\mu^{eq}_{h_1})$

$$\theta_{h_1,h_1}\left(B\begin{pmatrix} x_1+i0 & x_2 \\ h_1 & h_2 \end{pmatrix} + B\begin{pmatrix} x_1-i0 & x_2 \\ h_1 & h_2 \end{pmatrix}\right) + \sum_{h_1'\neq h_1} 2\theta_{h_1,h_1'} B\begin{pmatrix} x_1 & x_2 \\ h_1' & h_2 \end{pmatrix} = -\frac{\delta_{h_1,h_2}}{(x_1-x_2)^2}$$

Topological recursion (=TR)

Once $w := W_{1;0}$ and $B := W_{2;0}$ have been found,

TR reconstructs by a universal procedure

using analytic properties, residue formulae, …

functions $W_{k;g}$, inductively on $2g - 2 + k > 0$

Chekhov, Eynard, Orantin, B., Marchal (05 - …)

They give the coefficients in the asymptotic expansion of k-point correlators

$$W_k = \sum_{g \geqslant 0} N^{2-2g-k} W_{k;g} + O(N^{-\infty})$$

4

The Riemann-Hilbert problem

IV - The RHP: setting

Data $\alpha_1 < \beta_1 < \cdots < \alpha_H < \beta_H$

θ symmetric HxH matrix, $\geqslant 0$, diagonal > 0

$\phi_1, \ldots, \phi_H$ rational functions

Problem Find functions f_h holomorphic on $\mathbb{C} \setminus [\alpha_h, \beta_h]$ $h \in \{1, \ldots, H\}$

- subjected to growth conditions at $\infty, \alpha_1, \ldots, \beta_H$
- admitting continous boundary values on $(\alpha_h, \beta_h) \pm i0$
- obeying $\forall x \in (\alpha_h, \beta_h)$

$$\theta_{h,h}\big(f_h(x + i0) + f_h(x - i0)\big) + \sum_{h' \neq h} 2\theta_{h,h'} f_{h'}(x) = \phi_h(x)$$

IV - The RHP: solutions

Theorem (vague formulation) in progress

(uses positivity, and the matrix model as a controlled approximation)

1. The solution always exist and is unique
2. There exists a unique solution with sources $\phi_h(x) = -\frac{\delta_{h,h_2}}{(x-x_2)^2}$ (considering x_2 and h_2 fixed) denoted $B\begin{pmatrix} x & x_2 \\ h & h_2 \end{pmatrix}$
3. We have $B\begin{pmatrix} x_1 & x_2 \\ h_1 & h_2 \end{pmatrix} = B\begin{pmatrix} x_2 & x_1 \\ h_2 & h_1 \end{pmatrix}$
4. All solutions can be constructed from B

But does not give formulae …

IV - The RHP: reflection group

Reformulation (homogeneous case) $$\boldsymbol{v} \cdot \mathrm{f}(x) = \sum_{h=1}^{H} v_h f_h(x)$$

$$\forall x \in (\alpha_h, \beta_h) \qquad \boldsymbol{v} \cdot \mathbf{f}(x + i0) = T^{(h)}(\boldsymbol{v}) \cdot \mathbf{f}(x - i0)$$

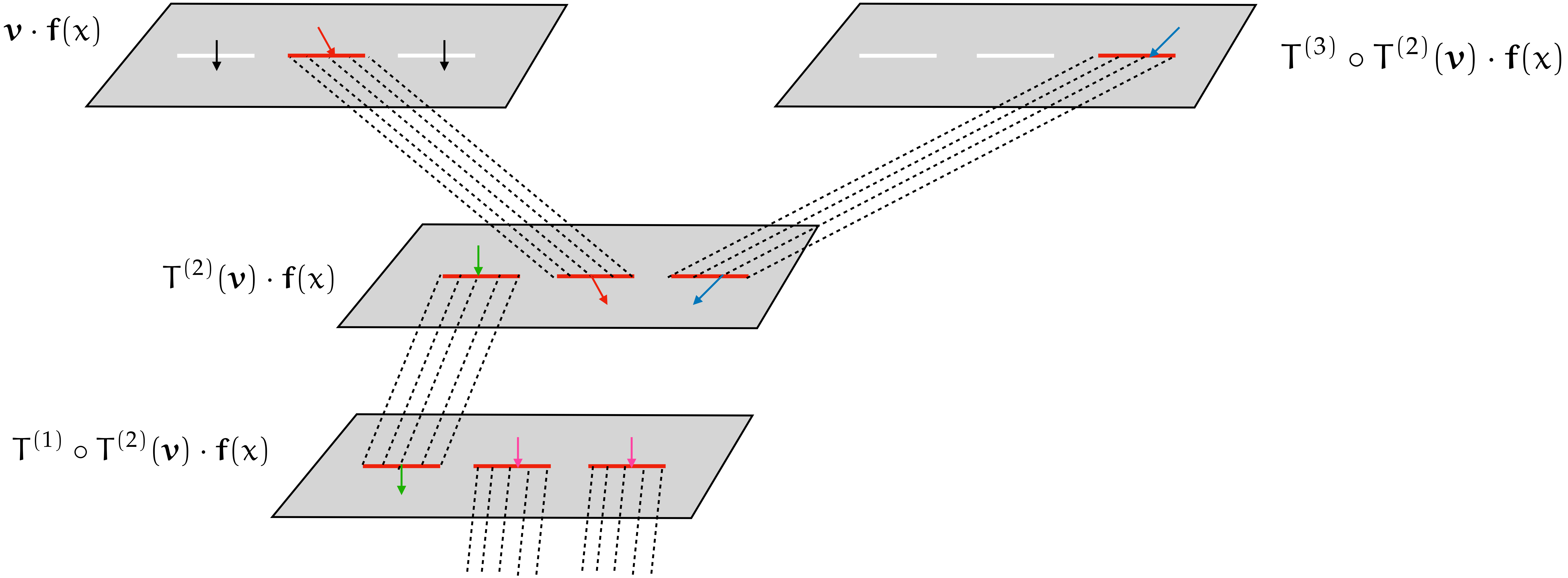
with $T^{(h)}(\boldsymbol{v}) = \boldsymbol{v} - \frac{2v_h}{\theta_{h,h}} \sum_{g=1}^{H} \theta_{g,h} \boldsymbol{e}^{(g)}$ pseudoreflections

- Introduce the group $G = \langle T^{(1)}, \ldots, T^{(H)} \rangle \subset GL(\mathbb{R}^H)$
- $\boldsymbol{v} \cdot f(x)$ continues analytically to a meromorphic function

on the Riemann surface $\Sigma_{\boldsymbol{v}} = \bigsqcup_{\boldsymbol{v}' \in G.\boldsymbol{v}} \widehat{\mathbb{C}} / \sim$

IV - The RHP: reflection group

Construction of the spectral curve $\Sigma_{\boldsymbol{v}} = \bigsqcup_{\boldsymbol{v}' \in G.\boldsymbol{v}} \widehat{\mathbb{C}} / \sim$



etc.

IV - The RHP: reflection group

- We are interested in orbits of $G = \langle T^{(1)}, \ldots, T^{(H)} \rangle \subset GL(\mathbb{R}^H)$

- $G.\boldsymbol{v}$ finite $\implies$ $\boldsymbol{v} \cdot f(x)$ algebraic
  determined by its singularities + period conditions

Observations

1. $\mathbb{R}^H = \mathrm{Ker}(\theta) \overset{\perp}{\oplus} \mathrm{Im}(\theta)$ and $E = \mathrm{Im}(\theta)$ is stable under G
2. The action of G on E is conjugate to action of a reflection group

   $\hat{G}_E \subseteq O(E, q) \qquad q(\boldsymbol{v}) = \boldsymbol{v} \cdot \theta(\boldsymbol{v})$
3. $\exists$ finite orbit $\neq \{0\} \iff \exists$ finite irreducible factor in $\hat{G}_E$

IV - The RHP: algebraic examples

Finite irreducible reflection groups are known (= Coxeter)

In the algebraic case one can use this to compute explicitly a spectral curve

- (H-cut) **random matrix theory** : $\Sigma_{(1,\ldots,1)}$ = hyperelliptic branched cover of $\widehat{\mathbb{C}}$

  $\theta_{h,h} = \beta > 0$

- **Chern-Simons theory on** $S(a_1, \ldots, a_r)$

  $\theta \geqslant 0 \Leftrightarrow 2 - r + \sum_{m=1}^{r} \frac{1}{a_m} \geqslant 0 \Leftrightarrow \hat{G}_E$ finite reflection group B., Brini, Eynard 15-17

  Classification: ADE+three affine A cases

One finds (for simple V) spectral curves of the relativistic Toda chain of type ADE

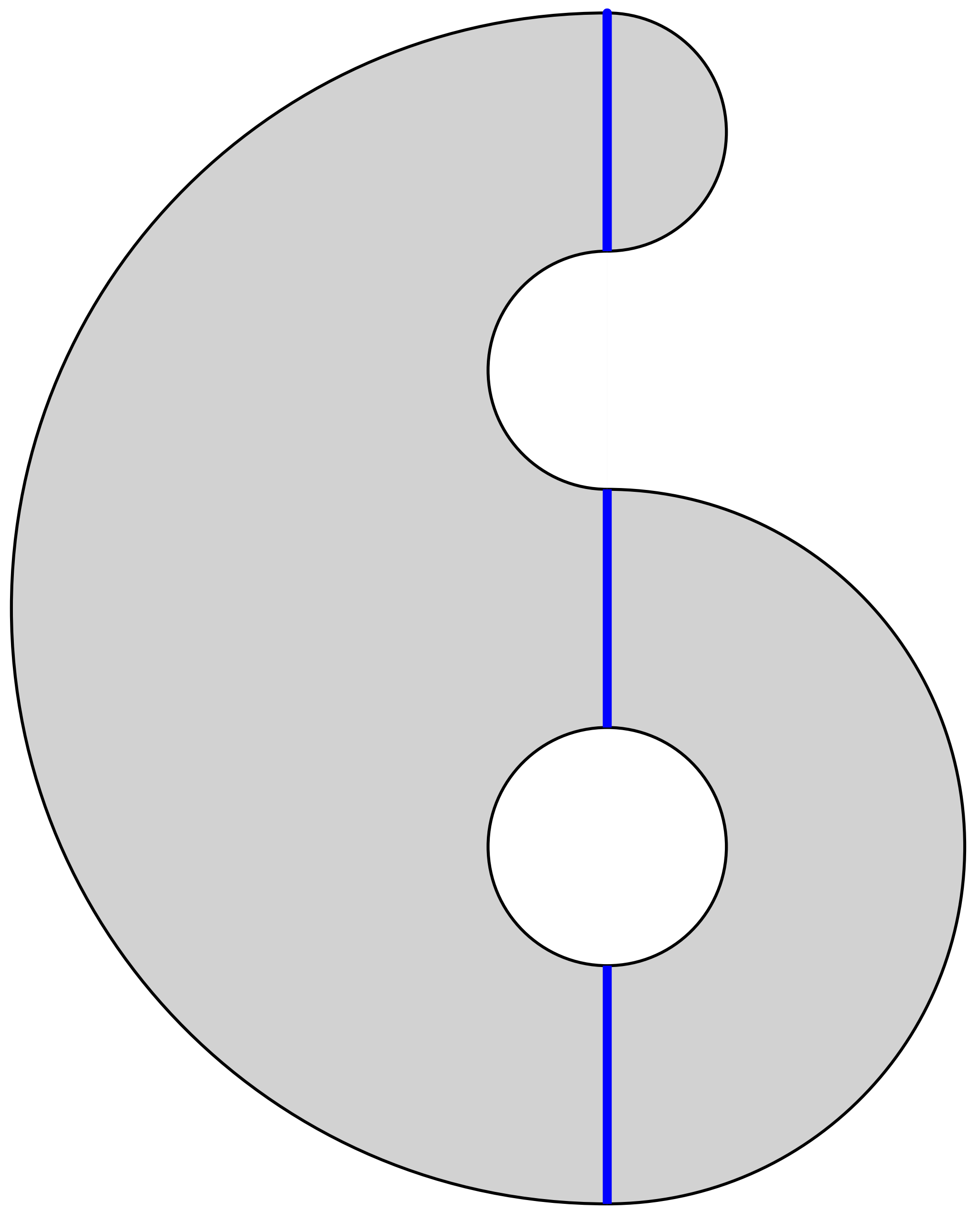
IV - The RHP: algebraic examples

Not much difference between discrete and continuous case for the RHP

Tiling model B., Guionnet, Gorin - in progress

$\hat{G}_E$ = Coxeter group of type A (bipartite) or D (non-bipartite)

- For good ν, Σ_ν is a doubling (along blue segments) of the tiled domain



- B can be related to the Green function of the tiled domain (fluctuations ~ Gaussian free field)

$$\theta = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

IV - The RHP: non-algebraic examples

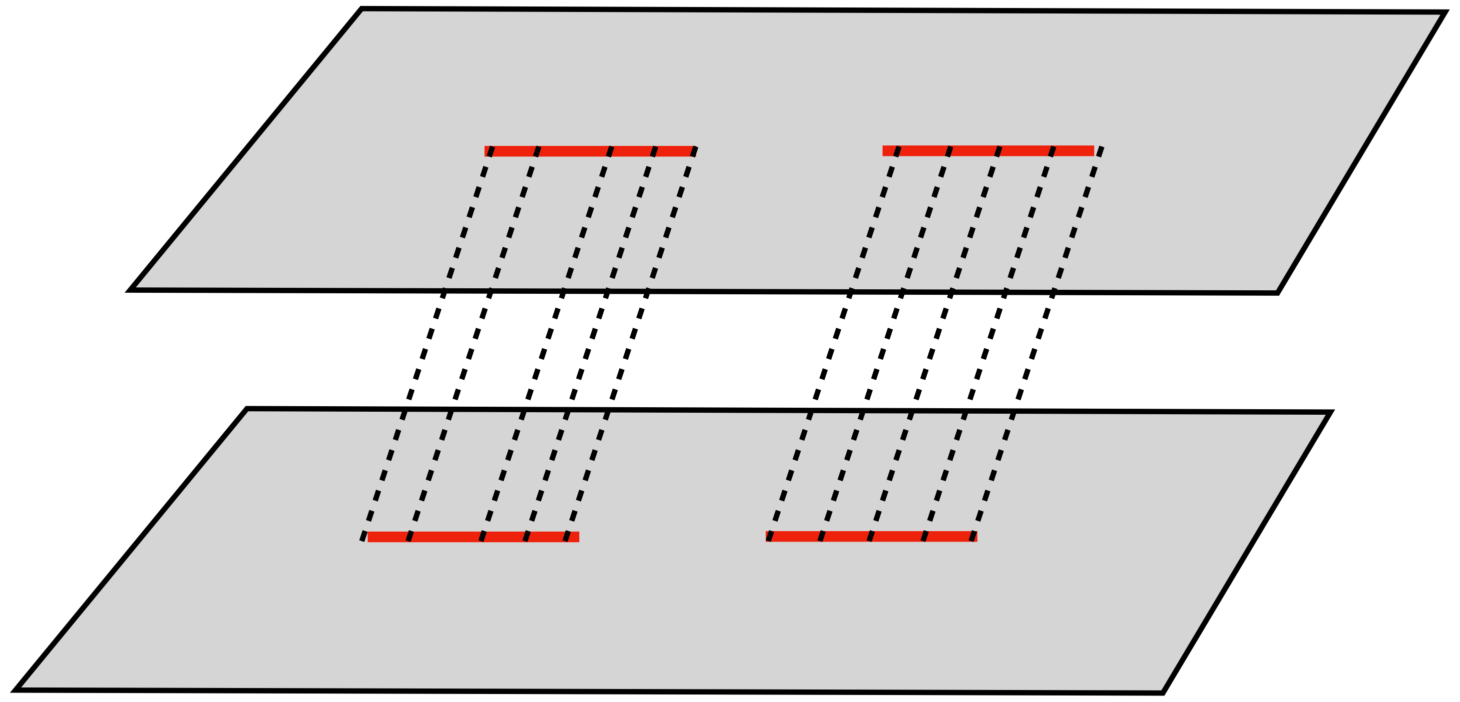
O(n) model on random surfaces

is essentially the only case with infinite $\hat{G}_E$

for which the explicit solution is known

$$\theta = \begin{pmatrix} 1 & -n/2 \\ -n/2 & 1 \end{pmatrix}$$

$$\theta \geqslant 0 \;\Leftrightarrow\; |n| \leqslant 2$$



$$\begin{cases} f(u+2\tau) + nf(u-\tau) + f(u) = \phi(x(u)) \\ f(u+1) = f(u) \end{cases}$$

solution in terms of elliptic theta functions

Open problems

1. Find other (non-algebraic) solvable cases !

2. Complex geometry :

- with $\theta \geqslant 0$: direct proof of symmetry of B ?
- without $\theta \geqslant 0$: existence / uniqueness of solutions of RHP ?

3. Asymptotic analysis : leading order + asymptotics without $\theta \geqslant 0$

4. Are Chern-Simons matrix model on spherical Seifert spaces (ADE) integrable ?