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Entangled states representation by deformed oscillators and some effective models

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Plan

1. Composite bosons (*quasibosons*) as entangled bipartite system. Entangled / bound states *of composite particles* – **effectively**, as deformed oscillators.
2. Representation of *composite* bosons operators by *deformed* oscillator ones.
3. The representation for "fermion + deformed boson" composite fermi-like particles (CFs).
   Using deformed constituents as well.
4. Entanglement measures, between constituents of composite boson (or fermionic analogs). Link with deformation parameter, and with energy.
5. Use of the representation (realization) when constructing deformed Bose gas models aimed at effective description of interaction/ compositeness of particles
6. Conclusions.

Motivation.

Composite (quasi)particles in diverse branches of physics:

- *Mesons, baryons, nuclei* in nuclear or subnuclear physics;
- *Excitons, biexcitons, dropletons* in semiconductors and nanostructures (quantum dots etc.);
- *Trions* ("2 electrons + hole" or "2 holes + electron") in semiconductors;
- *Cooper pairs* in superconductors, in the study of electron transport;
- *Bipolarons* in crystals, organic semiconductors;
- *Composite fermions* ("electron + magnetic flux quanta" bound states) appearing in fractional quantum Hall effect;
- *Biphotons* in quantum optics, quantum information;
- *Biphonons, triphonons, ...* in crystals;
- *Atoms, molecules, ...* .

Hydrogen atom viewed as quasiboson

The creation operator for hydrogen atom[1] with zero total momentum and quantum number n

$$A^{\dagger}_{\mathbf{0}n} = \frac{(2\pi\hbar)^{3/2}}{\sqrt{V}} \sum_{\mathbf{p}} \phi_{\mathbf{p}n} a^{(e)\dagger}_{\mathbf{p}} b^{(p)\dagger}_{-\mathbf{p}}, \tag{1}$$

where $a^{(e)\dagger}_{\mathbf{p}}$ and $b^{(p)\dagger}_{-\mathbf{p}}$ are the creation operators for electron and proton respectively taken with opposite momenta. The momentum-space wavefunction $\phi_{\mathbf{p}n}$ is determined by the Schrodinger equation:

$$\phi_{\mathbf{p}n} = \int \frac{e^{\frac{i}{\hbar}\mathbf{pr}}}{(2\pi\hbar)^{3/2}} \phi_n(\mathbf{r}) d^3\mathbf{r}; \quad -\frac{\hbar^2\nabla^2}{2m}\phi_n(\mathbf{r}) + U(\mathbf{r})\phi_n(\mathbf{r}) = E_n\phi_n(\mathbf{r}).$$

Expansion (1) can be viewed directly as the Schmidt decomposition for the state $A^{\dagger}_{\mathbf{0}n}|0\rangle$ with Schmidt coefficients $\lambda_{\mathbf{p}} = \frac{(2\pi\hbar)^{3/2}}{\sqrt{V}}\phi_{\mathbf{p}n}$.

[1] Note that similar ansatz is used for the excitonic creation operators

Composite particles

Creation & annihilation operators:

$$A_\alpha^\dagger = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger,$$

$$A_\alpha = \sum_{\mu\nu} \overline{\Phi}_\alpha^{\mu\nu} b_\nu a_\mu,$$

$$N_\alpha = N_\alpha(a_\mu^\dagger, a_\mu, b_\nu^\dagger, b_\nu).$$

Commutator:

$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha\beta} - \Delta_{\alpha\beta}\left[\Phi_\gamma^{\mu\nu} \middle| a_\mu, b_\nu\right].$$

Closed-form relation holds

$$[[A_\alpha, A_\beta^\dagger], A_\gamma^\dagger] = -\epsilon \sum_\delta C_{\alpha\beta\gamma}^\delta(\Phi)\, A_\delta^\dagger,$$

$C_{\alpha\beta\gamma}^\delta$ depend on wavefunctions Φ.

Deformed oscillators

Generalized oscillator algebra:

$$\begin{cases} \mathcal{A}_\alpha^\dagger \mathcal{A}_\alpha = \varphi(\mathcal{N}_\alpha), \\ [\mathcal{A}_\alpha, \mathcal{A}_\alpha^\dagger] = \varphi(\mathcal{N}_\alpha + 1) - \varphi(\mathcal{N}_\alpha), \\ [\mathcal{N}_\alpha, \mathcal{A}_\alpha^\dagger] = \mathcal{A}_\alpha^\dagger,\ [\mathcal{N}_\alpha, \mathcal{A}_\alpha] = -\mathcal{A}_\alpha. \end{cases}$$

Commutator:

$$[\mathcal{A}_\alpha, \mathcal{A}_\alpha^\dagger] = 1 + \delta_\varphi(\mathcal{N}_\alpha)$$

Example. Arik-Coon deformation

$$\varphi(\mathcal{N}_\alpha) = \left[\mathcal{N}_\alpha\right]_q \equiv \frac{q^{\mathcal{N}_\alpha} - 1}{q - 1},$$

$$[\mathcal{A}_\alpha, \mathcal{A}_\alpha^\dagger] = 1 - (q-1)\left[\mathcal{N}_\alpha\right]_q;$$

$$[[\mathcal{A}_\alpha, \mathcal{A}_\beta^\dagger], \mathcal{A}_\gamma^\dagger] = -\sum_\delta \widetilde{C}_{\alpha\beta\gamma}^\delta(q; \mathcal{N}_\delta)\, \mathcal{A}_\delta^\dagger.$$

Deformed oscillators

Defined as nonlinear generalization of ordinary quantum oscillator by deformation structure function $\varphi(N)$,

$$[a, a^\dagger] = 1 + \delta_\varphi(N), \quad \delta_\varphi(N) \equiv \varphi(N+1) - \varphi(N) - 1$$

with Fock space: $|n\rangle = \frac{1}{\sqrt{\varphi(n)!}}(a^\dagger)^n|0\rangle$.

The realization/modelling of composite bosons (e.g. mesons, excitons, cooperons etc.) by deformed oscillators allows to:

- "forget" the internal structure details;
- considerably simplify the calculations by applying the *theory of deformed oscillators* only.

Operators of composite particles *map* to the deformed oscillator operators. The internal structure information enters into deformation parameter(s).

Quasibosons' operators as Deformed oscillators' ones

Composite (quasi-)boson creation/annihilation operators $A^\dagger_\alpha$, A_α (mode α) are represented by deformed oscillator ones

$$A^\dagger_\alpha = \sum_{\mu\nu} \Phi^{\mu\nu}_\alpha a^\dagger_\mu b^\dagger_\nu \;\rightarrow\; \mathcal{A}^\dagger_\alpha\,, \quad A_\alpha = \sum_{\mu\nu} \overline{\Phi^{\mu\nu}_\alpha} b_\nu a_\mu \;\rightarrow\; \mathcal{A}_\alpha\,, \quad (2)$$

which obey standard definition:

$$\mathcal{A}^\dagger_\alpha \mathcal{A}_\alpha = \varphi(\mathcal{N}_\alpha), \qquad [\mathcal{A}_\alpha, \mathcal{A}^\dagger_\beta] = \delta_{\alpha\beta}\big(1 + \delta_\varphi(\mathcal{N}_\alpha)\big),$$

$\Phi^{\mu\nu}_\alpha$ is the composite's wavefunction, and constituent operators a_μ, b_ν are both fermionic (or both bosonic[2]).

We find *wavefunctions* & resp. operators $A_\alpha, A^\dagger_\alpha, N_\alpha \equiv \varphi^{-1}(A^\dagger_\alpha A_\alpha)$ behaving *on Co-boson states* as if for φ-deformed oscillator

$$\begin{aligned} &[A_\alpha, A^\dagger_\beta] = \delta_{\alpha\beta} - \epsilon\Delta_{\alpha\beta} = \varphi(N_\alpha + 1) - \varphi(N_\alpha) \quad \text{at } \alpha = \beta, \\ &[N_\alpha, A^\dagger_\alpha] = A^\dagger_\alpha, \quad [N_\alpha, A_\alpha] = -A_\alpha, \end{aligned} \quad (3)$$

with $\Delta_{\alpha\beta} \equiv \sum_{\mu\mu'} (\Phi_\beta \Phi^\dagger_\alpha)^{\mu'\mu} a^\dagger_{\mu'} a_\mu + \sum_{\nu\nu'} (\Phi^\dagger_\alpha \Phi_\beta)^{\nu\nu'} b^\dagger_{\nu'} b_\nu \leftrightarrow \Delta_\varphi(\mathcal{N}_\alpha)$.

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[2] $\epsilon = +1/-1$ – for boson/fermion constituents respectively

Realization conditions (3) reduce to equations for Φ_α and $\varphi(n)$:

$$\Phi_\beta \Phi^\dagger_\alpha \Phi_\gamma + \Phi_\gamma \Phi^\dagger_\alpha \Phi_\beta = 0, \quad \alpha \neq \beta,$$
$$\Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha = \tilde{\mu}\, \Phi_\alpha, \quad \tilde{\mu} = 1 - \frac{1}{2}\varphi(2),$$
$$\varphi(n+1) = \sum\nolimits_{k=0}^{n} (-1)^{n-k} C^k_{n+1} \varphi(k), \quad n \geq 2.$$

Their solution yields:

- *deformation structure function* $\varphi_{\tilde{\mu}}(N) = (1+\epsilon\tilde{\mu})N - \epsilon\tilde{\mu}N^2$

with

- *deformation parameter* $\tilde{\mu} = \frac{1}{m}$, $m \in \mathbb{N}$ (m–positive integer);
- *matrices*

$$\Phi_\alpha = U_1(d_a)\, \mathrm{diag}\{0..0, \sqrt{\tilde{\mu}}\, U_\alpha(m), 0..0\} U^\dagger_2(d_b), \qquad (4)$$

where $U_1(d_a)$, $U_2(d_b)$, $U_\alpha(m)$ are arbitrary unitary matrices of dimensions $d_a \times d_a$, $d_b \times d_b$ and $m \times m$.

Generalization to quasibosons, composed of q-fermions

Commutation relations for the constituent q-fermions:

$$a_\mu a^\dagger_{\mu'} + q^{\delta_{\mu\mu'}} a^\dagger_{\mu'} a_\mu = \delta_{\mu\mu'}, \qquad b_\nu b^\dagger_{\nu'} + q^{\delta_{\nu\nu'}} b^\dagger_{\nu'} b_\nu = \delta_{\nu\nu'},$$
$$a_\mu a_{\mu'} + a_{\mu'} a_\mu = 0, \quad \mu \neq \mu', \quad b_\nu b_{\nu'} + b_{\nu'} b_\nu = 0, \quad \nu \neq \nu'.$$

The nilpotency is absent (as opposed to the previous case):

$$q < 1 \quad \Rightarrow \quad (a^\dagger_\mu)^k \neq 0, \quad (b^\dagger_\nu)^k \neq 0, \quad k \geq 2. \qquad (5)$$

Solving the conditions analogous to (3) we obtain:

- Resulting expression for the structure function:

$$\varphi(n) = \left([n]_{-q}\right)^2 = \left(\frac{1-(-q)^n}{1+q}\right)^2, \quad q < 1. \qquad (6)$$

- Solution for matrices Φ_α at $q < 1$:

$$\Phi^{\mu\nu}_\alpha = \Phi^{\mu_0(\alpha)\nu_0(\alpha)}_\alpha \delta_{\mu\mu_0(\alpha)} \delta_{\nu\nu_0(\alpha)}, \qquad |\Phi^{\mu_0(\alpha)\nu_0(\alpha)}_\alpha| = 1.$$

This solution is NOT entangled for $q < 1$! Reason – overcompleteness of basis (5). $\Rightarrow$ *"Physics" subspace* should be reduced.

For more details see:

[1] A. Gavrilik, I. Kachurik, Yu. Mishchenko, J. Phys. A **44**, 475303 (2011).

"Fermion + Deformed boson" composite fermions

CFs' creation/annihilation operators are given by "ansatz" (2); $a^\dagger_\mu, a_\mu$ & $b^\dagger_\nu, b_\nu$ – operators for constituent def. bosons & fermions,

$$a^\dagger_\mu a_\mu = \chi(n^a_\mu), \quad [a_\mu, a^\dagger_{\mu'}] = \delta_{\mu\mu'}\big(\chi(n^a_\mu+1)-\chi(n^a_\mu)\big); \quad [a^\dagger_\mu, a^\dagger_{\mu'}] = 0.$$

$\Rightarrow$ *Fermionic nilpotency* $(A^\dagger_\alpha)^2 = 0$.

For *nondeformed constituents* ($\chi(n) \equiv n$) anticommutator yields

$$\{A_\alpha, A^\dagger_\beta\} = \delta_{\alpha\beta} + \sum\nolimits_{\mu\mu'}(\Phi_\beta\Phi^\dagger_\alpha)^{\mu'\mu} a^\dagger_{\mu'} a_\mu - \sum\nolimits_{\nu\nu'}(\Phi^\dagger_\alpha\Phi_\beta)^{\nu\nu'} b^\dagger_{\nu'} b_\nu,$$

So, the concerned CFs' operators are *realized by fermion operators*. The validity of the realization on one-CF states yields

$$\begin{aligned}(\Phi_\beta\Phi^\dagger_\alpha\Phi_\gamma)^{\mu\nu} - (\Phi_\gamma\Phi^\dagger_\alpha\Phi_\beta)^{\mu\nu} + \\ + \big(\chi(2)-2\big)\Big[(\Phi_\beta\Phi^\dagger_\alpha)^{\mu\mu}\Phi^{\mu\nu}_\gamma - (\Phi_\gamma\Phi^\dagger_\alpha)^{\mu\mu}\Phi^{\mu\nu}_\beta\Big] = 0. \quad (7)\end{aligned}$$

For non-deformed constituents:

$$\begin{cases} \mathrm{Tr}(\Phi_\beta\Phi^\dagger_\alpha) = \delta_{\alpha\beta}, \\ \Phi_\beta\Phi^\dagger_\alpha\Phi_\gamma - \Phi_\gamma\Phi^\dagger_\alpha\Phi_\beta = 0. \end{cases} \quad (8)$$

a) Single CF mode α case. General solution:

$$\Phi_\alpha = U_\alpha \,\mathrm{diag}\{\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, ...\} V_\alpha^\dagger$$

with $\lambda_i^{(\alpha)} \geq 0$, $\sum_i (\lambda_i^{(\alpha)})^2 = 1$, and arbitrary unitary U_α, V_α.

b) CFs in 2 modes & non-deformed constituents in 3 modes. The parametrization of two orthonormal vectors $(\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, \lambda_3^{(\alpha)})$, $\alpha = \overline{1,2}$ follows from $SU(3)$ parametrization, e.g.

$$\lambda_1^{(1)} = \cos\theta_1 \cos\theta_2, \quad \lambda_2^{(2)} = \cos\theta_1 \sin\theta_2, \quad \lambda_3^{(3)} = \sin\theta_1;$$

and
$$\begin{aligned}
\lambda_1^{(2)} &= -\sin\theta_1 \cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3 e^{i\gamma}, \\
\lambda_2^{(2)} &= \cos\theta_2 \sin\theta_3 e^{i\gamma} - \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
\lambda_3^{(2)} &= \cos\theta_1 \cos\theta_3, \qquad 0 \leq \theta_1, \theta_2, \theta_3 \leq \pi/2, \quad 0 \leq \gamma \leq 2\pi.
\end{aligned} \tag{9}$$

Solution for any number of modes (non-def. CF constituents).

$$\Phi_\alpha = U \,\mathrm{diag}\{\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, ...\} V^\dagger$$

where U, V are fixed (for any α) unitary matrices, $\lambda^{(\alpha)} = (\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, ...)$ are complex orthonormal vectors. Compare with (4).

Interpretation of the involved parameters. The concerned parameters θ_i, $i = \overline{1,3}$, and γ (the 3-mode case) should correspond to CF internal quantum numbers like spin, the ones determining CF binding energy, etc.

Possible applications: trions, baryons. The results involving nontrivial deformation χ may be applied to effective description of three-component composite particles e.g. when two constituents form a bound state (χ-deformed boson). For the modeling of composite *constituent boson*, the quasibosons realization could be applied. Proper combination of two realizations, – quasibosonic and CF ones, – can provide an alternative effective description of tripartite complexes like:

- trions (e.g. exciton-electron or "2 electrons + hole" composites),
- baryons (as either three quark or quark-diquark bound state).

These issues deserve special detailed study.

For composite (quasi-) fermions' realization see:

[2] A. Gavrilik, Yu. Mishchenko, Ukr. J. Phys. **64**(12), 1134 (2019).

Entanglement in composite boson vs deformation parameter [A. Gavrilik, Yu. Mishchenko, PLA **376**, 1596 (2012)]

The state of the quasiboson which can be realized by deformed oscillator is entangled (inter-component entanglement):

$$|\Psi_\alpha\rangle = \sum_{k=1}^{m} \frac{1}{\sqrt{m}} |v_k^\alpha\rangle \otimes |w_k^\alpha\rangle, \;\; |v_k^\alpha\rangle = U_1^{\mu k}|a_\mu\rangle, |w_k^\alpha\rangle = \tilde{U}_2^{k\nu}|b_\nu\rangle,$$

$$\lambda_k^\alpha = \lambda = \sqrt{\tilde{\mu}} = \sqrt{1/m}. \qquad (10)$$

Calculation of the entanglement characteristics yields:

- Schmidt rank $r = m$;
- Schmidt number (P – the purity of subsystems)

$$K = \left[\sum_k (\lambda_k^\alpha)^4\right]^{-1} = 1/P = m; \qquad (11)$$

- Entanglement entropy $S_{\text{entang}} = -\sum_k (\lambda_k^\alpha)^2 \ln(\lambda_k^\alpha)^2 = \ln(m)$;
- Concurrence $\quad C = \left[\frac{r}{r-1}\left(1 - \sum_k (\lambda_k^\alpha)^4\right)\right]^{1/2} = 1.$

Remark. Strongly entangled composite boson (high m) approaches standard boson at small quantum numbers n:

$$\varphi(n) \approx \varphi_{boson}(n) \equiv n, \quad n \ll m, \;\; m \gg 1.$$

Generalization to multi-quasiboson states

Example 1. Multi-quasiboson state, *one mode*

$$|n_\alpha\rangle = [\varphi(n_\alpha)!]^{-1/2}(A_\alpha^\dagger)^{n_\alpha}|0\rangle \tag{12}$$

(φ-factorial $\varphi(n)! \stackrel{def}{=} \varphi(1)...\varphi(n)$).
Entanglement characteristics for (12):

$$K_{\epsilon=+1} = C_m{}^{n_\alpha}, \; K_{\epsilon=-1} = C^{n_\alpha}_{m+n_\alpha-1};$$
$$S_{\rm entang}|_{\epsilon=+1} = \ln C_m{}^{n_\alpha}, \; S_{\rm entang}|_{\epsilon=-1} = \ln C^{n_\alpha}_{m+n_\alpha-1}.$$

Example 2. n-quasiboson Fock states with 1 quasiboson per mode:

$$|\Psi\rangle = A^\dagger_{\gamma_1} \cdot ... \cdot A^\dagger_{\gamma_n}|0\rangle, \quad \gamma_i \neq \gamma_j, \; i \neq j, \; i,j = 1,...,n.$$

Entanglement characteristics: $K_{\epsilon=+1} = K_{\epsilon=-1} = m^n$;

$$S_{\rm entang}|_{\epsilon=+1} = S_{\rm entang}|_{\epsilon=-1} = n\ln(m).$$

Example 3. For the coherent state (two bosonic constituents)

$$|\Psi_\alpha\rangle = \tilde{C}(\mathcal{A};m)\sum_{n=0}^{\infty}\frac{\mathcal{A}^n}{\varphi(n)!}(A_\alpha^\dagger)^n|0\rangle, \quad A_\alpha|\Psi_\alpha\rangle = \mathcal{A}_\alpha|\Psi_\alpha\rangle$$

Schmidt number and Entanglement entropy were also calculated.

Energy dependence of Quasiboson's Entanglement entropy

[A.M. Gavrilik, Yu.A. Mishchenko, J. Phys. A 46, 145301 (2013)]

Entanglement characteristics between the constituents of a quasiboson and their energy dependence are important in quantum information research: in quantum communication, entanglement production/enhancement, particle addition or subtraction etc.

We take the Hamiltonian of deformed oscillators system as

$$H = \frac{1}{2}\sum_{\alpha} \hbar\omega_{\alpha}\big(\varphi(N_{\alpha}) + \varphi(N_{\alpha}+1)\big). \tag{13}$$

Single quasiboson case

$$S_{\text{ent}}(E) = \ln\frac{\epsilon}{\frac{3}{2} - \frac{E}{\hbar\omega}} = \begin{cases} -\ln\Big(\frac{3}{2} - \frac{E}{\hbar\omega}\Big), & \epsilon = 1, \quad \frac{1}{2} \le \frac{E}{\hbar\omega} \le \frac{3}{2}, \\ -\ln\Big(\frac{E}{\hbar\omega} - \frac{3}{2}\Big), & \epsilon = -1, \quad \frac{3}{2} \le \frac{E}{\hbar\omega} \le \frac{5}{2}. \end{cases}$$

The corresponding plots are presented on Fig. 1, Fig. 2.

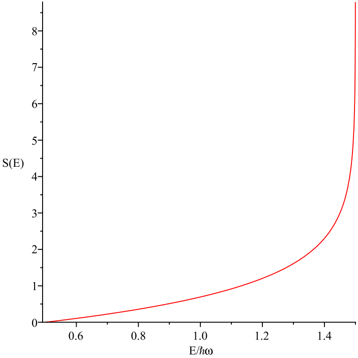


Figure 1: Dependence of the entanglement entropy S_{ent} on the energy E_α for a single composite boson in the case of fermionic components i.e. at $\epsilon = +1$.

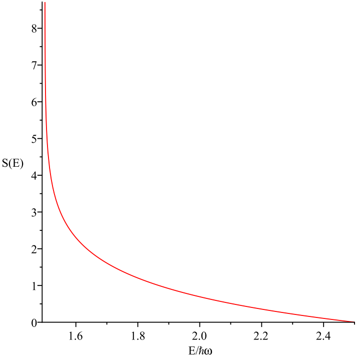


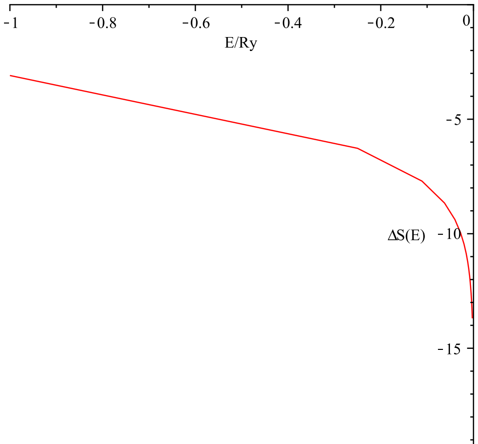
Figure 2: Dependence of the entanglement entropy S_{ent} on the energy E_α for a single composite boson in the case of bosonic components i.e. at $\epsilon = -1$.

The entanglement entropy for the hydrogen atom is given by

$$S_{\text{ent}} = -\sum_{\mathbf{p}} |\lambda_{\mathbf{p}}|^2 \ln |\lambda_{\mathbf{p}}|^2 = -\sum_{\mathbf{p}} \frac{(2\pi\hbar)^3}{V} |\phi_{\mathbf{p}n}|^2 \ln\Big(\frac{(2\pi\hbar)^3}{V} |\phi_{\mathbf{p}n}|^2 \Big).$$

(!) **Remark**: H-atom cannot be *exactly* realized by quadratically deformed oscillators.

Let us consider the simplest case of quantum numbers $l=0, m=0$.



Effective deformed models

Approach A. Deformed are *underlying relations* (microscopic, as usually) for a physical system:

$$R^{(m)}(A_k, B_k) = 0 \;\rightarrow\; R_q^{(m)}(A_k, B_k) = 0.$$

For instance, *deformed (nonlinear) oscillator* is defined by the structure function φ and relations

$$a^\dagger a = \varphi(N),\; aa^\dagger = \varphi(N+1) \quad\Rightarrow\quad [a, a^\dagger] = 1 + \delta_\varphi(N).$$

Approach B. Standard/nondeformed relations apply, but – to *someway deformed* physical quantities (usually macroscopic):

$$F_i \rightarrow F_i^{(q)}, \qquad R(F_i) = 0 \;\rightarrow\; R(F_i^{(q)}) = 0.$$

In q-Bose gas model of [N. Swamy, 2009] and others, the particle number is deformed say as: $N^{(q)} = z\mathcal{D}_z^{(q)} \ln Z^{(0)}$
using deformed (Jackson's) derivative $\mathcal{D}_z^{(q)}$. Then, other thermodynamic quantities could be derived.

Approach C. . . . (*other variants*)

$\tilde{\mu}, q$-deformed Bose gas model **#1**: deformation of thermodynamics

In our φ-Bose gas model (type **B**), total number of particles

$$N^{(\varphi)} = \varphi\Big(z\frac{d}{dz}\Big)\ \ln Z^{(0)} \equiv z\mathcal{D}_z^{(\varphi)}\ln Z^{(0)} = \frac{V}{\lambda^3}\sum_{n=1}^{\infty}\varphi(n)\frac{z^n}{n^{5/2}}$$

where $z = e^{\beta\mu}$ – fugacity, $\ln Z^{(0)} = -\sum_i \ln(1 - ze^{-\beta\varepsilon_i})$ and the φ-derivative $\mathcal{D}_z^{(\varphi)}$ is used (like Jackson's q-derivative):

$$z\frac{d}{dz} \to z\mathcal{D}_z^{(\varphi)} = \varphi\Big(z\frac{d}{dz}\Big).$$

(*Deformed*) Partition function $Z^{(\varphi)}$ is recovered from $N^{(\varphi)} = \big(z\frac{d}{dz}\big)\ln Z^{(\varphi)}$. So we obtain φ-deformed virial (λ^3/v)-expansion

$$\frac{P^{(\varphi)}v^{(\varphi)}}{k_{\rm B}\,T} = 1 - \frac{\varphi(2)}{2^{7/2}}\frac{\lambda^3}{v^{(\varphi)}} + \Big(\frac{\varphi(2)^2}{2^5} - \frac{2\varphi(3)}{3^{7/2}}\Big)\Big(\frac{\lambda^3}{v^{(\varphi)}}\Big)^2 + \dots$$

where $v^{(\varphi)} = \frac{V}{N^{(\varphi)}}$ is specific volume, $\lambda = h/\sqrt{2\pi m k_B T}$ – thermal wavelength.

Second virial coefficient for a gas with interaction.
As known,

$$V_2 - V_2^{(0)} = -8^{1/2} \sum_B e^{-\beta\varepsilon_B} - \frac{8^{1/2}}{\pi} \sum_l{}' (2l+1) \int_0^\infty e^{-\beta\frac{\hbar^2 k^2}{m}} \frac{\partial \delta_l(k)}{\partial k} dk$$

where B runs over bound states, l is the angular momentum, and $\delta_l(k)$ partial wave phaseshift. In low-energy approximation we retain only the $l = 0$ summand (s-*wave approximation*). Resp. phaseshift $\delta_0(k)$ is determined by Schrodinger eq. for a specified interaction.

The gas of non-interacting but composite bosons. Using the anzats $A_\alpha^\dagger = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger$ and the known formula

$$V_2 = \frac{1}{2!V} \Big[\Big(\mathrm{Tr}_1\, e^{-\beta H_1} \Big)^2 - \mathrm{Tr}_2\, e^{-\beta H_2} \Big]. \qquad (14)$$

if for all $(\mathbf{k}, n)$-modes $(A_{\mathbf{k},n}^\dagger)^2 |0\rangle \neq 0$ we obtain that in the absence of explicit interaction between *composite* bosons

$$V_2(T) - V_2^{(0)} = -\frac{1}{2^{5/2}} \Big(\sum\nolimits_n e^{-2\beta\varepsilon_n^{int}} - 1 \Big). \qquad (15)$$

Effective account for the **both** factors

[A. Gavrilik, Yu. Mishchenko, Phys. Rev. E **90**, 052147 (2014)].

Interparticle interaction. In [N. Swamy, J.Stat.Mech (2009)] the q-deformation given by structure function $[N]_q \equiv \frac{1-q^N}{1-q}$ was interpreted as incorporating the interparticle interaction.

Compositeness of particles. In [Gavrilik *et al*, J.Phys.A (2011)] composite (two-fermion or two-boson) quasi-bosons with creation/annihilation operators (2) were realized by def. bosons with quadratic SF $\varphi_{\tilde{\mu}}(N)$.

Unification of $\tilde{\mu}$- and q-deformations. The effective description of the *both* mentioned factors may be expected from a combination of SFs $[N]_q$ and $\varphi_{\tilde{\mu}}(N)$ (as certain approximation). The simplest (*but non-unique*) variant:

$$\varphi_{\tilde{\mu},q}(N) = \varphi_{\tilde{\mu}}\big([N]_q\big) = (1+\tilde{\mu})[N]_q - \tilde{\mu}[N]_q^2 \tag{16}$$

Deformed vs. **microscopic** 2^{nd} virial coefficient: *deformation parameter(s) explicitly*

We use the first deformed virial coefficients,

$$V_2^{(\tilde{\mu},q)} = -\frac{\varphi_{\tilde{\mu},q}(2)}{2^{7/2}}, \quad V_3^{(\tilde{\mu},q)} = \frac{\varphi_{\tilde{\mu},q}(2)^2}{2^5} - \frac{2\varphi_{\tilde{\mu},q}(3)}{3^{7/2}} \tag{17}$$

to juxtapose with resp. results in microscopic description. Parameter $\tilde{\mu}$ corresponds to the compositeness, q – to interparticle interaction.

Effective account for the compositeness up to $(\lambda^3/v)^2$-terms.
In the absence of explicit interaction between quasibosons, cf. (15),

$$V_2(T) = -\frac{1}{2^{5/2}} \sum_n e^{-2\beta\varepsilon_n^{int}}$$

On the other hand: $\quad V_2^{(\tilde{\mu},q)} - V_2^{(0)}|_{q=1} = \frac{2-\varphi_{\tilde{\mu},q}(2)}{2^{7/2}}|_{q=1} = \frac{\tilde{\mu}}{2^{5/2}}$.

After equating,

$$\Rightarrow \quad \tilde{\mu} = \tilde{\mu}(\varepsilon_n^{int}, \Phi_\alpha^{\mu\nu}, T) = 1 - \sum_n e^{-2\beta\varepsilon_n^{int}}. \tag{18}$$

$*$ Temperature dependence of def. parameter $\tilde{\mu}(\ldots, \underline{T})$ appears unexpected since in our interpretation $\tilde{\mu}$ characterizes *just* particles' substructure of deformed Bose gas.

$\tilde{\mu}, q$-deformed Bose gas model **#2**: deformation of *particle operators*

▷ Correlation function intercepts for certain $\tilde{\mu}, q$-deformed Bose gas were calculated, and compared to STAR (RHIC) experiment on $\pi\pi$-correlations.

▷ This $\tilde{\mu}, q$-deformed Bose gas model (type **A**) is based on $\tilde{\mu}, q$-deformation structure function $\varphi_{\tilde{\mu},q}(N)$,

$$\varphi_{\tilde{\mu},q}(N) = \varphi_{\tilde{\mu}}\big([N]_q\big) = (1+\tilde{\mu})[N]_q - \tilde{\mu}\big([N]_q\big)^2 \equiv [N]_{\tilde{\mu},q},$$

where $[N]_q \equiv \frac{1-q^N}{1-q}$.

▷ Detailed analysis of application of this model is given in

[A. M. Gavrilik, Yu. A. Mishchenko, Nucl. Phys. B **891**, 466 (2015)].

Conclusions

• For quasibosons built of 2 fermions, 2 bosons or 2 q-fermions their operator representation as deformed oscillators ('deformed bosons') with quadr. structure function is found. The "fermion + def. boson" composite fermi-particles were also treated.

• The deformation parameter turned out to be unambiguously determined by entanglement characteristics for realized composite bosons. Thus, inter-component entanglement reveals the physics meaning of the deformation parameter.

• The relation of 2nd virial coefficient of resp. $\tilde{\mu}, q$-deformed Bose gas model to the parameters of compositeness (interaction) is found. Def. parameter $\tilde{\mu}$ relates to *internal energy levels* of quasibosons (q – to the *scattering length and effective radius* of interaction).

Basing on Publications:

1. A. M. Gavrilik, I. I. Kachurik and Yu. A Mishchenko. *Quasi-bosons composed of two q-fermions: realization by deformed oscillators*. J. Phys. A: Math. Theor., **44**, 475303 (2011).
2. A. M. Gavrilik and Yu. A Mishchenko. *Entanglement in composite bosons realized by deformed oscillators*. Phys. Lett. A, **376**, P. 1596 (2012).
3. A. M. Gavrilik and Yu. A Mishchenko. *Energy dependence of the entanglement entropy of composite boson (quasiboson) systems*. J. Phys. A: Math. Theor., **46**, 145301 (2013).
4. A. M. Gavrilik and Yu. A Mishchenko. *Virial coefficients in the $(\tilde{\mu}, q)$-deformed Bose gas model related to compositeness of particles and their interaction: Temperature-dependence problem*. Phys. Rev. E **90**, 052147 (2014).
5. A. M. Gavrilik and Yu. A Mishchenko. *Composite Fermions as Deformed Oscillators: Wavefunctions and Entanglement*. Ukr. J. Phys. **64**(12), 1134 (2019).