The Kodaira dimension of $\overline{\mathcal{M}}_g$: latest progress on a century-old problem

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The moduli space of curves

• Smooth algebraic curves are compact Riemann surfaces, that is, 1-dimensional compact complex manifolds. A smooth algebraic curve C has a unique topological invariant, its genus $g = \frac{1}{2}\text{dim } H_1(C, \mathbb{Q})$.

• Riemann in 1857 came up with the idea of a space $\mathcal{M}_g$ parametrizing all smooth algebraic curves of genus g. He named this space the moduli space of curves and correctly computed its dimension to be $3g-3$, for $g \geq 2$.

• The existence of $\mathcal{M}_g$ as an algebraic variety is due to Mumford (after Clebsch, Grothendieck, Teichmüller etc.) in the 1960's. Deligne and Mumford in 1969 constructed the moduli space $\overline{\mathcal{M}}_g$ parametrizing stable stable algebraic curves of genus g. It is a projective (compact) algebraic variety with mild singularities of dimension $3g-3$. It contains $\mathcal{M}_g$ as a dense open subset.

• A stable curve is a connected curve with nodal singularities and finite automorphism group. It has the mildest possible singularities.

• The study of the geometry and topology of $\mathcal{M}_g$ has been a major theme of research in mathematics (and theoretical phyics) for 150 years (Mumford, Witten, Konstsevich, Okounkov, Mirzakhani).

A classical problem

• What type of space is $\overline{\mathcal{M}}_g$? What is its Kodaira dimension?

• For a (smooth) projective variety X its Kodaira dimension $\kappa(X)$ is characterized by

$$\dim H^0\big(X, K_X^{\otimes \ell}\big) \sim \ell^{\kappa(X)},$$

where $K_X = \Omega_X^{\mathrm{top}}$ is the canonical bundle of top holomorphic differentials. By definition $\kappa(X) \in \big\{-\infty, 0, \ldots, \dim(X)\big\}$.

• If X is unirational (i.e. $\exists$ a dominant map $\mathbf{P}^n \twoheadrightarrow X$), or more generally rationally connected, then $\kappa(X) = -\infty$. If $\kappa(X) = \dim(X)$, one says that X is of general type.

Theorem

(Severi 1915) The moduli space $\mathcal{M}_g$ of curves of genus g is unirational for $g \leq 10$.

• Severi used plane curves of minimal degree whose nodes are in general position. The result implies that one can write down explicitly the general curve of genus $g \leq 10$ in a family depending on free parameters.

• Severi's unirationality result predates the proof of the existence of $\mathcal{M}_g$!

• Sernesi, Chang-Ran (1980s), Verra (2005): $\mathcal{M}_g$ is unirational for $g = 11, 12, 13, 14$.

• Bruno-Verra (2005), Schreyer (2016): $\overline{\mathcal{M}}_{15}$ is rationally connected.

Will this hold for every g? This was Severi's Conjecture. Answer is: NO!

Theorem

(Harris, Mumford, Eisenbud 1982-87) $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

So for large g one cannot write down explicitly the general curve of genus g. For instance if C is a general curve of genus $g \geq 24$ and $S \supset C$ is a smooth algebraic surface, then C cannot move on S!

• To prove that $\overline{\mathcal{M}}_g$ is of general type one needs to construct lots of sections of $K_{\overline{\mathcal{M}}_g}$. This can be done via the algebraic geometry of the curves in question.

The Harris-Mumford approach

$$\overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{\lfloor \frac{g}{2} \rfloor},$$

where Δ_i are the irreducible boundary divisors (that is, hypersurfaces) in $\overline{\mathcal{M}}_g$. Precisely, $\Delta_0 := \{[C/p \sim q] : C$ of genus $g-1$ and $p, q \in C\}^-$ and for $i \geq 1$ $\Delta_i := \{[C_1 \cup C_2] : C_1$ of genus i, C_2 of genus $g-i\}^-$

- The Hodge class: $\lambda := [\mathbb{E}] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$, where $\mathbb{E} \to \overline{\mathcal{M}}_g$ is the Hodge bundle with fibres $\mathbb{E}[C] = \bigwedge^g H^0(C, \omega_C)$, for any stable curve C.

Theorem

(Harer, Arbarello-Cornalba) For $g \geq 3$, the group $CH^1(\overline{\mathcal{M}}_g)$ is freely generated by the classes $\lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$.

- Via a Riemann-Roch calculation on the universal curve, Harris and Mumford computed the canonical class of the moduli space:

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor \frac{g}{2} \rfloor}.$$

Since the singularities of $\overline{\mathcal{M}}_g$ are mild (Harris-Mumford), $\overline{\mathcal{M}}_g$ is of general type if and only if $K_{\overline{\mathcal{M}}_g}$ is big, i.e. a certain power of it has lots of global sections.

Strategy: Find an effective divisor (hypersurface) $D \subseteq \overline{\mathcal{M}}_g$ such that $[D] = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$, with $a, b_i \geq 0$ and its slope

$$s(D) := \frac{a}{\min_{i \geq 0} b_i} < s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}.$$

Then for $\alpha, \beta > 0$ we can write that

$$K_{\overline{\mathcal{M}}_g} = \alpha \cdot \lambda + \beta \cdot D + \{\text{non-negative combination of } \delta_i\}.$$

Since λ is big (Siegel modular forms), it follows that $K_{\overline{\mathcal{M}}_g}$ is big, that is, $\overline{\mathcal{M}}_g$ is of general type.

Summary: $\overline{\mathcal{M}}_g$ of general type $\Leftrightarrow$ there exists an effective divisor $D \subseteq \overline{\mathcal{M}}_g$ with $s(D) < \frac{13}{2}$.

• D can be chosen to be a geometric divisor, i.e. points of D can be characterized by a geometric property (existence of a linear system, a special syzygy)

• Brill-Noether Theorem: If C is a general curve of genus g, the variety

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C,L) \geq r+1\}$$

has dimension $\rho(g,r,d) = g - (r+1)(g-d+r)$.

Theorem

(Eisenbud-Harris 1987) If $\rho(g,r,d) = -1$, the locus $\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : W^r_d(C) \neq \emptyset\}$ is a divisor in $\mathcal{M}_g$. The class of its closure in $\overline{\mathcal{M}}_g$ is $[\overline{\mathcal{M}}^r_{g,d}] = c\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\Big)$.

Thus $s(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g+1} < \frac{13}{2}$ for $g \geq 24$. This proves the Harris-Mumford-Eisenbud theorem (at least when $g+1$ is composite).

• Can one construct divisors of slope $< 6 + \frac{12}{g+1}$ (Slope Conjecture)?

Theorem

(Farkas-Popa 2003) If $D \subseteq \overline{\mathcal{M}}_g$ is an effective divisor with $s(D) < 6 + \frac{12}{g+1}$, then

$$D \supseteq \mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}.$$

The Kodaira dimension of $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$

Theorem

(Farkas-Jensen-Payne) Both moduli spaces $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type.

Fix $r = 6$ and $d = g + 3$. $\begin{cases} g = 22, d = 25 : C \xrightarrow{|L|} \mathbf{P}^6, \ \rho = 1 \\ g = 23, d = 26 : C \xrightarrow{|L|} \mathbf{P}^6, \ \rho = 2 \end{cases}$

Denote by $\mathcal{G}_g$ the parameter space of pairs $[C, L]$, where C is a curve of genus g and $L \in W^6_{g+3}(C)$. One has a forgetful map $\sigma \colon \mathcal{G}_g \to \mathcal{M}_g$ whose fibres are ρ-dimensional. The degeneracy locus of those $[C, L] \in \mathcal{G}_g$ for which the map

$$\phi_L \colon \mathrm{Sym}^2 H^0(C, L) \longrightarrow H^0(C, L^2)$$

is not injective has expected codimension $g - 20$, that is, 2 if $g = 22$ respectively 3 if $g = 23$. Consequently

$$\mathcal{D}_g := \big\{[C] \in \mathcal{M}_g : \exists L \in W^6_{g+3}(C) \text{ with } \phi_L \text{ not injective}\big\}$$

is a virtual divisor on $\mathcal{M}_g$.

Since $\mathcal{K}_g \subseteq \mathcal{D}_g$ (sections of $K3$ surfaces behave differently w.r.t. quadrics!), $\mathcal{D}_g$ is a prime candidates to show that $\overline{\mathcal{M}}_g$ is of general type.

- Issues: (i) Transversality and (ii) compactifying $\mathcal{D}_g$.
- We choose a partial compactification $\widetilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ differing from $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$ only in codimension 2 and a proper extension

$$\sigma \colon \widetilde{\mathcal{G}}_g \longrightarrow \widetilde{\mathcal{M}}_g,$$

from the stack of limit linear series.

- We construct locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ over $\widetilde{\mathcal{G}}_g$ with $\mathsf{rk}(\mathcal{E}) = 7$ and $\mathsf{rk}(\mathcal{F}) = g + 7 (= 2d + 1 - g)$, together with a morphism

$$\phi \colon \mathsf{Sym}^2 \mathcal{E} \to \mathcal{F},$$

s.t. for $[C, L] \in \mathcal{G}^6_{g+3}$, we have $\mathcal{E}(C, L) = H^0(L)$, $\mathcal{F}(L) = H^0(L^2)$ and

$$\phi_{[C,L]} \colon \mathsf{Sym}^2 H^0(C, L) \to H^0(C, L^2)$$

is the usual multiplication of sections.

Definition

Define the virtual class $[\widetilde{\mathcal{D}}_g]^{\mathrm{vir}} := \sigma_* \left(c_{g-20} \big(\mathcal{F} - \mathsf{Sym}^2 \mathcal{E} \big) \right) \in CH^1(\widetilde{\mathcal{M}}_g)$.

• If the degeneracy locus of ϕ has the expected codimension 2 for $g = 22$ and 3 for $g = 23$ for $g = 23$, then $[\widetilde{\mathcal{D}}_g]^{\mathrm{vir}} = [\widetilde{\mathcal{D}}_g]$.

• If the degeneracy locus $\mathcal{U}$ of ϕ: $\mathrm{Sym}^2\mathcal{E} \to \mathcal{F}$ has the expected codimension 2 for $g = 22$, respectively 3 for $g = 23$, then $[\widetilde{\mathcal{D}}_g]^{\mathrm{vir}} = [\widetilde{\mathcal{D}}_g]$.

Theorem

(FJP) The virtual classes of the MRC divisors have the following slopes:

$$s\big([\widetilde{\mathcal{D}}_{22}]^{\mathrm{vir}}\big) = \frac{17121}{2636} = 6.495\ldots \text{ and } s\big([\widetilde{\mathcal{D}}_{23}]^{\mathrm{vir}}\big) = \frac{470749}{72725} = 6.473\ldots.$$

Conclusion: Both virtual slopes are $< \frac{13}{2}$! Transversality issues:

Theorem

(FJP) For a general C of genus 22 or 23, the multiplication map ϕ_L is injective for every $L \in W^6_{g+3}(C)$.

• This shows that $\widetilde{\mathcal{D}}_g \neq \widetilde{\mathcal{M}}_g$. The proof uses tropical geometry.